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Decision-Making Model to Portfolio Selection Using Analytic Hierarchy Process (AHP) With Expert Knowledge

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ABSTRACT On the premise of ensuring profits, how to give a relatively dispersed portfolio selection result reasonably and rapidly is a challenging problem in both theory and practice. Although the use of optimization models to make decision has been shown to be an essential approach towards portfolio selection, there still has an acute need for developing a knowledge-based expert model for portfolio selection so that this model can achieve better performance in reliability and real time, especially in leading more distributed investments. In this paper, a knowledge-based expert model is proposed for portfolio selection with the aid of analytic hierarchy process (AHP) and fuzzy sets. In the proposed model, the expert knowledge which can reflect the investment attitude and experience of different investors is mainly integrated into the criterion layer and represented by a reciprocal matrix, and the scheme layer is abstracted to a strictly consistent matrix by comparing and analyzing the state characteristics of investment objects. In order to characterize the state characteristics of investment objects under fuzzy environment, their corresponding time series data are quantified as fuzzy variables in advance. Experiments involving synthetic and real-world data demonstrate that the proposed model produces better performance than other typical portfolio selection models and gives more distributed investments.

INDEX TERMS Decision-making, portfolio selection, analytic hierarchy process (AHP), consistency, expert knowledge.

I. INTRODUCTION

With the increasing of the uncertainty of modern financial markets, portfolio selection analysis, whose aim is to spread risk, has been acting as an important part of modern investment theory. Although investment managers and economists have long recognized the need to consider both returns and risks [1], they have ignored the contradiction between the diversification of investment and the maximization of expected returns. In order to solve this contradiction, Markowitz first puts forward the mean-variance framework [2]. Under the mean-variance framework, it can be known from a game between mean and variance that the expected return of a portfolio is determined by its own variance. However, with in-depth research of technical means and social progress, scholars have found that variance [3] can be replaced by some more reliable and convincing quantitative indicators, such as semi-variance [4], entropy [5], [6], or semi-entropy [7]. On the basis of above facts, Markowitz's mean-variance framework has evolved rapidly and been extended to the mean-risk framework.

Through the above analysis and the existed literatures [8]–[10], it is clear that portfolio selection can be abstracted as a multi-criteria decision-making (MCDM) problem in essence, and the expected return and different types of risks are two typical types of evaluation criteria of it. For MCDM problems, in order to deal with the challenges of new factors in the digital age, several novel approaches and techniques have been presented and extended based on some underlying theories, including fuzzy sets [11]–[14], interval theory [15], [16], evidential reasoning [17], [18], multi-perspective framework [19], and group decision-making strategy [20]–[22]. According to whether the decision space is continuous or discrete, MCDM can be subdivided into two parallel

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contents: multi-objective decision-making (MODM) and multi-attribute decision-making (MADM). The most representative model in MODM is the optimization model [23]. Generally, after having characterized the return of each investment object as a random or fuzzy variable, most of the portfolio selection frameworks mentioned above are constructed by the optimization model. However, owing to the nonlinearity of these moments, it is difficult to find the analytic solution of these optimization-based portfolio selection models [7], [24]–[26]. Consequently, only some optimization algorithms such as genetic algorithm (GA) [27], differential evolution algorithm (DE) [28], or particle swarm optimization algorithm (PSO) [29] can be used to solve these models, but it leads to a slow speed in the model-solving process and an unstable solving result [30]. Even if so, it is usually not guaranteed to find the global optimal solution of an optimization model, and in most cases, only a local optimal solution can be found. In order to avoid the above disadvantages brought by the optimization model, some models for MADM should be taken into account to deal with the portfolio selection problem. In recent years, some MADM models such as evidential correlation coefficient (ECC), balancing and ranking method (BR), multi-perspective MADM (MPMADM), and analytic hierarchy process (AHP) have made remarkable progress in uncertainty analysis. Xiao [18] presented a novel ECC method to optimally manage the conflicts of multiple pieces of evidence in an uncertainty environment. Gitinavard *et al.* [12] proposed the hesitant fuzzy balancing and ranking (HF-BR) method by integrating hesitant fuzzy sets into BR, which links incompatible and uncertain attributes with pair-wise comparisons of the possible alternatives. Chen *et al.* [19] built a generalized MPMADM framework to generating reliable weight vectors, which can effectively manage the uncertainty and ambiguity in the decision process. Ebrahimnejad *et al.* [15] combined AHP technique with incomplete interval-valued information to assess the risks, which decreases the judgment errors towards the problem of group decision-making. Karaşan *et al.* [31] provided a novel Pythagorean fuzzy AHP method to weaken the subjectivity from linguistic factors, which produces informative outcomes with better consistency. Nevertheless, MADM models overemphasize the importance of historical data when solving final results, so they usually ignore the preferences and experiences from investors (known as the expert knowledge [32]) in the calculation process. Affected by the inaccuracy of historical data, this must make the investment proportion of some alternatives too high and that of other alternatives too low without the experience correction given by investors, which means that the portfolio result cannot spread risk effectively.

In order to attach the importance to both historical data and expert knowledge, a new trail is blazed for portfolio selection and a new model under the mean-risk framework is proposed with the aid of AHP [33] and fuzzy sets [34]. Compared with the optimization-based portfolio selection model, the proposed AHP-based portfolio selection model can be solved analytically, which means that the portfolio selection model no longer needs to be solved by numerical optimization algorithms. Therefore, the real-time performance and result stability of the proposed model are much better than that of the optimization model. Moreover, following the modeling approach given in this paper, not only the expert knowledge but the objective judgment result given by investors can be integrated into AHP directly and effectively, so the proposed model can make decisions according to both current investment situations and past investment experiences. Just because of the expert knowledge and experience included in the proposed AHP-based portfolio selection model, the quantitative results given by the proposed model are more consistent with those of qualitative analysis and more in line with the objective decision-making law of investors compared with the optimization-based portfolio selection model. Most of all, by integrating investors' preferences and experiences into the criteria layer of AHP, the proposed model leads to more distributed investments than other portfolio selection models, which perfectly practices the investment principle that "never put all your eggs in one basket". In brief, the main contributions of this paper are summarized as follows:

1) The proposed approach gives a simple and light way for searching non-inferior alternatives, whose rationality is guaranteed by Theorem 1. This means that a set of Pareto non-inferior solutions can be rapidly constructed by the weighted combination of these non-inferior alternatives.
2) On the premise of ensuring the acceptable consistency of the pairwise comparison matrix in the criterion layer, investors' preferences and experiences are integrated into the construction of criterion layer to better depict and reflect the investment attitude of investors. Its generality is illustrated by Theorem 2.
3) In the construction of scheme layer, an analytic method for building consistency matrix is given and elaborated by Theorem 3, which overcomes the challenge of constructing a pairwise comparison matrix in the face of a large number ($N > 7$) of alternatives. Moreover, using analytic calculation to construct the matrix can achieve absolute consistency in parallel with avoiding the interference of subjective factors.

The main contents of this paper are organized as follows. In Section II, some brief reviews of the credibility measure and AHP, which are applied to the later modeling process, are given. In Section III, some basic state characteristics of portfolio to be used in modeling are analyzed. In Section IV, the detailed modeling process is presented and the calculation method of corresponding parameters is determined. Then three comparative experiments are given in Section V. Finally, some conclusions are listed in Section VI.

II. PRELIMINARIES

In order to carry out portfolio analysis in fuzzy environment, a quantitative description for each portfolio alternative is needed in advance. In this section, a brief review of the credibility measure, which is used to quantify the fuzzy uncertainty of an investment market, is given. After that, some key points of AHP are introduced.

A. CREDIBILITY MEASURE

Since the pioneering work done by Zadeh on fuzzy sets and possibility theory, the research on credibility measure, which is the basis of quantitative description for portfolio alternatives, has been constantly improved and become a powerful tool for dealing with incomplete and uncertain situations.

Consider a nonempty set Θ having finite elements, and let Γ denote a σ-algebra over Θ. In general, each element from Γ can be treated as a basic event. For a specific event $A \in \Gamma$, its corresponding credibility measure, which can be denoted as $\mathrm{Cr}\{A\} \in [0, 1]$, is used to reflect the semantic fuzziness of A. Li and Liu [35] have pointed out that the credibility measure should satisfy the following four axioms:

Axiom 1: Normalization, i.e., $\mathrm{Cr}\{\Theta\} = 1$.

Axiom 2: Monotonicity, i.e., $\mathrm{Cr}\{A_1\} \le \mathrm{Cr}\{A_2\}$ whenever $A_1 \subseteq A_2$.

Axiom 3: Duality, i.e., $\mathrm{Cr}\{A\} + \mathrm{Cr}\{\complement_\Gamma A\} = 1$ for any basic event A.

Axiom 4: Generalized countable additivity, i.e., for any combination of events $\{A_i\}$ with $\sup_i \mathrm{Cr}\{A_i\} \le 0.5$, there has $\mathrm{Cr}\{\bigcup_i A_i\} = \sup_i \mathrm{Cr}\{A_i\}$.

Suppose that there is a fuzzy variable ξ with the membership function μ. Let $B \subset \mathbb{R}$, then the credibility measure [36] of $\xi \in B$ can be calculated as follows

$$\mathrm{Cr}\{\xi \in B\} = \frac{1}{2}\big(\mathrm{Pos}\{\xi \in B\} - \mathrm{Nec}\{\xi \in B\}\big), \tag{1}$$

where $\mathrm{Pos}\{\xi \in B\}$ and $\mathrm{Nec}\{\xi \in B\}$ represent the possibility degree and necessity degree of $\xi \in B$ respectively, and they are defined by the following mathematical expressions

$$\mathrm{Pos}\{\xi \in B\} = \sup_{x \in B} \mu(x), \tag{2a}$$

$$\mathrm{Nec}\{\xi \in B\} = 1 - \sup_{x \in \complement_\Gamma B} \mu(x). \tag{2b}$$

B. ANALYTIC HIERARCHY PROCESS (AHP)

Consider a financial market having several investment alternatives waiting to be selected, say $p_1, p_2, \ldots, p_n$. In order to maximize returns in a dynamic risk environment, these alternatives need to be analyzed and processed under an evaluation paradigm.

The core idea of AHP is to decompose a MCDM problem into multiple objectives or criteria, and then obtain the decision-making results by the pairwise comparison between two alternatives. Evidently, the core of AHP is the construction of pairwise comparison matrix. Saaty has indicated that the pairwise comparison matrix $\boldsymbol{R} = \left(r_{ij}\right)_{n\times n}$ should be a reciprocal matrix [37], so the elements in the matrix $\boldsymbol{R}$ must satisfy the following relationship

$$r_{ij} = \begin{cases} 1, & \text{if } i = j \\ r_{ji}^{-1}, & \text{if } i \ne j, \end{cases} \tag{3}$$

TABLE 1. Relationship Between Random Consistency Index and Reciprocal Matrix Order.

n	1	2	3	4	5
$RI(n)$	0.00	0.00	0.58	0.90	1.12
n	6	7	8	9	10
$RI(n)$	1.24	1.32	1.41	1.45	1.49

where r_{ij} denotes the pairwise comparison result of p_i to p_j.

In pairwise comparison, the relative scale is used to quantify the comparison results, which can simplify the comparison process between different alternatives and enhance the rationality of a pairwise comparison result. However, due to the conflict between decision preferences, the pairwise comparison matrix constructed in a certain decision-making process may lack consistency. In general, the consistency ratio $CR(\boldsymbol{R})$ is encountered in various decision scenarios to judge whether the matrix $\boldsymbol{R}$ is consistent or not. The consistency ratio $CR(\boldsymbol{R})$ is defined as follows

$$CR(\boldsymbol{R}) = \frac{CI(\boldsymbol{R})}{RI(n)}, \tag{4}$$

where $CI(\boldsymbol{R})$ represents the consistency index and $RI(n)$ represents the random consistency index. Here, the definition of the consistency index is given as follows

$$CI(\boldsymbol{R}) = \frac{\lambda_{\max} - n}{n - 1}, \tag{5}$$

and the detailed values of the random consistency index with different reciprocal matrix orders are given in Table 1.

Classically, if the pairwise comparison matrix $\boldsymbol{R}$ meets the requirement such that $CR(\boldsymbol{R}) < 0.10$, it can be considered to pass the consistency test. Otherwise, the pairwise comparison matrix $\boldsymbol{R}$ needs to be rebuilt until it has gratifying consistency. Once an acceptable pairwise comparison matrix $\hat{\boldsymbol{R}}$ is found, its normalized eigenvector $\boldsymbol{\omega}_{\lambda_{\max}}$, which is corresponding to the largest eigenvalue $\lambda_{\max}$, can be selected as a weight vector and applied in the subsequent calculations.

III. STATE CHARACTERISTICS OF PORTFOLIO

Considering a given investment market having N securities. Let ξ_i denote the fuzzy return of the i-th security, and x_i stand for the investment proportion of the corresponding security ξ_i for $i = 1, 2, \ldots, N$. In order to evaluate the acceptability of a portfolio investment $\boldsymbol{X} = (x_1, x_2, \ldots, x_N)^T$ to these N alternatives such that $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_N)^T$, some state characteristics of a specific portfolio project can be selected as evaluation criteria.

In the early portfolio analysis, the expected value and variance are considered to be the two most important state characteristics. On the basis of expected value and variance, the alternatives available for investment and their corresponding optimal allocation proportion of finite investment capital can be determined by the mean-variance framework [38]. Subsequently, some new state characteristics such as entropy and other higher-order moments have been noticed with going deep into the research. In this paper, five state characteristics are taken into account under fuzzy environment, including fuzzy expected value, fuzzy variance, fuzzy entropy, and two higher-order fuzzy moments. And they are elaborated in more detail in the following part.

A. FUZZY EXPECTED VALUE

In a specific portfolio selection, its future return can be quantified by the fuzzy expected value. Fuzzy expected value is a weighted sum of all available values of a fuzzy return in accordance with the corresponding membership degree, so it can reflect the average value of fuzzy returns. Therefore, when the historical portfolio data of a specific security is converted to a fuzzy return, its fuzzy expected value best represents the return of this security in the future.

The fuzzy expected value $E[\xi]$ of a fuzzy return ξ can be defined as follows [36]

$$E[\xi] = \int_0^{\infty} \mathrm{Cr}\{\xi \geq x\}\, dx - \int_{-\infty}^{0} \mathrm{Cr}\{\xi \leq x\}\, dx, \tag{6}$$

where $\mathrm{Cr}\{\cdot\}$ is the credibility measure mentioned in Section II. In particular, for a triangular fuzzy return $\xi_t = (a, b, c)$, its fuzzy expected value is given by the following expression

$$E[\xi_t] = \frac{a+2b+c}{4}. \tag{7}$$

Consider an arbitrary portfolio selection $\boldsymbol{X}$. Since

$$E\Big[\sum_{i=1}^{N} x_i \xi_{t,i}\Big] = \sum_{i=1}^{N} x_i E[\xi_{t,i}],$$

let

$$E\left[\boldsymbol{X}; \boldsymbol{\xi}_t\right] = \sum_{i=1}^{N} x_i E[\xi_{t,i}], \tag{8}$$

then $E\left[\boldsymbol{X}; \boldsymbol{\xi}_t\right]$ reflects the overall profitability of a portfolio selection in a dynamic and uncertainty environment.

B. FUZZY VARIANCE AND ENTROPY

In a specific portfolio selection, both fuzzy variance and entropy can be used to quantify the investment risk.

1) FUZZY VARIANCE

The essence of fuzzy variance is a second-order central moment of the fuzzy return, which can reflect its dispersion. The more dispersive the distribution of a fuzzy return (i.e., available values fluctuate violently around its fuzzy expected value), the greater is the square sum of the difference between each value and its fuzzy expected value, the greater is the fuzzy variance. On the contrary, when the distribution of a fuzzy return is relatively concentrated, its corresponding fuzzy variance is relatively small. It can be seen that fuzzy variance objectively quantifies the volatility of original return series. Therefore, in most hedging models, fuzzy variance is taken to measure risk.

For a specific fuzzy return ξ, the detailed definition of its fuzzy variance [36] is given as follows

$$V[\xi] = \int_0^{\infty} \mathrm{Cr}\{(x-e)^2 \geq x\}\, dx. \tag{9}$$

And for a triangular fuzzy return, its corresponding fuzzy variance can be calculated directly with

$$V[\xi_t] = \frac{5b_-^2 + 5b_+^2 + 6b_- b_+}{48}, \tag{10}$$

where $b_- = b - a$ and $b_+ = c - b$.

It is easy to verify that $V[\xi_t]$ is a multivariate convex function. Hence the following inequality can be obtained by using Jensen's inequality on multivariate function

$$\begin{aligned} V\Big[\sum_{i=1}^{N} x_i \xi_{t,i}\Big] &= V\Big(\sum_{i=1}^{N} x_i b_{-,i}, \sum_{i=1}^{N} x_i b_{+,i}\Big) \\ &\leq \sum_{i=1}^{N} x_i V\big(b_{-,i}, b_{+,i}\big) = \sum_{i=1}^{N} x_i V[\xi_{t,i}]. \end{aligned}$$

Let

$$V\left[\boldsymbol{X}; \boldsymbol{\xi}_t\right] = \sum_{i=1}^{N} x_i V[\xi_{t,i}], \tag{11}$$

then $V\left[\boldsymbol{X}; \boldsymbol{\xi}_t\right]$ can be utilized to quantify the risk ceiling of a portfolio selection.

2) FUZZY ENTROPY

Aimed at the mean-variance framework, Maasoumi and Racine [5] have indicated that entropy is more suitable than variance to measure the uncertainty of wealth allocation strategy when a clear understanding of the specific distribution of financial markets is not available. Moreover, by extending the concept of entropy, Xiao [39] has confirmed that the entropy-based model has great potential in knowledge representation and uncertainty measure, even if in complex-valued distributions. Consequently, as another risk measure, entropy can describe the average uncertainty of probability distribution in the whole value space, and reflect the size of loss distribution. That is, entropy expresses the disorder and uncertainty of both low and high extreme returns, so it can sacrifice higher extreme returns to avoid risks.

Within the framework of credibility theory, Li and Liu [40] provide an original definition of entropy for both discrete and continuous fuzzy variables, which is shown as follows

$$H\left[\xi\right] = \int_{-\infty}^{\infty} S\big(\mathrm{Cr}\left\{\xi = x\right\}\big)\, dx, \tag{12}$$

where $S\left(t\right)$ represents the Shannon-like entropy such that $S\left(t\right) = -t \ln t - (1-t)\ln(1-t)$. For a triangular fuzzy return ξ_t, its corresponding fuzzy entropy can be calculated as follows

$$H[\xi_t] = \frac{b_- + b_+}{2}, \tag{13}$$

where $b_- = b - a$ and $b_+ = c - b$.

For an arbitrary portfolio selection $\boldsymbol{X}$, from (13) it can be readily verified that

$$H\Big[\sum_{i=1}^{N} x_i \xi_{t,i}\Big] = \sum_{i=1}^{N} x_i H[\xi_{t,i}].$$

Let

$$H\left[\boldsymbol{X}; \boldsymbol{\xi}_t\right] = \sum_{i=1}^{N} x_i H[\xi_{t,i}], \tag{14}$$

then $H\left[\boldsymbol{X}; \boldsymbol{\xi}_t\right]$ can be used to quantify the risk ceiling of a portfolio selection from the aspect of entropy.

C. HIGHER-ORDER FUZZY MOMENTS

In portfolio selection, higher-order fuzzy moments (e.g. fuzzy skewness, fuzzy kurtosis) are closely related to the decision-making result given by investors. The variance or entropy-based portfolio selection model depends only on the first- and second-order moments of return distributions. Many researchers (see [24], [41]–[44]) have argued that higher-order moments cannot be ignored unless there has enough evidence to prove that the return of each alternative is symmetrically distributed (e.g. normal) or that the investors' decision is independent to higher-order moments (e.g. skewness [24], [42]–[44], kurtosis [41], [44]). Especially when the first- and second-order moments of investment alternatives are the same, the higher-order moments are bound to become the decisive index, and almost all the investors will choose the portfolio with larger third-order moment or smaller fourth-order moment. Consequently, it is necessary to utilize higher-order fuzzy moments to evaluate the supplementary risk of a portfolio selection result.

1) FUZZY SKEWNESS

When the distributions of security returns are asymmetric, both fuzzy variance and entropy become insufficient indexes to measure the investment risk. This is because a portfolio based on variance or entropy may sacrifice too much expected return when eliminating low and high return extremes [42]. In order to overcome this limitation, the third-order moment named fuzzy skewness needs to be taken into account.

Definition 1 (Li-Qin-Kar [42], [43]): Suppose that there is a continuous fuzzy return ξ with a differentiable membership function $\mu(x)$ and a finite fuzzy expected value $E[\xi]$. Then its fuzzy skewness $S[\xi]$ is defined as

$$S[\xi] = E\big[(\xi - E[\xi])^3\big]. \tag{15}$$

In general, the bigger the absolute value of the fuzzy skewness $S[\xi]$ means the bigger the deviation of the fuzzy return ξ. And the fuzzy return is in positive skew distribution in the case of $S[\xi] > 0$, while the fuzzy return is in negative skew distribution in the condition of $S[\xi] < 0$.

Proposition 1: Let $\xi_t = (a, b, c)$ be a triangular fuzzy return. Then its fuzzy skewness $S[\xi_t]$ can be calculated as follows

$$S[\xi_t] = \frac{(c-a)^2}{8}\big(E[\xi_t] - b\big). \tag{16}$$

Remark 1: It can be readily checked from (16) that a triangular fuzzy return ξ is in positive skew distribution when its fuzzy expected value $E[\xi_t]$ is greater than its peak value b, i.e., $E[\xi_t] > b$. And a triangular fuzzy return ξ is in negative skew distribution if $E[\xi_t] < b$. In addition, a triangular fuzzy return ξ presents symmetrical distribution if and only if $E[\xi_t] = b$.

2) FUZZY KURTOSIS

The current hype over hedge funds provides a compelling evidence [41], [45] of how dangerous it can be to ignore the fourth-order central moment, i.e., kurtosis. Therefore, kurtosis risk should be integrated into the decision-making approach. To achieve this, this paper gives a definition of kurtosis to fuzzy returns as follows.

Definition 2: Suppose that there is a continuous fuzzy return ξ with a differentiable membership function $\mu(x)$ and a finite fuzzy expected value $E[\xi]$. Then its fuzzy kurtosis $K[\xi]$ is defined as

$$K[\xi] = E\big[(\xi - E[\xi])^4\big]. \tag{17}$$

Fuzzy kurtosis, which is similar to fuzzy skewness, is another higher-order fuzzy moment used to describe the shape and structure of fuzzy returns. The difference is that fuzzy kurtosis reflects the steepness and slowness of all the distribution values stored in fuzzy returns. From an investment portfolio perspective, fuzzy kurtosis can be considered as a measure of the gambling ingredient of an investment, and the higher the membership of extreme results, the more likely this investment is to be a pure gambling game.

Proposition 2: Let $\xi_t = (a, b, c)$ be a triangular fuzzy return. Then its fuzzy kurtosis $K[\xi_t]$ can be calculated as follows

$$K[\xi_t] = \frac{16(b_-^2 + 3b_+^2)(3b_-^2 + b_+^2) - 15(b_+ - b_-)^4}{1280}, \tag{18}$$

where $b_- = b - a$ and $b_+ = c - b$.

Remark 2: Based on the above proposition, it can be concluded that for a triangular fuzzy return ξ_t, its fuzzy kurtosis has nonnegativity, i.e., $K[\xi_t] \geq 0$ and symmetry, i.e., $K[\xi_t] = K[-\xi_t]$. In fact, it can be proved that the nonnegativity and symmetry of fuzzy kurtosis holds for all fuzzy returns, even those variables that have no symmetry of their own. In addition, for a symmetric triangular fuzzy return, it is easy to verify that the calculation of its fuzzy kurtosis can be simplified as $K[\xi_t] = b_-^4/5 = b_+^4/5$.

IV. PORTFOLIO SELECTION MODEL USING ANALYTIC HIERARCHY PROCESS WITH EXPERT KNOWLEDGE

Under general financial rules, in order to maximize their profits, investors face a decision-making problem of choosing and optimizing projects by means of state characteristics indicated in Section III. Next, the detailed modeling process is presented and elaborated step by step to solve this problem.

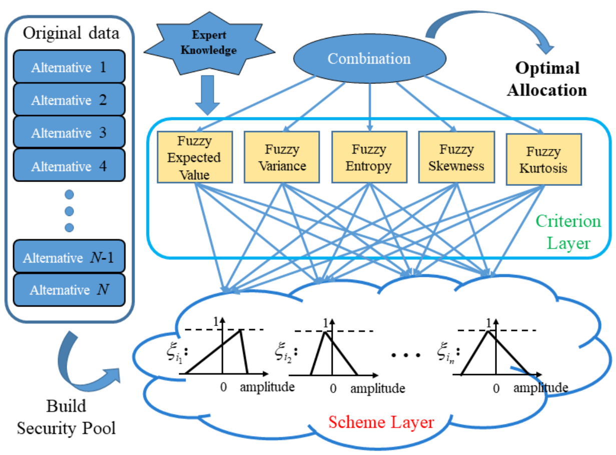


FIGURE 1. Decision-making model to portfolio selection using AHP with expert knowledge: an overview.

A. BUILD SECURITY POOL

Before making investment decisions, investors should first determine their investment object based on their own conditions and changes from inventory. The security pool refers to a selection of some investment objects with a broad investment prospect from all the alternative objects $\boldsymbol{\xi}$. In order to build the security pool within all the securities, the following theorem is give first.

Theorem 1: Let SC_1 and SC_2 be two state characteristics such that $(SC_1, SC_2) \in \{(E, V), (E, H), (V, E), (H, E)\}$. For a specific security p, if there exists a security q ($p \neq q$) in the security pool, and both $SC_1[\xi_{t,q}]$ and $SC_2[\xi_{t,q}]$ are better than those of $\xi_{t,p}$, then the security p does not belong to the security pool.

Proof: There are four cases should be discussed.

Let us consider the first case of $(SC_1, SC_2) = (E, V)$. In this case, without loss of generality, suppose that $\hat{\boldsymbol{X}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N)$ is the optimal allocation such that $\hat{x}_i > 0$ for all $i = 1, 2, \ldots, N$.

For a specific security p, let $\xi_{t,p}$ denote its corresponding fuzzy return. If there exists a security q such that

$$SC_1[\xi_{t,p}] < SC_1[\xi_{t,q}],\ SC_2[\xi_{t,p}] > SC_2[\xi_{t,q}],$$

then based on $\hat{\boldsymbol{X}}$, another investment allocation denoted as $\tilde{\boldsymbol{X}} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N)$ can be constructed as follows

$$\tilde{x}_i = \begin{cases} 0, & \text{if } i = p \\ \hat{x}_p + \hat{x}_q, & \text{if } i = q \\ \hat{x}_i, & \text{otherwise} \end{cases}.$$

Based on the work in Section III, since

$$SC_1[\tilde{\boldsymbol{X}}; \boldsymbol{\xi}_t] = \left(\hat{x}_p + \hat{x}_q\right) SC_1[\xi_{t,q}] + \sum_{i=1, i\neq q}^{N} \tilde{x}_i SC_1[\xi_{t,i}]$$
$$> \sum_{i=1}^{N} \hat{x}_i SC_1[\xi_{t,i}] = SC_1[\hat{\boldsymbol{X}}; \boldsymbol{\xi}_t],$$

and

$$SC_2[\tilde{\boldsymbol{X}}; \boldsymbol{\xi}_t] = \left(\hat{x}_p + \hat{x}_q\right) SC_2[\xi_{t,q}] + \sum_{i=1, i\neq q}^{N} \tilde{x}_i SC_2[\xi_{t,i}]$$
$$< \sum_{i=1}^{N} \hat{x}_i SC_2[\xi_{t,i}] = SC_2[\hat{\boldsymbol{X}}; \boldsymbol{\xi}_t],$$

it can be known that $\tilde{\boldsymbol{X}}$ is better than $\hat{\boldsymbol{X}}$, which is contradict to the above assumption. Hence, if $\hat{\boldsymbol{X}}$ is the optimal allocation, there must have $\hat{x}_p = 0$.

Similarly, the other three cases can be proved through the above processes. The proof is complete. ■

Remark 3: The above theorem reflects a portfolio strategy that is to diversify the capital into several high-return, high-risk and low-return, low-risk securities during the security pool construction. Further more, after having combined the return preference with risk preference for analysis, Theorem 3 presents an efficient method to remove those securities with low-return, high-risk from the investment market.

For the given N securities $\xi_1, \xi_2, \ldots, \xi_N$, suppose that n securities $\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_n}$ among them are selected in the security pool, and then define the optimal solution vector $\boldsymbol{x} = \left(x_{i_1}, x_{i_2}, \ldots, x_{i_n}\right)^T$ such that $x_{i_k} \neq 0$ for all $k = 1, 2, \ldots, n$. Once the optimal solution vector $\boldsymbol{x}$ is solved, the portfolio selection $\boldsymbol{X} = (x_1, x_2, \ldots, x_N)^T$ can be obtained as an augmented solution vector of $\boldsymbol{x}$, where x_i is determined by

$$x_i = \begin{cases} x_{i_k}, & \text{if } i = i_k, k = 1, 2, \ldots, n \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

Next, the detailed modeling method which uses AHP associate with expert knowledge to solve the optimal solution vector $\boldsymbol{x}$ is elaborated.

B. CONSTRUCT CRITERION LAYER

As an integral part in AHP, the criterion layer can systematically help decision makers to balance priorities between different criteria, which can represent their preferences in a specific situation. According to Section III, there are five state characteristics that can directly or potentially reflect the quality of a portfolio when investors make decision in their own portfolio selection. Consequently, the construction of the criterion layer should use all or part of these state characteristics as the evaluation criteria.

Let $\bar{D}$ denote the set of these state characteristics such that $\bar{D} = \{E, V, H, S, K\}$, then, the process of pairwise comparisons between these evaluation criteria with different preferences can be summarized in the following matrix

$$\bar{\boldsymbol{R}} = \left(\begin{array}{c|ccccc} & SC_{\sigma_1} & SC_{\sigma_2} & SC_{\sigma_3} & SC_{\sigma_4} & SC_{\sigma_5} \\ \hline SC_{\sigma_1} & r_{\sigma_1\sigma_1} & r_{\sigma_1\sigma_2} & r_{\sigma_1\sigma_3} & r_{\sigma_1\sigma_4} & r_{\sigma_1\sigma_5} \\ SC_{\sigma_2} & r_{\sigma_2\sigma_1} & r_{\sigma_2\sigma_2} & r_{\sigma_2\sigma_3} & r_{\sigma_2\sigma_4} & r_{\sigma_2\sigma_5} \\ SC_{\sigma_3} & r_{\sigma_3\sigma_1} & r_{\sigma_3\sigma_2} & r_{\sigma_3\sigma_3} & r_{\sigma_3\sigma_4} & r_{\sigma_3\sigma_5} \\ SC_{\sigma_4} & r_{\sigma_4\sigma_1} & r_{\sigma_4\sigma_2} & r_{\sigma_4\sigma_3} & r_{\sigma_4\sigma_4} & r_{\sigma_4\sigma_5} \\ SC_{\sigma_5} & r_{\sigma_5\sigma_1} & r_{\sigma_5\sigma_2} & r_{\sigma_5\sigma_3} & r_{\sigma_5\sigma_4} & r_{\sigma_5\sigma_5} \end{array}\right), \tag{20}$$

where $SC_{\sigma_i} \in \bar{D}$ for all $i = 1, 2, \ldots, 5$, and σ denotes a certain permutation about the elements in $\bar{D}$. Here, each $r_{\sigma_i\sigma_j}$ is filled with a positive number to represent the importance of the σ_i-th state characteristic to the σ_j-th. And once $r_{\sigma_i\sigma_j}(\sigma_i \neq \sigma_j)$ is determined, $r_{\sigma_j\sigma_i}$ can be calculated according to the principle of pairwise comparison such that $r_{\sigma_i\sigma_j}r_{\sigma_j\sigma_i} = 1$. Let $D \in P\left(\bar{D}\right)$ denote the state characteristics that an investor selects to make decision, then those $r_{\sigma_i\sigma_j}(i = j)$ can be defined as follows

$$r_{\sigma_i\sigma_i} = \begin{cases} +1, & \text{if } SC_{\sigma_i} \in D \\ -1, & \text{if } SC_{\sigma_i} \notin D. \end{cases} \tag{21}$$

It can be seen from (21) that the value of $r_{\sigma_i\sigma_i}$ depends on whether its corresponding state characteristic SC_{σ_i} should be considered in the decision-making process or not. Moreover, it is worth mentioning that in these five state characteristics, only the fuzzy expected value E can be employed to estimate the future return of a portfolio selection, while the other four are all utilized to quantify risks. Consequently, according to the mean-risk rule [2] proposed by Markowitz, E should always be considered in the decision-making process, i.e., $E \in D$ for any $D \in P\left(\bar{D}\right)$. It can be verified that (21) is still suitable even in the condition of $SC_{\sigma_i} = E$. Then the criterion layer can be described by a reciprocal matrix $\boldsymbol{R}$ such that

$$\boldsymbol{R} = \boldsymbol{Q}^T\bar{\boldsymbol{R}}\boldsymbol{Q}, \tag{22}$$

where $\boldsymbol{Q} = \left(\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_5\right)^T$ is a matrix used to distinguish different portfolio frameworks. And each column vector $\boldsymbol{q}_i$ of the matrix $\boldsymbol{Q}$ is accomplished by letting

$$\boldsymbol{q}_i = \begin{cases} \boldsymbol{e}_k, & \text{if } r_{\sigma_i(k)\sigma_i(k)} = 1, k = 1, 2, \ldots, d \\ \boldsymbol{0}, & \text{otherwise}, \end{cases} \tag{23}$$

where $\boldsymbol{e}_k$ is the k-th column of the identity matrix $\boldsymbol{E}_d$. Here, the parameter d, which denotes the rank of $\boldsymbol{E}_d$, is determined by the number of elements in the set D, so it can be represented as $d = \text{Card}(D)$, where $\text{Card}(\cdot)$ is the function of cardinality.

As mentioned above, the matrix $\bar{\boldsymbol{R}}$ is directly related to the order of these five state characteristics, hence, the criterion layer described by the reciprocal matrix $\boldsymbol{R}$ encounters the same condition according to (22). In order to illustrate the generality of the modeling process in the criterion layer, the following theorem is formulated.

Theorem 2: The weight distribution $\boldsymbol{p}$ *of each state characteristic* $SC_{\sigma_i} \in \bar{D}$, $i = 1, 2, \ldots, 5$ *in the criterion layer is independent from the permutation* σ.

Proof: Without loss of generality, let $SC_{\sigma'_j} \in \bar{D}$, $j = 1, 2, \ldots, 5$ be a new arrangement order of the state characteristics and different from that $SC_{\sigma_i} \in \bar{D}$, $i = 1, 2, \ldots, 5$. Then performing the above process yields $\bar{\boldsymbol{R}}' = \left\{r_{\sigma'_i\sigma'_j}\right\}_{5\times5}$. For the new permutation σ', there must exist an integral number σ'_j such that $SC_{\sigma_i} = SC_{\sigma'_j} \in \bar{D}$ for all $i = 1, 2, \ldots, 5$, and hence it can be readily established that

$$\bar{\boldsymbol{R}}' = \boldsymbol{P}_{5\times5}^T\bar{\boldsymbol{R}}\boldsymbol{P}_{5\times5}, \ \boldsymbol{Q}' = \boldsymbol{P}_{5\times5}^T\boldsymbol{Q}\boldsymbol{P}_{d\times d},$$

where $\boldsymbol{P}_{5\times5}$ and $\boldsymbol{P}_{d\times d}$ are two orthogonal matrices such that

$$\boldsymbol{P}_{5\times5} = \prod_{i=1}^{5}\boldsymbol{P}_{5\times5}(\sigma_i, \cdot), \ \boldsymbol{P}_{d\times d} = \prod_{k=1}^{d}\boldsymbol{P}_{d\times d}(k, \cdot),$$

and $\boldsymbol{P}_{m\times m}(u, v)$ denotes a m-order elementary matrix used to interchange the u-th row (column) and the v-th row (column). In addition, the uniqueness of $\boldsymbol{P}_{5\times5}$, $\boldsymbol{P}_{d\times d}$, and their factorizations mention above can be easily proved.

For this new permutation σ', it can be got from (22) that

$$\boldsymbol{R}' = \boldsymbol{Q}'^T\bar{\boldsymbol{R}}'\boldsymbol{Q}',$$

then substituting the above $\bar{\boldsymbol{R}}'$ and $\boldsymbol{Q}'$ with $\bar{\boldsymbol{R}}$ and $\boldsymbol{Q}$ yields

$$\boldsymbol{R}' = \boldsymbol{P}_{d\times d}^T\boldsymbol{Q}^T\bar{\boldsymbol{R}}\boldsymbol{Q}\boldsymbol{P}_{d\times d} = \boldsymbol{P}_{d\times d}^T\boldsymbol{R}\boldsymbol{P}_{d\times d}.$$

Suppose that $\lambda_{\max}$ and $\lambda'_{\max}$ are the largest eigenvalues of $\boldsymbol{R}$ and $\boldsymbol{R}'$ respectively, and their corresponding weight eigenvectors are $\boldsymbol{\alpha}(\|\boldsymbol{\alpha}\|_1 = 1)$ and $\boldsymbol{\alpha}'(\|\boldsymbol{\alpha}'\|_1 = 1)$, then

$$\boldsymbol{R}\boldsymbol{\alpha} = \lambda_{\max}\boldsymbol{\alpha}, \ \boldsymbol{R}'\boldsymbol{\alpha}' = \lambda'_{\max}\boldsymbol{\alpha}'.$$

Since $\lambda_{\max} = \lambda'_{\max}$ and

$$\boldsymbol{P}_{d\times d}^T\boldsymbol{R}\boldsymbol{P}_{d\times d}\boldsymbol{\alpha}' = \lambda'_{\max}\boldsymbol{\alpha}' \Leftrightarrow \boldsymbol{R}\boldsymbol{P}_{d\times d}\boldsymbol{\alpha}' = \lambda'_{\max}\boldsymbol{P}_{d\times d}\boldsymbol{\alpha}',$$

there has

$$\boldsymbol{\alpha} = \boldsymbol{P}_{d\times d}\boldsymbol{\alpha}'.$$

On noting that $p(SC_\sigma)$ is the determined weight of a state characteristic SC_σ, it can be established that

$$\boldsymbol{p}' = \boldsymbol{P}_{d\times d}^T\boldsymbol{p},$$

where $\boldsymbol{p} = \left(p(SC_{\sigma_i(1)}), p(SC_{\sigma_i(2)}), \ldots, p(SC_{\sigma_i(d)})\right)^T$ and $\boldsymbol{p}' = \left(p(SC_{\sigma'_j(1)}), p(SC_{\sigma'_j(2)}), \ldots, p(SC_{\sigma'_j(d)})\right)^T$.

From $\boldsymbol{\alpha} = \boldsymbol{P}_{d\times d}\boldsymbol{\alpha}'$ and $\boldsymbol{p}' = \boldsymbol{P}_{d\times d}^T\boldsymbol{p}$ it can be obtained that

$$\boldsymbol{p} = \boldsymbol{P}_{d\times d}\boldsymbol{p}' = \boldsymbol{P}_{d\times d}\boldsymbol{P}_{d\times d}^T\boldsymbol{\alpha} = \boldsymbol{\alpha}.$$

That is, the weight distribution $\boldsymbol{p}$ is independent from the permutation σ. The proof is complete. ■

Remark 4: The above theorem reveals that the weight distribution calculated from the criterion layer has no relationship with the order of these five state characteristics arranged in (20)*, which shows the general applicability of the modeling process for the criterion layer. In fact, it can be obtained that the weight distribution only depends on the importance comparison results between two state characteristics.*

As is known to all, investors can be divided into two different groups having the completely opposite investment attitude: aggressive and conservative. Aimed at these two different investment attitudes, two specific forms, which correspond to the aggressive and conservative investors respectively, are given to describe the criterion layer by analyzing the parameter constraints. In order to make the following analysis more focused, the following matrix (24) is utilized as a canonical form instead of (20) to elaborate the parameter calculating of the criterion layer when facing the investors having an aggressive or conservative attitude, i.e.,

$$\bar{\boldsymbol{R}}^{\star}=\left(\begin{array}{c|ccccc} & E & V & H & S & K \\ \hline E & \boxed{r_{ee}} & r_{ev} & r_{eh} & r_{es} & r_{ek} \\ V & r_{ve} & \boxed{r_{vv}} & r_{vh} & r_{vs} & r_{vk} \\ H & r_{he} & r_{hv} & \boxed{r_{hh}} & r_{hs} & r_{hk} \\ S & r_{se} & r_{sv} & r_{sh} & \boxed{r_{ss}} & r_{sk} \\ K & r_{ke} & r_{kv} & r_{kh} & r_{ks} & \boxed{r_{kk}} \end{array}\right), \tag{24}$$

and its generality is guaranteed by Theorem 2.

All the parameters listed in (24) can be divided into two parts: one can be denoted as $S_0 = \left\{r_{xy}|X, Y \in \bar{D}, X = Y\right\}$ and another is $S = \left\{r_{xy}|X, Y \in \bar{D}, X \neq Y\right\}$. For the parameter $r_{xy} \in S_0$ (framed in (24)), its value is given by the expression (21). And for another set S, it can be known that only half of the parameters in S are independent from the restrictive relationship of $r_{xy}r_{yx} = 1$. For the sake of convenience, let S_A denote a typical set of the independent parameters

$$S_A = S_A^{(e)} \cup S_A^{(v)} \cup S_A^{(h)} \cup S_A^{(s)}, \tag{25}$$

where $S_A^{(e)} = \{r_{ev}, r_{eh}, r_{es}, r_{ek}\}$, $S_A^{(v)} = \{r_{vh}, r_{vs}, r_{vk}\}$, $S_A^{(h)} = \{r_{hs}, r_{hk}\}$, and $S_A^{(s)} = \{r_{sk}\}$. Similarly, another typical set of the rest independent parameters can be denoted as S_C such that

$$S_C = S_C^{(e)} \cup S_C^{(v)} \cup S_C^{(h)} \cup S_C^{(s)}, \tag{26}$$

where $S_C^{(e)} = \{r_{ve}, r_{he}, r_{se}, r_{ke}\}$, $S_C^{(v)} = \{r_{hv}, r_{sv}, r_{kv}\}$, $S_C^{(h)} = \{r_{sh}, r_{kh}\}$, and $S_C^{(S)} = \{r_{ks}\}$. It can be readily verified that the above two sets satisfy the following relationship

$$S_A \cup S_C = S. \tag{27}$$

Let us recall the fuzzy variance, entropy, skewness, and kurtosis introduced in Section III. As the aforementioned studies, these four state characteristics can be used to measure both detectable and latent risk. For aggressive investors, they are more inclined to control risks within a higher range of profits and keep the risks as small as possible. So compared to the risk, aggressive investors think the investment return is more important. For the above-mentioned reason, each r_{ex} should satisfy the following constraint

$$r_{ex} \geq 1 \text{ (or } r_{xe} \leq 1), \quad \forall X \in \{V, H, S, K\}. \tag{28}$$

Afterwards, compared to the fuzzy variance and entropy, the meaning of higher-order fuzzy moments is drawing into a new supplement on the basis of detectable risk, especially when the values of the fuzzy variance or entropy to several portfolio alternatives are almost the same [7], [24], [41]–[44]. It is then enough to show that the fuzzy variance and entropy are two more important impact factors than those higher-order fuzzy moments. Therefore, the results of pairwise comparison in S_A should come in the following relationship

$$r_{ey} \leq r_{ez}, \quad \forall Y \in \{V, H\}, Z \in \{S, K\}. \tag{29}$$

TABLE 2. Semantic interpretation of Different Aggressive Levels.

β	Investment Attitude	Abbreviation
1	Lightly Aggressive	LA
2	Between Lightly and Somewhat	BLSA
3	Somewhat Aggressive	SA
4	Between Somewhat and Moderately	BSMA
5	Moderately Aggressive	MA
6	Between Moderately and Very	BMVA
7	Very Aggressive	VA
8	Between Very and Extremely	BVEA
9	Extremely Aggressive	EA

Next, the importance orders between not only fuzzy variance and entropy, but also the mentioned two higher-order fuzzy moments should be refined. Consider that fuzzy variance is better than fuzzy entropy in reflecting market risk [46], and the fourth-order moment is generally a supplement to the third-order moment [47], the fuzzy variance V is more important than the fuzzy entropy H and the fuzzy skewness S is more important than the fuzzy kurtosis K, which can be represented as follows

$$r_{ev} \leq r_{eh}, \quad r_{es} \leq r_{ek}. \tag{30}$$

Prior to making any assessment of importance, since the scale given by AHP is finite with its maximum value 9, the following constraint can be obtained by integrating (28), (29), and (30)

$$1 \leq r_{ev}(\beta) \leq r_{eh}(\beta) \leq r_{es}(\beta) \leq r_{ek}(\beta) \leq 9, \tag{31}$$

where β denotes the aggressive level of aggressive investors. Here, the aggressive level β is used to quantify the investment attitude of aggressive investors. Generally speaking, the bigger the value of β, the more aggressive attitude the investor has. In order to be consistent with the scale in (31), β should be limited to the positive integers in-between 1 and 9 [33], which is the most frequently used scale in AHP. In addition, the detailed semantic interpretation of different aggressive levels and the corresponding abbreviations are shown in Table 2.

Through the above analyses, it is clear that the properties and rules of the criterion layer are mainly determined by the four parameters r_{ev}, r_{eh}, r_{es}, and r_{ek} in an aggressive investment. For convenience of calculations, these four parameters can be rewritten into a vector form denoted as $I_A(\beta)$ and express it as the coming formula

$$\begin{aligned} I_A(\beta) &= \left(r_{ev}(\beta), r_{eh}(\beta), r_{es}(\beta), r_{ek}(\beta)\right) \\ &= \left(\lceil k_1\beta \rceil, \lceil k_2\beta \rceil, \lceil k_3\beta \rceil, \lceil 1\beta \rceil\right), \end{aligned} \tag{32}$$

where $k_1, k_2, k_3 \in (0, 1]$, $k_1 \leq k_2 \leq k_3$, and $\lceil\cdot\rceil$ represents the ceiling function. For those $r_{yz} \in S_A \setminus S_A^{(e)}$, in order to make the aggressive reciprocal matrix $\boldsymbol{R}_A(\beta; k_1, k_2, k_3)$ have better

TABLE 3. Semantic interpretation of Different Conservative Levels.

γ	Investment Attitude	Abbreviation
1	Lightly Conservative	LC
2	Between Lightly and Somewhat	BLSC
3	Somewhat Conservative	SC
4	Between Somewhat and Moderately	BSMC
5	Moderately Conservative	MC
6	Between Moderately and Very	BMVC
7	Very Conservative	VC
8	Between Very and Extremely	BVEC
9	Extremely Conservative	EC

consistency, the consistency conditions, i.e. $r_{xz} = r_{xy}r_{yz}$, can be used here as a basis for development. Then the following computation rules are given as follows

$$r_{yz} = \begin{cases} \kappa\left(\dfrac{r_{ez}}{r_{ey}}\right), & \text{if } r_{yz} \in S_A^{(v)} \\ \kappa\left(\dfrac{r_{vz}}{r_{vy}}\right), & \text{if } r_{yz} \in S_A^{(h)} \\ \kappa\left(\dfrac{r_{hz}}{r_{hy}}\right), & \text{if } r_{yz} \in S_A^{(s)}, \end{cases} \tag{33}$$

where $\kappa\,(x)$ denotes an integral function having the following expression

$$\kappa\,(x) = \begin{cases} \lfloor x \rfloor\,, & \text{if } x \leq \big(\lfloor x \rfloor + \lceil x \rceil\big)/2 \\ \lceil x \rceil\,, & \text{if } x > \big(\lfloor x \rfloor + \lceil x \rceil\big)/2. \end{cases} \tag{34}$$

In the same way, let us consider the conservative investment behavior. Different from the aggressive investors, conservative investors tend to pursue the maximization of profits within the controllable risk range, which means that risk is their first consideration. Therefore, each r_{xe} should meet the following requirement

$$r_{xe} \geq 1 \text{ (or } r_{ex} \leq 1), \quad \forall\, X \in \{V, H, S, K\}\,. \tag{35}$$

In other hand, it should be indicated that whether the investment attitude is aggressive or conservative would not change the importance relationship among these four state characteristics. Consequently, on noting that γ is the conservative level of conservative investors, from the foregoing analysis it follows

$$1 \leq r_{ke}\,(\gamma) \leq r_{se}\,(\gamma) \leq r_{he}\,(\gamma) \leq r_{ve}\,(\gamma) \leq 9, \tag{36}$$

where the value of γ is limited to positive integers in-between 1 and 9. The detailed semantic interpretation of different conservative levels is shown in Table 3. Similarly, on noting that $I_C(\gamma)$ is a column vector of r_{ve}, r_{he}, r_{se}, and r_{ke}, then it can be calculated by

$$\begin{aligned} I_C(\gamma) &= \big(r_{ve}\,(\gamma)\,, r_{he}\,(\gamma)\,, r_{se}\,(\gamma)\,, r_{ke}\,(\gamma)\,\big)^T \\ &= \big(\,\lceil 1\gamma \rceil\,, \lceil k_3\gamma \rceil\,, \lceil k_2\gamma \rceil\,, \lceil k_1\gamma \rceil\,\big)^T. \end{aligned} \tag{37}$$

And for those $r_{yz} \in S_C \setminus S_C^{(e)}$, in order to make the aggressive reciprocal matrix $\boldsymbol{R}_C(\gamma; k_1, k_2, k_3)$ have better consistency, the specific computation rules are designed as follows

$$r_{yz} = \begin{cases} \kappa\left(\dfrac{r_{ze}}{r_{ye}}\right), & \text{if } r_{yz} \in S_C^{(v)} \\ \kappa\left(\dfrac{r_{zv}}{r_{yv}}\right), & \text{if } r_{yz} \in S_C^{(h)} \\ \kappa\left(\dfrac{r_{zh}}{r_{yh}}\right), & \text{if } r_{yz} \in S_C^{(s)} \end{cases}. \tag{38}$$

C. CONSTRUCT SCHEME LAYER

Under a certain evaluation criteria employed by the criterion layer, scheme layer is used to weigh and describe the basic parameters of the given securities (alternatives). Here, suppose that there are no interaction and coupling between any two different alternatives.

As stated earlier, consider that there are n securities $\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_n}$ selected in the security pool, and their qualities can be judged by the state characteristics listed in the criterion layer each by each. For a specific evaluation criterion $Z \in \{E, V, H, S, K\}$ employed in the criterion layer, the quality of each alternative is quantified as a number (i.e., the evaluation value $Z[\xi_{i_k}]$, $k = 1, 2, \ldots, n$) objectively. Here, in order to avoid the interference from subjective factors, the construction of the reciprocal matrix $\boldsymbol{R}(Z)$ corresponding to these n alternatives should follow the objective evaluation values $Z[\xi_{i_1}], Z[\xi_{i_2}], \ldots$, and $Z[\xi_{i_n}]$ strictly, i.e.,

$$\boldsymbol{R}(Z) = \left(\begin{array}{c|cccc} Z & \xi_{i_1} & \xi_{i_2} & \cdots & \xi_{i_n} \\ \hline \xi_{i_1} & r_{i_1 i_1} & r_{i_1 i_2} & \cdots & r_{i_1 i_n} \\ \xi_{i_2} & r_{i_2 i_1} & r_{i_2 i_2} & \cdots & r_{i_2 i_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_{i_n} & r_{i_n i_1} & r_{i_n i_2} & \cdots & r_{i_n i_n} \end{array}\right), \tag{39}$$

where the element $r_{i_p i_q}$ is determined by a bivariate function

$$r_{i_p i_q} = G_Z\big(Z[\xi_{i_p}], Z[\xi_{i_q}]\big). \tag{40}$$

In our proposed model, it can be seen from (40) that the pairwise comparison result between any two alternatives only depends on the quantitative values evaluated by the criteria Z. With this, the construction of the scheme layer can be formed into searching a proper pairwise comparison function $G_Z\,(x, y)$. Next, by analyzing the properties that the function $G_Z\,(x, y)$ should satisfy, the detailed derivation process of this function is elaborated, and its specific expressions with necessary proofs are given.

First, notice that $\boldsymbol{R}(Z)$ is a reciprocal matrix, hence according to the definition of the reciprocal matrix there has

$$r_{i_p i_q} = \begin{cases} 1, & \text{if } p = q \\ r_{i_q i_p}^{-1}, & \text{if } p \neq q. \end{cases} \tag{41}$$

Substituting (40) into (41) yields

$$\begin{cases} G_Z\,(x, x) = 1 \\ G_Z\,(x, y) = G_Z^{-1}\,(y, x)\,. \end{cases} \tag{42}$$

Then, let us take focus on the consistency of the reciprocal matrix $\boldsymbol{R}(Z)$. Generally, since there is no fixed reference to quantify the quality of alternatives, human beings may make some inconsistent judgments in their subjective comparison process. This means that the reciprocal matrix $\boldsymbol{R}(Z)$ constructed by human beings subjectively is difficult to satisfy the consistency condition such that $r_{i_r i_t} = r_{i_r i_s} r_{i_s i_t}$. But fortunately, different from the general situation, the alternatives here are securities, and their quality evaluations in any evaluation criteria are all the specific values, which can be regard as a fixed reference. Hence it is possible to make $\boldsymbol{R}(Z)$ a consistent matrix. In order to achieve the above purpose, it should break with the limitation of the original scale which has seventeen fixed values only (i.e., $1^{\pm 1}, 2^{\pm 1}, \ldots, 9^{\pm 1}$ [33]), and use the interval $[9^{-1}, 9^{1}]$ instead. Then, by taking the consistency condition into account, there has

$$G_Z(x, z) = G_Z(x, y)\, G_Z(y, z)\,. \tag{43}$$

In what follows, the specific expression of the pairwise comparison function $G_Z(x, y)$ is given by the following theorem.

Theorem 3: Let $G(x, y) : \mathbb{D} \subseteq \mathbb{R}^2 \rightarrow [9^{-1}, 9^{1}]$ be a bivariate function which satisfies the following properties

1) *G-normality: $G(x, x) = 1$,*
2) *G-reciprocity: $G(x, y) = G^{-1}(y, x)$,*
3) *G-consistency: $G(x, z) = G(x, y)\, G(y, z)$.*

Then $G(x, y)$ has only two expressions such that

$$G(x, y) = \frac{\phi(x) + c}{\phi(y) + c}, \tag{44}$$

$$G(x, y) = \frac{\phi(y) + c}{\phi(x) + c}, \tag{45}$$

where $\phi(\cdot)$ called the kernel of $G(x, y)$ can be any unary function and c is a constant determined by the domain $\mathbb{D}$.

Proof: Without loss of generality, let $a \in \mathbb{R}$ be a real number, from the property of G-consistency there has

$$G(x, y) = G(x, a)\, G(a, y)\,.$$

With the above equation, two first-order partial derivatives of $G(x, y)$ with respect to x and y can be calculated as follows

$$\frac{\partial G(x, y)}{\partial x} = \frac{dG(x, a)}{dx} G(a, y)\,,$$
$$\frac{\partial G(x, y)}{\partial y} = G(x, a) \frac{dG(a, y)}{dy},$$

and from which the second-order mixed derivative follows such that

$$\frac{\partial G(x, y)}{\partial x \partial y} = \frac{dG(x, a)}{dx} \frac{dG(a, y)}{dy}.$$

Consider the calculation results of these partial derivatives and further the condition of $G(x, y) = G(x, a)\, G(a, y)$, then it can be readily checked that

$$\frac{\partial G(x, y)}{\partial x} \frac{\partial G(x, y)}{\partial y} = \frac{\partial G(x, y)}{\partial x \partial y} G(x, y)\,. \tag{46}$$

With the aid of the theorem given by Scott [48], (46) reveals that the bivariate function $G(x, y)$ is separable-variable. Hence, there must exist $g_1(\cdot)$ and $g_2(\cdot)$ such that

$$G(x, y) = g_1(x)\, g_2(y)\,. \tag{47}$$

From (47) it should be clear that the property of G-reciprocity to the function $G(x, y)$ can be equally represented as

$$g_1(x)\, g_2(y) = \big(g_1(y)\, g_2(x)\big)^{-1}.$$

If we however let $y = x$, then there has

$$\begin{aligned} g_1(x)\, g_2(x) &= \big(g_1(x)\, g_2(x)\big)^{-1} \\ &\Leftrightarrow \big(g_1(x)\, g_2(x)\big)^2 = 1 \Leftrightarrow g_1(x)\, g_2(x) = \pm 1. \end{aligned}$$

According to the property of G-normality, it can be known that only one of the last two equations is correct, i.e.,

$$g_1(\cdot)\, g_2(\cdot) = 1. \tag{48}$$

Eventually, the only two expressions of $G(x, y)$ can be obtained by substituting (48) into (47), which is shown as follows

$$G(x, y) = \frac{g_1(x)}{g_1(y)} \text{ or } G(x, y) = \frac{g_2(y)}{g_2(x)}.$$

For the convenience of analysis and application, separate the constant term from $g_1(\cdot)$ and $g_2(\cdot)$, then yield

$$G(x, y) = \frac{\phi(x) + c}{\phi(y) + c} \text{ or } G(x, y) = \frac{\phi(y) + c}{\phi(x) + c}.$$

The proof is complete. ■

Remark 5: The above theorem gives a general form to the pairwise comparison function, which can be directly used to construct the consistent matrix. And it can be seen from (44) *and* (45) *that the selection of $\phi(x)$ has a considerable degree of freedom. However, in practical applications, there is a monotonic relationship between the quality of alternatives and the quantitative value given by most evaluation criteria. Hence, a monotone function should be first considered to be chosen as the kernel function $\phi(x)$.*

In what follows, the calculation method of the constant c in (44) or (45) is elaborated in detail. It is noteworthy that for a specific criterion layer, the state characteristics included in it can be divided into two different types from the aspect of being an evaluation index. More concretely, the fuzzy expected value and skewness are the same type, because both of them are positive indicators. On the contrary, the fuzzy entropy, variance, and kurtosis are negative indicators. For a positive evaluation criterion Z_p, if ξ_u is better than ξ_v, i.e., $Z_p[\xi_u] > Z_p[\xi_v]$, there should be $G_{Z_p}\big(Z_p[\xi_u], Z_p[\xi_v]\big) > 1$. And in order to make the difference between different alternatives as large as possible, the pairwise comparison result between the best one ξ_b and the worst one ξ_w should reach the highest scale $G_{Z_p}\big(Z_p[\xi_b], Z_p[\xi_w]\big) = 9$. To achieve the above requirements, if (44) is selected to construct the reciprocal matrix $\boldsymbol{R}(Z_p)$, then the kernel function $\phi_p(x)$ must be a monotonic

TABLE 4. Fuzzy Returns of Ten Securities and Their Corresponding State Characteristics.

Security i	Fuzzy return $\xi_{t,i}$	Expected value E	Variance V	Entropy H	Skewness S	Kurtosis K
1	$(-0.2000, 2.1000, 2.5000)$	1.6250	0.6827	1.3500	-0.4328	1.0034
2	$(-0.1000, 1.9000, 3.0000)$	1.6750	0.8177	1.5500	-0.2703	1.2522
3	$(-0.4000, 3.0000, 4.0000)$	2.4000	1.7333	2.2000	-1.4520	6.1050
4	$(-0.1000, 2.0000, 2.5000)$	1.6000	0.6167	1.3000	-0.3380	0.7927
5	$(-0.6000, 3.0000, 4.0000)$	2.3500	1.9042	2.3000	-1.7193	7.4205
6	$(-0.2000, 2.5000, 3.0000)$	1.9500	0.9542	1.6000	-0.7040	1.9485
7	$(-0.2000, 3.0000, 3.5000)$	2.3250	1.2927	1.8500	-1.1551	3.6317
8	$(-0.4000, 2.5000, 4.0000)$	2.1500	1.6542	2.2000	-0.8470	5.1624
9	$(-0.3000, 2.8000, 3.2000)$	2.1250	1.1727	1.7500	-1.0336	3.0336
10	$(-0.3000, 2.0000, 2.5000)$	1.5500	0.7208	1.4000	-0.4410	1.0940

increasing function, and the constant c_p can be calculated as follows

$$c_p = \frac{1}{8}\left(\phi_p(\overline{Z}_p) - 9\phi_p(\underline{Z}_p)\right), \tag{49}$$

where $\overline{Z}_p = \max_{k=1,2,\ldots,n} Z_p[\xi_{i_k}]$ and $\underline{Z}_p = \min_{k=1,2,\ldots,n} Z_p[\xi_{i_k}]$. If (45) is selected to construct the reciprocal matrix $\boldsymbol{R}(Z_n)$, then the kernel function $\phi_n(x)$ must be a monotonic decreasing function, and the constant c_n can be calculated as follows

$$c_n = \frac{1}{8}\left(\phi_n(\overline{Z}_n) - 9\phi_n(\underline{Z}_n)\right), \tag{50}$$

where $\overline{Z}_n = \max_{k=1,2,\ldots,n} Z_n[\xi_{i_k}]$ and $\underline{Z}_n = \min_{k=1,2,\ldots,n} Z_n[\xi_{i_k}]$.

D. OVERALL PROCEDURE OF AHP-BASED PORTFOLIO SELECTION

In order to express the method more concisely and clearly, the overall procedure of the proposed method is summarized in the form of Algorithm 1. In the process of building the security pool, Algorithm 1 utilizes fuzzy expected value and fuzzy variance to measure the return and risk of portfolio. It is noteworthy that according to Theorem 1, the fuzzy variance V can be replaced by the fuzzy entropy H in the process of building a security pool.

Algorithm 1 Overall Procedure for Portfolio Selection

Input: The N investment objects, $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_N)^T$.
Output: The investment result, $\boldsymbol{X} = (x_1, x_2, \ldots, x_N)^T$.

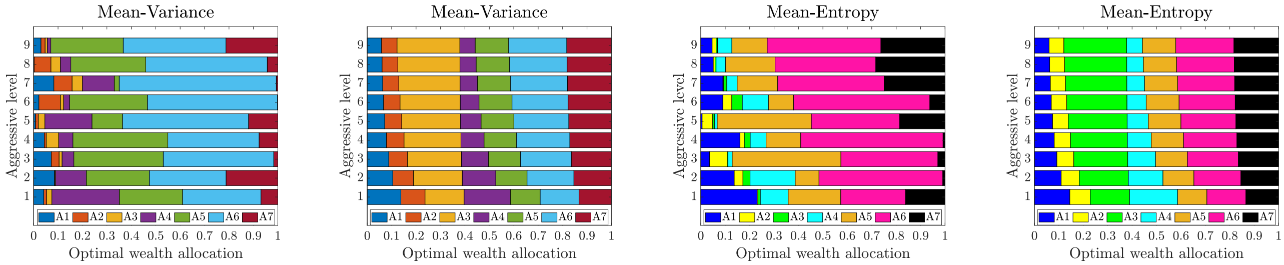
Put $\xi_1, \xi_2, \ldots, \xi_N$ into the security pool;
for $i = 1 : N$ **do**
  for $j = i : N$ **do**
    if $E[\xi_{t,i}] < E[\xi_{t,j}]$ && $V[\xi_{t,i}] > V[\xi_{t,j}]$ **then**
      Remove ξ_i from the security pool;
      Break the loop;
    end if
  end for
end for
Select the needed state characteristics into the set D;
Construct the criterion layer with the aid of (22);
Calculate the weight vector of the criterion layer;
for $k = 1 : \mathrm{Card}(D)$ **do**
  Construct the k-th scheme layer with the aid of (39);
  Calculate the weight vector of the k-th scheme layer;
end for
Calculate the total sort weight vector of the overall hierarchical structure;
Determine the investment result with (19).
return Outputs

V. ILLUSTRATIVE EXAMPLES

In this section, three examples are presented to illustrate the validity and advantages of the proposed AHP-based portfolio selection model. In order to make comparisons with those optimization models (see [6], [42], [49]) and other typical MCDM models, the securities given by Huang's work [6] are cited as available investment alternatives to the first and second examples. These securities are shown in Table 4. Besides, the value of the five state characteristics associated with each security is also reported in the same table. In the last example, the time series data of 280 securities are randomly chosen from the China Shanghai Stock Exchange to further illustrate the effectiveness of the proposed model. Here, the returns of all the securities are modeled by triangular fuzzy variables $\xi_{t,i} = (a_i, b_i, c_i)$ with $i = 1, 2, \ldots, N$.

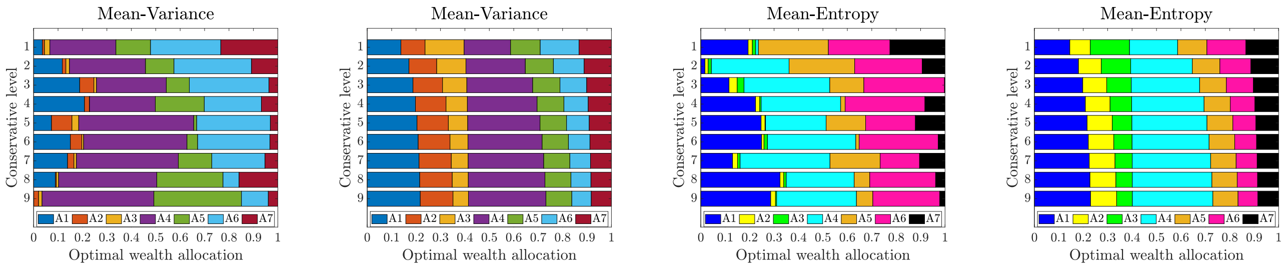
A. EXAMPLE 1

In this example, the mean-variance framework and mean-entropy framework, which are the two most basic and simple portfolio frameworks, are adopted here to verify the practicability and validity of the proposed model. Under these two frameworks, it is easy to check that the reciprocal matrix in the criterion layer must be a consistent matrix. Accordingly, there is no need to check the consistency of them. Moreover, the consistency of the reciprocal matrix in the scheme layer is guaranteed by Theorem 3.



(a) Optimization model (b) Proposed model (c) Optimization model (d) Proposed model

FIGURE 2. **Comparisons of aggressive portfolio decision-making results between optimization model and proposed model under different frameworks.**



(a) Optimization model (b) Proposed model (c) Optimization model (d) Proposed model

FIGURE 3. **Comparisons of conservative portfolio decision-making results between optimization model and proposed model under different frameworks.**

Since the optimization model is the most commonly used model in portfolio selection under an established framework, the portfolio selection results given by the optimization model and the proposed model are compared and discussed firstly. It can be obtained from Theorem 1 that there are 7 securities (alternatives) in the security pool. For aggressive and conservative investors, the detailed portfolio selection results using the optimization model and the proposed model under the mean-variance and mean-entropy framework are shown in Fig. 2 and Fig. 3 respectively, where each color represents one alternative and the length of the visible color bar represents the allocation proportion of different alternatives. It can be seen from Fig. 2 or Fig. 3 that the proposed model gives stabler portfolio decision results than that given by the optimization model, and the decision results given by the proposed model show obvious gradual change with the change of aggressive or conservative level, which primarily demonstrates the stability and validity of the proposed model.

Furthermore, in order to further examine the effectiveness of the proposed model, not only the optimization model, but also the other three typical MCDM models, which are the standard AHP [33], fuzzy comprehensive evaluation (fuzzy CE) [50], and fuzzy AHP [31], are employed as the objects of comparison. An acceptable portfolio selection result should be relatively more diversified and balanced to disperse risk, which means that the allocation proportion of each investment object should not be too much or too little. Consequently, the dispersion degree of portfolio selection results can be reflected by its corresponding standard deviation. After getting different optimal results given by the MCDM models mentioned above, the standard deviations of these portfolio selection results in different conditions are shown in Table 5 and Table 6. Since the expert practical experience are incorporated into the reciprocal matrix in the criterion layer, it can be seen that the standard deviations of the results given by the proposed model are smaller than those given by the optimization model and other three MCDM models, which means the proposed model leads to more distributed investments. In addition, from the aspect of calculation speed, the detailed comparison results of running time for the above models are shown in Fig. 4. It can be known from Fig. 4 that the running time of the optimization model changes significantly with the investment attitude, while that of the proposed model and other three MCDM models has little to do with the investment attitude. This is because the decision-making processes given by the proposed model and other three MCDM models are highly systematized and structured, and optimization algorithms are unnecessary in the process of solving the model. Besides that, compared with the other three MCDM models, Theorem 3 indicates that the evaluation process of the proposed model can be completed automatically according to a specific evaluation criterion. Fig. 4 shows that the proposed model can lead to a faster decision-making speed, which confirms that the proposed model has higher decision efficiency in both aggressive and conservative investment.

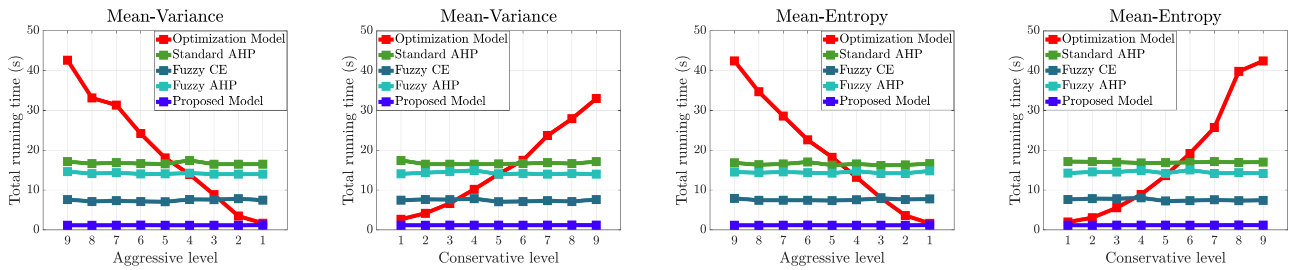
Finally, in order to study the practicability of the proposed model, a sensitivity analysis is conducted with respect to the investment attitude of investors. The weights of the used

TABLE 5. Comparisons of Standard Deviation of Portfolio Results among Different MCDM Models under Mean-variance Framework.

		Value of β (or γ)	1	2	3	4	5	6	7	8	9
$\sigma(\boldsymbol{x})$	Aggressive	Optimization model	0.1358	0.1229	0.1848	0.1642	0.1787	0.2048	0.2245	0.1866	0.1676
		Standard AHP	0.0896	0.0945	0.1048	0.1102	0.1216	0.1328	0.1243	0.1167	0.0983
		Fuzzy CE	0.0607	0.0784	0.0852	0.0976	0.1070	0.1103	0.1085	0.0907	0.0875
		Fuzzy AHP	0.0563	0.0645	0.0778	0.0823	0.0886	0.0934	0.0965	0.1019	0.1041
		Proposed model	**0.0298**	**0.0427**	**0.0567**	**0.0659**	**0.0721**	**0.0767**	**0.0802**	**0.0829**	**0.0851**
	Conservative	Optimization model	0.1222	0.1254	0.1253	0.1102	0.1752	0.1590	0.1416	0.1472	0.1869
		Standard AHP	0.0774	0.0879	0.0945	0.1065	0.1130	0.1236	0.1341	0.1268	0.1167
		Fuzzy CE	0.0657	0.0764	0.0792	0.0859	0.1072	0.1167	0.1193	0.1148	0.1103
		Fuzzy AHP	0.0579	0.0678	0.0739	0.0803	0.0883	0.0952	0.1002	0.1089	0.1145
		Proposed model	**0.0298**	**0.0488**	**0.0636**	**0.0730**	**0.0795**	**0.0841**	**0.0876**	**0.0904**	**0.0926**

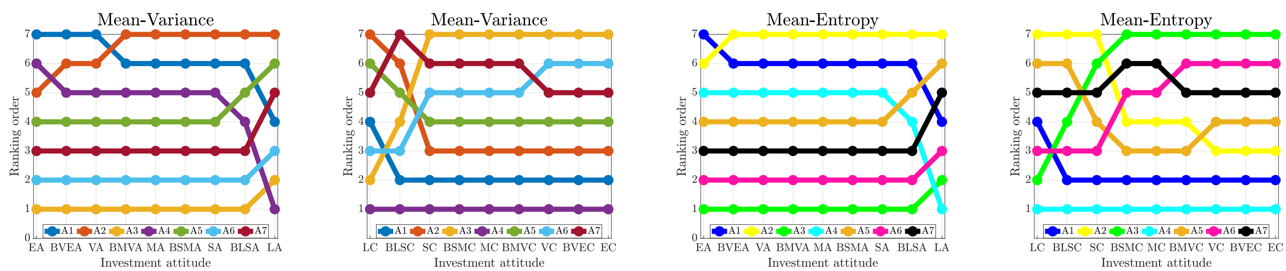
TABLE 6. Comparisons of Standard Deviation of Portfolio Results among Different MCDM Models under Mean-entropy Framework.

		Value of β (or γ)	1	2	3	4	5	6	7	8	9
$\sigma(\boldsymbol{x})$	Aggressive	Optimization model	0.1051	0.1726	0.1917	0.2032	0.1698	0.1853	0.1565	0.1596	0.1681
		Standard AHP	0.0975	0.0988	0.1092	0.1168	0.1345	0.1256	0.1168	0.1098	0.0874
		Fuzzy CE	0.0789	0.0835	0.0882	0.0935	0.1085	0.1192	0.1046	0.0895	0.0894
		Fuzzy AHP	0.0694	0.0785	0.0894	0.1074	0.1237	0.1365	0.1167	0.1108	0.0974
		Proposed model	**0.0357**	**0.0448**	**0.0576**	**0.0664**	**0.0725**	**0.0770**	**0.0804**	**0.0831**	**0.0852**
	Conservative	Optimization model	0.1237	0.1393	0.1439	0.1458	0.1019	0.1613	0.1218	0.1404	0.1452
		Standard AHP	0.0847	0.0942	0.0967	0.1035	0.1175	0.1293	0.1321	0.1223	0.1128
		Fuzzy CE	0.0867	0.0981	0.1063	0.1123	0.1178	0.1257	0.1287	0.1184	0.1121
		Fuzzy AHP	0.0805	0.0864	0.0908	0.0957	0.1025	0.1094	0.1104	0.1178	0.1235
		Proposed model	**0.0357**	**0.0553**	**0.0699**	**0.0794**	**0.0857**	**0.0904**	**0.0939**	**0.0967**	**0.0989**



(a) Aggressive portfolio (b) Conservative portfolio (c) Aggressive portfolio (d) Conservative portfolio

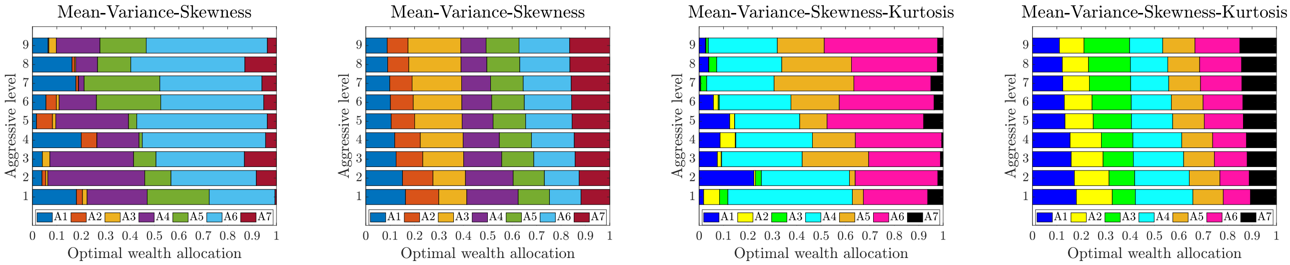
FIGURE 4. Comparisons of portfolio decision-making running time among different MCDM models under different frameworks.



(a) Aggressive portfolio (b) Conservative portfolio (c) Aggressive portfolio (d) Conservative portfolio

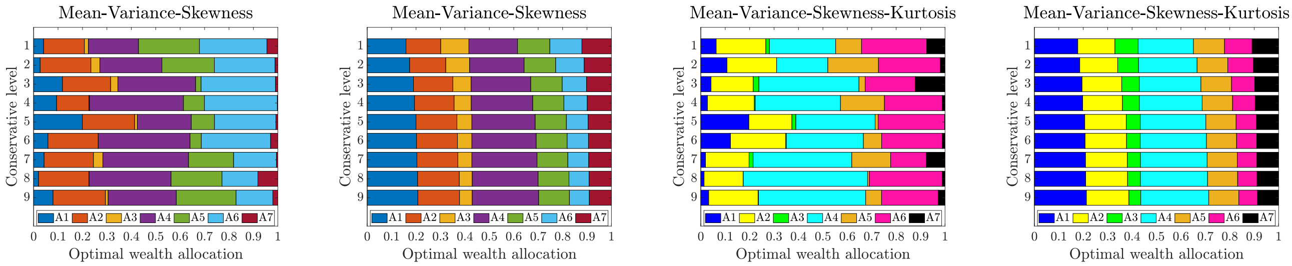
FIGURE 5. Sensitivity analysis results of proposed model under mean-variance framework and mean-entropy framework.

evolution criteria are changed gradually with each of the linguistic terms given in Table 2 or Table 3. And the results of the sensitivity analysis are shown in Fig 5. Fig. 5 shows that with the decrease of aggressive level (or the increase of conservative level), investors are more and more inclined to choose the alternative with less risk and lower return. On the contrary, with the decrease of conservative level (or the increase of aggressive level), investors tend to choose the alternative with higher return and more risk. It is worth noticing that when the quantitative value of investment attitude is at a low level,



(a) Optimization model (b) Proposed model (c) Optimization model (d) Proposed model

FIGURE 6. **Comparisons of aggressive portfolio decision-making results between optimization model and proposed model under different frameworks.**



(a) Optimization model (b) Proposed model (c) Optimization model (d) Proposed model

FIGURE 7. **Comparisons of conservative portfolio decision-making results between optimization model and proposed model under different frameworks.**

the ranking orders of the alternatives are easier to change with the change of the investment attitude. This is because at this stage, the result comes from a fierce game of return and risk, and neither of them has gained an overwhelming position. According to these results, the proposed model can highly simulate the process of human cognition and decision making in a robust and reliable way.

B. EXAMPLE 2

To further assess the generality and flexibility of the proposed model in other frameworks, in this example, the mean-variance-skewness framework and mean-variance-skewness-kurtosis framework, which add higher-order moments of return distribution into the mean-variance framework, need to be discussed. In the ensuing experiments, let $k_1 = 0.42$ for the mean-variance-skewness framework and $k_1 = 0.23$, $k_2 = 0.48$ for the mean-variance-skewness-kurtosis framework. It can be readily verified that the reciprocal matrices constructed in both criterion layer and scheme layer have acceptable consistency with the above parameters.

Consider that numerous optimization models have been proposed for portfolio selection under these two frameworks, it is necessary to make comparisons with the proposed model. After having solved the optimization model and the proposed model, the intuitive investment portfolio results are shown in Fig. 6 and Fig. 7. It can be seen from Fig. 6 and Fig. 7 that the security pool has 7 securities (alternatives) in total and the allocation proportion of each alternative is represented by the length of its corresponding color bar. for the optimization model, both Fig. 6 and Fig. 7 show that small changes in investment attitude always lead to irregular and drastic changes in investment results, which means the optimization model lack stability. But for the proposed model, no matter the investor's investment attitude is aggressive or conservative, the investment results given by the proposed model changes regularly with the change of investment attitude level, which reveals that the proposed model gives more reasonable and stable decision results.

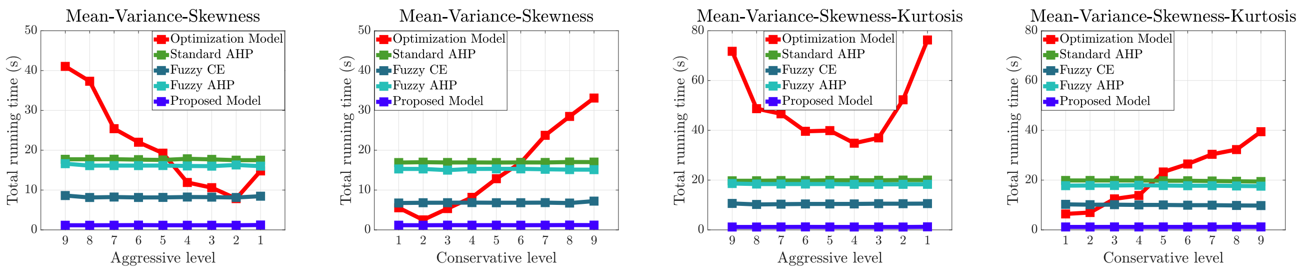
In order to make a more comprehensive comparison, besides the optimization model, the other three MCDM models mentioned in the previous example are employed as the objects of comparison. The standard deviations of these portfolio selection results in different conditions are shown in Table 7 and Table 8. From Table 7 and Table 8 it can be known that the proposed model leads more distributed decision results, which verifies that the proposed model can use expert experiences to disperse risks more effectively and reasonably. Meanwhile, since some potential risks can be identified by the higher-order fuzzy moments, the proposed model leads more balanced portfolio selection results compared to the mean-variance and mean-entropy framework. However, excessive consideration of risk factors would inevitably weaken the effect of return, and then make the investment results prefer to focus on those alternatives with less risk and lower return. This is the reason why the proposed model the mean-variance-skewness framework leads more distributed portfolio selection results than that under the mean-variance-skewness-kurtosis framework.

TABLE 7. **Comparisons of Standard Deviation of Portfolio Results among Different MCDM Models under Mean-variance-skewness Framework.**

Value of β (or γ)			1	2	3	4	5	6	7	8	9
$\sigma(\boldsymbol{x})$	Aggressive	Optimization model	0.1215	0.1620	0.1496	0.1771	0.1998	0.1505	0.1655	0.1557	0.1722
		Standard AHP	0.1034	0.1188	0.1273	0.1363	0.1487	0.1561	0.1450	0.1274	0.1149
		Fuzzy CE	0.0746	0.0824	0.0813	0.0957	0.1028	0.0975	0.0921	0.0886	0.0832
		Fuzzy AHP	0.0673	0.0728	0.0792	0.0864	0.0921	0.1093	0.1134	0.1039	0.0948
		Proposed model	**0.0338**	**0.0251**	**0.0226**	**0.0268**	**0.0388**	**0.0420**	**0.0446**	**0.0529**	**0.0548**
	Conservative	Optimization model	0.1078	0.1118	0.1328	0.1461	0.1035	0.1459	0.1220	0.1188	0.1085
		Standard AHP	0.1142	0.1074	0.1002	0.0945	0.0867	0.0825	0.0894	0.0931	0.0983
		Fuzzy CE	0.0783	0.0832	0.0870	0.0916	0.0934	0.0953	0.0982	0.1001	0.1064
		Fuzzy AHP	0.0653	0.0632	0.0675	0.0836	0.1058	0.1134	0.1096	0.1063	0.0982
		Proposed model	**0.0286**	**0.0435**	**0.0594**	**0.0640**	**0.0701**	**0.0724**	**0.0743**	**0.0770**	**0.0782**

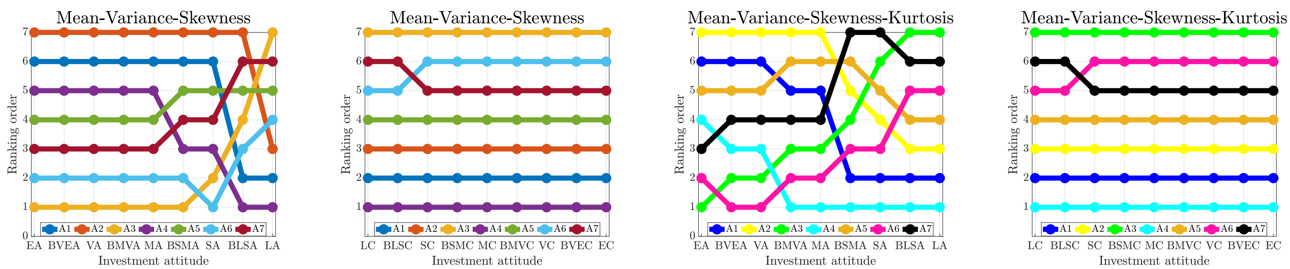
TABLE 8. **Comparisons of Standard Deviation of Portfolio Results among Different MCDM Models under Mean-variance-skewness-kurtosis Framework.**

Value of β (or γ)			1	2	3	4	5	6	7	8	9
$\sigma(\boldsymbol{x})$	Aggressive	Optimization model	0.1820	0.1604	0.1487	0.1434	0.1413	0.1521	0.1543	0.1510	0.1786
		Standard AHP	0.1375	0.1231	0.1192	0.1127	0.1064	0.1014	0.1105	0.1286	0.1364
		Fuzzy CE	0.1190	0.1123	0.1052	0.0938	0.0901	0.0833	0.0784	0.0736	0.0721
		Fuzzy AHP	0.1082	0.1072	0.1015	0.0986	0.0914	0.0845	0.0764	0.0682	0.0608
		Proposed model	**0.0500**	**0.0420**	**0.0307**	**0.0273**	**0.0192**	**0.0200**	**0.0232**	**0.0245**	**0.0333**
	Conservative	Optimization model	0.1036	0.1013	0.1383	0.1324	0.1336	0.1226	0.1326	0.1964	0.1585
		Standard AHP	0.1021	0.1163	0.1268	0.1302	0.1367	0.1417	0.1386	0.1278	0.1146
		Fuzzy CE	0.1094	0.1173	0.1293	0.1346	0.1280	0.1178	0.1167	0.1128	0.1084
		Fuzzy AHP	0.1127	0.1273	0.1321	0.1284	0.1208	0.1147	0.1091	0.1025	0.0948
		Proposed model	**0.0464**	**0.0548**	**0.0642**	**0.0671**	**0.0758**	**0.0772**	**0.0790**	**0.0799**	**0.0828**



(a) Aggressive portfolio (b) Conservative portfolio (c) Aggressive portfolio (d) Conservative portfolio

FIGURE 8. **Comparisons of portfolio decision-making running time among different MCDM models under different frameworks.**



(a) Aggressive portfolio (b) Conservative portfolio (c) Aggressive portfolio (d) Conservative portfolio

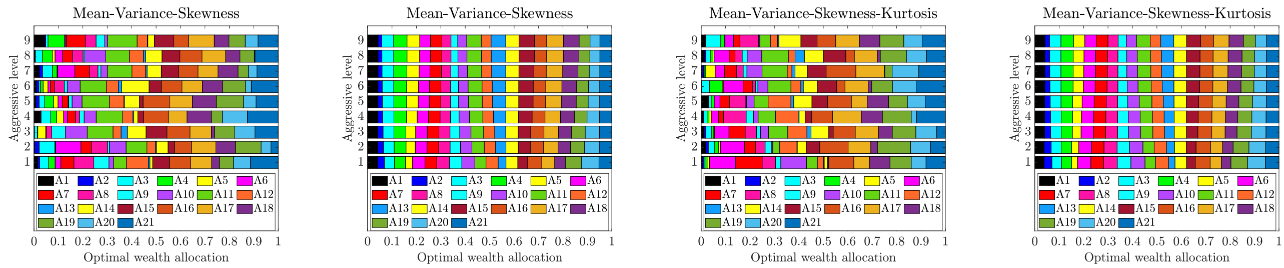
FIGURE 9. **Sensitivity analysis results of proposed model under mean-variance-skewness framework and mean-variance-skewness-kurtosis framework.**

In addition, from the perspective of the speed and efficiency for a decision-making model, since the evaluation process of the proposed model can be completed automatically, it achieves the fastest speed in the decision-making of portfolio selection according to Fig. 8.

Finally, in order to show the practicability of the proposed model under these two frameworks, a sensitivity analysis is conducted with respect to the investment attitude of investors. The results of the sensitivity analysis are shown in Fig 9, where the linguistic terms corresponding to different

TABLE 9. Fuzzy Returns of 280 Securities from China Shanghai Stock Exchange and Their Corresponding State Characteristics.

Security i	Security Code	Fuzzy return $\xi_{t,i}$	Expected value E	Variance V	Entropy H	Skewness S	Kurtosis K
1	600141	$(-2.0958, 0.2600, 4.1423)$	0.6416	3.2914	3.1190	1.8563	20.0663
2	600160	$(-2.9008, 1.2400, 2.4483)$	0.5069	2.5636	2.6745	−2.6221	13.3672
3	600163	$(-3.0285, 0.3000, 2.9640)$	0.1339	3.0017	2.9962	−0.7457	16.3175
...	...	...	...	...	...	...	...
279	600780	$(-1.7791, 0.0000, 2.5080)$	0.1822	1.5427	2.1435	0.4186	4.3447
280	600789	$(-4.5156, 1.4000, 2.2246)$	0.1272	4.3258	3.3701	−7.2277	41.0421



(a) Optimization model (b) Proposed model (c) Optimization model (d) Proposed model

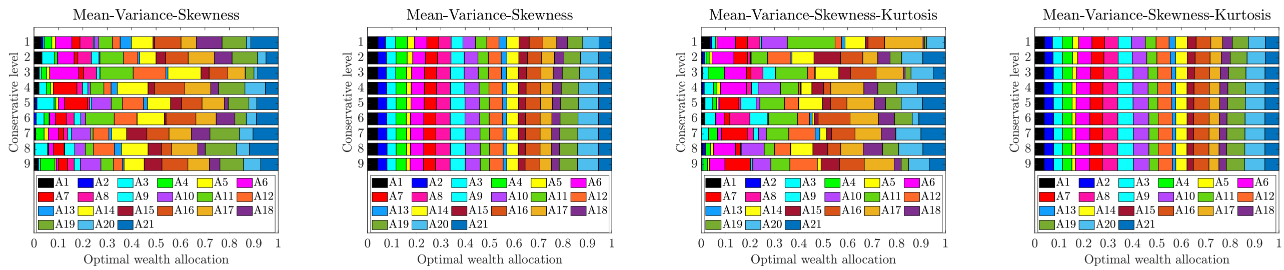
FIGURE 10. Comparisons of aggressive portfolio decision-making results between optimization model and proposed model under different frameworks.

aggressive or conservative levels are given in Table 2 or Table 3. Consistent with the previous example, Fig. 9 shows that investors tend to choose the alternative with less risk and lower return when the aggressive level decreases or the conservative level increases. Instead, when the aggressive level decreases or the conservative level increases, investors tend to choose the alternative with less risk and lower return. However, different from the previous example, the ranking orders of the alternatives are easier to change with the change of the investment attitude for aggressive investors. This is because the addition of supplementary risk shakes their investment attitude and makes them find a new balance through a game between return and risk. But for conservative investors, the consideration of supplementary risk deepened their understanding of the importance of market risk, so there are few changes on ranking in all variations of the investment attitude. According to these results, the proposed model can highly simulate the process of human cognition and decision making in a robust and reliable way.

C. EXAMPLE 3

In this example, a case study of using a real-world data set is given to illustrate the effectiveness of the proposed model. This real-world data set randomly select 280 stocks from China Shanghai Stock Exchange, and part of these stocks are listed in Table 9. After that, considering the variability and complexity of the real market, the mean-variance-skewness and mean-variance-skewness-kurtosis framework are employed here to comprehensively grasp the possible risk in the security market.

First of all, among these 280 stocks, it can be known from Theorem 1 that only 21 securities have the investment value. Then, the proposed model is implemented with $k_1 = 0.65$ for the mean-variance-skewness framework and $k_1 = 0.35$, $k_2 = 0.75$ for the mean-variance-skewness-kurtosis framework. It can be checked that the reciprocal matrices constructed in both criterion layer and scheme layer have acceptable consistency with the above parameters. Since the most commonly used model of portfolio is the optimization model, it is necessary to take it as reference to show the superiority of the proposed model. The detailed investment portfolio results given by the optimization model and the proposed model are shown in Fig. 10 and Fig. 11 intuitively. It can be seen from Fig. 10 and Fig. 11 that the results given by the optimization model cannot achieve the smooth transition associated with the change of aggressive or conservative level, while the proposed model acts well in this aspect, which is more in line with the actual investment done by human beings. Furthermore, the practicability and efficiency of a portfolio selection model can be evaluated by the standard deviation of investment results and the running time of decision-making processes. In order to further verify the superiority of the proposed model, the standard deviation of investment results and the running time are considered. Here, four typical MCDM models are implemented to make comparisons, including the optimization model and other three MCDM models. The detailed standard deviation of each portfolio result is calculated and shown in Table 10 and Table 11. Compared to the optimization model and other three MCDM models, both Table 10 and Table 11 imply that the results given by the proposed model have smaller standard deviations, which means that it can spread risk more effectively. Moreover, compared to the optimization model and other three MCDM models, Fig. 12 reveals that the proposed



(a) Optimization model (b) Proposed model (c) Optimization model (d) Proposed model

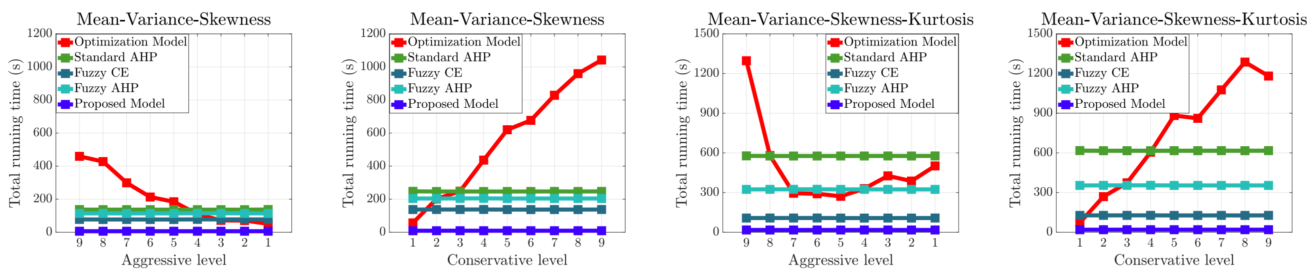
FIGURE 11. Comparisons of conservative portfolio decision-making results between optimization model and proposed model under different frameworks.

TABLE 10. Comparisons of Standard Deviation of Portfolio Results Between Optimization Model and Proposed Model under Mean-variance-skewness Framework.

Value of β (or γ)			1	2	3	4	5	6	7	8	9
$\sigma(x)$	Aggressive	Optimization model	0.0329	0.0351	0.0391	0.0381	0.0393	0.0387	0.0297	0.0340	0.0347
		Standard AHP	0.0311	0.0316	0.0325	0.0333	0.0341	0.0339	0.0332	0.0327	0.0325
		Fuzzy CE	0.0302	0.0307	0.0313	0.0315	0.0318	0.0324	0.0321	0.0318	0.0312
		Fuzzy AHP	0.0282	0.0287	0.0291	0.0296	0.0301	0.0293	0.0287	0.0284	0.0268
		Proposed model	**0.0106**	**0.0079**	**0.0080**	**0.0089**	**0.0101**	**0.0105**	**0.0116**	**0.0124**	**0.0127**
	Conservative	Optimization model	0.0367	0.0344	0.0481	0.0418	0.0361	0.0407	0.0377	0.0403	0.0337
		Standard AHP	0.0345	0.0367	0.0373	0.0362	0.0357	0.0352	0.0348	0.0341	0.0331
		Fuzzy CE	0.0338	0.0343	0.0359	0.0364	0.0352	0.0346	0.0341	0.0338	0.0332
		Fuzzy AHP	0.0302	0.0310	0.0318	0.0326	0.0330	0.0334	0.0332	0.0329	0.0322
		Proposed model	**0.0093**	**0.0127**	**0.0141**	**0.0153**	**0.0160**	**0.0165**	**0.0169**	**0.0172**	**0.0175**

TABLE 11. Comparisons of Standard Deviation of Portfolio Results Between Optimization Model and Proposed Model under Mean-variance-skewness-kurtosis Framework.

Value of β (or γ)			1	2	3	4	5	6	7	8	9
$\sigma(x)$	Aggressive	Optimization model	0.0403	0.0385	0.0325	0.0424	0.0333	0.0370	0.0421	0.0355	0.0364
		Standard AHP	0.0347	0.0352	0.0356	0.0360	0.0363	0.0358	0.0354	0.0351	0.0349
		Fuzzy CE	0.0312	0.0318	0.0325	0.0327	0.0328	0.0322	0.0313	0.0304	0.0297
		Fuzzy AHP	0.0223	0.0236	0.0247	0.0252	0.0251	0.0243	0.0238	0.0234	0.0230
		Proposed model	**0.0129**	**0.0106**	**0.0080**	**0.0078**	**0.0077**	**0.0081**	**0.0085**	**0.0087**	**0.0096**
	Conservative	Optimization model	0.0533	0.0390	0.0417	0.0410	0.0373	0.0434	0.0413	0.0378	0.0425
		Standard AHP	0.0364	0.0372	0.0376	0.0379	0.0362	0.0353	0.0345	0.0341	0.0339
		Fuzzy CE	0.0316	0.0321	0.0325	0.0331	0.0328	0.0323	0.0314	0.0306	0.0302
		Fuzzy AHP	0.0202	0.0208	0.0215	0.0223	0.0231	0.0235	0.0239	0.0243	0.0241
		Proposed model	**0.0121**	**0.0143**	**0.0163**	**0.0168**	**0.0172**	**0.0179**	**0.0181**	**0.0183**	**0.0186**



(a) Aggressive portfolio (b) Conservative portfolio (c) Aggressive portfolio (d) Conservative portfolio

FIGURE 12. Comparisons of portfolio decision-making running time between optimization model and proposed model under different frameworks.

model has higher decision efficiency and lower decision cost in both aggressive and conservative investment.

VI. CONCLUSION AND FUTURE WORK

In this paper, different from the traditional optimization model, a new decision-making model is proposed for portfolio selection with the aid of AHP. In the proposed model, in order to reflect the investment attitude and experience of different investors, the expert knowledge is integrated into the criterion layer and the structural method to construct a reciprocal matrix with acceptable consistency is given. Moreover, in order to avoid the influence of other non-empirical subjective factors on the decision-making result, this paper breaks with the limitation of the original scale which has seventeen fixed values only and give a calculation method of building strictly consistency matrix, which ensures that the decision result is completely driven by the state characteristics of a portfolio itself. Additionally, three experiments involving synthetic and real-world data show that the proposed model produces better performance in rationality and timeliness than the optimization model and gives more distributed investments. The approach presented here has strong generality and can be applied to other MCDM problems.

The present research is merely the first step in using AHP to make portfolio decisions, and it is limited in the two following aspects. First, only the five most representative state characteristics of portfolio and the relative importance relationship between two of them are considered, but the interaction and coupling among them are not accounted for. Second, the proposed method is mainly focused on the problem of individual decision, so the differences of opinions (judgments) expressed by individual members in group decision cannot be reconciled. Therefore, in further studies, underlying causal relationships or association analysis between different state characteristics and the effect brought about by the change of causal strength cause for concern. In addition, to achieve an increasing level of consistency within the group towards the problem of portfolio selection, aggregation mechanisms can be used in AHP by admitting membership degrees for different investors.

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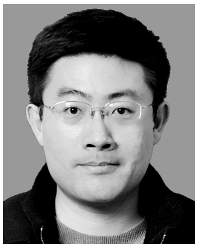

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