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Metric and Fault-Tolerant Metric Dimension of Hollow Coronoid

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ABSTRACT Coronoid systems actually arrangements of hexagons into six sides of benzenoids. By nature, it is an organic chemical structure. Hollow coronoids are primitive and catacondensed coronoids. It is also known as polycyclic conjugated hydrocarbons. The mathematical study of chemicals is of great interest to different specialties researchers. While graph theory always played a significant role to make chemical structures understandable and blessed with applications also. After transforming the chemical structure into a graph, one can implement different theoretical and implicative studies on structures. Metric dimension is considered as one of the most studied and implicative parameters of graph theory. In this concept, few suggested vertices are chosen such as the remaining vertices have unique locations or identifications. In this study, we discussed different metric-based parameters for the hollow coronoid structure.

INDEX TERMS Hollow coronoid, metric dimension, resolving set, fault-tolerant metric dimension.

I. INTRODUCTION

Chemical graph theory is considered a varied field and combination of chemistry and mathematics. It is actually an application of mathematics and therefore, it is known as mathematical chemistry. It deals with the study of different chemical structures, networks, and their topologies in the form of a graph (usually vertex and edge). This applied mathematics contributed a lot, it either solved complex mysteries or provides the tools to make them understandable. By drawing a graph or by transforming it to a graph, a chemical structure (network) shows its easiest or understandable topology. In different past studies shows that while transforming to graph, a chemical network's atom becomes vertex, and the line (bond) joining between atoms renamed as edges.

There are many different ways to analyze and study the electric circuits in terms of graph theory, we are moving to presents some graph-theoretical parameters as an application in electronics. In 1975, [17], [45] introduced an effective concept of visualization of a network, in this idea few principal nodes are selected so that one can attain the complete set of principal nodes in a unique identification in terms of distance vector, it is known as resolving set or metric basis. This concept lays down the foundation of many graph-theoretical parameters which are also implemented in different engineering, chemical, electrical, and other sciences. The fault-tolerant resolving set is another way of studying a network, in which the failure of a single principal node from resolving set can be tolerated and the entire set of principal nodes still have a unique position, it is defined by [6]. When the entire set of principal nodes divided into subsets and putting the condition of getting the unique position of a set of principal nodes is called a partition resolving set [7]. The above concepts are the topics of well-known metric-based resolvability parameters and studied for different graphs, networks, and structures.

The authors of [8], [18], [26], addressed the computational complexity and proved that all the resolvability parameter studied in this paper belongs to the problem of NP-hardness. The researchers are motivated by the fact that the metric dimension has a variety of functional uses in our everyday lives, and it has been extensively researched. Metric dimension is applied on different topics of fields such as weighing problem of coins [46], robot navigation also related to this concept by [25], pharmaceutical chemistry always been a part of the application of graph theory and particular to this topic discussed in [8], different studies conducted about the combinatorial optimization in [42] also related to this concept, coastguard Loran, sonar and facility location problems related to this concept in the seminal paper by [45], computer networks [28], for more detailed study of the applications of

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this parameter [32], [33]. Partition dimension is also falling in different applications of real-world, for example, for the strategies, decoding, and coding of games and especially the mastermind games brief in [11], the popular relation which is named as Djokovic-Winkler linked to this concept [6], the piloting or the guidance of a robot also associated with this unique idea [25], the procedure of discovering a network and also its verification related to this concept by [4]. For more usage and applications of these resolvability parameters, we refer to see [8], [17], [22], [23], [30].

As we discussed above that this has numerous use in the chemical field, much work has done with graph prospectives and metric dimension also consider important to study different structures with it, like the structure of H-Naphtalenic and VC_5C_7 nano-tubes discussed with metric concept [20], some upper bounds of cellulose network considering metric dimension as a point of discussion [43], metric of silicate star are computed in [44], a two-dimensional lattice of α-boron nanotubes discussed with specific applications in terms of metric dimension in [19]. For the partition dimension, a graph with $n-3$, partition dimension discussed in [3], $(4, 6)$ is a special type of fullerene structure and it is also studied by [29] with the concept of partition dimension, There are few very recent research on the bounded partition dimension, we encourage to have a look the articles [10], [27], [31]. The bounds of partitioning on the specific type of nanotube are studied in [39]. All the resolvability parameters of graph theory on the polycyclic aromatic hydrocarbons are discussed in [2]. For the fault-tolerant concept discussed in [21] for basic graphs, [34] for different interconnection networks along with implementation of its applications, some recent work can be acquired by the references [35], [36], [41].

In different applications of the usage of graph theory, the terminologies changed accordingly. Like, when an electrical circuit transformed into a graph, the current sources (or open circuits) renamed as vertices, while the voltage sources and passive elements replaced by edges. The vertices are known as principal nodes (or nodes) and edges become line segments (or branches) [47]. In this study, we can use any of the terms defined above, like principal nodes or simply nodes for the term of vertices, and line segments or branches may use for edges. Following are some very basic concepts and preliminary mathematical definitions especially useful for understanding the research work that has been done here.

Definition 1 [31]: Suppose $\aleph\left(V(\aleph), E(\aleph)\right)$ is an undirected graph of a chemical structure (network) with $V(\aleph)$ is called as set of principal nodes (vertex set) and $E(\aleph)$ is the set of branches (edge set). The distance between two principal nodes $\zeta_1, \zeta_2 \in V(\aleph)$, denoted as $d(\zeta_1, \zeta_2)$ is the minimum count of branches between $\zeta_1-\zeta_2$ path.

Definition 2 [45]: Suppose $R \subset V(\aleph)$ is the subset of principal nodes set and defined as $R = \{\zeta_1, \zeta_2, \ldots, \zeta_s\}$, and let a principal node $\zeta \in V(\aleph)$. The identification or locations $r(\zeta|R)$ of a principal node ζ with respect to R is actually a $s-$ordered distances $(d(\zeta, \zeta_1), d(\zeta, \zeta_2), \ldots, d(\zeta, \zeta_s))$. If each principal node from $V(\aleph)$ have unique identification according to the ordered subset R, then this subset renamed as a resolving set of network $\aleph$. The minimum numbers of the elements in the subset R is actually the metric dimension of $\aleph$ and it is denoted by the term $dim(\aleph)$.

Definition 3 [6]: A particular chosen ordered subset which were actually resolving set symbolize by R of a network $\aleph$ is considered to be a fault-tolerant denoted by (R_f), now if for each member of $\zeta \in R$, with the condition $R\backslash\zeta$ is also remain a resolving set for the network $\aleph$. The minimum number of elements in the fault-tolerant resolving set is known as the fault-tolerant metric dimension and described as $dim_f(\aleph)$.

Definition 4 [7]: Let $R_p \subseteq V(\aleph)$ is the s-ordered proper set and $r\left(\zeta|R_p\right) = \{d(\zeta, R_{p1}), d(\zeta, R_{p2}), \ldots, d(\zeta, R_{ps})\}$, is the s-tuple distance identification of a principal node ζ with respect to R_p. If the entire set of principal nodes have unique identifications, then R_p is the partition resolving set of the set of principal node of a network $\aleph$. The minimum count of subsets in the partition resolving set of $V(\aleph)$ is defined as the partition dimension $(pd(\aleph))$ of $\aleph$.

Theorem 1 [9]: Let $dim(\aleph)$, $dim_f(\aleph)$ are the metric and fault-tolerant metric dimension of graph $\aleph$ respectively. Then

$$dim_f(\aleph) \geq dim(\aleph)+1.$$

II. CONSTRUCTION OF HOLLOW CORONOID $HC(p, q, s)$

In 1987, the term coronoid was devised by [5], due to its possible relation with benzenoid. Because a benzenoid with a hole in the center is known as a coronoid. Coronoid is rooted in organic chemistry and falls in the category of polyhex systems. The graph shown in Figure 1, is a hollow coronoid structure. Hollow coronoid is also contained Kekule with the specific values of parameters $(p = q = s = 3)$ [13]. Also, the circumcoronene is known as peri-condensed benzenoids and related with hollow coronoids [39]. Actually a hollow coronoid contained six sides p, q, s, p', q', s' as shown in Figure 1. It is considered a sub-cluster of primitive coronoids and is further categorized as catacondensed coronoids [14].

For a new variety of hollow coronoid topologies, one can find in [38]. In which authors discussed some topological properties of this structure and its relevance also. There is another name of this structure which is called a zigzag-edge coronoid [15]. The authors discussed some useful properties related to the topology of hollow coronoid. For the mathematical study of coronoid and related structures, we refer to see [12]. For the polynomial study of hollow coronoid, the recent research work is [1]. In which authors computed some types of polynomials. The link of hollow coronoid with polyhex discussed in [16]. In this study, the authors discussed the mathematical chemistry of coronoid structure. For a better understanding of hollow coronoid and its generalizations, we refer to see the lecture notes and books [13], [14].

In this work, we consider a hollow coronoid with $p = p'$, $q = q'$, $s = s'$ with total six sides, but the three (p, q, s) sides are symmetric to other three (p', q', s'). We label this

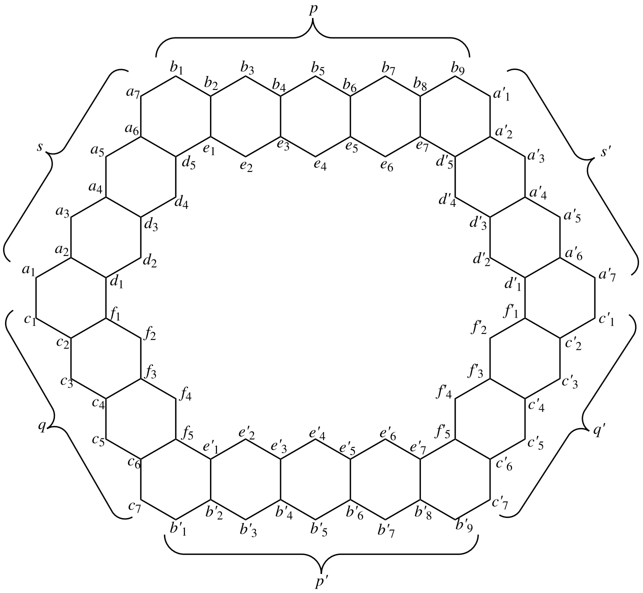

FIGURE 1. Hollow coronoid with $p = 5, q = s = 4$..

hollow coronoid as $HC(p, q, s)$, with $p, q, s \geq 2$. The structure contained total $8(p+q+s-3)$ vertices or nodes, in which $4(p+q+s-3)$ are with degree two and the same amount of vertices with degree three. The total count of edges (branches, line segments or bonds) are $10(p+q+s-3)$. The vertex and edge set of hollow coronoid structure $HC(p, q, s)$, are describe as:

$$
\begin{aligned}
V(HC(p,q,s)) = {} & \{a_i,\ a'_i,\ :\ 1 \le i \le 2s-1\} \\
& \cup\{b_i,\ b'_i,\ :\ 1 \le i \le 2p-1\} \\
& \cup\{c_i,\ c'_i,\ :\ 1 \le i \le 2q-1\} \\
& \cup\{d_i,\ d'_i,\ :\ 1 \le i \le 2s-3\} \\
& \cup\{e_i,\ e'_i,\ :\ 1 \le i \le 2p-3\} \\
& \cup\{f_i,\ f'_i,\ :\ 1 \le i \le 2q-3\}, \\
E(HC(p,q,s)) = {} & \{a_ia_{i+1},\ a'_ia'_{i+1},\ :\ 1 \le i \le 2s-2\} \\
& \cup\{b_ib_{i+1}, b'_ib'_{i+1}, :\ 1 \le i \le 2p-2\} \\
& \cup\{c_ic_{i+1}, c'_ic'_{i+1}, :\ 1 \le i \le 2q-2\} \\
& \cup\{d_id_{i+1}, d'_id'_{i+1}, :\ 1 \le i \le 2s-4\} \\
& \cup\{e_ie_{i+1}, e'_ie'_{i+1}, :\ 1 \le i \le 2p-4\} \\
& \cup\{f_if_{i+1}, f'_if'_{i+1}, :\ 1 \le i \le 2q-4\} \\
& \cup\{a_{i+1}d_i, a'_{i+1}d'_i, :\ 1 \le i\,(\text{odd}) \le 2s-3\} \\
& \cup\{b_{i+1}e_i, b'_{i+1}e'_i,\ :\ 1 \le i\,(\text{odd}) \le 2p-3\} \\
& \cup\{c_{i+1}f_i, c'_{i+1}f'_i,\ :\ 1 \le i\,(\text{odd}) \le 2q-3\} \\
& \cup\{a_1c_1,\ a'_1c'_1,\ a_{2s-1}b_1,\ a'_{2s-1}b_{2p-1}, \\
& b'_{2p-1}c'_{2q-1}, b'_1c_{2q-1}, d_1f_1, d'_1f'_1, d_{2s-3}e_1, \\
& d'_{2s-3}e_{2p-3},\ e'_{2p-3}f'_{2q-3}, \\
& e'_{2p-3}f'_{2q-3},\ e'_1f_{2q-3}\}.
\end{aligned}
$$

To make it understandable, we assigned the specific callings to different clusters of vertices. Like, outside vertices of hollow coronoid are labeled as $a_i, a'_i, b_i, b'_i, c_i$ and c'_i. While, the vertices inside the hollow coronoid are $d_i, d'_i, e_i, e'_i, f_i$ and f'_i. We can do the same assignments for the edges as well.

Furthermore, the vertice marking used in the findings described in Figure 1, and the generalize $HC(p, q, s)$, can be made by combining the vertex and edge sets of $HC(p, q, s)$, defined above.

III. RESULTS ON THE RESOLVABILITY OF HOLLOW CORONOID

Following is the core of this study in Lemma 1. It is chosen a suitable subset for resolving set from $V(HC(p, q, s))$.

Lemma 1: If $HC(p, q, s)$ be the graph of hollow coronoid with $p, q, s \geq 2$, then the cardinality of its resolving set is 3.

Proof: Let $R = \{a_1,\ b_1,\ b_{2p-1}\}$, is the ordered subset and to show that the R is one of the candidate of resolving set of $HC(p, q, s)$, with cardinality 3. Now, given below are the identifications of the entire set of nodes of $HC(p, q, s)$, with respect to the nodes in R.

For $i = 1, 2, \ldots, 2s-1$, the $r(a_i|R)$ and $r(a'_i|R)$, are following;

$$
r(a_i|R) = (i-1,\ 2s-i,\ 2(s+p-1)-i),
$$

$$
r(a'_i|R) = \begin{cases} (2(2s+p)-5-i,\ 2(p+s-1)-i,\ 2s-i), \\ \quad \text{if } i = 1, 2, \ldots, 2s-2; \\ (2(s+p-1),\ 2(p+s-1)-i,\ 2s-i), \\ \quad \text{if } i = 2s-1. \end{cases}
$$

For $i = 1, 2, \ldots, 2p-1$, the $r(b_i|R)$ and $r(b'_i|R)$, are following;

$$
r(b_i|R) = (2s-2+i,\ i-1,\ 2p-1-i),
$$

$$
r(b'_i|R) = \begin{cases} (2q-1+i,\ 2(q+s)-1,\ 2(q+s+p-2)-i), \\ \quad \text{if } i = 1; \\ (2q-1+i,\ 2(q+s-2)+i,\ 2(q+s+p-2)-i), \\ \quad \text{if } i = 2, 3, \ldots, 2p-1. \end{cases}
$$

For $i = 1, 2, \ldots, 2q-1$, the $r(c_i|R)$ and $r(c'_i|R)$, are following;

$$
r(c_i|R) = (i,\ 2s-i+1,\ 2(s+p)-3+i),
$$

$$
r(c'_i|R) = \begin{cases} (2(2q+p-2)-i,\ 2(s+p)-3+i,\ 2s-1+i), \\ \quad \text{if } i = 1, 2, \ldots, 2q-2; \\ (2(q+p)-1,\ 2(s+p)-3+i,\ 2s-1+i), \\ \quad \text{if } i = 2q-1. \end{cases}
$$

We can see that all the outside vertices of $HC(p, q, s)$ have unique locations (identifications). Now we can find the locations for inside vertices of $HC(p, q, s)$.

For $i = 1, 2, \ldots, 2s-3$, the $r(d_i|R)$ and $r(d'_i|R)$, are following;

$$
r(d_i|R) = (i+1,\ 2s-i,\ 2(s+p-2)-i),
$$

$$
r(d'_i|R) = (2(2s+p)-7-i,\ 2(p+s-2)-i,\ 2s-i).
$$

For $i = 1, 2, \ldots, 2p-3$, the $r(e_i|R)$ and $r(e_i'|R)$, are following;

$$r(e_i|R) = (2(s-1)+i,\ i+1,\ 2p-1-i),$$
$$r(e_i'|R) = (2q-1+i,\ 2(q+s-2)+i,\ 2(s+p+q-3)-i).$$

For $i = 1, 2, \ldots, 2q-3$, the $r(f_i|R)$ and $r(f_i'|R)$, are following;

$$r(f_i|R) = (2+i,\ i-1+2s,\ 2(p+s)-5+i),$$
$$r(f_i'|R) = (2(2q+p-3)-i,\ 2(p+s)+i-5,\ 2s-1+i).$$

Identifications of the whole group of nodes of $HC(p, q, s)$, emerged from the above data, we can observe that all the principal nodes possessed the unique identifications and meet the concept of a resolving set by concluding that $|R| = 3$. □

Theorem 2: Let the graph of hollow coronoid $HC(p, q, s)$, with $p, q, s \geq 2$. Then

$$dim(HC(p, q, s)) = 3.$$

Proof: To prove that the metric dimension of $HC(p, q, s)$, is 3, by the method of double inequality and referring the Lemma 1, which is already proved that $R = \{a_1,\ b_1,\ b_{2p-1}\}$, is a candidate of the resolving set has the cardinality 3.

Now we will prove that $dim(HC(p, q, s)) \geq 3$. On contrary assume that $dim(HC(p, q, s)) = 2$, and R' is the resolving set with cardinality two is possible. Given below are some cases to check that $|R'| = 2$ is possible or not.

Let the index notations, $S_1 = \{1, 2, \ldots, 2s-1\}$, $S_2 = \{1, 2, \ldots, 2p-1\}$, $S_3 = \{1, 2, \ldots, 2q-1\}$, $S_4 = \{1, 2, \ldots, 2s-3\}$, $S_5 = \{1, 2, \ldots, 2p-3\}$ and $S_6 = \{1, 2, \ldots, 2q-3\}$.

Case 1: For $|R'| = 2$, and $R' \subset \{a_i : 1 \leq i \leq 2s-1\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(a_l|R') = r(d_m|R')$, where $l \in S_1$ and $m \in S_4$.

Case 2: For $|R'| = 2$, and $R' \subset \{b_i : 1 \leq i \leq 2p-1\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(b_l|R') = r(e_m|R')$, where $l \in S_2$ and $m \in S_5$.

Case 3: For $|R'| = 2$, and $R' \subset \{c_i : 1 \leq i \leq 2q-1\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(a_l|R') = r(d_m|R')$, where $l \in S_1$ and $m \in S_4$.

Case 4: For $|R'| = 2$, and $R' \subset \{a_i' : 1 \leq i \leq 2s-1\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(a_l|R') = r(d_m|R')$, where $l \in S_1$ and $m \in S_4$.

Case 5: For $|R'| = 2$, and $R' \subset \{b_i' : 1 \leq i \leq 2p-1\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(b_l|R') = r(e_m|R')$, where $l \in S_2$ and $m \in S_5$.

Case 6: For $|R'| = 2$, and $R' \subset \{c_i' : 1 \leq i \leq 2q-1\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(b_l|R') = r(e_m|R')$, where $l \in S_2$ and $m \in S_5$.

Case 7: For $|R'| = 2$, and $R' \subset \{d_i : 1 \leq i \leq 2s-3\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(a_l|R') = r(d_m|R')$, where $l \in S_1$ and $m \in S_4$.

Case 8: For $|R'| = 2$, and $R' \subset \{e_i : 1 \leq i \leq 2p-3\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(a_l|R') = r(d_m|R')$, where $l \in S_1$ and $m \in S_4$.

Case 9: For $|R'| = 2$, and $R' \subset \{f_i : 1 \leq i \leq 2q-3\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(a_l|R') = r(d_m|R')$, where $l \in S_1$ and $m \in S_4$.

Case 10: For $|R'| = 2$, and $R' \subset \{d_i' : 1 \leq i \leq 2s-3\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(a_l|R') = r(f_m|R')$, where $l \in S_1$ and $m \in S_6$.

Case 11: For $|R'| = 2$, and $R' \subset \{e_i' : 1 \leq i \leq 2p-3\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(a_l|R') = r(e_m|R')$, where $l \in S_1$ and $m \in S_5$.

Case 12: For $|R'| = 2$, and $R' \subset \{f_i' : 1 \leq i \leq 2q-3\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(f_l|R') = r(c_m|R')$, where $l \in S_6$ and $m \in S_3$.

Case 13: For $|R'| = 2$, and $R' \subset \{a_i,\ b_j : 1 \leq i \leq 2s-1,\ 1 \leq i \leq 2p-1\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(a_l|R') = r(d_m|R')$, where $l \in S_1$ and $m \in S_4$.

Case 14: For $|R'| = 2$, and $R' \subset \{a_i,\ c_j : 1 \leq i \leq 2s-1,\ 1 \leq i \leq 2q-1\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(b_l|R') = r(e_m|R')$, where $l \in S_2$ and $m \in S_5$.

Case 15: For $|R'| = 2$, and $R' \subset \{a_i,\ a_j' : 1 \leq i,j \leq 2s-1\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(a_l'|R') = r(d_m'|R')$, where $l \in S_1$ and $m \in S_4$.

Case 16: For $|R'| = 2$, and $R' \subset \{a_i,\ b_j' : 1 \leq i \leq 2s-1,\ 1 \leq i \leq 2p-1\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(c_l|R') = r(f_m|R')$, where $l \in S_3$ and $m \in S_6$.

Case 17: For $|R'| = 2$, and $R' \subset \{a_i,\ c_j' : 1 \leq i \leq 2s-1,\ 1 \leq i \leq 2q-1\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(c_l|R') = r(f_m|R')$, where $l \in S_3$ and $m \in S_6$.

Case 18: For $|R'| = 2$, and $R' \subset \{a_i,\ d_j : 1 \leq i \leq 2s-1,\ 1 \leq i \leq 2s-3\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(b_l'|R') = r(e_m'|R')$, where $l \in S_2$ and $m \in S_5$.

Case 19: For $|R'| = 2$, and $R' \subset \{a_i,\ e_j : 1 \leq i \leq 2s-1,\ 1 \leq i \leq 2p-3\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(b_l'|R') = r(e_m'|R')$, where $l \in S_2$ and $m \in S_5$.

Case 20: For $|R'| = 2$, and $R' \subset \{a_i,\ f_j : 1 \leq i \leq 2s-1,\ 1 \leq i \leq 2q-3\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(c_l'|R') = r(f_m'|R')$, where $l \in S_3$ and $m \in S_6$.

Case 21: For $|R'| = 2$, and $R' \subset \{a_i,\ d_j' : 1 \leq i \leq 2s-1,\ 1 \leq i \leq 2s-3\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(c_l|R') = r(f_m|R')$, where $l \in S_3$ and $m \in S_6$.

Case 22: For $|R'| = 2$, and $R' \subset \{a_i,\ e_j' : 1 \leq i \leq 2s-1,\ 1 \leq i \leq 2p-3\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(c_l'|R') = r(f_m'|R')$, where $l \in S_3$ and $m \in S_6$.

Case 23: For $|R'| = 2$, and $R' \subset \{a_i,\ f_j' : 1 \leq i \leq 2s-1,\ 1 \leq i \leq 2q-3\} \subset V(HC(p, q, s))$. Such case implied that $|R'| \neq 2$, because $r(a_l|R') = r(c_m|R')$, where $l \in S_1$ and $m \in S_3$.

All the cases for $|R'| = 2$, enclosed that there does not exist a set with cardinality two which is the candidate for resolving set and it further indicated that two metric dimension of $HC(p, q, s)$ is not possible. Hence; $dim(HC(p, q, s)) \geq 3$.

Furthermore,

$$dim(HC(p, q, s)) = 3.$$

□

Lemma 2: Let the graph of hollow coronoid $HC(p, q, s)$, with $p, q, s \geq 2$. Then the cardinality of fault-tolerant resolving set of $HC(p, q, s)$, is 4.

Proof: To show that the graph $HC(p, q, s)$, has a fault-tolerant resolving set of cardinality 4, let R_f is a candidate for the fault-tolerant resolving set and it is taken as, $R_f = \{a_1, b_1, b_{2p-1}, e'_{2p-3}\}$. Now, given below are the identifications of the entire set of nodes of $HC(p, q, s)$, with respect to the nodes in R_f.

We divide the proof into two following cases on the values of p, q and s.

Case 1: When $p = 2, 3$ and $q = s = 2$.

For $i = 1, 2, \ldots, 2s-1$, the $r(a_i|R_f)$ and $r(a'_i|R_f)$, are following;

$$r(a_i|R_f) = \begin{cases} (i-1,\ 2s-i,\ 2(s+p-1)-i,\ 2p+2), \\ \text{if } i = 1, 3; \\ (1,\ 2(s-1),\ 2(s+p-2),\ 2p+1), \\ \text{if } i = 2. \end{cases}$$

$$r(a'_i|R_f) = \begin{cases} (2(2s+p)-5-i,\ 2(p+s-1)-i, \\ 2s-i,\ i-1), \\ \text{if } i = 1, 2, \ldots, 2s-2; \\ (2(s+p-1),\ 2(p+s-1)-i,\ 2s-i, \\ i-1), \\ \text{if } i = 2s-1. \end{cases}$$

For $i = 1, 2, \ldots, 2p-1$, the $r(b_i|R_f)$ and $r(b'_i|R_f)$, are following;

$$r(b_i|R_f) = (2s-2+i,\ i-1,\ 2p-1-i,\ 2(p+1)-i),$$

$$r(b'_i|R_f) = \begin{cases} (2q-1+i,\ 2(q+s)-1,\ 2(q+s+p-2) \\ -i,\ 2p+3-i), \\ \text{if } i = 1; \\ (2q-1+i,\ 2(q+s-2)+i,\ 2(q+s+p-2) \\ -i,\ 2p+3-i), \\ \text{if } i = 2, 3, \ldots, 2p-1. \end{cases}$$

For $i = 1, 2, \ldots, 2q-1$, the $r(c_i|R_f)$ and $r(c'_i|R_f)$, are following;

$$r(c_i|R_f) = \begin{cases} (i,\ 2s-i+1,\ 2(s+p)-3+i,\ 2p+3), \\ \text{if } i = 1, 3; \\ (i,\ 2s-i+1,\ 2(s+p)-3+i,\ 2p+2), \\ \text{if } i = 2 \end{cases}$$

$$r(c'_i|R_f) = \begin{cases} (2(2q+p-2)-i,\ 2(s+p)-3+i, \\ 2s-1+i,\ i), \\ \text{if } i = 1, 2, \ldots, 2q-2; \\ (2(q+p)-1,\ 2(s+p)-3+i,\ 2s-1+i,\ i), \\ \text{if } i = 2q-1. \end{cases}$$

We can see that all the outside vertices of $HC(p, q, s)$ have unique locations and it will not effect if we deleted any of the member of R_f. Now we can find the locations for inside vertices of $HC(p, q, s)$.

For $i = 1, 2, \ldots, 2s-3$, the $r(d_i|R_f)$ and $r(d'_i|R_f)$, are following;

$$r(d_i|R_f) = (i+1,\ 2s-i,\ 2(s+p-2)-i,\ 2p),$$
$$r(d'_i|R_f) = (2(2s+p)-7-i,\ 2(p+s-2)-i,\ 2s-i,\ 2).$$

For $i = 1, 2, \ldots, 2p-3$, the $r(e_i|R_f)$ and $r(e'_i|R_f)$, are following;

$$r(e_i|R_f) = (2(s-1)+i,\ i+1,\ 2p-1-i,\ 2p-i),$$
$$r(e'_i|R_f) = (2q-1+i,\ 2(q+s-2)+i,\ 2(s+p+q-3) \\ -i,\ 2p+1-i).$$

For $i = 1, 2, \ldots, 2q-3$, the $r(f_i|R_f)$ and $r(f'_i|R_f)$, are following;

$$r(f_i|R_f) = (2+i,\ i-1+2s,\ 2(p+s)-5+i,\ 2p+1),$$
$$r(f'_i|R_f) = (2(2q+p-3)-i,\ 2(p+s)+i-5,\ 2s-1+i,\ 3).$$

Case 2: When $p, q, s \geq 3$.

For $i = 1, 2, \ldots, 2s-1$, the $r(a_i|R_f)$ and $r(a'_i|R_f)$, are following;

$$r(a_i|R_f) = \begin{cases} (i-1,\ 2s-i,\ 2(s+p-1)-i,\ 2(q+p-2)), \\ \text{if } i = 1; \\ (i-1,\ 2s-i,\ 2(s+p-1)-i,\ 2(q+p)-7+i), \\ \text{if } i = 2, 3, \ldots, 2s-1. \end{cases}$$

$$r(a'_i|R_f) = \begin{cases} (2(2s+p)-5-i,\ 2(p+s-1)-i,\ 2s-i,\ 2q), \\ \text{if } i = 1; \\ (2(2s+p)-5-i,\ 2(p+s-1)-i,\ 2s-i,\ 2q, \\ 2q-3+i), \\ \text{if } i = 2, 3, \ldots, 2s-2; \\ (2(s+p-1),\ 2(p+s-1)-i,\ 2s-i, \\ 2q-3+i), \\ \text{if } i = 2s-1. \end{cases}$$

For $i = 1, 2, \ldots, 2p-1$, the $r(b_i|R_f)$ and $r(b'_i|R_f)$, are following;

$$r(b_i|R_f) = \begin{cases} (2s-2+i,\ i-1,\ 2p-1-i, \\ 2(s+p+q-3)-i), \\ \text{if } i = 1, 2, \ldots, 2p-2; \\ (2s-2+i,\ i-1,\ 2p-1-i,\ 2(s+q)-3), \\ \text{if } i = 2p-1. \end{cases}$$

$$r\left(b_i'|R_f\right)=\begin{cases}(2q-1+i,\ 2(q+s)-1,\ 2(q+s+p-2)\\ -i,\ 2p-1-i), \\ \text{if } i=1;\\ (2q-1+i,\ 2(q+s-2)+i,\ 2(q+s+p-2)\\ -i,\ 2p-1-i),\\ \text{if } i=2,3,\ldots,2p-2;\\ (2q-1+i,\ 2(q+s-2)+i,\\ 2(q+s+p-2)-i,\ 2),\\ \text{if } i=2p-1.\end{cases}$$

For $i = 1, 2, \ldots, 2q-1$, the $r\left(c_i|R_f\right)$ and $r\left(c_i'|R_f\right)$, are following;

$$r\left(c_i|R_f\right)=\begin{cases}(i,\ 2s-i+1,\ 2(s+p)-3+i,\\ 2(q+p-2)-i),\\ \text{if } i=1,2,\ldots,2q-2;\\ (i,\ 2s-i+1,\ 2(s+p)-3+i,\ 2p-1),\\ \text{if } i=2q-1\end{cases}$$

$$r\left(c_i'|R_f\right)=\begin{cases}(2(2q+p-2)-i,\ 2(s+p)-3+i,\\ 2s-1+i,\ 2q-i),\\ \text{if } i=1,2,\ldots,2q-2;\\ (2(q+p)-1,\ 2(s+p)-3+i,\\ 2s-1+i,\ 3),\\ \text{if } i=2q-1.\end{cases}$$

We can see that all the outside vertices of $HC(p, q, s)$ have unique locations and it will not effect if we deleted any of the member of R_f. Now we can find the locations for inside vertices of $HC(p, q, s)$.

For $i = 1, 2, \ldots, 2s-3$, the $r\left(d_i|R_f\right)$ and $r\left(d_i'|R_f\right)$, are following;

$$\begin{aligned}r\left(d_i|R_f\right) &= (i+1,\ 2s-i,\ 2(s+p-2)-i,\\ &\quad 2(q+p-3)+i),\\ r\left(d_i'|R_f\right) &= (2(2s+p)-7-i,\ 2(p+s-2)-i,\\ &\quad 2s-i,\ 2q-3+i).\end{aligned}$$

For $i = 1, 2, \ldots, 2p-3$, the $r\left(e_i|R_f\right)$ and $r\left(e_i'|R_f\right)$, are following;

$$\begin{aligned}r\left(e_i|R_f\right) &= (2(s-1)+i,\ i+1,\ 2p-1-i,\\ &\quad 2(p+q+s-4)-i),\\ r\left(e_i'|R_f\right) &= (2q-1+i,\ 2(q+s-2)+i,\\ &\quad 2(s+p+q-3)-i,\ 2p-3-i).\end{aligned}$$

For $i = 1, 2, \ldots, 2q-3$, the $r\left(f_i|R_f\right)$ and $r\left(f_i'|R_f\right)$, are following;

$$\begin{aligned}r\left(f_i|R_f\right) &= (2+i,\ i-1+2s,\ 2(p+s)-5+i,\\ &\quad 2(q+p-3)-i),\\ r\left(f_i'|R_f\right) &= (2(2q+p-3)-i,\ 2(p+s)+i-5,\\ &\quad 2s-1+i,\ 2(q-1)-i).\end{aligned}$$

Identifications of the complete vertex set of hollow coronoid $HC(p, q, s)$, emerged from the above data in the form of $r\left(.|R_f\right)$, we can observe that all the principal nodes possessed a unique identifications and meet the concept of fault-tolerant resolving set and further implied that $\left|R_f\right| = 4$. □

Theorem 3: If the graph of hollow coronoid $HC(p, q, s)$, with $p, q, s \geq 2$, then

$$dim_f(HC(p, q, s)) = 4.$$

Proof: To show that the graph of hollow coronoid $HC(p, q, s)$, has 4, fault-tolerant metric dimension, by the implementation of method of double inequality, for $dim_f(HC(p, q, s)) \leq 4$, we are referring the Lemma 2, which is already proved that the fault-tolerant resolving set R_f is a candidate with cardinality 4 and it can be settled as $R_f = \{a_1,\ b_1,\ b_{2p-1},\ e'_{2p-3}\}$.

Now for $dim_f(HC(p, q, s)) \geq 4$, by contradiction we get $dim_f(HC(p, q, s)) = 3$, which is not possible by Theorems 1 and 2. It is implied that the graph $HC(p, q, s)$ does not have $dim_f(HC(p, q, s)) = 3$. Hence; $dim_f(HC(p, q, s)) \geq 4$.

Now by relating both inequalities, end up with conclusion that

$$dim_f(HC(p, q, s)) = 4.$$

□

Lemma 3: Let the graph of hollow coronoid $HC(p, q, s)$, with $p, q, s \geq 2$. Then the cardinality of partition resolving set of $HC(p, q, s)$, is at most 4.

Proof: To show that the graph of hollow coronoid $HC(p, q, s)$, has a candidate for the partition resolving set with cardinality 4, and it is taken as $R_p = \{R_{p1},\ R_{p2},\ R_{p3},\ R_{p4}\}$, where $R_{p1} = \{a_1\}$, $R_{p2} = \{b_2\}$, $R_{p3} = \{b_{2p-1}\}$, $R_{p4} = V(HC(p, q, s)) \setminus \{a_1,\ b_1,\ b_{2p-1}\}$. Now the given below are the identifications of the complete set of nodes of $HC(p, q, s)$, with respect to the R_p.

For $i = 1, 2, \ldots, 2s-1$, the $r\left(a_i|R_p\right)$ and $r\left(a_i'|R_p\right)$, are following;

$$r\left(a_i|R_p\right) = (i-1,\ 2s-i,\ 2(s+p-1)-i,\ z_1),$$

$$r\left(a_i'|R_p\right)=\begin{cases}(2(2s+p)-5-i,\ 2(p+s-1)-i,\ 2s-i,\ 0),\\ \text{if } i=1,2,\ldots,2s-2;\\ (2(s+p-1),\ 2(p+s-1)-i,\ 2s-i,\ 0),\\ \text{if } i=2s-1.\end{cases}$$

$$z_1=\begin{cases}1 & \text{if } i=1;\\ 0 & \text{otherwise.}\end{cases}$$

For $i = 1, 2, \ldots, 2p-1$, the $r\left(b_i|R_p\right)$ and $r\left(b_i'|R_p\right)$, are following;

$$r\left(b_i|R_p\right) = (2s-2+i,\ i-1,\ 2p-1-i,\ z_2),$$

$$r\left(b_i'|R_p\right)=\begin{cases}(2q-1+i,\ 2(q+s)-1,\\ 2(q+s+p-2)-i,\ 0),\\ \text{if } i=1;\\ (2q-1+i,\ 2(q+s-2)+i,\\ 2(q+s+p-2)-i,\ 0),\\ \text{if } i=2,3,\ldots,2p-1.\end{cases}$$

$$z_2=\begin{cases}1 & \text{if } i=1,2p-1;\\ 0 & \text{otherwise.}\end{cases}$$

For $i = 1, 2, \ldots, 2q-1$, the $r\left(c_i|R_p\right)$ and $r\left(c_i'|R_p\right)$, are following;

$$r\left(c_i|R_p\right) = (i,\ 2s-i+1,\ 2(s+p)-3+i,\ 0),$$

$$r\left(c_i'|R_p\right) = \begin{cases} (2(2q+p-2)-i,\ 2(s+p)-3+i, \\ 2s-1+i,\ 0), \\ \text{if } i = 1, 2, \ldots, 2q-2; \\ (2(q+p)-1,\ 2(s+p)-3+i, \\ 2s-1+i,\ 0), \\ \text{if } i = 2q-1. \end{cases}$$

We can see that all the outside vertices of $HC(p, q, s)$ have unique locations. Now we can find the locations for inside vertices of $HC(p, q, s)$.

For $i = 1, 2, \ldots, 2s-3$, the $r\left(d_i|R_p\right)$ and $r\left(d_i'|R_p\right)$, are following;

$$r\left(d_i|R_p\right) = (i+1,\ 2s-i,\ 2(s+p-2)-i,\ 0),$$
$$r\left(d_i'|R_p\right) = (2(2s+p)-7-i,\ 2(p+s-2)-i,\ 2s-i,\ 0).$$

For $i = 1, 2, \ldots, 2p-3$, the $r\left(e_i|R_p\right)$ and $r\left(e_i'|R_p\right)$, are following;

$$r\left(e_i|R_p\right) = (2(s-1)+i,\ i+1,\ 2p-1-i,\ 0),$$
$$r\left(e_i'|R_p\right) = (2q-1+i,\ 2(q+s-2)+i,\ 2(s+p+q-3)-i,\ 0).$$

For $i = 1, 2, \ldots, 2q-3$, the $r\left(f_i|R_p\right)$ and $r\left(f_i'|R_p\right)$, are following;

$$r\left(f_i|R_p\right) = (2+i,\ i-1+2s,\ 2(p+s)-5+i,\ 0),$$
$$r\left(f_i'|R_p\right) = (2(2q+p-3)-i,\ 2(p+s)+i-5,\ 2s-1+i,\ 0).$$

Identifications of the complete set of principal nodes of $HC(p, q, s)$, we can observe that all the principal nodes possesses a unique identifications and meet the defined concept of partition resolving set and stating that $\left|R_p\right| = 4$. □

Theorem 4: Let the graph of hollow coronoid $HC(p, q, s)$, with $p, q, s \geq 2$. Then

$$pd\,(HC(p, q, s)) \leq 4.$$

Proof: To show that $HC(p, q, s)$, has the partition dimension which is 4. From Lemma 3 given above shows that there is a candidate of the partition resolving set with cardinality 4 and it is been taken as, $R_p = \{R_{p1},\ R_{p2},\ R_{p3},\ R_{p4}\}$, where $R_{p1} = \{a_1\}$, $R_{p2} = \{b_2\}$, $R_{p3} = \{b_{2p-1}\}$, $R_{p4} = V(HC(p, q, s)) \setminus \{a_1,\ b_1,\ b_{2p-1}\}$. By using Lemma 3, it is concluded that

$$pd\,(HC(p, q, s)) \leq 4.$$

□

IV. CONCLUSION

Studying chemical structures in the terminologies of graph theory is an attractive and very useful concept. It helps chemical researchers to study different chemical networks and topologies in more accurate and easiest form. We pursue such motivation and studied a very impressive cluster of organic chemistry. We studied hollow coronoid topology with six sides and computed its metric, fault-tolerant metric, and its generalization which is known as partition dimension. We proved that the above parameters are constant and do not depend on the number of vertices of the hollow coronoid.

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