IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received May 25, 2021, accepted June 11, 2021, date of publication June 18, 2021, date of current version June 28, 2021.

Digital Object Identifier 10.1109/ACCESS.2021.3090649

Informative Order-Reduction of Underdamped Third-Order Systems

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This work was supported by the Jordan University of Science and Technology under Grant 20180352.

ABSTRACT Model order reduction simplifies the understanding of a given system and minimizes the simulation studies computational burden. A new order reduction method that depends on a predetermined normalized error of the transient performance indices is introduced. Ten percent and five percent error criteria in modeling and analyzing the transient performance of the third-order system are considered to have an accurate study. All sufficient special conditions and general rules required to achieve precise order reduction are determined. This work focuses on underdamped third-order systems without zeros. Third-order systems with three real poles are also analyzed for the study completeness. The relationships between the characteristic equation parameters are identified and the range in which the reduction is accurately valid is clearly specified. Each approximation or order reduction is studied separately in terms of the transient response characteristics: rise time, settling time and percentage overshoot. The comparison shows the effectiveness of the proposed method.

INDEX TERMS Order reduction, underdamped response, third-order, transient response.

NOMENICLATURE

ζ	The damping ratio of the second-order system part
ω_n	The natural frequency of the second-order part
σ	The damping factor of the second-order system
ω_d	The damping frequency of the second-order system
s_1	The first pole of the third-order system
s_2	The second pole of the third-order system
s_3	The third pole of the third-order system
P_1	The closest pole to the origin when all poles are real
α	The ratio between the targeted pole (P_1) and the second real pole s_2/s_1
β	The ratio between the targeted pole and the third real pole s_3/s_1
γ	The ratio between the third pole and the real part of the complex conjugated poles $\lvert s_3\rvert /\zeta\omega_n$
OS	percent overshoot
t_s	settling time
t_r	rise time
$\omega_n t_s$	normalized settling time
$\omega_n t_r$	normalized rise time
ε	the normalized error
ε_{Max}	The maximum acceptable normalized error
G_o	the transfer function of the original system
$\hat{G}_M$	the reduced transfer function using Matlab toolbox
$\hat{G}_P$	The reduced transfer function using the proposed work
$\hat{G}_{PSO}$	the reduced transfer function using particle swarm optimization
$\hat{G}_{GA}$	The reduced transfer function using genetic algorithm
G_3	The general expression of the third-order system under study
K_{dc}	The dc gain
τ	The time constant
K	The gain of the amplifier in the presented practical systems

The associate editor coordinating the review of this manuscript and approving it for publication was Salman Ahmed.

I. INTRODUCTION

Mathematical models are essential in the development, design, and control processes of the systems. In linearized models, one way to relate the relationship between variables is the transfer function. Many systems are modeled as first and second-order models for simplicity and existing mathematical design, analysis, and direct formulations. This is valid based on some acceptable assumptions. However, working at operation regions where the initial assumptions do not coincide with the actual conditions will lead to inaccurate representation. Reduced-order models received more attention in the last decade to work with higher-order models [1].

Unlike second-order differential equations, the third-order differential equations don't have a unique general solution; instead, they have a procedure that must be followed to reach the answers. Unfortunately, these roots can't be expressed in available formulas. This limits the usage of third-order representation in control systems to numerically defined parameters. In literature, it is acceptable to reduce the third-order system to a second-order system when the third real pole is at least ten times higher than the closer pole. Mathematically, this condition is not verified, and the impact of this condition is not analyzed. It is hard to accurately estimate the system parameters in some cases [2], and eventually, it is hard to guarantee this condition.

In real life, many control systems have third-order model. Generally, obtaining the analytical solution of a general third-order system is not direct. For example, the system frequency response model in [3] is a third-order system, and finding an analytical solution of the maximum frequency deviation was difficult, and a mechanism analysis was used to handle that. Moreover, many practical systems are recommended to be modeled as third-order systems. The boost converter conventional control is a third-order system [4]. Additionally, authors of [4] proposed a strategy to reduce the order to a second-order to enhance the stability of the system. In renewable energy resources, the doubly-fed induction generator based wind turbine is actually a nonlinear model that can be linearized to a third-order model [3] and the fuel cell is a third-order system [5] as well. The control lens actuation system is a third-order system [6]. The system can be represented by a second-order model by ignoring the inductance value and the back electromotive force. Yet, the model accuracy has been compromised. Authors of [7] showed that the model of the closed-loop current control of the three-terminal PWM switch is preferable to be a third-order. Anyways, it is usually represented in first or second-order models. In [8], the active disturbance rejection control, which is traditionally applied for a second-order system, is tuned for third-order systems and higher. The transfer function that describes the output voltage of the thermal accelerometer to the signal acceleration is a third-order system [9]. In addition, it was found that the second-order model is not sufficient to accurately represent the dynamics of intrinsic ankle mechanics system at all frequencies and it is recommended to be modeled as a third-order system [10], [11]. Even for simulation analysis, third-order models are usually used [12].

Model order reduction concept is widely used in many engineering applications and research [1], [13], [14]. The authors in [15] reduced the order of active magnetic bearing and developed new control strategy. In [16], the model of self-excited induction generator is reduced based on complex transformation to avoid reaching nonpractical model. A reduced-order and enhanced observer-based control strategy for dc tot dc converter is presented in [17] to have better performance in terms of disturbance rejection and sensitivity to parameter variation. A reduced-order active disturbance rejection controller is presented in [18] with a new tuning technique to have better performance.

In this paper, the sufficient conditions needed to reduce the underdamped third-order system into first or second-order systems are investigated in detail. Unlike other order reduction techniques, the proposed work keeps the original system symbolic parameters without any modification and gives the maximum attainable normalized error after reducing the order. Essential time-domain indices are considered in this study. The error in normalized settling time, normalized rise time and percent overshoot, which represent the transient response of the system, are considered as indicators to accept or reject the order reduction. This work is establishing a solid mathematical representation for the third-order system. Unlike second-order system, third-order system is not sufficiently studied and no solid mathematical analysis is presented in literature. As this paper is the first step toward this goal, the proposed work shows the necessary conditions to accurately reduce the order to first or second-order systems. All possible cases are simulated. The possibility of acceptable reduction to a first or second-order system is checked for all possible values based on the essential time-domain performance indices error. Thus, the sufficient conditions needed to obtain acceptable order reduction are investigated.

This paper is organized as follows: third-order system mathematical representation is presented in Section 2. In Sections 3 and 4, order-reduction sufficient constrants are presented when two poles are real or complex, respectively. The effectiveness of the proposed method is presented in Section 5 with an insightful comparison. Finally, the paper is concluded in Section 6.

II. THE PROPOSED PROCEDURE

Regardless of the types of system poles (real or complex-conjugated, coincident or distinct), the general transfer function of the normalized stable third-order system under study is:

$$G_3(s) = \frac{s_1 s_2 s_3}{(s+s_1)(s+s_2)(s+s_3)} = \begin{cases} \dfrac{\alpha\beta P_1}{(s+P_1)(s+\alpha P_1)(s+\beta P_1)} \\ \zeta = 0 \text{ or } \zeta \geq 1 \\ \dfrac{\gamma\zeta\omega_n^3}{(s^2+2\zeta\omega_n s+\omega_n^2)(s+\gamma\zeta\omega_n)} \\ 0 < \zeta < 1 \end{cases} \tag{1}$$

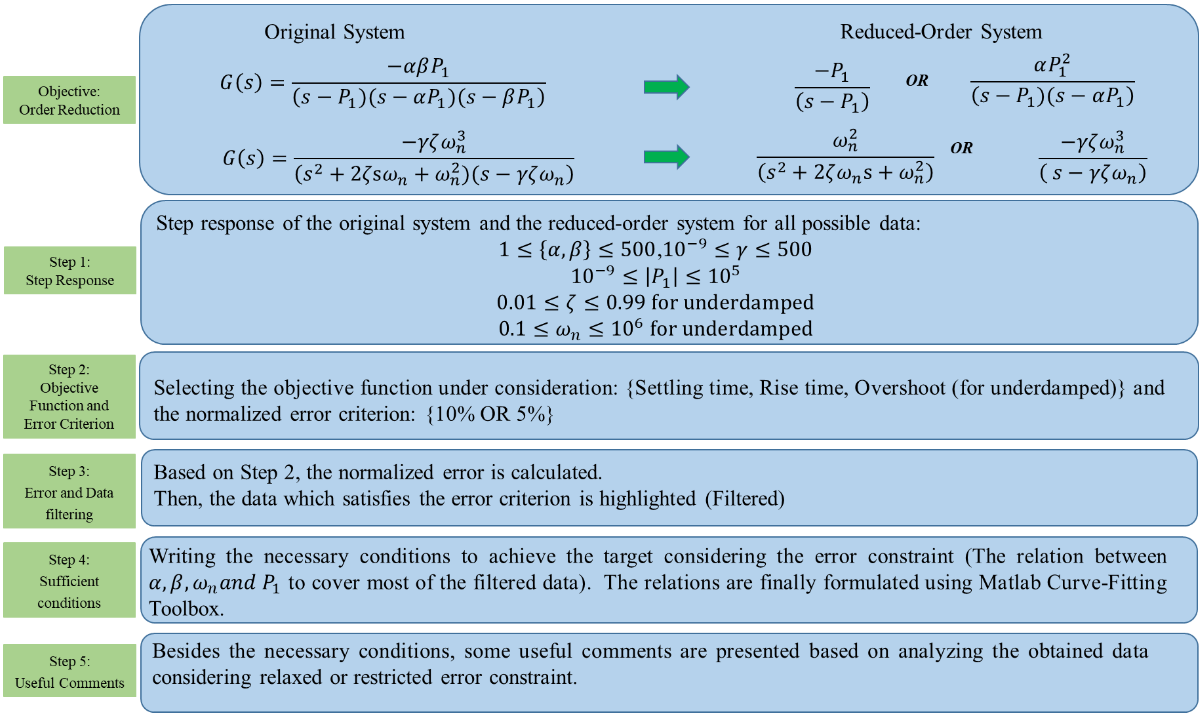

FIGURE 1. Proposed order reduction flow chart.

where ζ is the damping ratio of the second-order system part, ω_n is the natural frequency of the second-order part and (s_1, s_2, s_3) are the poles of the third-order system under study. It is important to highlight here that the terms {undamped ($\zeta = 0$), underdamped ($0 < \zeta < 1$), critically-damped ($\zeta = 1$), overdamped ($\zeta > 1$)} are defined for second-order system. There are no-zeros in the system in this work, which keeps these terms also applicable for the third-order system. On the other hand, when $\zeta = 0$ or when ($\alpha = \beta = 1$), there is no importance of order reduction and such cases are excluded in this study. $\{\alpha, \beta, \gamma\}$ are the ratios between the targeted pole (P_1) and the other real poles where, $\alpha = s_2/s_1$, $\beta = s_3/s_1$ and $\alpha \leq \beta$. The targeted pole P_1 must be closest to the origin when all poles are real and as a result: $\alpha \& \beta \geq 1$. For underdamped system, $\gamma = |s_3| / \zeta\omega_n$ which is defined as the ratio between the third pole and the real part of the complex conjugated poles: ($s_{1,2} = -\sigma \pm j\omega_d$). σ is the damping factor of the second-order system ($\sigma = \zeta\omega_n$) and ω_d is the damping frequency of the second-order system. As the third pole can be less than σ, the value of γ can be less than one ($0 \leq \gamma$), unlike $\{\alpha, \beta\}$.

Third-order system reduction can result in first-order or second-order systems, if possible, based on a particular objective function. The similarity between the original and the reduced-order (approximation) models are evaluated using different objective functions, which are: percent overshoot (OS), normalized settling time ($\omega_n t_s$) and the normalized rise time ($\omega_n t_r$). It is known that OS is limited to underdamped system as the system doesn't have any zeros.

The general procedure used in the proposed order reduction is summarized in Fig.1. In step 3, the filtration process depends on the eye screening of the data and logical outcomes. These data were filtered to have useful mathematical relations. For any variable (λ), the normalized error of any variable ($\varepsilon(\lambda)$) is calculated using:

$$\varepsilon(\lambda) = \left|\frac{\lambda - \hat{\lambda}}{\lambda}\right| \tag{2}$$

where λ can be any of the three decision parameters ($OS, \omega_n t_s, \omega_n t_r$) of the original third-order system, $\hat{\lambda}$ is the corresponding parameter of the reduced-order system.

The maximum acceptable normalized error (ε_{Max}) considered in this work is 10%. ε_{Max} is introduced to find the boundaries or the starting point at which ε is at most 10%. There is no specific reason behind selecting the 10% other than it is commonly used in many approximations. For more accurate order reduction, the sufficient conditions for $\varepsilon_{Max} = 5\%$ are additionally listed in this work.

Unlike other reduction methods, the proposed work introduced sufficient off-line constraints, thus, eliminating the need for commercial/iterative software to execute any algorithms. As the proposed work is the initial step toward having a complete analytical solution to third-order systems, the off-line constraints coincide with this goal. Simply, matching the case with the sufficient conditions is the only step to directly reduce the order of the system without any computational effort or any initial guess.

III. ORDER REDUCTION FOR $\zeta \geq 1$

A. REDUCTION TO FIRST-ORDER FOR $\zeta \geq 1$

If the error in t_s (2% criterion) is the objective function, Table 1 summaries the sufficient conditions to reduce the order to first. For $\varepsilon_{Max} = 10\%$, the must condition is ($\alpha \& \beta \geq 2.9$) which is sufficient to have acceptable order reduction. For example, if one of α or β is greater than 84 ($\alpha or \beta > 84$) while the other factor satisfies the must condition ($\alpha \& \beta \geq 2.9$), the reduction is acceptable for $\varepsilon_{Max} = 10\%$ (i.e., $\varepsilon \leq 10\%$). This condition is useful when only one pole is far away from the dominant pole while the second pole is closer to the dominant pole. For example, if the denominator of the transfer function is $(s + 1)(s + 3)(s + 100)$, the system can be reduced to first-order $(1/(s + 1))$ even if the second pole is only three-times the dominant pole.

TABLE 1. Sufficient conditions to obtain first-order reduction considering $\omega_n t_s$ as objective function and $\zeta \geq 1$.

Objective		Settling Time
Reduction		$\frac{\alpha\beta P_1^3}{(s+P_1)(s+\alpha P_1)(s+\beta P_1)}$ To $\frac{P_1}{(s+P_1)}$
Must Condition		(α & $\beta \geq 2.9$) for ε_{Max} =10% (α & $\beta \geq 5.5$) for ε_{Max} =5%
	ε_{Max}	Condition
Extreme	10%	(α *OR* $\beta > 84$) & (α & $\beta \geq 2.9$)
	5%	(α *OR* $\beta > 190$) & (α &$\beta \geq 5.5$)
$\alpha = 10$	11.89%	(α *OR* $\beta \geq 10$) & (α & $\beta \geq 2.9$)
	5.11%	(α & $\beta \geq 10$)
Ineq.	10%	($\alpha + \beta \geq 15.5$) & (α &$\beta \geq 3.4$)
	10%	($\alpha\beta^2 + 45.73\,\alpha\beta - 138.7\alpha - 177.4\beta \geq -261.2$) & ($\alpha$ &$\beta \geq 3.4$)
Relaxed 10%	11.51%	(α *or* $\beta > 12$) & (α &$\beta \geq 2.9$)
	11.27%	(α *or* $\beta > 13$) & (α &$\beta \geq 2.9$)
	11.17%	(α *or* $\beta > 14$) & (α &$\beta \geq 2.9$)
	10.35%	(α *or* $\beta > 34$) & (α &$\beta \geq 2.9$)
Ineq.	5%	($\alpha + \beta \geq 23.6$) & (α &$\beta \geq 7.6$)
	5%	($\alpha\beta - 5.392\alpha - 5.39\beta \geq -5.628$) & ($\alpha$ &$\beta \geq 7.6$)
Relaxed 5%	6.26%	(α *or* $\beta > 16$) & (α & $\beta \geq 5.5$)
	5.77%	(α *or* $\beta > 25$) & (α & $\beta \geq 5.5$)
	5.54%	(α *or* $\beta > 34$) & (α & $\beta \geq 5.5$)

As shown from Fig. 2, the boundaries of the filtered data which satisfy the error criteria for small values of α and β are plotted to have useful relation. Then, the points (α, β) which fulfill the error criterion are included above the linear plane ($\alpha + \beta \geq 15.5$). The linear plane is preferred due to its simplicity but it excludes some acceptable points or factors (α, β). For this case, a nonlinear inequality is introduced to cover almost all of the acceptable factors ($\alpha\beta^2 + 45.73\alpha\beta - 138.7\alpha - 177.4\beta + 261.2 \geq 0$). This non-linear inequality can also be used to have piece-wise linearized inequalities without going back to the filtered data. It is important to emphasize here that both the inequality and the "must condition" must be satisfied to have acceptable order reduction and this "ANDing" with the "must condition" is mandatory for all other relations. As listed in Table 1, If the

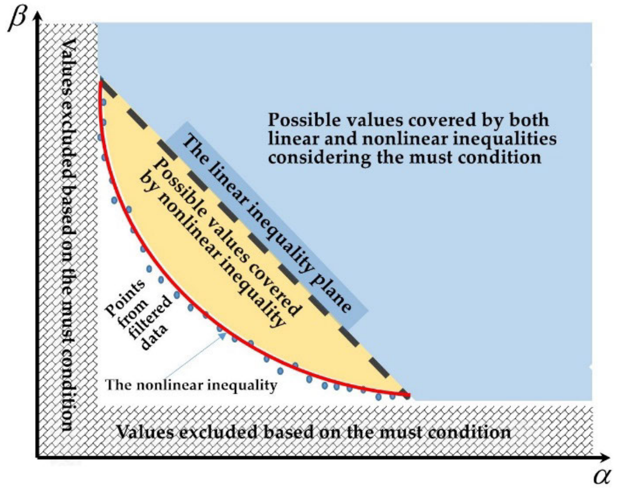

FIGURE 2. Generalized description of the excluded ranges and the ranges covered by the linear and the nonlinear inequalities.

error criterion can be relaxed in any control problem, the 84-times condition (α *or* $\beta > 84$) can be reduced to 10-,12-, 13-,14-, or 34-times for ε_{Max} = 11.90%, 11.52%, 11.27%, 11.17%, and 10.35%, respectively. The results for $\varepsilon_{Max} = 5\%$ are also listed in Table 1. It is important to highlight the status of 10-times condition, which is highlighted with all results throughout the paper. If $\alpha = \beta = 10$, then $\varepsilon_{Max} = 5.11\%$.

Following the same procedure but assuming that the designer concern is mainly about the system t_r, the reduction is attainable only if ($\alpha \& \beta > 2.6$) for $\varepsilon_{Max} = 10\%$. In other words, ($\alpha \& \beta \geq 2.6$) is the must condition for this objective and the error in t_r is always greater than 10% if $\alpha or \beta$ is less than 2.6. As summarized in Table 2, the sufficient conditions are easier and both the must condition and the inequality

TABLE 2. Sufficient conditions to obtain first-order reduction system considering $\omega_n t_r$ as objective function and $\zeta \geq 1$.

Objective		Rise Time
Reduction		$\frac{\alpha\beta P_1^3}{(s+P_1)(s+\alpha P_1)(s+\beta P_1)}$ To $\frac{P_1}{(s+P_1)}$
Must Condition		(α & $\beta \geq 2.7$) for ε_{Max} =10% (α &$\beta \geq 4.0$) for ε_{Max} =5%
	ε_{Max}	Condition
Extrem	10%	(α *OR* $\beta > 11.7$) & (α & $\beta \geq 2.7$)
	5%	(α *OR* $\beta > 46$) & (α &$\beta \geq 4.0$)
10-times	10.15%	(α *OR* $\beta \geq 10$) & (α &$\beta \geq 2.7$)
	5.62%	(α *OR* $\beta \geq 10$) & (α &$\beta \geq 4.0$)
	1.63%	(α & $\beta \geq 10$)
Ineq.	10%	($\alpha + \beta \geq 14.5$) & (α &$\beta \geq 2.7$)
	10%	($\alpha\beta - 2.558\alpha - 2.576\beta \geq 5.279$) & ($\alpha$ &$\beta \geq 2.7$)
Ineq.	5%	($\alpha + \beta \geq 21.1$) & (α &$\beta \geq 4.0$)
	5%	($\alpha\beta - 3.89\alpha - 3.968\beta \geq -12.71$) & ($\alpha$ &$\beta \geq 4.0$)
Relaxed 5%	5.15%	(α *or* $\beta > 17$) & (α &$\beta \geq 4.2$)
	5.03%	(α *or* $\beta > 30$) & (α &$\beta \geq 4.2$)

constraints are covered by the conditions for t_s. Also, the 10-times condition is superb in terms of t_r where $\varepsilon_{Max} \leq 1.63\%$. The 10-times condition is acceptable as the ε_{Max} is 5.11% and 1.63% for t_s and t_r, respectively. On the other hand, the order reduction can be accepted without satisfying this condition. For example, if $\alpha = 7.5$ and $\beta = 8.0$, the 10-times role does not permit to ignore these two poles or any of them to reduce the transfer function to a second or first-order system. Checking the simple linear inequalities ($\alpha + \beta \geq 15.5$) and ($\alpha + \beta \geq 14.5$), the errors in t_s and t_r are $\leq$10%. The importance of thinking outside the 10-times condition becomes more crucial with underdamped system as in the next sections.

B. REDUCTION TO SECOND-ORDER SYSTEM FOR $\zeta \geq 1$

As $1 \leq \alpha \leq \beta$ and with conservation of replacing α by β which was clear in the previous section, reducing the system to second-order will focus on the factor β. From Tables 3 and 4, it is clear that the reduction to second-order is more comfortable. From the extreme conditions, β must be at least 2.8 to have 10% maximum normalized error in both t_s and t_r. As this range is much narrower than that in first-order reduction, the necessity of presenting the inequalities that cover the sufficient ranges of β are not included by the extreme condition becomes less. For the sake of completeness, the linear and the nonlinear inequality conditions are listed for $\varepsilon_{Max} = 5\%$ and 10% in t_s and t_r. The 10-times condition is relatively hard condition for such order reduction as the $\varepsilon_{Max} = 2.55\%$ and 0.81% for t_s and t_r, respectively. In terms of both t_s and t_r, the general constraint for reducing the overdamped system to second-order is to have $\beta \geq 2.8$ and $\beta \geq 5.3$ for $\varepsilon_{Max} = 10\%$ and 5%, respectively.

TABLE 3. Sufficient conditions to obtain second-order reduction system considering $\omega_n t_s$ as objective function and $\zeta \geq 1$.

Objective		Settling Time
Reduction		$\frac{\alpha\beta P_1^3}{(s+P_1)(s+\alpha P_1)(s+\beta P_1)}$ To $\frac{\alpha P_1^2}{(s+P_1)(s+\alpha P_1)}$
Must Condition		($\beta \geq 2$) for ε_{Max} =10% ($\beta \geq 4.7$) for ε_{Max} =5%
ε_{Max}		Condition
Extreme	10%	($\beta \geq 2.8$)
Extreme	5%	($\beta \geq 5.3$)
10-tim	2.55%	($\beta = 10$) & ($\alpha \leq 10$)
Ineq. 10%	10%	($\beta \geq 0.1176\alpha + 2.382$)
Ineq. 10%	10%	($\alpha\beta - 0.3349\beta - 2.833\alpha \geq -1.503$)
Ineq. 5%	5%	($\beta \geq 0.1818\alpha + 4.364$)
Ineq. 5%	5%	($\alpha\beta - 0.2978\beta - 5.375\alpha \geq -2.81$)

IV. ORDER REDUCTION FOR $\zeta < 1$

For underdamped third-order system, the reduction of order may lead to undamped first-order system or standard underdamped second-order system. Usually, the researcher focuses on reducing the order to second. In this section, the sufficient conditions to have acceptable order reduction to both first and second-order systems are presented.

TABLE 4. Sufficient conditions to obtain first-order reduction system considering $\omega_n t_r$ as objective function and $\zeta < 1$.

Objective		Rise Time
Reduction		$\frac{\gamma\zeta\omega_n^3}{(s^2+2\zeta\omega_n+\omega_n^2)(s+\gamma\zeta\omega_n)}$ To $\frac{\gamma\zeta\omega_n}{(s+\gamma\zeta\omega_n)}$
Must Condition		($\gamma \geq 2$) for ε_{Max} =10% and if $\zeta > 0.6$ ($\gamma \geq 3.2$) for ε_{Max} = 5% and if $\zeta > 0.65$
ε_{Max}		Condition
Extreme	10%	($\gamma < 0.28$)
Extreme	10%	($\zeta \leq 0.60$) & ($\gamma < 66$)
Extreme	5%	($\gamma < 0.19$)
Extreme	5%	($\zeta \leq 0.44$) & ($\gamma \leq 0.44$)
Extreme	5%	($0.45 \leq \zeta \leq 0.64$) & ($\gamma \leq 0.57$)
Ineq. 10%		**For ($0.99 \geq \zeta > 0.60$) & ($\gamma < 1.5$)**
Ineq. 10%	10%	$\gamma - 6.873\,\zeta^2 + 13.91\zeta - 7.312 \leq 0$
Ineq. 10%	10%	$\gamma - 0.2725\ \zeta^{-3.384} \leq 0$
Ineq. 5%		**For ($0.99 \geq \zeta \geq 0.65$) & ($\gamma < 1.1$)**
Ineq. 5%	5%	$\gamma - 9.267\zeta^2 + 17.68\zeta - 8.626 \leq 0$
Ineq. 5%	5%	$\gamma - 0.1502\ \zeta^{-4.596} \leq 0$

A. REDUCTION TO FIRST-ORDER SYSTEM

Considering t_s for both 2% and 5% criteria, the sufficient conditions are summarized in Table 5. For example; if 2% criterion is adopted and $\varepsilon_{Max} = 10\%$, the reduction to first-order is only acceptable if $\gamma \leq 3.4$ (must-condition). This also means that the reduction may be acceptable even if the third pole is greater than the real part of the complex conjugate poles. For $\varepsilon_{Max} = 10\%$, the simple check is to have $\gamma < 0.20$, which means that the third pole is five-times closer to the origin than the real part of the complex poles. For small values of ζ (i.e., $\zeta < 0.40$), the condition on the third pole becomes less than 120% of the real part of the complex poles ($\zeta < 0.40 \& \gamma < 1.2$). The 10-times condition (i.e., $\gamma < 0.10$) is sufficient to have $\varepsilon_{Max} = 5\%$. For further acceptable ranges that satisfy the following condition: $(0.99 \geq \zeta \geq 0.40) \& (\gamma < 1.1)$, the linear inequality constraint ($\gamma \leq 1.5690 - 1.496\zeta$) can be used to check if the reduction keeps ε_{Max} below the set value. The same discussion is applicable for 5% criterion as shown in Table 5 but the ranges and the constraints are clearly relaxed.

If the aim is directed towards t_r, the new results are listed in Table 6. This means that ε_{Max} is greater than 10% if the real pole is at least 150% of the real part of the complex poles. Generally, and for $\varepsilon_{Max} = 10\%$, the reduction to first-order is acceptable if $\gamma < 0.28$ regardless of the value of ζ. On the other hand, if $\zeta \leq 0.6$, this reduction is acceptable even if γ is up to 65 ($\varepsilon_{Max} = 10\%$). In this case the must condition ($\gamma < 1.5$) is valid only for higher values of ζ ($\zeta > 0.60$). The sufficient inequality constraints for ($\zeta > 0.60$) are listed in Table 6. Moreover, the results considering $\varepsilon_{Max} = 5\%$ are also listed in Table 6 which has the same logic and more restricted conditions. It is important to highlight here that

TABLE 5. Sufficient conditions to obtain first-order reduction system considering $\omega_n t_s$ as objective function and $\zeta < 1$ (2% and 5% criterions).

Objective		Settling Time (2% Criterion)
Reduction		$\frac{\gamma\zeta\omega_n^3}{(s^2+2\zeta\omega_n+\omega_n^2)(s+\gamma\zeta\omega_n)}$ To $\frac{\gamma\zeta\omega_n}{(s+\gamma\zeta\omega_n)}$
Must Condition		$(\gamma \le 3.4)$ for ε_{Max} =10% $(\gamma \le 2.8)$ for ε_{Max} =5%
	ε_{Max}	Condition
Extreme	10%	$(\gamma < 0.20)$ OR $(\zeta < 0.40$ & $\gamma < 1.2)$
Extreme	5%	$(\gamma < 0.10)$ OR $(\zeta < 0.45$ & $\gamma < 0.8)$
Inequalitie s		For $(0.99 \ge \zeta \ge 0.40)$ & $(\gamma < 1.1)$
Inequalitie s	10%	$\gamma \le 1.569 - 1.496\,\zeta$
Inequalitie s	10%	$\gamma\zeta + 0.4343\zeta - 0.6257 \le 0$
Inequalities		For $(0.99 \ge \zeta > 0.45)$ & $(\gamma < 0.62)$
Inequalities	5%	$\gamma - 2.164\,\zeta^2 + 3.953\zeta - 1.914 \le 0$
Inequalities	5%	$\gamma\zeta - 0.3467\gamma - 0.07275 \le 0$
Objective		Settling Time (5% Criterion)
Must Condition		$(\gamma < 4.7)$ for ε_{Max} =10% $(\gamma < 3.3)$ for ε_{Max} =5%
	ε_{Max}	Condition
Extreme	10%	$(\gamma < 0.16)$ OR $(\zeta < 0.35$ & $\gamma < 1.1)$
Extreme	5%	$(\gamma < 0.08)$ OR $(\zeta < 0.39$ & $\gamma < 0.79)$
Inequalitie s		For $(0.99 \ge \zeta \ge 0.35)$ & $(\gamma < 1)$
Inequalitie s	10%	$\gamma \le 1.333 - 1.31\zeta$
Inequalitie s	10%	$\gamma\zeta - 0.1768\gamma - 0.1915 \le 0$
Inequalities		For $(0.99 \ge \zeta \ge 0.39)$ & $(\gamma < 0.76)$
Inequalities	5%	$\gamma \le 2.785\zeta^2 - 4.729\zeta + 2.082$
Inequalities	5%	$\gamma\zeta - 0.314\gamma - 0.06421 \le 0$

the reduction to first-order is not only depends on closeness to the origin which is represented here by the factor γ as commonly discussed. The value ζ becomes a key player when it is relatively very low. For example, the normalized error is only 5% when $\zeta = 0.02$ even if $\gamma = 81$. The normalized error becomes 5% when the third pole is 60-times the real part of the complex poles if $\zeta = 0.03$.

Reducing the order of the system to first produces a system without overshoot. If the control engineer accepts to have a peak value which exceeds the final value by the 5% or 10%, this reduction has a value. In other words, the *OS* of the underdamped third-order system is less than or equal to 10% and 5% if the conditions summarized in Table 7 is satisfied.

B. REDUCTION TO SECOND-ORDER SYSTEM

Reducing the system to second-order is more common. In order to keep the values of the natural frequency and the damping ratio of the system from altering, the impact of the third real pole is studied. This part is usually discussed based on the distance between the real pole and the real part of the complex poles to determine the dominant poles. Following the same steps in Fig. 1 and considering the three transient characteristics, the third-order system is extensively studied for this important part.

TABLE 6. Sufficient conditions to obtain second-order reduction system considering $\omega_n t_r$ as objective function and $\zeta \ge 1$.

Objective		Rise Time
Reduction		$\frac{\alpha\beta P_1^3}{(s+P_1)(s+\alpha P_1)(s+\beta P_1)}$ To $\frac{\alpha P_1^2}{(s+P_1)(s+\alpha P_1)}$
Must Condition		$(\beta \ge 2)$ for ε_{Max} =10% $(\beta \ge 3.2)$ for ε_{Max} =5%
	ε_{Max}	Condition
Extrem e	10%	$(\beta \ge 2.6)$
Extrem e	5%	$(\beta \ge 4.0)$
10-tim	0.81%	$(\beta = 10)$ & $(\alpha \le 10)$
Ineq. 10%	10%	$(\beta \ge 0.2\alpha + 1.8)$
Ineq. 10%	10%	$(\alpha\beta - 3.766\beta - 3.12\alpha - 6.393 \ge 0)$
Ineq. 5%	5%	$(\beta \ge 0.1613\alpha + 2.981)$
Ineq. 5%	5%	$(\alpha\beta + 0.1045\beta - 4.135\alpha + 1.524 \ge 0)$

TABLE 7. Sufficient conditions to obtain first-order reduction system considering *OS* as objective function and $\zeta < 1$.

Objective		Over Shoot
Reduction		$\frac{\gamma\zeta\omega_n^3}{(s^2+2\zeta\omega_n+\omega_n^2)(s+\gamma\zeta\omega_n)}$ To $\frac{\gamma\zeta\omega_n}{(s+\gamma\zeta\omega_n)}$
Must Condition		$(\gamma \le 2.6)$ for ε_{Max} =10% if $\zeta < 0.91$ $(\gamma \le 2.4)$ for ε_{Max} = 5% if $\zeta < 0.93$
	ε_{Max}	Condition
Extreme	10%	$(\gamma \le 1.0)$
Extreme	10%	$(\zeta \ge 0.91)$
Extreme	5%	$(\gamma \le 1.0)$
Extreme	5%	$(\zeta \ge 0.93)$
Ineq. 10%		For $(\zeta \le 0.62)$ & $(1.1 \le \gamma \le 2.6)$
Ineq. 10%	10%	$\gamma - 0.9795\ \zeta^{-0.2071} \le 0$
Ineq. 10%		For $(0.8 \le \zeta \le 0.9)$ & $(1.1 \le \gamma \le 1.8)$
Ineq. 10%	10%	$\gamma - 108\zeta^2 + 177.3\zeta - 73.85 \le 0$
Ineq. 5%		For $(\zeta \le 0.62)$ & $(1.1 \le \gamma \le 2.4)$
Ineq. 5%	5%	$\gamma - 0.9731\ \zeta^{-0.1885} \le 0$
Ineq. 5%		$(0.92 \ge \zeta \ge 0.84)$ & $(1.1 \le \gamma \le 2.4)$
Ineq. 5%	5%	$\gamma - 115.6\zeta^2 + 197\zeta - 85.04 \le 0$

Starting with t_s, the third-order system can be reduced directly to the standard second-order, based on the general rules in Table 8. It is important to clarify that the restricted 2% and 5% criteria lead to high normalized error in a wide range. In many cases, the response of the reduced-order model is neatly close to the exact model. The reduced-order model exceeds the envelope of the 2% or 5% criteria by small value and the exact model keeps damping inside this envelope. This will cause a big difference between the two settling times, even for a minimal value in terms of magnitude. Considering that any approximation is linked with accepting small error. To reduce this problem, the criteria are slightly relaxed to be 2.25%. This step does not affect the accuracy of the presented relations as the maximum normalized error is unchanged. For example, when $\zeta = 0.38$ and $\gamma = 9.9$, the responses of the original third-order system and the

TABLE 8. Sufficient conditions to obtain second-order reduction system considering $\omega_n t_s$ as objective function and for $\zeta < 1$. (2% and 5% criterions).

	Settling Time (2% Criterion)
Reduction	$\frac{\gamma\zeta\omega_n^3}{(s^2+2\zeta\omega_n+\omega_n^2)(s+\gamma\zeta\omega_n)}$ To $\frac{\omega_n^2}{(s^2+2\zeta\omega_n+\omega_n^2)}$
ε_{Max}	Condition
10%	$(0.99 \geq \zeta \geq 0.38)$ & $(\gamma \geq 4.0)$
10%	$(0.38 > \zeta \geq 0.10)$ & $(\gamma \geq 7.0)$
5%	$(0.99 \geq \zeta \geq 0.18)$ & $(\gamma \geq 7.6)$
5%	$(0.18 > \zeta \geq 0.05)$ & $(\gamma \geq 23.1)$
Objective	Settling Time (5% Criterion)
ε_{Max}	Condition
10%	$(0.99 \geq \zeta \geq 0.30)$ & $(\gamma > 5.6)$
10%	$(0.30 > \zeta \geq 0.03)$ & $(\gamma > 15)$
5%	$(0.99 \geq \zeta \geq 0.16)$ & $(\gamma > 11.5)$
5%	$(0.16 > \zeta \geq 0.04)$ & $(\gamma > 34)$

approximated second-order system are presented in Fig. 3. Relaxing the criteria of t_s to be 2.25% and 5.5% results in a new normalized settling time for the second-order system (8.3502 *rad*). From Fig. 3, it is clear that the two responses are very close and the second-order system is sufficient to be adopted. Without relaxing the set criteria, having a simple general condition to directly reduce the system's order to its second-order system will not be possible for wide range of γ.

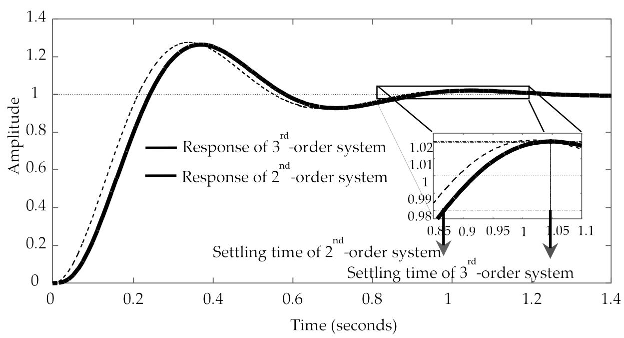

FIGURE 3. Step response of third and second order systems when $\zeta = 0.38$, $\omega_n = 1$, and $\gamma = 9.9$.

Usually this order reduction relies totally on the 10-times condition ($\gamma \geq 10$). It is clear from Table 8 that the reduction can be conditionally accepted for $\gamma \geq 4$. On the other hand, the condition $\gamma \geq 10$ is not acceptable for very low value of ζ. For example, if $\zeta = 0.01$, the lowest value of γ needed to have $\varepsilon_{Max} = 10\%$ in the normalized rise time is 76 and 99 for 5% and 2% criteria, respectively. If $\zeta = 0.01$ and $\varepsilon_{Max} = 5\%$, then γ must be 72 and 80 for 5% and 2% criteria, respectively.

As shown in Table 9 with considering the normalized rise time, the value of ζ plays more important role than that in the normalized settling time. This is clear from the extreme

TABLE 9. Sufficient conditions to obtain second-order reduction system considering $\omega_n t_r$ as objective function and $\zeta < 1$.

Objective		RiseTime
Reduction		$\frac{\gamma\zeta\omega_n^3}{(s^2+2\zeta\omega_n+\omega_n^2)(s+\gamma\zeta\omega_n)}$ To $\frac{\omega_n^2}{(s^2+2\zeta\omega_n+\omega_n^2)}$
Must Condition		$(\gamma > 1.66)$ for ε_{Max} =10% $(\gamma > 2.4)$ for ε_{Max} = 5%
ε_{Max}		Condition
Extreme	10%	$(\gamma \geq 220)$
	5%	$(\gamma \geq 340)$
Ineq. 10%		For $(0.18 > \zeta \geq 0.04)$
	10%	$\gamma \geq 3029\zeta^2 - 900.5\zeta + 81.3$
	10%	$\gamma\zeta - 2.172 \geq 0$
		For $(0.91 > \zeta \geq 0.18)$
	10%	$\gamma \geq 22.79\zeta^2 - 36.92\zeta + 17.47$
	10%	$\gamma\zeta \geq 2.342$
		For $(0.99 \geq \zeta \geq 0.91)$
	10%	$\gamma \geq 5.657 - 4.033\zeta$
Ineq. 5%		For $(0.31 > \zeta \geq 0.05)$
	5%	$\gamma \geq 1111\zeta^2 - 55.3\zeta + 83.14$
	5%	$\gamma\zeta \geq 3.534$
		For $(0.99 \geq \zeta \geq 0.31)$
	5%	$\gamma \geq 17.92\zeta^2 - 36.13\zeta + 21.07$
	5%	$\gamma\zeta \geq 3.53$

condition, which is extremely high for both $\varepsilon_{Max} = 5\%$ and 10%. This condition reflects the importance of this study and how the 10-times condition is not valid for low ζ. To have accurate relations, the range of ζ is divided into three ranges to have $\varepsilon_{Max} = 10\%$ and two ranges to have $\varepsilon_{Max} = 5\%$. For very low values of ζ, γ must be very large to attain the ε_{Max} condition. For $\varepsilon_{Max} = 10\%$, γ must be at least 220, 110 and 72, if ζ is 0.01, 0.02 or 0.03, respectively.

In second-order systems, the *OS* depends only on ζ. For the third-order system under study, both ζ and γ are important in determining the overshoot. Table 10 summaries the needed conditions to reduce the order of the system to second while keeping its physical parameters as it is for $\varepsilon_{Max} = 10\%$ and 5%. Starting with testing the 10-times condition, the reduction of order is only and conditionally accepted for $\gamma \geq 3.2$ and not only for $\gamma \geq 10$. The extreme condition is large regardless of the value of ζ. At the same time, if $\zeta \geq 0.94$, then $\varepsilon_{Max} = 10\%$ regardless of the value of γ. It is important to emphasize the effect of low values of ζ which is not considered in the 10-times condition. For $\varepsilon_{Max} = 10\%$, γ must be at least 220, 110, 74 and 56, when ζ is equal to 0.01, 0.02, 0,03 and 0.04, respectively. On the other hand, If ($\zeta \geq 0.95$), *OS* is negligible for both 3rd and 2nd order systems. At $\zeta = 0.94$, the error is ignored logically and not mathematically due to the extremely low *OS*.

V. COMPARISONS

For known system parameters, using Matlab Control System Toolbox is a common and effective choice. The built-in

TABLE 10. Sufficient conditions to obtain second-order reduction system considering OS as objective function and $\zeta < 1$.

Objective Reduction		Over Shoot $\frac{\gamma\zeta\omega_n^3}{(s^2+2\zeta\omega_n+\omega_n^2)(s+\gamma\zeta\omega_n)}$ To $\frac{\omega_n^2}{(s^2+2\zeta\omega_n+\omega_n^2)}$
Must Condition		($\gamma > 3.1$) for ε_{Max} =10% if $\zeta < 0.94$ ($\gamma > 4.1$) for ε_{Max} = 5% if $\zeta < 0.95$
	ε_{Max}	Condition
Extreme	10%	($\gamma \geq 220$)
	10%	($\zeta \geq 0.94$)
	5%	($\gamma \geq 320$)
	5%	($\zeta \geq 0.94$)
Ineq. 10%		($0.17 \geq \zeta \geq 0.05$)
	10%	$\gamma - 2398\zeta^2 + 758.2\zeta - 74.89 \geq 0$
	10%	$\gamma\zeta - 2.274 \geq 0$
		($0.93 \geq \zeta \geq 0.18$)
	10%	$\gamma - 20.69\zeta^2 + 32.8\zeta - 16.4 \geq 0$
	10%	$\gamma\zeta - 2.439 \geq 0$
		($0.99 \geq \zeta \geq 0.91$)
	10%	$\gamma + 4.033\zeta - 5.657 \geq 0$
Ineq. 5%		($0.25 \geq \zeta \geq 0.06$)
	5%	$\gamma - 1348\zeta^2 + 592.8\zeta - 80.05 \geq 0$
	5%	$\gamma\zeta - 3.242 \geq 0$
		($0.93 \geq \zeta \geq 0.26$)
	5%	$\gamma - 20.37\zeta^2 + 35.21\zeta - 19.68 \geq 0$
	5%	$\gamma\zeta - 3.53 \geq 0$

function "balred" with the default options is used to validate the effectiveness of the proposed work. It is important to highlight that the main advantage of this work is not the highest accuracy. It focuses on keeping the system parameters with an acceptable predefined maximum normalized error.

Underdamped systems are the most commonly used systems in industrial applications, while the standard underdamped second-order system is extensively studied in the literature and textbooks. This gives the underdamped third-order system the highest priority without losing the effectiveness of the proposed work in overdamped systems. All examples are not accepted using the 10-times condition.

A. EXAMPLE 1: REDUCTION TO FIRST-ORDER SYSTEM

In this example, the transfer function of the original system ($G_o(s)$) is presented in (3). This system has, $\gamma = 0.4$, $\zeta = 0.3$ and $\omega_n = 10$ rad/s.

$$G_o(s) = \frac{120}{(s+1.2)\left(s^2+6s+100\right)} \tag{3}$$

Using "balred" function, the reduced transfer function using Matlab toolbox ($\hat{G}_M(s)$) is presented in (4). It is clear that the new pole is not identical with the original real pole. Besides that, new zero is introduced in the system which may affect the frequency response of the system.

$$\hat{G}_M(s) = \frac{-0.098714\,(s-12.59)}{(s+1.243)} \tag{4}$$

This example matches the study in Tables 5, 6 and 7. For t_s, the system matches this condition "($\zeta < 0.45$& $\gamma < 0.8$)" which guarantees 5% maximum error. This same case is applicable to t_r and OS. The reduced transfer function using the proposed work ($\hat{G}_P(s)$) is:

$$\hat{G}_P(s) = \frac{1.2}{(s+1.2)} \tag{5}$$

The proposed reduction does not need any computational efforts like other reduction methods. The most important point is that the reduced system is part of the original system (i.e., same real pole). The comparison and the step response of the three transfer functions are shown in Fig. 4. The t_s using the proposed work is more accurate where the error is 1.49% compared to 4.91% for $\hat{G}_M$. The error in t_s is 3.86% which is higher than the results from "balred" (0.26%). All systems have no overshoot. The step response of the three systems is shown in Fig. 4.

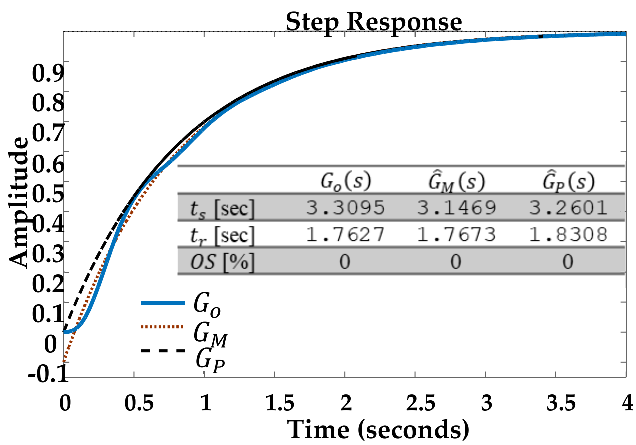

FIGURE 4. Step response of third-order system when $\gamma = 0.4$, $\zeta = 0.3$ and $\omega_n = 10$ rad/s and its two estimated transfer functions $\hat{G}_M$ and $\hat{G}_P$.

B. EXAMPLE 2: REDUCTION TO SECOND-ORDER SYSTEM

This example is used to have underdamped second-order system. The three transfer functions are shown below:

$$G_o(s) = \frac{5.6\times10^6}{(s+560)\left(s^2+140s+1\times10^4\right)} \tag{6}$$

$$\hat{G}_M(s) = \frac{0.015672\left(s^2-961.7s+6.205\times10^4\right)}{\left(s^2+138.2s+9724\right)} \tag{7}$$

$$\hat{G}_P(s) = \frac{1\times10^4}{\left(s^2+140s+1\times10^4\right)} \tag{8}$$

For this case, $\zeta = 0.7$, $\omega_n = 100$ rad/s and $\gamma = 8$. It is clear that the 10-times condition is not fulfilled while the accuracy of the system is stellar. The comparison between the three transfer functions is summarized in Table 11. The highest normalized error is 2.92%. The second-order numerator in $\hat{G}_M$ adds two complex-conjugated zeros on the right side of the s-plane. This complexity is introduced by the algorithms to have very accurate reduced model. Adopting the proposed

TABLE 11. Step-response indices of the three transfer functions in example 2.

	$G_o(s)$	$\hat{G}_M(s)$	$\hat{G}_P(s)$	Error %
t_s [sec]	0.0616	0.0622	0.0598	2.92
t_r [sec]	0.0217	0.0213	0.0213	1.84
OS [%]	4.5130	4.5599	4.5986	1.90

work results in a clearly simplified new reduction system with a natural error tradeoff like any other approximation.

VI. PRACTICAL EXAMPLES

In this section, three practical examples are considered to clarify the importance of this work.

A. MATHEMATICAL ANALYSIS

Fig. 5 shows the blood pressure control system block diagram [19]. The resulted transfer function is a third-order system as in (9) and the valve gain "K" is the design parameter.

$$G_o = \frac{Y(s)}{R(s)} = \frac{K}{s^3 + 4s^2 + 4s + K} \quad (9)$$

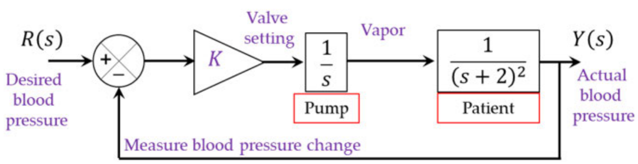

FIGURE 5. Blood pressure control system.

It is known that third-order systems analytical analysis is not handy to solve for K, one option for the designer is to assume that the third pole is nondominant (ten-time the real part of the complex poles) and consider the poles of the second-order terms. This converts the characteristic equation to be represented in the following form:

$$\begin{aligned} s^3 + 4s^2 + 4s + k &= \left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)(s + 10\zeta\omega_n) \\ &= s^3 + 12\zeta\omega_n s^2 + (\omega_n^2 + 20(\zeta\omega_n)^2)s + 10\zeta\omega_n^3 \end{aligned} \quad (10)$$

From (10), the value of $\zeta\omega_n$ must be (1/3). Consequently, there is a unique solution for this case in which $\omega_n = 4/3$ and $\zeta = 1/4$. This leads to a design value of $K = 160/27$. The increment in K leads to a lower value of ζ and, eventually, higher $O.S$. According to these parameters, the original transfer function $O.S = 40.8\%$ and 44.3% for the approximated second-order system.

Based on the proposed work as shown in Table 10, the sufficient condition (to have an acceptable order reduction) has two degrees of freedom ζ and γ ($\zeta\gamma \geq 3.533$). To explain that, the following two cases are discussed:

Case1: $\zeta\gamma = 4$ and $\zeta = 1.0$ (to eliminate the overshoot).

The new characteristic equation will be:

$$s^3 + 4s^2 + 4s + k = s^3 + 6\omega_n s^2 + 9\omega_n^2 s + 4\omega_n^3 \quad (11)$$

Then, $\omega_n = 2/3$ and $k = 32/27$. The new transfer function is:

$$G_o = \frac{32/27}{s^3 + 4s^2 + 4s + 32/27} = \frac{32/27}{\left(s^2 + 4s/3 + 4/9\right)(s + 8/3)} \quad (12)$$

Based on the proposed work, the approximated transfer function $\hat{G}_p$ will be:

$$\hat{G}_p = \frac{4/9}{\left(s^2 + 4s/3 + 4/9\right)} \quad (13)$$

Table 12 shows a comparison between the original and the proposed approximated functions.

TABLE 12. Validation of the proposed method using blood pressure control system.

	G_o	$\hat{G}_p$	Error%
Rise time	5.1352	5.0369	1.91%
Settling time	9.1707	8.7509	4.58%
Overshoot	0	0	N.A

Based on the proposed work, the freedom in ζ selection balances depth of control between $O.S$ and ω_n, where the latter controls the speed of the response. In the next case, the first target ($O.S$ control) is considered. Without the proposed work, these two degrees of freedom are not possible without using commercial software, and this shows the importance of this study as the stepping stone of the third-order systems analytics.

Case 2: $\zeta\gamma = 3.6$ and $\zeta = 0.8$ (to satisfy the deadbeat overshoot condition).

This case can't be analytically solved considering the 10-times condition and the desired poles are complex. This shows the importance of finding an alternative analytical analysis to third-order systems.

Following the same approach in case 1, $\omega_n = 4/5.2$ and $k = 3600/2197$. The new transfer function is:

$$\begin{aligned} G_o &= \frac{3600/2197}{s^3 + 4s^2 + 4s + 3600/2197} \\ &= \frac{3600/2197}{\left(s^2 + 16s/13 + 100/169\right)(s + 36/13)} \end{aligned} \quad (14)$$

The proposed approximated transfer function is:

$$\hat{G}_p = \frac{100/169}{\left(s^2 + 16s/13 + 100/169\right)} \quad (15)$$

The effectiveness of the proposed work is shown in the comparative results in Table 13. Ultimately, ζ flexible selection leads to a balanced performance.

Another example is considered to convert the system into a first-order system. Fig. 6 represents an automatic fluid dispenser [19]. The closed-loop transfer function is shown in (16).

$$G_o = \frac{K}{s^3 + 15s^2 + 50s + K} \quad (16)$$

TABLE 13. Transient response comparison considering blood pressure control system "case 2".

	G_o	$\hat{G}_p$	Error%
Rise time	3.328	3.208	3.62%
Settling time	5.318	4.883	8.18%
Overshoot	1.439	1.516	5.40%

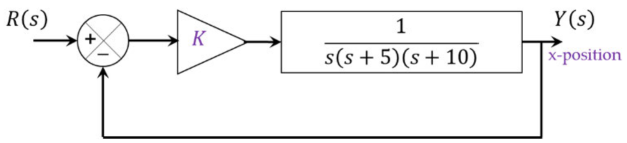

FIGURE 6. Automatic fluid dispenser.

It is hard to analytically link between the speed of the system and the value of the gain K. As in (16)-(17) and based on G_3 in (1) and for $\zeta \geq 1$, there are many possible solutions to the three poles of the system based on the value of K.

$$\frac{K}{s^3 + 15s^2 + 50s + K} = \frac{(s_1 s_2 s_3)}{(s + s_1)(s + s_2)(s + s_3)} \quad (17)$$

Then,

$$s_1 + s_2 + s_3 = -15 \quad (18)$$

$$s_1 s_2 + s_2 s_3 + s_3 s_1 = 50 \quad (19)$$

$$s_1 s_2 s_3 = -K \quad (20)$$

Based on the proposed work, the system may be reduced to first-order system "$K_{dc}/(\tau s + 1)$". Then, the time constant of the system can be directly approximated by $1/|s_1|$. The designer has the option to select the value of the time-constant as the dc-gain is known from the original system. The importance of this work can be explained by considering the sufficient condition to have acceptable order reduction. Based on Tables 1 and 2, essential mathematical experience, and considering 5% as maximum reduction error, α, which is the smaller ratio, must be at least 7.6. This conclusion can be expressed as new equation that can help in solving the previous set of equations (17)-(20), while granting an insight of mathematical-physical meanings.

$$\frac{s_2}{s_1} = 7.6 \quad (21)$$

The inequality is converted to equality in (21) to find the maximum value of K that keeps the reduction to first order acceptable. The lower the value of K is, the closer the first pole to the origin is. Solving the four equations for: $K = 22.5971$, $s_1 = -0.5346$, $s_2 = -4.0632$, and $s_3 = -10.4021$. Consequently, $\alpha = 7.6$ and $\beta = 19.4564$ which satisfy the two conditions $(\alpha + \beta \geq 23.6)$ & $(\alpha \& \beta \geq 7.6)$ in Table 1. The transient response of the original and the reduced order systems are shown in Table 14. The error in the rise time is much lower as the condition in Table 2 is relaxed "$(\alpha + \beta \geq 21.1)$ & $(\alpha \& \beta \geq 4.0)$".

Another analytical freedom that the proposed work grants to the designer is to select the time constant directly.

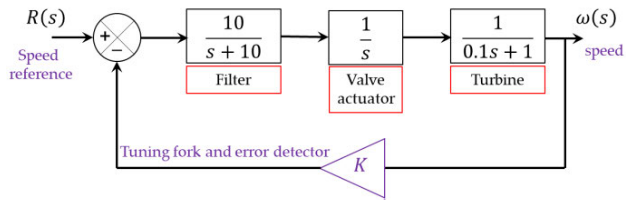

FIGURE 7. Steam turbine control system.

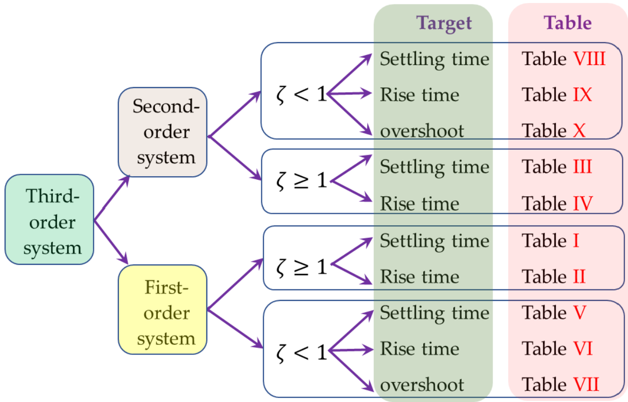

FIGURE 8. User-guide to the listed sufficient conditions.

For example, if the same mechanical system presented in Fig. 1 must be slower, and the time constant must be, for example, 2.5 s, the value of s_1 must be -0.4. Adding this relation ($s_1 = -0.4$) to the set of equations (17)-(20) ends by the following solution: $K = 17.664$, $s_1 = -0.4$, $s_2 = -4.2784$, and $s_3 = -10.3216$. Reducing the gain K has a greater impact on the closest pole to the origin as expected and noted in the first part of this example. As a result, $\alpha = 10.696$ and $\beta = 25.804$. The results in Table 14 show lower error as α and β are higher.

B. COMPARISON

The steam turbine control system [19] in Fig. 7 is considered in this section to show the effectiveness of the proposed work by comparing its outcomes with two different methods. The first method uses particle swarm optimization algorithm [20] while the other one uses genetic algorithm [21]. Both methods use the well-known integral squared error (ISE) of the transient response of the approximated model compared with the original model. The proposed work is superior to these methods since it is noniterative with off-line inequalities. Interestingly, the proposed work has the simplest model without losing accuracy. The results in Table 15 show that the error resulted from adopting the proposed work is closer to or better than the other two methods. Moreover, as the steam turbine is part of the thermal power plant, the proposed reduced order model has the simplest form, reducing the complexity of the overall system.

Based on the proposed work, the designer can reduce the order of the system with keeping at least one performance

TABLE 14. Comparison between the original and the reduced order model for automatic fluid dispenser in terms of the transient response indices.

Marginal value of the gain K			
	$G_o = \frac{22.5971}{s^3+15s^2+50s+22.5971}$	$\hat{G}_p = \frac{1}{1.87043s+1}$	Error%
Rise time	4.1783	4.1094	1.65%
Settling time	7.6798	7.3173	4.72%
Time constant = 2.5 s			
	$G_o = \frac{17.664}{s^3+15s^2+50s+17.664}$	$\hat{G}_p = \frac{1}{2.5s+1}$	Error%
Rise time	5.5362	5.4925	0.79%
Settling time	10.1243	9.7802	3.4%

TABLE 15. Comparative study considering steam turbine control system.

	Transfer Function	Rise Time (error%)	Settling Time (error%)	Overshoot (error%)
Original system	$\frac{10}{s^3+10.1s^2+s+0.1}$	3.7051	82.8210	63.5061
Proposed model ($\hat{G}_P$)	$\frac{0.999}{s^2+0.0899s+0.0999}$	3.6949 (0.2749%)	82.7313 (0.1084%)	63.6422 (0.0021%)
PSO-model ($\hat{G}_{PSO}$)	$\frac{0.0351s^2-0.1485s+1.0393}{s^2+0.107s+0.1}$	3.7254 (0.5476%)	72.2918 (12.713%)	58.1768 (0.0839%)
GA-model ($\hat{G}_{GA}$)	$\frac{0.0024s^2-0.0919+0.9947}{s^2+0.0889s+0.0997}$	3.6978 (0.1953%)	83.0283 (0.2502%)	63.8738 (0.0058%)

index close to the original system. Sometime, the designer can consider the rise time as an indicator to the speed of the control system while another can consider the settling time. Towards that end, all presented sufficient conditions are indexed in Fig. 8

VII. CONCLUSION

In the literature, third-order system does not have many mathematical studies. This paper studied the third-order system in terms of the transient response characteristics; settling time, rise time and percentage overshoot. The accurate mathematical models are extracted to represent the transient response characteristics. Based on a new procedure, simple inequalities are used to have order reduction depending on a predetermined normalized error. The third-order system is reduced to lower orders in terms of the transient response characteristics.

As a starting study that needs high accuracy, two acceptable approximation zones are presented. One of them is according to 10% normalized error and the other is according to 5% error to represent high and low errors, respectively. Each transient characteristic is studied separately and independently from the other characteristics.

The relationships between poles are set and exactly determined the intervals where the reduction is valid for both the first and second zones. The reduction criterion of the underdamped third-order systems is covered for all possible values of ζ regardless of the value of ω.

During this work, the well-known 10-time condition for reduction is checked for all the cases that are studied in terms of all transient response characteristics. After detailed analysis, it turned out that this condition is not necessarily always true to get an acceptable approximation. In some cases, this condition leads to a narrower space of reduction than the actual possible space. In other cases, this condition is not sufficient to have correct order reduction.

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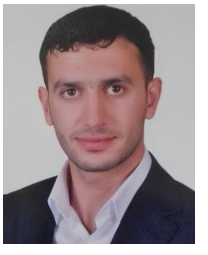

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