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# Inertial Derivative-Free Projection Method for Nonlinear Monotone Operator Equations With Convex Constraints

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**ABSTRACT** In this paper, we propose an inertial derivative-free projection method for solving convex constrained nonlinear monotone operator equations (CNME). The method incorporates the inertial step with an existing method called derivative-free projection (DFPI) method for solving CNME. The reason is to improve the convergence speed of DFPI as it has been shown and reported in several works that indeed the inertial step can speed up convergence. The global convergence of the proposed method is proved under some mild assumptions. Finally, numerical results reported clearly show that the proposed method is more efficient than the DFPI.

**INDEX TERMS** Monotone nonlinear operator, inertial algorithm, conjugate gradient, projection method.

## I. INTRODUCTION

Consider the problem of finding $y \in \mathbf{E}$ such that

$$T(y) = 0, \tag{1}$$

where $T : \mathbf{R}^n \to \mathbf{R}^n$ is a monotone and Lipschitz continuous operator and $\mathbf{E}$ is a nonempty, closed and convex subset of $\mathbf{R}^n$. This problem has recently received remarkable attention as it arises in a number of applicable problems. For example, in constrained neural networks [1], nonlinear compressed sensing [2], [3], phase retrieval [4], [5], power flow equations [6], economic and chemical equilibrium problems [7], [8], non-negative matrix factorisation [9], [10], forecasting of financial market, portfolio selection models, price returns [11]–[13] and many more. As such, recently several derivative-free methods such as the conjugate gradient (CG) method have been proposed for solving problem (1). Given an initial point $y_0$, the conjugate gradient method computes the next iterate as:

$$y_{k+1} = y_k + \alpha_k d_k, \quad k = 0, 1, 2, \ldots,$$

where $\alpha_k > 0$ is a step size and $d_k$ is called the CG direction of search defined as

$$d_k := \begin{cases} -T(y_k) & \text{if } k = 0, \\ -T(y_k) + \beta_k d_{k-1} & \text{if } k > 0. \end{cases}$$

The parameter $\beta_k$ is called the CG parameter. For more on derivative-free methods for solving (1), interested readers can refer to [14]–[35] and references therein.

Recently, several researchers are interested in how to improve the speed of convergence of existing iterative algorithms. One of the approach in this regard is the inertial extrapolation method where a new step called the inertial step is added to the existing step(s) of an iterative method. It has been shown that the inertial step enhance the speed of the existing methods such as methods for solving fixed

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point problems, variational inequality problems, equilibrium problems, split feasibility problems, and so on. By choosing two starting points $y_{-1}$ and $y_0$, the inertial term is defined as

$$v_k = y_k + \theta_k(y_k - y_{k-1}),$$

where $\{\theta_k\}_{k=1}^{\infty}$ is a sequence satisfying certain condition. Inertial extrapolation method has been employed successfully in improving the convergence of the sequence generated by various algorithms. However, to the best of our knowledge, there is no theoretical proof to justify that, indeed, all one can find is numerical justification using some examples. However, the choice of the parameter $\theta_k$ has an effect on the speed of convergence. For more on iterative methods with inertial extrapolation, the reader is referred to [36]–[41] and references therein.

Inspired by the inertial methods [36]–[41] and the derivative-free projection method proposed by Sun and Liu [17] which is an extension of the work of Cheng [42], we propose an inertial derivative-free projection method for finding solutions to problem (1). The method is based on the work of Sun and Liu [17], where the inertial term is incorporated in order speed up its convergence. The remaining part of this paper is organized as follows: the next section gives some preliminaries and the proposed algorithm, convergence results is provided in the third section, Numerical results in the fourth section and lastly the conclusion.

**Notation.** Unless otherwise stated, the symbol $\|\cdot\|$ stands for Euclidean norm on $\mathbf{R}^n$.

## II. PROPOSED ALGORITHM

*Definition 2.1:* Let $\mathbf{R}^n$ be an Euclidean space and $T : \mathbf{R}^n \to \mathbf{R}^n$ be a mapping. Then $T$ is

(i) Monotone, if

$$(T(y) - T(x))^T(y - x) \geq 0, \quad \forall y, x \in \mathbf{R}^n.$$

(ii) $L$-Lipschitz continuous, if there exists $L > 0$ such that

$$\|T(y) - T(x)\| \leq L\|y - x\|, \quad \forall y, x \in \mathbf{R}^n.$$

*Definition 2.2:* Let $\mathbf{E} \subset \mathbf{R}^n$ be closed and convex, the projection of $y \in \mathbf{R}^n$ onto $\mathbf{E}$ denoted by $P_{\mathbf{E}}(y)$, is defined as

$$P_{\mathbf{E}}(y) = \arg\min\{\|x - y\| \,|x \in \mathbf{E}\}.$$

*Lemma 2.3 ([43]):* Let $\mathbf{E} \subset \mathbf{R}^n$ be nonempty closed and convex. Then the following inequality hold:

$$\|P_{\mathbf{E}}(y) - P_{\mathbf{E}}(x)\| \leq \|y - x\|, \ \forall y, x \in \mathbf{R}^n$$

.

*Lemma 2.4 ([44]):* Let $y, x \in \mathbf{R}^n$. Then the following equality hold:

$$\|y + x\|^2 = \|y\|^2 + 2\, x^T(y + x).$$

*Lemma 2.5 ([45]):* Let $\{y_k\}$ and $\{x_k\}$ be sequences of nonnegative real number satisfying the following relation

$$y_{k+1} \leq y_k + x_k,$$

where $\sum_{k=1}^{\infty} x_k < \infty$, then $\lim_{k\to\infty} y_k$ exists.

*Lemma 2.6 ([46]):* A point $y^* \in \mathbf{SOL}(\mathbf{T}, \mathbf{E})$ if and only if $y^* = P_{\mathbf{E}}(y^* - \mu u)$ for some $u = T(y^*)$ and $\mu > 0$.

We make use of the following assumptions.

*Assumption 1:*

(a) The feasible set $\mathbf{E}$ is a nonempty closed and convex subset of the Euclidean space $\mathbf{R}^n$.

(b) $T : \mathbf{R}^n \to \mathbf{R}^n$ is monotone and L-Lipschitz continuous.

(c) The solution set $\mathbf{SOL}(\mathbf{T}, \mathbf{E})$ of (1) is nonempty.

*Assumption 2:* Let $\{\theta_k\}$ be a sequence of nonnegative real numbers satisfying the conditions:

$$\theta_k \in (0, 1), \sum_{k=1}^{\infty} \theta_k \|y_k - y_{k-1}\| < \infty.$$

Based on the Sun and Liu [17] derivative-free projection method for monotone nonlinear equation with convex constraints called DFPI, we present an inertial derivative-free projection method for finding solutions to problem (1).

*Algorithm 2.7 (Inertial Derivative-Free Method (IDFPI):)*

**(S.0)** Choose a sequence $\{\theta_k\}_{k=1}^{\infty}$ satisfying Assumption 2 and select the parameters: $Tol > 0$, $\rho \in (0, 1)$, $\zeta > 0$, $\sigma > 0$. Select arbitrary points $y_{-1}, y_0 \in \mathbf{E}$. Set $k := 0$.

**(S.1)** Set

$$v_k = y_k + \theta_k(y_k - y_{k-1})$$

**(S.2)** Compute $T(v_k)$. If $\|T(v_k)\| \leq Tol$, stop. Otherwise, generate the search direction $d_k$ by

$$d_k := \begin{cases} -T(v_k) & \text{if } k = 0, \\ -\left(1 + \beta_k \dfrac{T(v_k)^T d_{k-1}}{\|T(v_k)\|^2}\right) & \\ T(v_k) + \beta_k d_{k-1} & \text{if } k > 0, \end{cases} \tag{2}$$

where,

$$\beta_k := 0.01 \frac{\|T(v_k)\|}{\|d_{k-1}\|}. \tag{3}$$

**(S.3)** Compute a trial point $x_k = v_k + \alpha_k d_k$.

**(S.4)** Determine the step-size $\alpha_k = \zeta\rho^i$ where $i$ is the least nonnegative integer satisfying

$$-T(v_k + \alpha_k d_k)^T d_k \geq \sigma\alpha_k\|d_k\|^2. \tag{4}$$

**(S.5)** If $x_k \in \mathbf{E}$ and $\|T(x_k)\| \leq Tol$, stop. Otherwise,

$$y_{k+1} = P_{\mathbf{E}}\left[v_k - \gamma_k T(x_k)\right], \tag{5}$$

where

$$\gamma_k := \frac{T(x_k)^T(v_k - x_k)}{\|T(x_k)\|^2}.$$

**(S.6)** Set $k = k + 1$, and go back to (S.1).

*Remark 2.8:* Let $d_k$ be generated by (2)-(3) in Algorithm 2.7. Then

$$T(v_k)^T d_k = -\|T(v_k)\|^2. \tag{6}$$

## III. CONVERGENCE RESULT

*Lemma 3.1:* The line search condition (4) is well-defined. That is, for all $k \geq 0$, there exists a non negative integer $i$ satisfying (4).

*Proof:* Suppose there is $k_0 \geq 0$ for which (4) is not true for any non-negative integer $i$, i.e.,

$$-T(v_{k_0} + \zeta\rho^i d_{k_0})^T d_{k_0} < \sigma\zeta\rho^i \|d_{k_0}\|^2.$$

Using Assumption 1 (b) and allowing $i \to \infty$, we have that

$$-T(v_{k_0})^T d_{k_0} \leq 0. \tag{7}$$

On the other hand, from (6),

$$-T(v_{k_0})^T d_{k_0} = \|T(v_{k_0})\|^2 > 0,$$

which contradicts (7). Hence, (4) is well defined. ∎

*Lemma 3.2:* Let $\{y_k\}$ and $\{x_k\}$ be generated via Algorithm 2.7. If $y^* \in \mathbf{SOL(T, E)}$, then under Assumption 1 and 2, it holds that

$$\|y_{k+1} - y^*\|^2 \leq \|v_k - y^*\| - \sigma^2\|v_k - x_k\|^4.$$

Moreover, the sequence $\{y_k\}$ and $\{x_k\}$ are bounded and

$$\lim_{k\to\infty} \|v_k - x_k\| = 0. \tag{8}$$

*Proof:* By the monotonicity of the mapping $T$, we have

$$\begin{aligned} T(x_k)^T(v_k - y^*) &= T(x_k)^T(v_k - x_k) + T(x_k)^T(x_k - y^*) \\ &\geq T(x_k)^T(v_k - x_k) + T(y^*)^T(x_k - y^*) \\ &= T(x_k)^T(v_k - x_k) && (9) \\ &= T(x_k)^T(-\alpha_k d_k) \\ &= \sigma\alpha_k^2\|d_k\|^2 \\ &\geq \sigma\|v_k - x_k\|^2. && (10) \end{aligned}$$

By Lemma 2.3 (iii), (5), (9) and (10), it holds that for any $y^* \in \mathbf{SOL(T, E)}$,

$$\begin{aligned} \|y_{k+1} - y^*\|^2 &= \|P_{\mathbf{E}}(v_k - \gamma_k T(x_k)) - y^*\|^2 \\ &\leq \|v_k - \gamma_k T(x_k) - y^*\|^2 \\ &= \|v_k - y^*\|^2 - 2\gamma_k T(x_k)^T(v_k - y^*) \\ &\quad + \gamma_k^2\|T(x_k)\|^2 \\ &\leq \|v_k - y^*\|^2 - 2\gamma_k T(x_k)^T(v_k - x_k) \\ &\quad + \gamma_k^2\|T(x_k)\|^2 \\ &\leq \|v_k - y^*\|^2 - \frac{T(x_k)^T(v_k - x_k)^2}{\|T(x_k)\|^2} \\ &\leq \|v_k - y^*\|^2 - \frac{\sigma^2\|v_k - x_k\|^4}{\|T(x_k)\|^2}. && (11) \end{aligned}$$

From equation (11), we can deduce that

$$\begin{aligned} \|y_{k+1} - y^*\| &\leq \|v_k - y^*\| \\ &= \|y_k + \theta_k(y_k - y_{k-1}) - y^*\| \\ &\leq \|y_k - y^*\| + \theta_k\|y_k - y_{k-1}\|. && (12) \end{aligned}$$

Because $\sum_{k=1}^{\infty} \theta_k\|y_k - y_{k-1}\| < \infty$, then by Lemma 2.5, the limit of $\{y_k - y^*\}$ exists and hence it is bounded. This implies that for all $k$, there exist $M_0 > 0$ such that $\|y_k - y^*\| \leq M_0$. Therefore, for all $k$ we can deduce that

$$\|y_k\| \leq M_1, \tag{13}$$

and

$$\|y_k - y_{k-1}\| \leq M,$$

where $M_1 = M_0 + \|y^*\|$ and $M = 2M_1$.

Using the above relations, we can have

$$\|v_k\| \leq M_2, \quad \|v_k - y^*\| \leq M_2, \text{ where } M_2 = 2M.$$

Since $H$ is Lipschitz continuous, we have

$$\|T(v_k)\| = \|T(v_k) - T(y^*)\| \leq L\|v_k - y^*\| \leq LM_2. \tag{14}$$

Also, using (14) and the monotonicity of $T$,

$$\begin{aligned} T(x_k)^T(v_k - x_k) &= (T(x_k) - T(v_k))^T(v_k - x_k) \\ &\quad + T(v_k)^T(v_k - x_k) \\ &\leq T(v_k)^T(v_k - x_k) \\ &\leq \|T(v_k)\|\|v_k - x_k\| \\ &\leq LM_2\|v_k - x_k\|. \end{aligned}$$

This together with (9) and (10) implies that

$$\|v_k - x_k\| \leq \frac{LM_2}{\sigma}.$$

Then, we have

$$\|x_k\| \leq \frac{LM_2}{\sigma} + \|v_k\|.$$

Hence the sequence $\{x_k\}$ is bounded since $\{v_k\}$ is bounded. Moreover as $T$ is continuous and $\{x_k\}$ is bounded, then $\{T(x_k)\}$ is bounded. That is, there exists $N > 0$ such that $\|T(x_k)\| \leq N$.

By the definition of $v_k$ and (13) we have

$$\begin{aligned} \|v_k - y^*\|^2 &= \|y_k + \theta_k(y_k - y_{k-1}) - y^*\|^2 \\ &= \|y_k - y^*\|^2 \\ &\quad + 2\theta_k(y_k - y_{k-1})^T(y_k + \theta_k(y_k - y_{k-1}) - y^*) \\ &\leq \|y_k - y^*\|^2 + 2\theta_k\|y_k - y_{k-1}\|(\|y_k - y^*\| \\ &\quad + \theta_k\|y_k - y_{k-1}\|) \\ &\leq \|y_k - y^*\|^2 + 2M\theta_k\|y_k - y_{k-1}\| \\ &\quad + 2M\theta_k\|y_k - y_{k-1}\| \\ &= \|y_k - y^*\|^2 + 4M\theta_k\|y_k - y_{k-1}\|. && (15) \end{aligned}$$

Combining (15) with (11), we have

$$\begin{aligned} \|y_{k+1} - y^*\|^2 \leq \|y_k - y^*\|^2 + 4M\theta_k\|y_k - y_{k-1}\| \\ - \frac{\sigma^2\|v_k - x_k\|^4}{\|T(x_k)\|^2}. && (16) \end{aligned}$$

Thus, we have

$$\frac{\sigma^2\|v_k - x_k\|^4}{\|T(x_k)\|^2} \le \|y_k - y^*\|^2 + 4M\theta_k\|y_k - y_{k-1}\| - \|y_{k+1} - y^*\|^2. \quad (17)$$

Adding (17) for $k = 0, 1, 2, \ldots$ and the fact that $\{T(x_k)\}$ is bounded, we have

$$\frac{\sigma^2}{N^2}\sum_{k=0}^{\infty}\|v_k - x_k\|^4 \le \sum_{k=0}^{\infty}(\|y_k - y^*\|^2 + 4M\theta_k\|y_k - y_{k-1}\| - \|y_{k+1} - y^*\|^2). \quad (18)$$

Now, let $S_k = \sum_{n=0}^{k}\left(\|y_n - y^*\|^2 - \|y_{n+1} - y^*\|^2\right)$, then $S_k = \sum_{n=0}^{k}\left(\|y_0 - y^*\|^2 - \|y_{k+1} - y^*\|^2\right)$. As limit of $\{\|y_k - y^*\|\}$ exists from (12) with limit say $L_1$, then

$$\left(\lim_{k\to\infty} S_k = \|y_0 - y^*\|^2 - L_1\right) \in \mathbf{R}.$$

So,

$$\sum_{k=0}^{\infty}\left(\|y_k - y^*\|^2 - \|y_{k+1} - y^*\|^2\right) < \infty$$

$$\text{and } \sum_{k=0}^{\infty}\theta_k\|y_k - y_{k-1}\| < \infty.$$

Using (18) together with the above inequalities, we conclude that

$$\lim_{k\to\infty}\|v_k - x_k\| = 0.$$

∎

*Remark 3.3:* By the definition of $\{x_k\}$ and (8), we have

$$\lim_{k\to\infty}\alpha_k\|d_k\| = 0.$$

*Lemma 3.4:* Suppose Assumptions 1-2 hold and the sequence $\{y_k\}$ and $\{v_k\}$ are generated by Algorithm 2.7. Then

$$\lim_{k\to\infty}\|v_k - y_{k+1}\| = 0. \quad (19)$$

*Proof:*
Using definition of $v_k$,

$$\|y_k - v_k\| = \|y_k - (y_k + \theta_k(y_k - y_{k-1}))\| = \theta_k\|y_k - y_{k-1}\|.$$

This implies that

$$\lim_{k\to\infty}\|y_k - v_k\| = 0. \quad (20)$$

Also,

$$\|y_k - x_k\| = \|y_k - v_k + v_k - x_k\| \le \|y_k - v_k\| + \|v_k - x_k\|.$$

Using (8) and (20), we have

$$\lim_{k\to\infty}\|y_k - x_k\| = 0. \quad (21)$$

**TABLE 1.** Starting points.

| SP | $y_{-1}$ | $y_0$ |
|---|---|---|
| $y_1$ | $(0.2, \ldots, 0.2)^T$ | $(0.1, \ldots, 0.1)^T$ |
| $y_2$ | $(0.2, \ldots, 0.2)^T$ | $(0.2, \ldots, 0.2)^T$ |
| $y_3$ | $(0.5, \ldots, 0.5)^T$ | $(0.5, \ldots, 0.5)^T$ |
| $y_4$ | $(1.2, \ldots, 1.2)^T$ | $(1.2, \ldots, 1.2)^T$ |
| $y_5$ | $(1.5, \ldots, 1.5)^T$ | $(1.5, \ldots, 1.5)^T$ |
| $y_6$ | $(2, \ldots, 2)^T$ | $(2, \ldots, 2)^T$ |
| $y_7$ | rand$(n, 1)$ | rand$(n, 1)$ |

**Note.** For DFPI algorithm [17], the starting point is $y_0$.

By Lemma 2.3, we have

$$\begin{aligned}\|y_{k+1} - y_k\| &= \|P_{\mathbf{E}}[v_k - \gamma_k T(x_k)] - y_k\| \\ &\le \|v_k - \gamma_k T(x_k) - y_k\| \\ &\le \|v_k - y_k\| + \|\gamma_k E(z_k)\| \\ &= \|v_k - y_k\| + \left\|\frac{T(x_k)^T(v_k - x_k)}{\|T(x_k)\|^2}T(x_k)\right\| \\ &\le \|v_k - y_k\| + \|v_k - x_k\|. \end{aligned} \quad (22)$$

Thus, from (8) and (20), we have

$$\lim_{k\to\infty}\|y_{k+1} - y_k\| = 0. \quad (23)$$

Therefore,

$$\begin{aligned}\|y_{k+1} - v_k\| &= \|y_{k+1} - (y_k + \theta_k(y_k - y_{k-1}))\| \\ &\le \|y_{k+1} - y_k\| + \theta_k\|y_k - y_{k-1}\|.\end{aligned}$$

Using (23) and Assumption 2, the desired equation is obtained.

∎

*Theorem 3.5:* Let $\{y_k\}$ be a sequence generated via Algorithm 2.7. Using Assumption 1 and 2, then $\{y_k\}$ converge to an element of $\mathbf{SOL(T, E)}$.

*Proof:* We know that the sequence $\{y_k\}$ is bounded from (13). This implies that there exists a subsequence $\{y_{k_j}\}$ of $\{y_k\}$ such that $\{y_{k_j}\}$ converge to some point $\bar{y}$. Also, we have that

$$\|v_{k_j} - y_{k_j}\| = \theta_{k_j}\|y_{k_j} - y_{k_j-1}\| \to 0, \text{ as } j \to \infty. \quad (24)$$

*Claim:* $\bar{y} \in \mathbf{SOL(T, E)}$. Suppose on the contrary that $\bar{y} \notin \mathbf{SOL(T, E)}$. Then from (19) and (24), we have that

$$\lim_{j\to\infty} y_{k_j+1} = \lim_{j\to\infty} P_{\mathbf{E}}\left(v_{k_j} - \gamma_{k_j}T(x_{k_j})\right) = \lim_{j\to\infty} y_{k_j} = \bar{y}. \quad (25)$$

Without loss of generality, if $\gamma_{k_j} \to \gamma^*$ and $T(x_{k_j}) \to T(x^*)$. Then since $T$ is continuous, we have $T(x^*) = T(\bar{y})$. Therefore, from (25)

$$P_{\mathbf{E}}\left(\bar{y} - \gamma^*T(x^*)\right) = \bar{y}.$$

It then follows from Lemma 2.6 that $y^* \in \mathbf{SOL(T, E)}$, which is a contradiction. Hence, our claim holds. Substituting $y^*$ with $\bar{y}$ in (12), it is easy to see that $\lim_{k\to\infty}\|y_k - \bar{y}\|$ exists

**TABLE 2.** Numerical experiments with different coefficients of $\beta_k$ for problem 1-5 with n = 1000.

| | | 0.01 | | | 0.5 | | | 1.5 | | | 2 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **n=1000** | **SP** | **NOI** | **NFE** | **CPUT** | **NOI** | **NFE** | **CPUT** | **NOI** | **NFE** | **CPUT** | **NOI** | **NFE** | **CPUT** |
| Problem 1 | $y_1$ | 9 | 36 | 0.00568 | 36 | 144 | 0.021784 | 96 | 384 | 0.044784 | 188 | 752 | 0.10513 |
| | $y_2$ | 10 | 40 | 0.003477 | 25 | 100 | 0.011812 | 126 | 504 | 0.059805 | 282 | 1128 | 0.11457 |
| | $y_3$ | 6 | 24 | 0.003472 | 28 | 112 | 0.010859 | 84 | 336 | 0.033187 | 222 | 888 | 0.081797 |
| | $y_4$ | 7 | 28 | 0.003033 | 26 | 104 | 0.008285 | 87 | 348 | 0.033013 | 319 | 1276 | 0.1232 |
| | $y_5$ | 7 | 28 | 0.00311 | 18 | 72 | 0.008741 | 102 | 408 | 0.049486 | 91 | 364 | 0.043828 |
| | $y_6$ | 7 | 28 | 0.002948 | 28 | 112 | 0.010702 | 114 | 456 | 0.045376 | 175 | 700 | 0.094839 |
| | $y_7$ | 19 | 76 | 0.006554 | 26 | 104 | 0.010181 | 191 | 764 | 0.077315 | - | - | - |
| Problem 2 | $y_1$ | 11 | 43 | 0.069731 | 11 | 43 | 0.004505 | 11 | 43 | 0.004516 | 11 | 43 | 0.005377 |
| | $y_2$ | 13 | 51 | 0.004297 | 13 | 51 | 0.004481 | 13 | 51 | 0.00571 | 13 | 51 | 0.004752 |
| | $y_3$ | 12 | 46 | 0.005032 | 12 | 46 | 0.004509 | 12 | 46 | 0.004683 | 12 | 46 | 0.00444 |
| | $y_4$ | 13 | 49 | 0.005324 | 13 | 49 | 0.006814 | 13 | 49 | 0.004967 | 13 | 49 | 0.005646 |
| | $y_5$ | 14 | 53 | 0.005046 | 14 | 53 | 0.005374 | 14 | 53 | 0.005185 | 14 | 53 | 0.00466 |
| | $y_6$ | 12 | 44 | 0.004429 | 12 | 44 | 0.006134 | 12 | 44 | 0.004458 | 12 | 44 | 0.004317 |
| | $y_7$ | 14 | 53 | 0.00526 | 25 | 95 | 0.008445 | 179 | 713 | 0.090395 | | | |
| Problem 3 | $y_1$ | 13 | 52 | 0.063903 | 13 | 52 | 0.004482 | 13 | 52 | 0.00409 | 13 | 52 | 0.004058 |
| | $y_2$ | 13 | 52 | 0.004457 | 13 | 52 | 0.004565 | 13 | 52 | 0.004368 | 13 | 52 | 0.003968 |
| | $y_3$ | 14 | 56 | 0.004882 | 14 | 56 | 0.004492 | 14 | 56 | 0.003748 | 14 | 56 | 0.004461 |
| | $y_4$ | 14 | 56 | 0.005488 | 14 | 56 | 0.004633 | 14 | 56 | 0.004022 | 14 | 56 | 0.005487 |
| | $y_5$ | 14 | 56 | 0.004491 | 14 | 56 | 0.005229 | 14 | 56 | 0.004916 | 14 | 56 | 0.004376 |
| | $y_6$ | 15 | 60 | 0.005917 | 15 | 60 | 0.005052 | 15 | 60 | 0.003972 | 15 | 60 | 0.004675 |
| | $y_7$ | 14 | 56 | 0.004507 | 114 | 456 | 0.039936 | - | - | - | - | - | - |
| Problem 4 | $y_1$ | 13 | 52 | 0.016113 | 13 | 52 | 0.004009 | 13 | 52 | 0.005635 | 13 | 52 | 0.004117 |
| | $y_2$ | 13 | 52 | 0.00361 | 13 | 52 | 0.003692 | 13 | 52 | 0.003629 | 13 | 52 | 0.003991 |
| | $y_3$ | 13 | 52 | 0.004013 | 13 | 52 | 0.004321 | 13 | 52 | 0.004246 | 13 | 52 | 0.003573 |
| | $y_4$ | 13 | 52 | 0.003393 | 13 | 52 | 0.003838 | 13 | 52 | 0.004084 | 13 | 52 | 0.003938 |
| | $y_5$ | 14 | 56 | 0.005 | 14 | 56 | 0.006158 | 14 | 56 | 0.004706 | 14 | 56 | 0.00564 |
| | $y_6$ | 14 | 56 | 0.00404 | 14 | 56 | 0.005038 | 14 | 56 | 0.003816 | 14 | 56 | 0.003763 |
| | $y_7$ | 15 | 60 | 0.004885 | 23 | 92 | 0.006901 | 153 | 612 | 0.040723 | - | - | - |
| Problem 5 | $y_1$ | 22 | 83 | 0.02265 | 25 | 93 | 0.006098 | 185 | 734 | 0.053499 | - | - | - |
| | $y_2$ | 18 | 68 | 0.005 | 26 | 98 | 0.006317 | 144 | 567 | 0.039396 | - | - | - |
| | $y_3$ | 22 | 86 | 0.006487 | 25 | 98 | 0.007825 | 176 | 696 | 0.05854 | - | - | - |
| | $y_4$ | 20 | 80 | 0.006677 | 27 | 108 | 0.007371 | 142 | 568 | 0.036509 | - | - | - |
| | $y_5$ | 23 | 92 | 0.00758 | 28 | 112 | 0.007031 | 159 | 636 | 0.042945 | - | - | - |
| | $y_6$ | 33 | 132 | 0.015034 | 26 | 104 | 0.007602 | 142 | 568 | 0.038659 | - | - | - |
| | $y_7$ | 39 | 155 | 0.014079 | 34 | 134 | 0.011338 | 161 | 639 | 0.045031 | - | - | - |

by Lemma 2.5. Since $\bar{y}$ is an accumulation point of $\{y_k\}$, we obtain that $\{y_k\}$ converges to $\bar{y}$.

■

## IV. NUMERICAL EXAMPLES

By comparing the proposed inertial algorithm (Iner. DFPI) to the DFPI algorithm in [17], we show the numerical efficiency and computational advantage of the proposed inertial algorithm (Iner. DFPI) in this section. The MATLAB implementation of the algorithms was executed on a Windows 10 computer with Intel(R) Core(TM) i7 processor with 8.0GB of RAM and CPU of 2.30GHz using MATLAB R2019b software. The numerical experiment made use of the following test problems to measure the efficiency and robustness of the proposed inertial algorithm (Iner. DFPI).

*Problem 1*: Modified exponential function [47]

$$t_1(y) = e^{y_1} - 1$$
$$t_i(y) = e^{y_i} + y_i - 1, \quad i = 2, \dots, n,$$
$$\mathbf{E} = \mathbf{R}^n_+.$$

*Problem 2*: Logarithmic function [47]

$$t_i(y_i) = \log(y_i + 1) - \frac{y_i}{n}, \quad i = 1, 2, \dots, n,$$
$$\mathbf{E} = \mathbf{R}^n_+.$$

*Problem 3*: Nonsmooth function [48]

$$t_i(y) = 2y_i - \sin(|y_i|), \text{ for } i = 1, 2, \dots, n,$$
$$\mathbf{E} = \left\{ y \in \mathbf{R}^n_+ : y \geq 0, \quad \sum_{i=1}^{n} y_i \leq n \right\}.$$

*Problem 4*: Strictly convex function I [47]

$$t_i(y) = e^{y_i} - 1, \quad i = 1, 2, \dots, n,$$
$$\mathbf{E} = \mathcal{R}^n_+.$$

*Problem 5*: Strictly convex function II [47]

$$t_i(y) = \left(\frac{i}{n}\right) e^{y_i} - 1, \quad i = 1, 2, \dots, n,$$
$$\mathbf{E} = \mathbf{R}^n_+.$$

**TABLE 3.** Numerical experiments with different coefficients of $\beta_k$ for problem 6-10 with n = 1000.

| | | | 0.01 | | | 0.5 | | | 1.5 | | | 2 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| n=1000 | SP | NOI | NFE | CPUT | NOI | NFE | CPUT | NOI | NFE | CPUT | NOI | NFE | CPUT |
| Problem 6 | $y_1$ | 16 | 64 | 0.028171 | 16 | 64 | 0.008079 | 74 | 296 | 0.039752 | 530 | 2120 | 0.27322 |
| | $y_2$ | 16 | 64 | 0.006196 | 16 | 64 | 0.007641 | 73 | 292 | 0.038947 | 529 | 2116 | 0.26621 |
| | $y_3$ | 16 | 64 | 0.007024 | 16 | 64 | 0.007142 | 74 | 296 | 0.03997 | 524 | 2096 | 0.24974 |
| | $y_4$ | 15 | 60 | 0.007648 | 15 | 60 | 0.007185 | 72 | 288 | 0.031371 | 510 | 2040 | 0.2355 |
| | $y_5$ | 15 | 60 | 0.007387 | 15 | 60 | 0.007385 | 70 | 280 | 0.03291 | 496 | 1984 | 0.24424 |
| | $y_6$ | 15 | 60 | 0.0087 | 15 | 60 | 0.007901 | 65 | 260 | 0.031665 | 467 | 1868 | 0.2211 |
| | $y_7$ | 16 | 64 | 0.007314 | 16 | 64 | 0.007423 | 43 | 172 | 0.020312 | 489 | 1956 | 0.22663 |
| Problem 7 | $y_1$ | 8 | 32 | 0.024297 | 8 | 32 | 0.00364 | 8 | 32 | 0.004126 | 8 | 32 | 0.00538 |
| | $y_2$ | 7 | 28 | 0.00333 | 7 | 28 | 0.002719 | 7 | 28 | 0.002391 | 7 | 28 | 0.003311 |
| | $y_3$ | 6 | 24 | 0.01827 | 6 | 24 | 0.002887 | 6 | 24 | 0.0024 | 6 | 24 | 0.003263 |
| | $y_4$ | 7 | 28 | 0.004197 | 7 | 28 | 0.002811 | 7 | 28 | 0.003058 | 7 | 28 | 0.003245 |
| | $y_5$ | 8 | 32 | 0.004041 | 8 | 32 | 0.004093 | 8 | 32 | 0.00396 | 8 | 32 | 0.003798 |
| | $y_6$ | 8 | 31 | 0.003253 | 8 | 31 | 0.003851 | 8 | 31 | 0.003888 | 8 | 31 | 0.003131 |
| | $y_7$ | 9 | 36 | 0.004528 | 21 | 84 | 0.007788 | 456 | 1824 | 0.16334 | 410 | 1640 | 0.3094 |
| Problem 8 | $y_1$ | 9 | 32 | 0.018072 | 9 | 32 | 0.004243 | 9 | 32 | 0.004036 | 9 | 32 | 0.004227 |
| | $y_2$ | 9 | 32 | 0.003068 | 9 | 32 | 0.003102 | 9 | 32 | 0.003234 | 9 | 32 | 0.003993 |
| | $y_3$ | 9 | 32 | 0.003516 | 9 | 32 | 0.0029 | 9 | 32 | 0.00361 | 9 | 32 | 0.003437 |
| | $y_4$ | 9 | 32 | 0.00303 | 9 | 32 | 0.003161 | 9 | 32 | 0.00347 | 9 | 32 | 0.003267 |
| | $y_5$ | 9 | 32 | 0.003866 | 9 | 32 | 0.003489 | 9 | 32 | 0.003431 | 9 | 32 | 0.005727 |
| | $y_6$ | 9 | 32 | 0.003614 | 9 | 32 | 0.002902 | 9 | 32 | 0.00275 | 9 | 32 | 0.004086 |
| | $y_7$ | 87 | 317 | 0.020828 | - | - | - | - | - | - | - | - | - |
| Problem 9 | $y_1$ | 5 | 20 | 0.016153 | 5 | 20 | 0.001863 | 5 | 20 | 0.001487 | 5 | 20 | 0.002629 |
| | $y_2$ | 5 | 20 | 0.001592 | 5 | 20 | 0.00164 | 5 | 20 | 0.001465 | 5 | 20 | 0.001785 |
| | $y_3$ | 5 | 20 | 0.002472 | 5 | 20 | 0.001823 | 5 | 20 | 0.001626 | 5 | 20 | 0.002922 |
| | $y_4$ | 6 | 24 | 0.001959 | 6 | 24 | 0.001971 | 6 | 24 | 0.002312 | 6 | 24 | 0.002833 |
| | $y_5$ | 6 | 24 | 0.002435 | 6 | 24 | 0.002336 | 6 | 24 | 0.002423 | 6 | 24 | 0.003877 |
| | $y_6$ | 6 | 24 | 0.001855 | 6 | 24 | 0.001686 | 6 | 24 | 0.00168 | 6 | 24 | 0.00189 |
| | $y_7$ | 5 | 20 | 0.001481 | 5 | 20 | 0.002654 | 5 | 20 | 0.001523 | 5 | 20 | 0.001975 |
| Problem 10 | $y_1$ | 9 | 36 | 0.021723 | 9 | 36 | 0.006713 | 9 | 36 | 0.006291 | 9 | 36 | 0.010543 |
| | $y_2$ | 10 | 40 | 0.00665 | 10 | 40 | 0.006561 | 10 | 40 | 0.007961 | 10 | 40 | 0.009544 |
| | $y_3$ | 1 | 3 | 0.000977 | 1 | 3 | 0.001388 | 1 | 3 | 0.00115 | 1 | 3 | 0.002128 |
| | $y_4$ | 1 | 4 | 0.001237 | 1 | 4 | 0.001396 | 1 | 4 | 0.00149 | 1 | 4 | 0.001539 |
| | $y_5$ | 1 | 3 | 0.001488 | 1 | 3 | 0.001946 | 1 | 3 | 0.001396 | 1 | 3 | 0.002452 |
| | $y_6$ | 1 | 4 | 0.001899 | 1 | 4 | 0.002218 | 1 | 4 | 0.001877 | 1 | 4 | 0.002951 |
| | $y_7$ | 11 | 44 | 0.007231 | 45 | 180 | 0.028201 | 447 | 1788 | 0.24792 | 405 | 1620 | 0.67977 |

*Problem 6*: Tridiagonal exponential function [47]

$$
\begin{aligned}
t_1(y) &= y_1 - e^{\cos(l(y_1+y_2))}\\
t_i(y) &= y_i - e^{\cos(l(y_{i-1}+y_i+y_{i+1}))}, \quad i = 2, \dots, n-1,\\
t_n(y) &= y_n - e^{\cos(l(y_{n-1}+y_n))},\\
l &= \frac{1}{n+1} \text{ and } \mathbf{E} = \mathbf{R}^n_+.
\end{aligned}
$$

*Problem 7*: Nonsmooth function II [49]

$$
\begin{aligned}
t_i(y) &= y_i - \sin(|y_i - 1|), \text{ for } i = 1, 2, \dots, n,\\
\mathbf{E} &= \left\{ y \in \mathbf{R}^n_+ : y \geq -1, \sum_{i=1}^{n} y_i \leq n \right\}.
\end{aligned}
$$

*Problem 8*: Penalty function I [16]

$$
\begin{aligned}
\xi_i &= \sum_{i=1}^{n} y_i^2, \quad c = 10^{-5},\\
t_i(y) &= 2c(y_i - 1) + 4(\xi_i - 0.25)y_i, \quad i = 1, 2, \dots, n,\\
\mathbf{E} &= \mathbf{R}^n_+.
\end{aligned}
$$

*Problem 9*: Pursuit-Evasion problem [16]

$$
\begin{aligned}
t_i(y) &= 8^{0.5} y_i - 1, \quad i = 1, 2, \dots, n,\\
\mathbf{E} &= \mathbf{R}^n_+.
\end{aligned}
$$

*Problem 10*: Pursuit-Evasion problem [16]

$$
\begin{aligned}
t_i(y) &= e^{y_i^2} + 3 \sin y_i \cos y_i - 1, \quad i = 1, 2, \dots, n,\\
\mathbf{E} &= \mathbf{R}^n_+.
\end{aligned}
$$

Note that, the mapping $T$ is taken as

$$
T(y) = (t_1(y), t_2(y), \dots, t_n(y))^T,
$$

and recall that, the inertial-type algorithm is an iterative procedure in which subsequent iterates are obtained using the preceding two iterates. As such, for the proposed inertial algorithm (Iner. DFPI), the two preceding iterates used in obtaining the initial iterates are as follows:

**Note.** For DFPI algorithm [17], the starting point is $y_0$.

The above listed problems are solved with dimensions $n = 1000$, $5000$, $10,000$, $50,000$ and $100,000$. The parameters

**TABLE 4.** Numerical experiments with different sequences $\{\theta_k\}$ for problem 1-5 with n = 1000.

| | | $\theta_k = \frac{1}{(2k+5)^2}$ | | | $\theta_k = \frac{1}{\exp(k+1)^{k+1}}$ | | | $\theta_k = \frac{1}{(k+1)^2}$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $n = 1000$ | SP | NOI | NFE | CPUT | NOI | NFE | CPUT | NOI | NFE | CPUT |
| Problem 1 | $y_1$ | 9 | 36 | 0.009119 | 9 | 36 | 0.019343 | 0 | 0 | 0.003002 |
| | $y_2$ | 10 | 40 | 0.009186 | 10 | 40 | 0.013494 | 10 | 40 | 0.024327 |
| | $y_3$ | 6 | 24 | 0.007353 | 6 | 24 | 0.006644 | 6 | 24 | 0.016818 |
| | $y_4$ | 7 | 28 | 0.008871 | 7 | 28 | 0.009398 | 7 | 28 | 0.017854 |
| | $y_5$ | 7 | 28 | 0.003838 | 7 | 28 | 0.011266 | 7 | 28 | 0.011046 |
| | $y_6$ | 7 | 28 | 0.008816 | 7 | 28 | 0.008429 | 7 | 28 | 0.009511 |
| | $y_7$ | 19 | 76 | 0.020365 | 19 | 76 | 0.012149 | 20 | 80 | 0.01573 |
| Problem 2 | $y_1$ | 11 | 43 | 0.062537 | 11 | 43 | 0.011218 | 0 | 0 | 0.002089 |
| | $y_2$ | 13 | 51 | 0.009315 | 13 | 51 | 0.00942 | 13 | 51 | 0.013417 |
| | $y_3$ | 12 | 46 | 0.011573 | 12 | 46 | 0.008909 | 12 | 46 | 0.012356 |
| | $y_4$ | 13 | 49 | 0.011216 | 13 | 49 | 0.009268 | 13 | 49 | 0.016215 |
| | $y_5$ | 14 | 53 | 0.010668 | 14 | 53 | 0.029236 | 14 | 53 | 0.016043 |
| | $y_6$ | 12 | 44 | 0.008497 | 12 | 44 | 0.00864 | 12 | 44 | 0.013048 |
| | $y_7$ | 14 | 53 | 0.010962 | 14 | 53 | 0.01112 | 14 | 53 | 0.01145 |
| Problem 3 | $y_1$ | 13 | 52 | 0.081519 | 13 | 52 | 0.009919 | 0 | 0 | 0.001735 |
| | $y_2$ | 13 | 52 | 0.010098 | 13 | 52 | 0.009571 | 13 | 52 | 0.008503 |
| | $y_3$ | 14 | 56 | 0.009522 | 14 | 56 | 0.010984 | 14 | 56 | 0.011609 |
| | $y_4$ | 14 | 56 | 0.012593 | 14 | 56 | 0.013154 | 14 | 56 | 0.013062 |
| | $y_5$ | 14 | 56 | 0.010016 | 14 | 56 | 0.011638 | 14 | 56 | 0.010194 |
| | $y_6$ | 15 | 60 | 0.0169 | 15 | 60 | 0.011157 | 15 | 60 | 0.018817 |
| | $y_7$ | 14 | 56 | 0.015987 | 14 | 56 | 0.006418 | 14 | 56 | 0.013272 |
| Problem 4 | $y_1$ | 13 | 52 | 0.027212 | 12 | 48 | 0.007644 | 0 | 0 | 0.002682 |
| | $y_2$ | 13 | 52 | 0.008549 | 13 | 52 | 0.006964 | 13 | 52 | 0.00497 |
| | $y_3$ | 13 | 52 | 0.011045 | 13 | 52 | 0.006538 | 13 | 52 | 0.011129 |
| | $y_4$ | 13 | 52 | 0.008611 | 13 | 52 | 0.010887 | 13 | 52 | 0.008666 |
| | $y_5$ | 14 | 56 | 0.00786 | 14 | 56 | 0.009174 | 14 | 56 | 0.015178 |
| | $y_6$ | 14 | 56 | 0.008609 | 14 | 56 | 0.006611 | 14 | 56 | 0.00971 |
| | $y_7$ | 15 | 60 | 0.009977 | 15 | 60 | 0.009921 | 15 | 60 | 0.007715 |
| Problem 5 | $y_1$ | 22 | 83 | 0.014915 | 22 | 83 | 0.012003 | 23 | 87 | 0.008566 |
| | $y_2$ | 18 | 68 | 0.010097 | 18 | 68 | 0.011021 | 18 | 68 | 0.011365 |
| | $y_3$ | 22 | 86 | 0.009015 | 22 | 86 | 0.012862 | 22 | 86 | 0.009067 |
| | $y_4$ | 20 | 80 | 0.015665 | 20 | 80 | 0.017248 | 20 | 80 | 0.00942 |
| | $y_5$ | 23 | 92 | 0.01444 | 23 | 92 | 0.022297 | 23 | 92 | 0.012842 |
| | $y_6$ | 33 | 132 | 0.032199 | 33 | 132 | 0.038352 | 33 | 132 | 0.025196 |
| | $y_7$ | 32 | 127 | 0.020831 | 38 | 150 | 0.038326 | 36 | 143 | 0.026164 |

**TABLE 5.** Numerical experiments with different sequences $\{\theta_k\}$ for problem 6-10 with n = 1000.

| | | $\theta_k = \frac{1}{(2k+5)^2}$ | | | $\theta_k = \frac{1}{\exp(k+1)^{k+1}}$ | | | $\theta_k = \frac{1}{(k+1)^2}$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $n = 1000$ | SP | NOI | NFE | CPUT | NOI | NFE | CPUT | NOI | NFE | CPUT |
| Problem 6 | $y_1$ | 16 | 64 | 0.080299 | 16 | 64 | 0.017491 | 16 | 64 | 0.016929 |
| | $y_2$ | 16 | 64 | 0.013439 | 16 | 64 | 0.011797 | 16 | 64 | 0.011792 |
| | $y_3$ | 16 | 64 | 0.018681 | 16 | 64 | 0.012646 | 16 | 64 | 0.017659 |
| | $y_4$ | 15 | 60 | 0.014368 | 15 | 60 | 0.013669 | 15 | 60 | 0.016687 |
| | $y_5$ | 15 | 60 | 0.013089 | 15 | 60 | 0.017483 | 15 | 60 | 0.014834 |
| | $y_6$ | 15 | 60 | 0.014044 | 15 | 60 | 0.016498 | 15 | 60 | 0.024312 |
| | $y_7$ | 16 | 64 | 0.012709 | 16 | 64 | 0.021474 | 16 | 64 | 0.013924 |
| Problem 7 | $y_1$ | 8 | 32 | 0.032771 | 8 | 32 | 0.006973 | 8 | 32 | 0.013911 |
| | $y_2$ | 7 | 28 | 0.005828 | 7 | 28 | 0.008311 | 7 | 28 | 0.008 |
| | $y_3$ | 6 | 24 | 0.037642 | 6 | 24 | 0.005015 | 6 | 24 | 0.00695 |
| | $y_4$ | 7 | 28 | 0.006844 | 7 | 28 | 0.008496 | 7 | 28 | 0.011212 |
| | $y_5$ | 8 | 32 | 0.009869 | 8 | 32 | 0.006206 | 8 | 32 | 0.008115 |
| | $y_6$ | 8 | 31 | 0.014343 | 8 | 31 | 0.011633 | 8 | 31 | 0.014518 |
| | $y_7$ | 9 | 36 | 0.007226 | 9 | 36 | 0.006186 | 9 | 36 | 0.006638 |
| Problem 8 | $y_1$ | 10 | 34 | 0.03011 | 10 | 34 | 0.004109 | 9 | 31 | 0.010335 |
| | $y_2$ | 10 | 34 | 0.00805 | 10 | 34 | 0.004493 | 10 | 34 | 0.005206 |
| | $y_3$ | 10 | 34 | 0.00437 | 10 | 34 | 0.005042 | 10 | 34 | 0.00788 |
| | $y_4$ | 10 | 34 | 0.009151 | 10 | 34 | 0.007257 | 10 | 34 | 0.003722 |
| | $y_5$ | 10 | 34 | 0.008475 | 10 | 34 | 0.006482 | 10 | 34 | 0.00584 |
| | $y_6$ | 10 | 34 | 0.006564 | 10 | 34 | 0.005847 | 10 | 34 | 0.005513 |
| | $y_7$ | 12 | 39 | 0.010024 | 12 | 39 | 0.006773 | 13 | 42 | 0.008748 |
| Problem 9 | $y_1$ | 5 | 20 | 0.039191 | 5 | 20 | 0.007925 | 5 | 20 | 0.009406 |
| | $y_2$ | 5 | 20 | 0.004174 | 5 | 20 | 0.005295 | 5 | 20 | 0.005742 |
| | $y_3$ | 5 | 20 | 0.002224 | 5 | 20 | 0.003368 | 5 | 20 | 0.007404 |
| | $y_4$ | 6 | 24 | 0.007065 | 6 | 24 | 0.014478 | 6 | 24 | 0.007671 |
| | $y_5$ | 6 | 24 | 0.003812 | 6 | 24 | 0.004744 | 6 | 24 | 0.007709 |
| | $y_6$ | 6 | 24 | 0.002333 | 6 | 24 | 0.002989 | 6 | 24 | 0.004279 |
| | $y_7$ | 5 | 20 | 0.008178 | 5 | 20 | 0.006976 | 5 | 20 | 0.006137 |
| Problem 10 | $y_1$ | 9 | 36 | 0.089549 | 9 | 36 | 0.016183 | 0 | 0 | 0.00387 |
| | $y_2$ | 10 | 40 | 0.019257 | 10 | 40 | 0.018461 | 10 | 40 | 0.0118 |
| | $y_3$ | 1 | 3 | 0.002148 | 1 | 3 | 0.003585 | 1 | 3 | 0.004648 |
| | $y_4$ | 1 | 4 | 0.005166 | 1 | 4 | 0.00363 | 1 | 4 | 0.002651 |
| | $y_5$ | 1 | 3 | 0.004204 | 1 | 3 | 0.003815 | 1 | 3 | 0.004593 |
| | $y_6$ | 1 | 4 | 0.009407 | 1 | 4 | 0.0036 | 1 | 4 | 0.00679 |
| | $y_7$ | 11 | 44 | 0.024152 | 11 | 44 | 0.019788 | 11 | 44 | 0.021147 |

$\theta_k = \frac{1}{(2k+5)^2}$ $\zeta = 1$, $\rho = 0.7$, $\sigma = 0.01$ were chosen for the Iner. DFPI algorithm to obtain the best possible results.

**TABLE 6.** Numerical results for IDFPI and DFPI algorithms on problem 1.

| | | IDFPI | | | | DFPI | | | |
|---|---|---|---|---|---|---|---|---|---|
| n | SP | NOI | NFE | CPUT | LNORM | NOI | NFE | CPUT | LNORM |
| 1000 | $y_1$ | 9 | 36 | 0.017162 | 6.68E-07 | 76 | 403 | 0.024481 | 7.21E-07 |
| | $y_2$ | 10 | 40 | 0.012476 | 5.19E-07 | 74 | 394 | 0.024509 | 6.86E-07 |
| | $y_3$ | 6 | 24 | 0.004145 | 2.76E-07 | 76 | 405 | 0.024232 | 7.26E-07 |
| | $y_4$ | 7 | 28 | 0.007213 | 4.53E-07 | 76 | 406 | 0.026091 | 5.12E-07 |
| | $y_5$ | 7 | 28 | 0.009987 | 2.98E-07 | 49 | 270 | 0.019812 | 8.46E-07 |
| | $y_6$ | 7 | 28 | 0.005634 | 2.47E-07 | 90 | 478 | 0.036841 | 5.10E-07 |
| | $y_7$ | 19 | 76 | 0.009093 | 8.82E-07 | 75 | 400 | 0.024059 | 7.17E-07 |
| 5000 | $y_1$ | 7 | 28 | 0.010122 | 7.08E-07 | 82 | 434 | 0.10282 | 8.03E-07 |
| | $y_2$ | 8 | 32 | 0.01214 | 7.91E-07 | 72 | 384 | 0.079772 | 1.37E-10 |
| | $y_3$ | 6 | 24 | 0.010519 | 4.74E-07 | 87 | 461 | 0.09622 | 8.07E-07 |
| | $y_4$ | 7 | 28 | 0.011582 | 5.70E-07 | 73 | 392 | 0.082718 | 5.71E-07 |
| | $y_5$ | 7 | 28 | 0.010631 | 4.47E-07 | 58 | 316 | 0.069378 | 9.41E-07 |
| | $y_6$ | 7 | 28 | 0.012023 | 4.64E-07 | 88 | 469 | 0.1209 | 5.67E-07 |
| | $y_7$ | 21 | 84 | 0.028841 | 4.15E-07 | 78 | 416 | 0.098002 | 7.90E-07 |
| 10000 | $y_1$ | 7 | 28 | 0.019183 | 4.89E-07 | 81 | 429 | 0.17261 | 1.93E-14 |
| | $y_2$ | 7 | 28 | 0.016653 | 8.52E-07 | 83 | 441 | 0.18057 | 5.39E-07 |
| | $y_3$ | 6 | 24 | 0.014861 | 6.41E-07 | 89 | 472 | 0.21044 | 5.70E-07 |
| | $y_4$ | 7 | 28 | 0.017194 | 6.94E-07 | 70 | 377 | 0.17188 | 8.07E-07 |
| | $y_5$ | 7 | 28 | 0.023033 | 5.83E-07 | 90 | 479 | 0.19026 | 7.81E-07 |
| | $y_6$ | 7 | 28 | 0.021992 | 6.39E-07 | 89 | 476 | 0.22891 | 7.77E-15 |
| | $y_7$ | 21 | 84 | 0.054409 | 5.30E-07 | 93 | 492 | 0.19355 | 5.62E-07 |
| 50000 | $y_1$ | 6 | 24 | 0.067682 | 2.67E-07 | 75 | 399 | 0.68515 | 2.15E-14 |
| | $y_2$ | 6 | 24 | 0.054711 | 2.58E-07 | 75 | 399 | 0.69894 | 1.18E-13 |
| | $y_3$ | 7 | 28 | 0.12306 | 1.03E-07 | 75 | 400 | 0.67299 | 1.25E-13 |
| | $y_4$ | 8 | 32 | 0.094062 | 2.09E-07 | 72 | 389 | 0.96128 | 3.20E-10 |
| | $y_5$ | 8 | 32 | 0.074329 | 1.26E-07 | 82 | 445 | 0.80865 | 6.53E-07 |
| | $y_6$ | 8 | 32 | 0.10103 | 8.10E-08 | 84 | 459 | 0.86014 | 8.22E-15 |
| | $y_7$ | 22 | 88 | 0.25757 | 5.05E-07 | 90 | 478 | 1.0839 | 6.27E-07 |
| 100000 | $y_1$ | 6 | 24 | 0.12284 | 1.69E-07 | 73 | 389 | 1.6516 | 6.06E-14 |
| | $y_2$ | 6 | 24 | 0.10121 | 1.47E-07 | 81 | 431 | 1.4517 | 9.10E-15 |
| | $y_3$ | 7 | 28 | 0.11899 | 1.07E-07 | 82 | 438 | 1.3592 | 8.99E-07 |
| | $y_4$ | 8 | 32 | 0.31637 | 2.11E-07 | 79 | 430 | 1.2955 | 1.80E-13 |
| | $y_5$ | 8 | 32 | 0.17021 | 1.29E-07 | 87 | 475 | 3.2752 | 8.66E-15 |
| | $y_6$ | 8 | 32 | 0.17119 | 8.56E-08 | 86 | 480 | 1.5122 | 8.04E-07 |
| | $y_7$ | 22 | 88 | 0.60812 | 7.15E-07 | 91 | 485 | 1.4229 | 7.04E-07 |

**TABLE 7.** Numerical results for IDFPI and DFPI algorithms on problem 2.

| | | IDFPI | | | | DFPI | | | |
|---|---|---|---|---|---|---|---|---|---|
| n | SP | NOI | NFE | CPUT | LNORM | NOI | NFE | CPUT | LNORM |
| 1000 | $y_1$ | 11 | 43 | 0.005001 | 8.52E-07 | 3 | 9 | 0.058465 | 5.17E-07 |
| | $y_2$ | 13 | 51 | 0.004356 | 3.18E-07 | 4 | 12 | 0.002939 | 6.04E-09 |
| | $y_3$ | 12 | 46 | 0.00557 | 8.35E-07 | 4 | 12 | 0.002548 | 4.37E-07 |
| | $y_4$ | 13 | 49 | 0.007236 | 4.32E-07 | 5 | 15 | 0.002855 | 1.52E-07 |
| | $y_5$ | 14 | 53 | 0.007473 | 4.27E-07 | 6 | 18 | 0.009373 | 1.10E-09 |
| | $y_6$ | 12 | 44 | 0.004463 | 9.51E-07 | 6 | 18 | 0.002567 | 1.74E-08 |
| | $y_7$ | 14 | 53 | 0.005317 | 7.52E-07 | 106 | 318 | 0.060475 | 9.65E-07 |
| 5000 | $y_1$ | 12 | 47 | 0.017058 | 5.52E-07 | 3 | 9 | 0.003876 | 1.75E-07 |
| | $y_2$ | 13 | 51 | 0.016004 | 6.89E-07 | 4 | 12 | 0.005113 | 6.27E-10 |
| | $y_3$ | 13 | 50 | 0.015614 | 5.37E-07 | 4 | 12 | 0.005662 | 1.42E-07 |
| | $y_4$ | 13 | 49 | 0.015854 | 9.11E-07 | 5 | 15 | 0.007849 | 3.94E-08 |
| | $y_5$ | 14 | 53 | 0.016588 | 9.11E-07 | 5 | 15 | 0.007635 | 4.05E-07 |
| | $y_6$ | 13 | 48 | 0.017176 | 5.76E-07 | 6 | 18 | 0.007015 | 2.36E-09 |
| | $y_7$ | 15 | 57 | 0.021617 | 5.08E-07 | 365 | 1095 | 0.45786 | 9.84E-07 |
| 10000 | $y_1$ | 12 | 47 | 0.024133 | 7.77E-07 | 3 | 9 | 0.009625 | 1.21E-07 |
| | $y_2$ | 13 | 51 | 0.027397 | 9.71E-07 | 4 | 12 | 0.008537 | 2.79E-10 |
| | $y_3$ | 13 | 50 | 0.031404 | 7.55E-07 | 4 | 12 | 0.009399 | 9.73E-08 |
| | $y_4$ | 14 | 53 | 0.031945 | 3.83E-07 | 5 | 15 | 0.011722 | 2.56E-08 |
| | $y_5$ | 15 | 57 | 0.03006 | 3.84E-07 | 5 | 15 | 0.013383 | 2.93E-07 |
| | $y_6$ | 13 | 48 | 0.025585 | 8.04E-07 | 6 | 20 | 0.016129 | 3.62E-09 |
| | $y_7$ | 15 | 57 | 0.034146 | 7.18E-07 | 416 | 1248 | 0.93838 | 9.70E-07 |
| 50000 | $y_1$ | 13 | 51 | 0.15665 | 5.19E-07 | 3 | 9 | 0.025051 | 6.32E-08 |
| | $y_2$ | 14 | 55 | 0.10992 | 6.49E-07 | 4 | 12 | 0.060209 | 6.75E-11 |
| | $y_3$ | 14 | 54 | 0.10742 | 5.04E-07 | 4 | 12 | 0.033312 | 4.87E-08 |
| | $y_4$ | 14 | 53 | 0.12007 | 8.52E-07 | 6 | 22 | 0.076411 | 5.75E-08 |
| | $y_5$ | 15 | 57 | 0.14337 | 8.55E-07 | 7 | 28 | 0.070887 | 9.31E-09 |
| | $y_6$ | 14 | 52 | 0.1029 | 5.33E-07 | 7 | 28 | 0.078715 | 9.31E-09 |
| | $y_7$ | 16 | 61 | 0.16827 | 4.84E-07 | 365 | 1095 | 3.4659 | 8.63E-07 |
| 100000 | $y_1$ | 13 | 51 | 0.26108 | 7.34E-07 | 3 | 9 | 0.051374 | 5.40E-08 |
| | $y_2$ | 14 | 55 | 0.2078 | 9.18E-07 | 4 | 12 | 0.064736 | 4.27E-11 |
| | $y_3$ | 14 | 54 | 0.24524 | 7.12E-07 | 5 | 17 | 0.084643 | 7.37E-10 |
| | $y_4$ | 15 | 57 | 0.27307 | 3.61E-07 | 8 | 34 | 0.19686 | 1.10E-09 |
| | $y_5$ | 16 | 61 | 0.23156 | 3.62E-07 | 8 | 34 | 0.16312 | 1.10E-09 |
| | $y_6$ | 14 | 52 | 0.33684 | 7.53E-07 | 8 | 34 | 0.15642 | 1.10E-09 |
| | $y_7$ | 16 | 61 | 0.30503 | 6.84E-07 | 346 | 1042 | 10.5937 | 5.00E-07 |

For the compared method (DFPI), its parameters were set as reported in [17]. All iterative procedure terminate when $\|T(v_k)\| < 10^{-6}$ is fulfilled. If this condition is not satisfied after 1000 iterations, failure is declared.

**TABLE 8.** Numerical results for IDFPI and DFPI algorithms on problem 3.

| | | IDFPI | | | | DFPI | | | |
|---|---|---|---|---|---|---|---|---|---|
| n | SP | NOI | NFE | CPUT | LNORM | NOI | NFE | CPUT | LNORM |
| 1000 | $y_1$ | 13 | 52 | 0.006685 | 4.82E-07 | 22 | 110 | 0.071091 | 7.52E-07 |
| | $y_2$ | 13 | 52 | 0.005691 | 9.91E-07 | 23 | 115 | 0.007863 | 7.47E-07 |
| | $y_3$ | 14 | 56 | 0.003863 | 6.78E-07 | 24 | 120 | 0.008379 | 8.92E-07 |
| | $y_4$ | 14 | 56 | 0.005027 | 8.58E-07 | 25 | 125 | 0.009312 | 8.38E-07 |
| | $y_5$ | 14 | 56 | 0.006172 | 4.93E-07 | 25 | 125 | 0.012101 | 8.90E-07 |
| | $y_6$ | 15 | 60 | 0.00505 | 6.63E-07 | 25 | 125 | 0.009162 | 8.19E-07 |
| | $y_7$ | 14 | 56 | 0.007421 | 7.33E-07 | 24 | 120 | 0.014291 | 9.51E-07 |
| 5000 | $y_1$ | 14 | 56 | 0.017961 | 3.23E-07 | 23 | 115 | 0.039916 | 8.41E-07 |
| | $y_2$ | 14 | 56 | 0.013373 | 6.65E-07 | 24 | 120 | 0.042328 | 8.36E-07 |
| | $y_3$ | 15 | 60 | 0.017098 | 4.55E-07 | 25 | 125 | 0.042063 | 9.98E-07 |
| | $y_4$ | 15 | 60 | 0.018924 | 5.76E-07 | 26 | 130 | 0.040312 | 9.36E-07 |
| | $y_5$ | 15 | 60 | 0.015824 | 3.31E-07 | 26 | 130 | 0.040585 | 9.95E-07 |
| | $y_6$ | 16 | 64 | 0.016746 | 4.45E-07 | 27 | 134 | 0.042017 | 0 |
| | $y_7$ | 15 | 60 | 0.018791 | 4.99E-07 | 26 | 130 | 0.045348 | 5.35E-07 |
| 10000 | $y_1$ | 14 | 56 | 0.026742 | 4.57E-07 | 24 | 118 | 0.082392 | 0 |
| | $y_2$ | 14 | 56 | 0.030263 | 9.40E-07 | 24 | 118 | 0.066964 | 0 |
| | $y_3$ | 15 | 60 | 0.033116 | 6.43E-07 | 26 | 128 | 0.068461 | 0 |
| | $y_4$ | 15 | 60 | 0.029508 | 8.14E-07 | 27 | 133 | 0.065645 | 0 |
| | $y_5$ | 15 | 60 | 0.030222 | 4.68E-07 | 28 | 139 | 0.065254 | 0 |
| | $y_6$ | 16 | 64 | 0.033895 | 6.29E-07 | 28 | 139 | 0.062041 | 0 |
| | $y_7$ | 15 | 60 | 0.035151 | 7.06E-07 | 26 | 130 | 0.073951 | 7.60E-07 |
| 50000 | $y_1$ | 15 | 60 | 0.10247 | 3.07E-07 | 25 | 123 | 0.20488 | 0 |
| | $y_2$ | 15 | 60 | 0.1067 | 6.31E-07 | 25 | 123 | 0.20469 | 3.70E-22 |
| | $y_3$ | 16 | 64 | 0.20393 | 4.32E-07 | 25 | 123 | 0.20252 | 0 |
| | $y_4$ | 16 | 64 | 0.10433 | 5.46E-07 | 29 | 147 | 0.24523 | 9.26E-07 |
| | $y_5$ | 16 | 64 | 0.1076 | 3.14E-07 | 28 | 141 | 0.27392 | 0 |
| | $y_6$ | 17 | 68 | 0.27182 | 4.22E-07 | 28 | 141 | 0.23675 | 0 |
| | $y_7$ | 16 | 64 | 0.14651 | 4.73E-07 | 27 | 133 | 0.28916 | 4.88E-23 |
| 100000 | $y_1$ | 15 | 60 | 0.21037 | 4.34E-07 | 25 | 125 | 0.61769 | 9.40E-07 |
| | $y_2$ | 15 | 60 | 0.31285 | 8.92E-07 | 25 | 123 | 0.40009 | 0 |
| | $y_3$ | 16 | 64 | 0.23906 | 6.10E-07 | 26 | 128 | 0.40665 | 0 |
| | $y_4$ | 16 | 64 | 0.26002 | 7.73E-07 | 28 | 142 | 0.81468 | 0 |
| | $y_5$ | 16 | 64 | 0.25336 | 4.44E-07 | 28 | 142 | 0.52701 | 0 |
| | $y_6$ | 17 | 68 | 0.22276 | 5.97E-07 | 28 | 143 | 0.45238 | 0 |
| | $y_7$ | 16 | 64 | 0.4129 | 6.68E-07 | 27 | 134 | 0.56451 | 1.32E-22 |

**TABLE 9.** Numerical results for IDFPI and DFPI algorithms on problem 4.

| | | IDFPI | | | | DFPI | | | |
|---|---|---|---|---|---|---|---|---|---|
| n | SP | NOI | NFE | CPUT | LNORM | NOI | NFE | CPUT | LNORM |
| 1000 | $y_1$ | 13 | 52 | 0.004035 | 4.10E-07 | 22 | 110 | 0.022312 | 6.82E-07 |
| | $y_2$ | 13 | 52 | 0.00492 | 7.01E-07 | 23 | 115 | 0.006507 | 6.15E-07 |
| | $y_3$ | 13 | 52 | 0.004044 | 7.14E-07 | 24 | 120 | 0.006548 | 5.54E-07 |
| | $y_4$ | 13 | 52 | 0.004554 | 9.53E-07 | 22 | 110 | 0.010051 | 5.79E-07 |
| | $y_5$ | 14 | 56 | 0.005693 | 8.47E-07 | 25 | 126 | 0.008113 | 5.94E-07 |
| | $y_6$ | 14 | 56 | 0.003309 | 7.79E-07 | 24 | 121 | 0.007482 | 9.98E-07 |
| | $y_7$ | 15 | 60 | 0.005889 | 3.52E-07 | 24 | 120 | 0.012266 | 5.25E-07 |
| 5000 | $y_1$ | 13 | 52 | 0.012538 | 9.17E-07 | 23 | 115 | 0.022903 | 7.62E-07 |
| | $y_2$ | 14 | 56 | 0.012422 | 4.70E-07 | 24 | 120 | 0.024653 | 6.88E-07 |
| | $y_3$ | 14 | 56 | 0.010397 | 4.79E-07 | 25 | 125 | 0.023646 | 6.20E-07 |
| | $y_4$ | 14 | 56 | 0.011449 | 6.39E-07 | 23 | 115 | 0.02075 | 6.47E-07 |
| | $y_5$ | 15 | 60 | 0.015644 | 5.68E-07 | 26 | 131 | 0.024758 | 6.64E-07 |
| | $y_6$ | 15 | 60 | 0.015205 | 5.22E-07 | 24 | 122 | 0.024551 | 6.33E-07 |
| | $y_7$ | 15 | 60 | 0.013931 | 8.16E-07 | 25 | 125 | 0.022842 | 5.91E-07 |
| 10000 | $y_1$ | 14 | 56 | 0.018642 | 3.89E-07 | 24 | 120 | 0.041772 | 5.39E-07 |
| | $y_2$ | 14 | 56 | 0.020051 | 6.65E-07 | 24 | 120 | 0.039254 | 9.73E-07 |
| | $y_3$ | 14 | 56 | 0.022804 | 6.77E-07 | 25 | 125 | 0.043002 | 8.76E-07 |
| | $y_4$ | 14 | 56 | 0.020417 | 9.04E-07 | 26 | 131 | 0.051036 | 9.36E-07 |
| | $y_5$ | 15 | 60 | 0.023572 | 8.04E-07 | 26 | 131 | 0.043001 | 9.39E-07 |
| | $y_6$ | 15 | 60 | 0.022412 | 7.39E-07 | 27 | 138 | 0.044418 | 9.36E-07 |
| | $y_7$ | 16 | 64 | 0.023027 | 3.49E-07 | 25 | 125 | 0.051145 | 8.37E-07 |
| 50000 | $y_1$ | 14 | 56 | 0.13458 | 8.70E-07 | 25 | 125 | 0.1406 | 6.03E-07 |
| | $y_2$ | 15 | 60 | 0.076105 | 4.46E-07 | 26 | 128 | 0.14453 | 0 |
| | $y_3$ | 15 | 60 | 0.074626 | 4.54E-07 | 26 | 130 | 0.3076 | 9.80E-07 |
| | $y_4$ | 15 | 60 | 0.078944 | 6.06E-07 | - | - | - | - |
| | $y_5$ | 16 | 64 | 0.087076 | 5.39E-07 | - | - | - | - |
| | $y_6$ | 16 | 64 | 0.090052 | 4.96E-07 | 30 | 157 | 0.18098 | 0 |
| | $y_7$ | 16 | 64 | 0.092506 | 7.71E-07 | 27 | 136 | 0.14182 | 7.33E-07 |
| 100000 | $y_1$ | 15 | 60 | 0.22368 | 3.69E-07 | 25 | 125 | 0.25496 | 8.52E-07 |
| | $y_2$ | 15 | 60 | 0.14734 | 6.31E-07 | - | - | - | - |
| | $y_3$ | 15 | 60 | 0.14545 | 6.42E-07 | 26 | 129 | 0.66636 | 0 |
| | $y_4$ | 15 | 60 | 0.21704 | 8.58E-07 | - | - | - | - |
| | $y_5$ | 16 | 64 | 0.16062 | 7.63E-07 | 32 | 169 | 0.91603 | 0 |
| | $y_6$ | 16 | 64 | 0.16709 | 7.01E-07 | 34 | 186 | 0.41102 | 0 |
| | $y_7$ | 17 | 68 | 0.26252 | 3.25E-07 | 28 | 141 | 0.29815 | 5.18E-07 |

To illustrate in detail the efficiency and robustness of Iner. DFPI, we start by performing some numerical experiments with different coefficients of the parameter $\beta_k$ and the results are reported in Table 2 and 3. It can be observed from the tables that the coefficient 0.01 is a good choice. In addition, we performed another numerical experiments with different sequences $\{\theta_k\}$ and the results are reported in Table 4 and 5. It can be observed from the tables that the sequence $\theta_k = \frac{1}{(2k+5)^2}$ is a good choice. We further employ the performance

**TABLE 10.** Numerical results for IDFPI and DFPI algorithms on problem 5.

| | | IDFPI | | | | DFPI | | | |
|---|---|---|---|---|---|---|---|---|---|
| n | SP | NOI | NFE | CPUT | LNORM | NOI | NFE | CPUT | LNORM |
| 1000 | $y_1$ | 22 | 83 | 0.006723 | 3.36E-07 | 27 | 125 | 0.025308 | 7.64E-07 |
| | $y_2$ | 18 | 68 | 0.004817 | 3.58E-07 | 27 | 127 | 0.013897 | 5.68E-07 |
| | $y_3$ | 22 | 86 | 0.007747 | 3.16E-07 | 30 | 142 | 0.009465 | 5.36E-07 |
| | $y_4$ | 20 | 80 | 0.007736 | 9.38E-07 | 26 | 130 | 0.00958 | 5.96E-07 |
| | $y_5$ | 23 | 92 | 0.007882 | 5.80E-07 | 27 | 134 | 0.010308 | 9.66E-07 |
| | $y_6$ | 33 | 132 | 0.013091 | 3.68E-07 | 26 | 132 | 0.011363 | 7.39E-07 |
| | $y_7$ | 36 | 143 | 0.015021 | 9.85E-07 | 33 | 173 | 0.012412 | 7.77E-07 |
| 5000 | $y_1$ | 22 | 83 | 0.023041 | 3.36E-07 | 27 | 125 | 0.028866 | 9.11E-07 |
| | $y_2$ | 18 | 68 | 0.017824 | 7.77E-07 | 28 | 132 | 0.031948 | 6.88E-07 |
| | $y_3$ | 23 | 90 | 0.019053 | 6.89E-07 | 32 | 150 | 0.033963 | 6.55E-07 |
| | $y_4$ | 21 | 84 | 0.018336 | 6.11E-07 | 27 | 135 | 0.040092 | 8.11E-07 |
| | $y_5$ | 27 | 108 | 0.033345 | 3.04E-07 | 30 | 147 | 0.03638 | 6.11E-07 |
| | $y_6$ | 46 | 184 | 0.077517 | 3.86E-07 | 27 | 137 | 0.032325 | 9.59E-07 |
| | $y_7$ | 51 | 203 | 0.074716 | 5.05E-07 | 43 | 243 | 0.052204 | 7.93E-07 |
| 10000 | $y_1$ | 25 | 95 | 0.042456 | 8.93E-07 | 29 | 138 | 0.050406 | 9.96E-07 |
| | $y_2$ | 19 | 72 | 0.033543 | 3.32E-07 | 28 | 132 | 0.047778 | 9.91E-07 |
| | $y_3$ | 23 | 90 | 0.040511 | 8.91E-07 | 32 | 150 | 0.053794 | 9.38E-07 |
| | $y_4$ | 21 | 84 | 0.031568 | 9.96E-07 | 32 | 155 | 0.0541 | 5.90E-07 |
| | $y_5$ | 28 | 112 | 0.05547 | 9.14E-07 | 30 | 147 | 0.062782 | 8.87E-07 |
| | $y_6$ | 50 | 200 | 0.18221 | 3.37E-07 | 31 | 154 | 0.055032 | 5.36E-07 |
| | $y_7$ | 60 | 239 | 0.18747 | 5.93E-07 | 38 | 205 | 0.074963 | 9.59E-07 |
| 50000 | $y_1$ | 48 | 187 | 0.60608 | 3.42E-07 | 39 | 219 | 0.33566 | 7.35E-07 |
| | $y_2$ | 19 | 72 | 0.12914 | 7.62E-07 | 37 | 194 | 0.26823 | 5.90E-07 |
| | $y_3$ | 26 | 102 | 0.1525 | 3.17E-07 | 33 | 157 | 0.20113 | 5.81E-07 |
| | $y_4$ | 34 | 136 | 0.3022 | 8.46E-07 | 34 | 163 | 0.22611 | 6.85E-07 |
| | $y_5$ | 35 | 140 | 0.37456 | 8.13E-07 | 33 | 162 | 0.22718 | 9.63E-07 |
| | $y_6$ | 75 | 300 | 1.3335 | 6.68E-07 | 33 | 162 | 0.21739 | 6.33E-07 |
| | $y_7$ | 88 | 351 | 1.4975 | 5.17E-07 | 35 | 167 | 0.2218 | 5.56E-07 |
| 100000 | $y_1$ | 64 | 251 | 1.7805 | 8.73E-07 | 37 | 181 | 0.4819 | 8.65E-07 |
| | $y_2$ | 20 | 76 | 0.2093 | 3.27E-07 | 37 | 183 | 0.44101 | 5.07E-07 |
| | $y_3$ | 26 | 102 | 0.37068 | 5.05E-07 | 33 | 159 | 0.40582 | 5.19E-07 |
| | $y_4$ | 35 | 140 | 0.72236 | 5.66E-07 | 34 | 166 | 0.42738 | 7.31E-07 |
| | $y_5$ | 40 | 160 | 0.77446 | 6.02E-07 | 35 | 173 | 0.47046 | 5.55E-07 |
| | $y_6$ | 75 | 300 | 2.5582 | 4.79E-07 | 35 | 177 | 0.56666 | 8.85E-07 |
| | $y_7$ | 98 | 391 | 3.6313 | 3.21E-07 | 35 | 168 | 0.43766 | 7.18E-07 |

**TABLE 11.** Numerical results for IDFPI and DFPI algorithms on problem 6.

| | | IDFPI | | | | DFPI | | | |
|---|---|---|---|---|---|---|---|---|---|
| n | SP | NOI | NFE | CPUT | LNORM | NOI | NFE | CPUT | LNORM |
| 1000 | $y_1$ | 16 | 64 | 0.010483 | 3.56E-07 | 27 | 135 | 0.033919 | 6.16E-07 |
| | $y_2$ | 16 | 64 | 0.009273 | 3.42E-07 | 27 | 135 | 0.014032 | 5.92E-07 |
| | $y_3$ | 16 | 64 | 0.008048 | 3.01E-07 | 27 | 135 | 0.013855 | 5.22E-07 |
| | $y_4$ | 15 | 60 | 0.007367 | 6.87E-07 | 26 | 130 | 0.01828 | 7.14E-07 |
| | $y_5$ | 15 | 60 | 0.005634 | 5.52E-07 | 26 | 130 | 0.014308 | 5.73E-07 |
| | $y_6$ | 15 | 60 | 0.006965 | 3.25E-07 | 25 | 125 | 0.019782 | 6.76E-07 |
| | $y_7$ | 16 | 64 | 0.010309 | 3.05E-07 | 27 | 135 | 0.016674 | 5.22E-07 |
| 5000 | $y_1$ | 16 | 64 | 0.029797 | 7.98E-07 | 28 | 140 | 0.062147 | 6.90E-07 |
| | $y_2$ | 16 | 64 | 0.027649 | 7.66E-07 | 28 | 140 | 0.053856 | 6.63E-07 |
| | $y_3$ | 16 | 64 | 0.025924 | 6.75E-07 | 28 | 140 | 0.053509 | 5.84E-07 |
| | $y_4$ | 16 | 64 | 0.02971 | 4.62E-07 | 27 | 135 | 0.051204 | 8.00E-07 |
| | $y_5$ | 16 | 64 | 0.025478 | 3.71E-07 | 27 | 135 | 0.056674 | 6.42E-07 |
| | $y_6$ | 15 | 60 | 0.023231 | 7.29E-07 | 26 | 130 | 0.050872 | 7.57E-07 |
| | $y_7$ | 16 | 64 | 0.026982 | 6.80E-07 | 28 | 140 | 0.071807 | 5.90E-07 |
| 10000 | $y_1$ | 17 | 68 | 0.060317 | 3.39E-07 | 29 | 146 | 0.11459 | 7.32E-07 |
| | $y_2$ | 17 | 68 | 0.049807 | 3.25E-07 | 29 | 146 | 0.11246 | 7.04E-07 |
| | $y_3$ | 16 | 64 | 0.047161 | 9.55E-07 | 29 | 146 | 0.11393 | 6.20E-07 |
| | $y_4$ | 16 | 64 | 0.045383 | 6.54E-07 | 28 | 140 | 0.11162 | 5.66E-07 |
| | $y_5$ | 16 | 64 | 0.055288 | 5.24E-07 | 27 | 135 | 0.10514 | 9.08E-07 |
| | $y_6$ | 16 | 64 | 0.04654 | 3.09E-07 | 27 | 135 | 0.10359 | 5.35E-07 |
| | $y_7$ | 16 | 64 | 0.047746 | 9.63E-07 | 29 | 146 | 0.10954 | 6.25E-07 |
| 50000 | $y_1$ | 17 | 68 | 0.39191 | 7.57E-07 | 33 | 172 | 0.77122 | 0 |
| | $y_2$ | 17 | 68 | 0.18194 | 7.27E-07 | 33 | 172 | 0.49267 | 0 |
| | $y_3$ | 17 | 68 | 0.18105 | 6.41E-07 | 32 | 165 | 0.74557 | 0 |
| | $y_4$ | 17 | 68 | 0.29297 | 4.38E-07 | 29 | 145 | 0.44817 | 0 |
| | $y_5$ | 17 | 68 | 0.18165 | 3.52E-07 | 29 | 145 | 0.42038 | 0 |
| | $y_6$ | 16 | 64 | 0.18681 | 6.91E-07 | 27 | 133 | 0.38266 | 0 |
| | $y_7$ | 17 | 68 | 0.22631 | 6.46E-07 | 32 | 165 | 0.49133 | 0 |
| 100000 | $y_1$ | 18 | 72 | 0.44716 | 3.21E-07 | 35 | 189 | 1.4108 | 0 |
| | $y_2$ | 18 | 72 | 0.52194 | 3.09E-07 | 35 | 188 | 1.9585 | 0 |
| | $y_3$ | 17 | 68 | 0.40182 | 9.06E-07 | 34 | 181 | 1.1235 | 0 |
| | $y_4$ | 17 | 68 | 0.53042 | 6.20E-07 | 31 | 160 | 1.0369 | 0 |
| | $y_5$ | 17 | 68 | 0.50865 | 4.98E-07 | 29 | 146 | 1.0626 | 0 |
| | $y_6$ | 16 | 64 | 0.43562 | 9.78E-07 | 27 | 134 | 0.95013 | 0 |
| | $y_7$ | 17 | 68 | 0.6718 | 9.13E-07 | 34 | 181 | 1.2456 | 0 |

**TABLE 12.** Numerical results for IDFPI and DFPI algorithms on problem 7.

| | | IDFPI | | | | DFPI | | | |
|---|---|---|---|---|---|---|---|---|---|
| n | SP | NOI | NFE | CPUT | LNORM | NOI | NFE | CPUT | LNORM |
| 1000 | $y_1$ | 8 | 32 | 0.003653 | 9.31E-08 | 7 | 35 | 0.088016 | 2.09E-07 |
| | $y_2$ | 7 | 28 | 0.002498 | 7.06E-07 | 7 | 35 | 0.002876 | 1.30E-07 |
| | $y_3$ | 6 | 24 | 0.002711 | 2.03E-07 | 5 | 25 | 0.002588 | 6.75E-07 |
| | $y_4$ | 7 | 28 | 0.004252 | 1.86E-07 | 7 | 35 | 0.003619 | 5.39E-07 |
| | $y_5$ | 8 | 32 | 0.004847 | 2.91E-07 | 7 | 35 | 0.005417 | 6.75E-07 |
| | $y_6$ | 8 | 31 | 0.002989 | 3.19E-07 | 7 | 33 | 0.003767 | 6.55E-07 |
| | $y_7$ | 9 | 36 | 0.003879 | 2.83E-07 | 10 | 50 | 0.005564 | 4.13E-07 |
| 5000 | $y_1$ | 8 | 32 | 0.011319 | 2.08E-07 | 7 | 35 | 0.012628 | 4.68E-07 |
| | $y_2$ | 8 | 32 | 0.010446 | 1.30E-07 | 7 | 35 | 0.010796 | 2.90E-07 |
| | $y_3$ | 6 | 24 | 0.007387 | 4.53E-07 | 6 | 30 | 0.009275 | 9.64E-08 |
| | $y_4$ | 7 | 28 | 0.008646 | 4.15E-07 | 8 | 40 | 0.011531 | 7.69E-08 |
| | $y_5$ | 8 | 32 | 0.011077 | 6.51E-07 | 8 | 40 | 0.012222 | 9.64E-08 |
| | $y_6$ | 8 | 31 | 0.011479 | 7.14E-07 | 8 | 38 | 0.012377 | 9.35E-08 |
| | $y_7$ | 9 | 36 | 0.012983 | 6.29E-07 | 10 | 50 | 0.016196 | 9.76E-07 |
| 10000 | $y_1$ | 8 | 32 | 0.016198 | 2.95E-07 | 7 | 35 | 0.019634 | 6.62E-07 |
| | $y_2$ | 8 | 32 | 0.017598 | 1.84E-07 | 7 | 35 | 0.018425 | 4.10E-07 |
| | $y_3$ | 6 | 24 | 0.015246 | 6.41E-07 | 6 | 30 | 0.017956 | 1.36E-07 |
| | $y_4$ | 7 | 28 | 0.014948 | 5.87E-07 | 8 | 40 | 0.021969 | 1.09E-07 |
| | $y_5$ | 8 | 32 | 0.018776 | 9.20E-07 | 8 | 40 | 0.020505 | 1.36E-07 |
| | $y_6$ | 9 | 35 | 0.019481 | 8.34E-08 | 8 | 40 | 0.024966 | 1.36E-07 |
| | $y_7$ | 9 | 36 | 0.025324 | 9.03E-07 | 11 | 55 | 0.033891 | 1.81E-07 |
| 50000 | $y_1$ | 8 | 32 | 0.074381 | 6.59E-07 | 8 | 40 | 0.078752 | 9.46E-08 |
| | $y_2$ | 8 | 32 | 0.057285 | 4.12E-07 | 7 | 35 | 0.06381 | 9.17E-07 |
| | $y_3$ | 7 | 28 | 0.05958 | 1.18E-07 | 6 | 30 | 0.071067 | 3.05E-07 |
| | $y_4$ | 8 | 32 | 0.11846 | 1.08E-07 | 9 | 47 | 0.08954 | 2.58E-07 |
| | $y_5$ | 9 | 36 | 0.076652 | 1.70E-07 | 9 | 47 | 0.087936 | 2.69E-07 |
| | $y_6$ | 9 | 35 | 0.0716 | 1.87E-07 | 9 | 47 | 0.091278 | 2.69E-07 |
| | $y_7$ | 10 | 40 | 0.20552 | 1.66E-07 | 11 | 55 | 0.12051 | 4.06E-07 |
| 100000 | $y_1$ | 8 | 32 | 0.11275 | 9.31E-07 | 8 | 41 | 0.14657 | 8.58E-07 |
| | $y_2$ | 8 | 32 | 0.11906 | 5.83E-07 | 8 | 40 | 0.1458 | 8.28E-08 |
| | $y_3$ | 7 | 28 | 0.22704 | 1.67E-07 | 6 | 30 | 0.10972 | 4.31E-07 |
| | $y_4$ | 8 | 32 | 0.12258 | 1.53E-07 | 9 | 47 | 0.17732 | 3.64E-07 |
| | $y_5$ | 9 | 36 | 0.13371 | 2.40E-07 | 9 | 47 | 0.2031 | 3.81E-07 |
| | $y_6$ | 9 | 35 | 0.24326 | 2.64E-07 | 9 | 47 | 0.16742 | 3.81E-07 |
| | $y_7$ | 10 | 40 | 0.16384 | 2.36E-07 | 11 | 56 | 0.24561 | 4.69E-07 |

**TABLE 13.** Numerical results for IDFPI and DFPI algorithms on problem 8.

| | | IDFPI | | | | DFPI | | | |
|---|---|---|---|---|---|---|---|---|---|
| n | SP | NOI | NFE | CPUT | LNORM | NOI | NFE | CPUT | LNORM |
| 1000 | $y_1$ | 9 | 32 | 0.004774 | 6.76E-08 | 14 | 84 | 0.047306 | 9.29E-07 |
| | $y_2$ | 9 | 32 | 0.002911 | 6.76E-08 | 14 | 86 | 0.007059 | 9.29E-07 |
| | $y_3$ | 9 | 32 | 0.004649 | 6.76E-08 | 14 | 89 | 0.006651 | 9.29E-07 |
| | $y_4$ | 9 | 32 | 0.005168 | 6.76E-08 | 14 | 91 | 0.007693 | 9.29E-07 |
| | $y_5$ | 9 | 32 | 0.003576 | 6.76E-08 | 14 | 92 | 0.012985 | 9.29E-07 |
| | $y_6$ | 9 | 32 | 0.002819 | 6.76E-08 | 15 | 99 | 0.012506 | 9.29E-07 |
| | $y_7$ | 93 | 340 | 0.033585 | 9.32E-07 | - | - | - | - |
| 5000 | $y_1$ | 11 | 41 | 0.017931 | 2.99E-07 | 16 | 107 | 0.04506 | 4.92E-07 |
| | $y_2$ | 11 | 41 | 0.014838 | 2.99E-07 | 16 | 109 | 0.032913 | 4.92E-07 |
| | $y_3$ | 11 | 41 | 0.015599 | 2.99E-07 | 16 | 111 | 0.10903 | 4.92E-07 |
| | $y_4$ | 11 | 41 | 0.072565 | 2.99E-07 | 17 | 129 | 0.038967 | 4.92E-07 |
| | $y_5$ | 11 | 41 | 0.026755 | 2.99E-07 | 17 | 126 | 0.051708 | 4.92E-07 |
| | $y_6$ | 11 | 41 | 0.020458 | 2.99E-07 | 18 | 142 | 0.04588 | 4.92E-07 |
| | $y_7$ | 32 | 112 | 0.033263 | 8.56E-07 | - | - | - | - |
| 10000 | $y_1$ | 8 | 29 | 0.051059 | 4.42E-07 | 11 | 75 | 0.052377 | 6.02E-07 |
| | $y_2$ | 8 | 29 | 0.027061 | 4.42E-07 | 11 | 77 | 0.049877 | 6.02E-07 |
| | $y_3$ | 8 | 29 | 0.023059 | 4.42E-07 | 11 | 80 | 0.10928 | 6.02E-07 |
| | $y_4$ | 8 | 29 | 0.026803 | 4.42E-07 | 12 | 99 | 0.080173 | 6.02E-07 |
| | $y_5$ | 8 | 29 | 0.028654 | 4.42E-07 | 13 | 107 | 0.079948 | 6.02E-07 |
| | $y_6$ | 8 | 29 | 0.026353 | 4.42E-07 | 14 | 132 | 0.093111 | 6.02E-07 |
| | $y_7$ | 27 | 102 | 0.065685 | 7.89E-07 | - | - | - | - |
| 50000 | $y_1$ | 6 | 22 | 0.075446 | 1.89E-07 | 10 | 76 | 0.34388 | 4.68E-07 |
| | $y_2$ | 6 | 22 | 0.15044 | 1.89E-07 | 10 | 78 | 0.2456 | 4.68E-07 |
| | $y_3$ | 6 | 22 | 0.070266 | 1.89E-07 | 11 | 97 | 0.31595 | 4.68E-07 |
| | $y_4$ | 6 | 22 | 0.069065 | 1.89E-07 | 14 | 154 | 0.54484 | 4.68E-07 |
| | $y_5$ | 6 | 22 | 0.10305 | 1.89E-07 | 14 | 151 | 0.4634 | 4.68E-07 |
| | $y_6$ | 6 | 22 | 0.11076 | 1.89E-07 | 16 | 201 | 0.75152 | 4.68E-07 |
| | $y_7$ | 9 | 33 | 0.1275 | 7.76E-07 | 11 | 97 | 0.28241 | 4.68E-07 |
| 100000 | $y_1$ | 8 | 31 | 0.22729 | 1.95E-07 | 8 | 65 | 0.40538 | 1.73E-07 |
| | $y_2$ | 8 | 31 | 0.24876 | 1.95E-07 | 8 | 67 | 0.41492 | 1.73E-07 |
| | $y_3$ | 8 | 31 | 0.21451 | 1.95E-07 | 10 | 104 | 0.72037 | 1.73E-07 |
| | $y_4$ | 8 | 31 | 0.35928 | 1.95E-07 | 13 | 176 | 1.3032 | 1.73E-07 |
| | $y_5$ | 8 | 31 | 0.21663 | 1.95E-07 | 14 | 202 | 1.4652 | 1.73E-07 |
| | $y_6$ | 8 | 31 | 0.22028 | 1.95E-07 | 16 | 255 | 1.7952 | 1.73E-07 |
| | $y_7$ | 11 | 43 | 0.36981 | 6.82E-07 | 11 | 115 | 0.85792 | 1.73E-07 |

profile proposed by Dolan and Morè in [50] in order to summarize Table 6-15. The profile is defined as follows:

$$\rho(\tau) := \frac{1}{|T_P|}\left|\left\{t_p \in T_P : \log_2\left(\frac{t_{p,q}}{\min\{t_{p,q} : q \in Q\}}\right) \leq \tau\right\}\right|,$$

**TABLE 14.** Numerical results for IDFPI and DFPI algorithms on problem 9.

| | | IDFPI | | | | DFPI | | | |
|---|---|---|---|---|---|---|---|---|---|
| n | SP | NOI | NFE | CPUT | LNORM | NOI | NFE | CPUT | LNORM |
| 1000 | $y_1$ | 5 | 20 | 0.002775 | 5.46E-07 | 14 | 84 | 0.025842 | 7.75E-07 |
| | $y_2$ | 5 | 20 | 0.001703 | 3.25E-07 | 14 | 84 | 0.00484 | 4.70E-07 |
| | $y_3$ | 5 | 20 | 0.00257 | 3.10E-07 | 14 | 84 | 0.006867 | 4.48E-07 |
| | $y_4$ | 6 | 24 | 0.003117 | 5.36E-08 | 15 | 90 | 0.004959 | 7.58E-07 |
| | $y_5$ | 6 | 24 | 0.002628 | 7.25E-08 | 16 | 96 | 0.006006 | 3.01E-07 |
| | $y_6$ | 6 | 24 | 0.001932 | 1.04E-07 | 16 | 96 | 0.005295 | 4.32E-07 |
| | $y_7$ | 5 | 20 | 0.001605 | 6.86E-07 | 15 | 90 | 0.007065 | 2.93E-07 |
| 5000 | $y_1$ | 6 | 24 | 0.032784 | 3.64E-08 | 15 | 90 | 0.023128 | 5.08E-07 |
| | $y_2$ | 5 | 20 | 0.004843 | 7.28E-07 | 15 | 90 | 0.013479 | 3.08E-07 |
| | $y_3$ | 5 | 20 | 0.00474 | 6.94E-07 | 15 | 90 | 0.019939 | 2.93E-07 |
| | $y_4$ | 6 | 24 | 0.005001 | 1.20E-07 | 16 | 96 | 0.018103 | 4.97E-07 |
| | $y_5$ | 6 | 24 | 0.028177 | 1.62E-07 | 16 | 96 | 0.059113 | 6.73E-07 |
| | $y_6$ | 6 | 24 | 0.005417 | 2.33E-07 | 16 | 96 | 0.02335 | 9.66E-07 |
| | $y_7$ | 6 | 24 | 0.004567 | 4.61E-08 | 15 | 90 | 0.018177 | 6.49E-07 |
| 10000 | $y_1$ | 6 | 24 | 0.009786 | 5.15E-08 | 15 | 90 | 0.035418 | 7.18E-07 |
| | $y_2$ | 6 | 24 | 0.013561 | 3.07E-08 | 15 | 90 | 0.033533 | 4.35E-07 |
| | $y_3$ | 5 | 20 | 0.009086 | 9.82E-07 | 15 | 90 | 0.036702 | 4.15E-07 |
| | $y_4$ | 6 | 24 | 0.009481 | 1.69E-07 | 16 | 96 | 0.039716 | 7.02E-07 |
| | $y_5$ | 6 | 24 | 0.02489 | 2.29E-07 | 16 | 96 | 0.033306 | 9.51E-07 |
| | $y_6$ | 6 | 24 | 0.012618 | 3.29E-07 | 17 | 103 | 0.034423 | 8.83E-07 |
| | $y_7$ | 6 | 24 | 0.009643 | 6.49E-08 | 15 | 90 | 0.03156 | 9.16E-07 |
| 50000 | $y_1$ | 6 | 24 | 0.040512 | 1.15E-07 | 16 | 96 | 0.22563 | 4.70E-07 |
| | $y_2$ | 6 | 24 | 0.038022 | 6.87E-08 | 15 | 90 | 0.1461 | 9.73E-07 |
| | $y_3$ | 6 | 24 | 0.04902 | 6.55E-08 | 15 | 90 | 0.11495 | 9.28E-07 |
| | $y_4$ | 6 | 24 | 0.039638 | 3.79E-07 | 18 | 109 | 0.22775 | 2.97E-07 |
| | $y_5$ | 6 | 24 | 0.034356 | 5.13E-07 | 18 | 110 | 0.18641 | 8.89E-07 |
| | $y_6$ | 6 | 24 | 0.039451 | 7.37E-07 | 20 | 126 | 0.17079 | 8.65E-07 |
| | $y_7$ | 6 | 24 | 0.047944 | 1.44E-07 | 16 | 96 | 0.21042 | 5.99E-07 |
| 100000 | $y_1$ | 6 | 24 | 0.070738 | 1.63E-07 | 16 | 96 | 0.30649 | 6.65E-07 |
| | $y_2$ | 6 | 24 | 0.096217 | 9.71E-08 | 16 | 96 | 0.26029 | 4.03E-07 |
| | $y_3$ | 6 | 24 | 0.1065 | 9.27E-08 | 16 | 96 | 0.41442 | 3.84E-07 |
| | $y_4$ | 6 | 24 | 0.084986 | 5.36E-07 | 18 | 110 | 0.33314 | 9.28E-07 |
| | $y_5$ | 6 | 24 | 0.071928 | 7.25E-07 | 20 | 126 | 0.32012 | 8.52E-07 |
| | $y_6$ | 7 | 28 | 0.089085 | 3.11E-08 | 22 | 142 | 0.53318 | 8.29E-07 |
| | $y_7$ | 6 | 24 | 0.096997 | 2.05E-07 | 16 | 96 | 0.2514 | 8.50E-07 |

**TABLE 15.** Numerical results for IDFPI and DFPI algorithms on problem 10.

| | | IDFPI | | | | DFPI | | | |
|---|---|---|---|---|---|---|---|---|---|
| n | SP | NOI | NFE | CPUT | LNORM | NOI | NFE | CPUT | LNORM |
| 1000 | $y_1$ | 9 | 36 | 0.007359 | 7.97E-07 | 17 | 119 | 0.050153 | 8.17E-07 |
| | $y_2$ | 10 | 40 | 0.006139 | 1.95E-07 | 18 | 126 | 0.017728 | 5.55E-07 |
| | $y_3$ | 1 | 3 | 0.001581 | 0 | 1 | 3 | 0.00131 | 0 |
| | $y_4$ | 1 | 4 | 0.001474 | 0 | 2 | 11 | 0.003299 | 0 |
| | $y_5$ | 1 | 3 | 0.001559 | 0 | 19 | 134 | 0.018093 | 7.43E-07 |
| | $y_6$ | 1 | 4 | 0.001463 | 0 | 19 | 135 | 0.026692 | 8.51E-07 |
| | $y_7$ | 11 | 44 | 0.011139 | 9.03E-07 | 19 | 133 | 0.019643 | 6.11E-07 |
| 5000 | $y_1$ | 10 | 40 | 0.026382 | 2.84E-07 | 18 | 126 | 0.056932 | 6.85E-07 |
| | $y_2$ | 10 | 40 | 0.025512 | 4.35E-07 | 19 | 133 | 0.060359 | 4.65E-07 |
| | $y_3$ | 1 | 3 | 0.002706 | 0 | 19 | 133 | 0.077922 | 8.49E-07 |
| | $y_4$ | 1 | 4 | 0.00444 | 0 | 20 | 141 | 0.085714 | 8.56E-07 |
| | $y_5$ | 1 | 3 | 0.002336 | 0 | 20 | 141 | 0.065807 | 6.23E-07 |
| | $y_6$ | 1 | 4 | 0.004291 | 0 | 21 | 151 | 0.06585 | 8.59E-07 |
| | $y_7$ | 12 | 48 | 0.03001 | 3.47E-07 | 20 | 140 | 0.072448 | 4.97E-07 |
| 10000 | $y_1$ | 10 | 40 | 0.047915 | 4.02E-07 | 18 | 126 | 0.12476 | 9.69E-07 |
| | $y_2$ | 10 | 40 | 0.053406 | 6.15E-07 | 19 | 133 | 0.12265 | 6.58E-07 |
| | $y_3$ | 1 | 3 | 0.004908 | 0 | 20 | 140 | 0.12326 | 4.51E-07 |
| | $y_4$ | 1 | 4 | 0.005123 | 0 | 21 | 148 | 0.15724 | 4.54E-07 |
| | $y_5$ | 1 | 3 | 0.003785 | 0 | 20 | 142 | 0.14843 | 8.59E-07 |
| | $y_6$ | 1 | 4 | 0.010065 | 0 | 22 | 158 | 0.15989 | 4.56E-07 |
| | $y_7$ | 12 | 48 | 0.060441 | 4.85E-07 | 20 | 140 | 0.13272 | 7.04E-07 |
| 50000 | $y_1$ | 10 | 40 | 0.15997 | 9.00E-07 | 19 | 133 | 0.59465 | 8.13E-07 |
| | $y_2$ | 11 | 44 | 0.41328 | 2.20E-07 | 20 | 140 | 0.49973 | 5.51E-07 |
| | $y_3$ | 1 | 3 | 0.019508 | 0 | 21 | 147 | 0.63901 | 3.78E-07 |
| | $y_4$ | 1 | 4 | 0.023778 | 0 | 22 | 157 | 0.64155 | 9.89E-07 |
| | $y_5$ | 1 | 3 | 0.017874 | 0 | 23 | 167 | 0.69792 | 9.89E-07 |
| | $y_6$ | 1 | 4 | 0.033341 | 0 | 24 | 178 | 0.8938 | 9.93E-07 |
| | $y_7$ | 13 | 52 | 0.24602 | 1.77E-07 | 21 | 148 | 0.5043 | 7.42E-07 |
| 100000 | $y_1$ | 11 | 44 | 0.48433 | 2.03E-07 | 20 | 140 | 1.171 | 4.31E-07 |
| | $y_2$ | 11 | 44 | 0.33993 | 3.11E-07 | 20 | 140 | 1.2389 | 7.80E-07 |
| | $y_3$ | 1 | 3 | 0.023354 | 0 | 22 | 155 | 1.0994 | 4.10E-07 |
| | $y_4$ | 1 | 4 | 0.058607 | 0 | 25 | 183 | 1.356 | 4.59E-07 |
| | $y_5$ | 1 | 3 | 0.030501 | 0 | 26 | 193 | 1.4711 | 4.59E-07 |
| | $y_6$ | 1 | 4 | 0.057797 | 0 | 28 | 217 | 1.6932 | 4.72E-07 |
| | $y_7$ | 13 | 52 | 0.61981 | 2.49E-07 | 22 | 155 | 1.1673 | 3.93E-07 |

where $T_P$ is the test set, $|T_P|$ is the number of problems in the test set $T_P$, $Q$ is the set of optimization solvers, and $t_{p,q}$ is the number of iterations (or the number of the function evaluations, or the CPU time (in seconds)) for $t_p \in T_P$ and

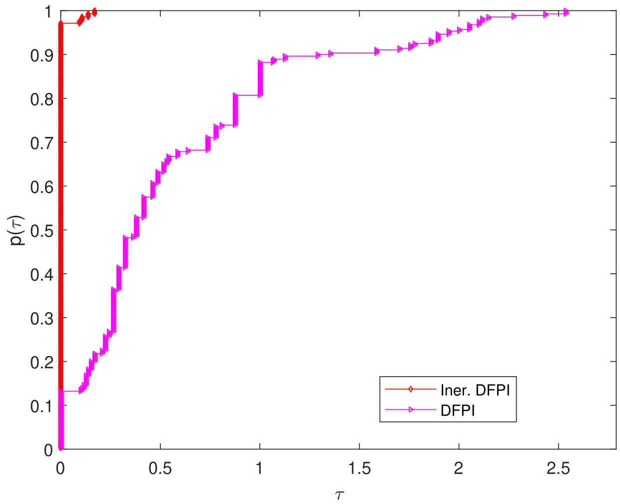


**FIGURE 1.** Performance profiles for the number of iterations.

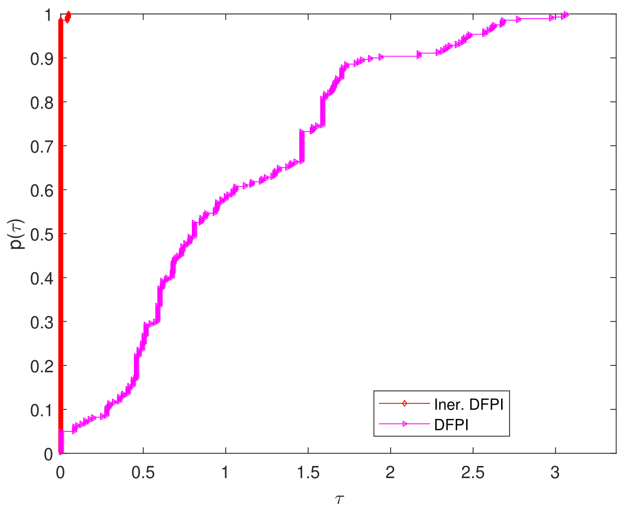


**FIGURE 2.** Performance profiles for the number of function evaluations.

$q \in Q$. The performance profile tells the percentage of win by each solver. Figures 1, 2 and 3 illustrate the performance of the two solvers (Iner. DFPI and DFPI) where the performance indices are the number of iterations, the number of function evaluations and the CPU time in seconds as reported in Tables 2-11. It can be observed from the figures that Iner. DFPI algorithm performs better with a higher percentage win of at least 90% in all the three metrics, i.e., number of iterations, the number of function evaluations and the CPU time. As a consequence, we can conclude that Iner. DFPI algorithm is an efficient solver. It is worth mentioning that the good numerical performance of the Iner. DFPI algorithm is as a result of the inertial term $v_k$, suitable control parameters such as $\rho$, $\sigma$ and the sequence $\{\theta_k\}$.

A detailed report of our numerical experiments is reported in Table 6-15 in the appendix section. The abbreviations on the tables can be read as follows:

n: denotes the dimension of the problem
SP: denotes the starting points
NOI: denotes the number of iterations

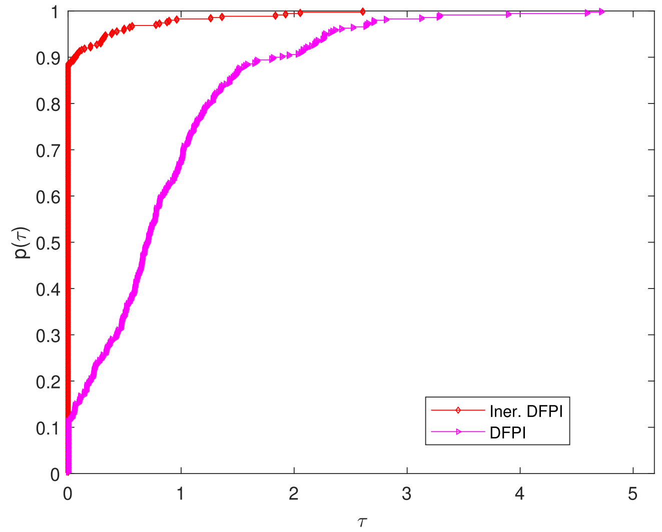


**FIGURE 3.** Performance profiles for the CPU time (in seconds).

NFE: denotes the number of function evaluations
CPUT: denotes the CPU time in seconds
LNORM: denotes the final norm

## V. CONCLUSION

In this paper, we suggested an inertial derivative-free method for solving nonlinear monotone operator equation. Based on the DFPI method, an inertial term was added to it in order to speed up its convergence. We used some mild assumptions to establish the global convergence of the proposed inertial method. To support the theoretical results, we perform some numerical experiments on some benchmark test problems with the proposed method and the DFPI. The results indicate that the proposed inertial method is faster than DFPI.

## APPENDIX

See Tables 2–15.

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