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On Graph Structures in Fuzzy Environment Using Optimization Parameter

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ABSTRACT This paper comprises the introduction of weighted mean products of fuzzy graph structures (*FGSs*) to construct weighted mean fuzzy graph structures (*WMFGSs*) with the help of optimization parameter, and establish some novel results after validating with examples, accordingly. The notions of regular and $m\mu_k$-regular *FGSs* are described, where $m \in]0, 1]$ represents the degree of all vertices in $\widetilde{G}$ under mapping μ_k, and develop certain properties of regular *WMFGSs*. In addition, we create a flowchart to present common application procedures of fuzzy graphical frameworks to classify the advanced city out of some important Pakistani cities subject to certain parameters.

INDEX TERMS Fuzzy graph structure, weighted mean product, vertex degree, vertex total degree, regular fuzzy graph structure.

I. INTRODUCTION

A graph is used to represent mathematical networks that define the association between vertices and edges. A vertex can be used to symbolize a workstation, while the edges denote the association between stations. The graph theory helps explain physical structures, e.g., positioning a certain number of cars at different corners to be observed at each corner only once, can be solved across networks where the positions are interlinked, and the navigation agent moves from one corner to another within the network.

Graphs often do not reflect many physical processes adequately because the complexity of various properties of the structures is obvious. Many real-world phenomena have emphasized the concept of fuzzy graphs. In 1973, Kaufmann [1] presented the first picture of a fuzzy graph under the use of Zadeh's [2] fuzzy relation. In 1975, a more detailed description was credited to Rosenfeld [3] who introduced the fuzzy graph theory by considering fuzzy relations on fuzzy sets. He established some relations regarding properties of path graph, trees and various graphs. Bhattacharya introduced the notion of fuzzy cut nodes and fuzzy bridges in [4]. A fuzzy graph is a generalization of a crisp graph. Hence, there exist many similar properties between them, but they also deviate at several places.

Mordeson and Nair [5] defined some fuzzy graphs and fuzzy hypergraphs. Mordeson and Chang-Shyh [6] discussed some operations on the fuzzy graphs. Nagoor Gani and Radha [7], [8] defined some properties of regular fuzzy graphs and the degree of a vertex in some fuzzy graphs. Dinesh [9] presented the concept of fuzzy graph structures and described certainly associated notions. Fuzzy graphical Frameworks are much more desirable than crisp graph structures since they cope with the imprecision and uncertainty of different real-life situations. Ramakrishnan and Dinesh [10] focused on the generalization of fuzzy graph structures. Sahoo and Pal [11] generalized the intuitionistic fuzzy competition graphs into k-competition and p-competition intuitionistic fuzzy graphs. Sahoo *et al.* [12] proposed the notions of covering and matching in an intuitionistic fuzzy graph based on strong arcs and introduced several useful properties on it. Sahoo and Pal [13] introduced the edge irregular intuitionistic fuzzy graphs, edge irregular intuitionistic fuzzy graphs, highly edge irregular intuitionistic fuzzy graphs and highly edge irregular intuitionistic fuzzy graphs. Akram *et al.* [14] have presented some theories about certain fuzzy graph

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structures, fuzzy graph structures and m-polar fuzzy graph structures. Jia *et al.* [15] proposed a consensus-based multi-person decision making (MPDM) procedure using consistency graphs (additive consistent and order consistent) in a fuzzy environment. Soumitra and Ganesh [16] introduced the Wiener index for a bipolar fuzzy graph and explained their properties. The Wiener absolute index is created based on the total accurate connectivity between all the pair of vertices and the whole bipolar fuzzy graph. Soumitra and Ganesh [17] improved some important results on different types of operations of bipolar fuzzy graphs. They explained some important theorems about the degree of composition, tensor product, and normal product of two bipolar fuzzy graphs using examples.

In this paper, we introduce a generalized framework to handle uncertain data with the help of fuzzy sets and graphical structure. We investigate and present some notions of associated properties after introducing weighted mean fuzzy graph structure (*WMFGS*). Furthermore, the notions of regular and $m\mu_k$-regular *FGSs* are described, and we develop certain properties of regular *WMFGSs*.

II. WEIGHTED MEAN PRODUCT

This section provides the basic notions concerning graph structure, *FGSs* including weighted mean product of two *FGSs* and regular *FGSs* with a few related properties. We have also defined the degree and total degree of vertex in *WMFGSs* and discussed some properties with examples.

Definition 1: A pair $G = (V, E)$ of sets of vertices and edges, respectively, is a graph structure (GS) if set E of edges carries mutually disjoint subsets E_i i.e., $E_i \subset E$, $1 \leq i \leq n$, associated with symmetric and irreflexive mapping f_i. Conveniently, a graph structure can be expressed just like a simple graph labled with f_i, $1 \leq i \leq n$.

Example 1: let $V = \{u, v, w, x\}$ be a set of vertices for an undirected graph G, and $E_1 = \{uv, uw, wx\}$, $E_2 = \{ux, vw, vx\}$ are two relations defined on V associated with the functions f_1 and f_2, respectively. One can easily observe that E_1 and E_2 are irreflexive because they do not have elements as uu, vv, ww, xx, moreover, these relations are symmetric being in undirected graph. Therefore, V alongwith E_1 and E_2 establish a graph structure as shown in following Figure (1).

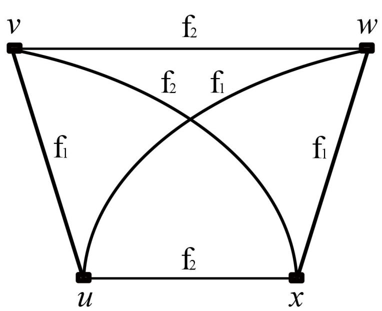


FIGURE 1. **A graph structure $G = (V, E_1, E_2)$.**

Definition 2: The structure $\widetilde{G} = (V, E_i, \lambda, \mu_i)$, $1 \leq i \leq n$, is an FGS with core GS, $G = (V, E_i)$, $1 \leq i \leq n$, where non empty set V is associated with fuzzy membership function λ i.e., $\lambda : V \to [0, 1]$, and E_i are completely described by μ_i i.e., $\mu_i : E_i \to [0, 1]$, $1 \leq i \leq n$, determined by $\mu_i(uv) \leq \lambda(u) \wedge \lambda(v)$ for all $u, v \in V$.

Example 2: Let us define a fuzzy set $V = \{(u, 0.4), (v, 0.4), (w, 0.6), (x, 0.7)\}$, associated with $\lambda : V \to [0, 1]$, in GS shown in figure (1). Then we can define fuzzy sets $E_1 = \{(uv, 0.35), (uw, 0.4), (wx, 0.5)\}$ and $E_2 = \{(ux, 0.4), (vw, 0.35), (vx, 0.3)\}$ in G associated with mappings $\mu_1 : E_1 \to [0, 1]$ and $\mu_2 : E_2 \to [0, 1]$, respectively. Hence, one can easily observe that $\widetilde{G} = (V, E_i, \lambda, \mu_i)$, $i = 1, 2$ is an FGS as given in following Figure (2).

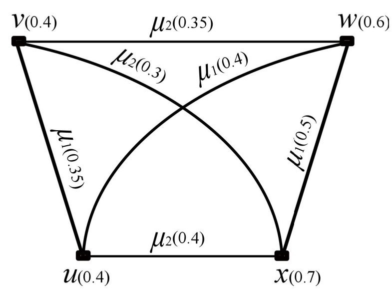


FIGURE 2. **A fuzzy graph structure $\widetilde{G} = (V, E_i, \lambda, \mu_i)$, $i = 1, 2$.**

Definition 3: If $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$ and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$, $1 \leq i \leq n$ and $1 \leq j \leq n$, are the two FGSs with core graph structures $G_1 = (V_1, E_{1i})$ and $G_2 = (V_2, E_{2j})$, respectively. Then $\widetilde{G} = \widetilde{G}_1 * \widetilde{G}_2 = (V, E_i, \lambda, \mu_i)$, $1 \leq i \leq n$, is called WMFGS with core graph structure $G = (V, E_i)$, $1 \leq i \leq n$, where $V = V_1 \times V_2$ and $E_i = \{(u_1, v_1)(u_2, v_2) \mid u_1 = u_2$ and $v_1v_2 \in E_{2j}$ or $v_1 = v_2$, and $u_1u_2 \in E_{1i}\}$ for all $u_1, u_2 \in V_1$ and $v_1, v_2 \in V_2$. Fuzzy membership functions λ and μ_i are defined as:

$$\lambda(uv) = \xi(\lambda_1(u) \vee \lambda_2(v)) + (1 - \xi)(\lambda_1(u) \wedge \lambda_2(v)),$$

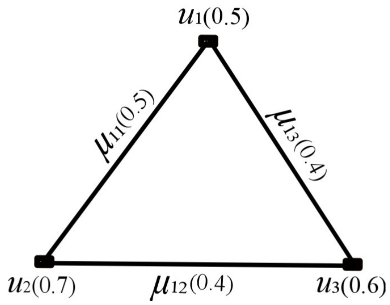
and

$$\mu_i((u_1, v_1)(u_2, v_2)) = \begin{cases} \xi(\lambda_1(u_1) \vee \mu_{2j}(v_1v_2)) + (1 - \xi)(\lambda_1(u_1) \wedge \mu_{2i}(v_1v_2)), \\ \qquad \text{such that } u_1 = u_2,\ v_1v_2 \in E_{2j}; \\ \xi(\lambda_2(v_1) \vee \mu_{1i}(u_1u_2)) + (1 - \xi)(\lambda_2(v_1) \wedge \mu_{1i}(u_1u_2)), \\ \qquad \text{such that } v_1 = v_2,\ u_1u_2 \in E_{1i}. \end{cases}$$

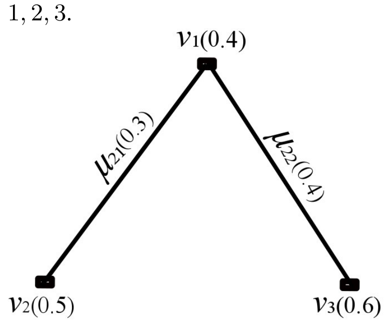
for $i = 1, 2, \ldots, n$, and $\xi \in [0, 1]$ is known as optimization parameter.

Example 3: Let us consider two FGSs $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$, $i = 1, 2, 3$, and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$, $j = 1, 2$, as shown in Figure (3). The weighted mean product of $\widetilde{G}_1$ and $\widetilde{G}_2$, for $\xi = 0$, can be constructed as shown in Figure (4).

Definition 4: An FGS $\widetilde{G} = (V, E_i, \lambda, \mu_i)$, $1 \leq i \leq n$, is called a strong fuzzy graph structure (SFGS) if $\mu_i(uv) = \lambda(u) \wedge \lambda(v)$, for all $u, v \in V$ and $uv \in E_i$. An SFGS is also known as μ_i-strong FGS.



(a) $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i}),\ i = 1, 2, 3.$



(b) $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j}),\ j = 1, 2.$

FIGURE 3. Two *FGSs* $\widetilde{G}_1$ and $\widetilde{G}$.

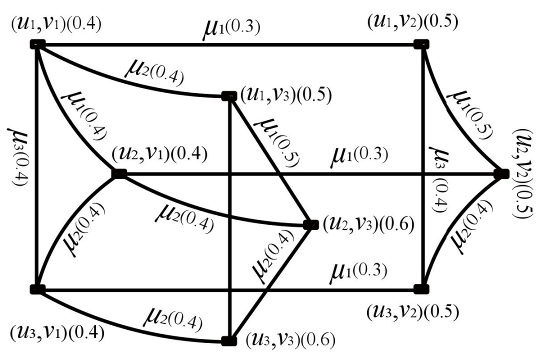


FIGURE 4. The *WMFGS* of $\widetilde{G}_1$ and $\widetilde{G}$ i.e., $\widetilde{G} = \widetilde{G}_1 * \widetilde{G}$.

Theorem 1: The product of two SFGSs under weighted mean operation, results in an SFGS.

Proof: Let $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$, $1 \leq i \leq n$, and $\widetilde{G}_2 = (V_1, E_{2j}, \lambda_2, \mu_{2j})$, $1 \leq j \leq n$, be two *SFGSs* then we have $\mu_{1i}(u_1u_2) = \lambda_1(u_1) \wedge \lambda_1(u_2)$ for any $u_1u_2 \in E_{1i}$ and $\mu_{2j}(v_1v_2) = \lambda_2(v_1) \wedge \lambda_2(v_2)$ for any $v_1v_2 \in E_{2j}$. Therefore, using definition of weighted mean product we get:

i. If $u_1 = u_2,\ v_1v_2 \in E_{2j}$

$$
\begin{aligned}
&\mu_i((u_1, v_1)(u_2, v_2))\\
&= \xi\left[\lambda_1(u_1) \vee \mu_{2j}(v_1v_2)\right] + (1-\xi)\left[\lambda_1(u_1) \wedge \mu_{2j}(v_1v_2)\right]\\
&= \xi\,[\lambda_1(u_1) \vee (\lambda_2(v_1) \wedge \lambda_2(v_2))] + (1-\xi)\,[\lambda_1(u_1) \wedge (\lambda_2(\\
&\quad \times\ v_1) \wedge \lambda_2(v_2))]\\
&= \xi\,[(\lambda_1(u_1) \vee \lambda_2(v_1)) \wedge (\lambda_1(u_1) \vee \lambda_2(v_2))] + (1-\xi)\,[\lambda_1(\\
&\quad \times\ u_1) \wedge \lambda_2(v_1) \wedge \lambda_1(u_1) \wedge \lambda_2(v_2)]\\
&= \xi\,[(\lambda_1(u_1) \vee \lambda_2(v_1))] + (1-\xi)\,[(\lambda_1(u_1) \wedge \lambda_2(v_1))] \wedge \xi\,[\\
&\quad \times\ \lambda_1(u_1) \vee \lambda_2(v_2)] + (1-\xi)\,[\lambda_1(u_1) \wedge \lambda_2(v_2)]\\
&= \xi\,[(\lambda_1(u_1) \vee \lambda_2(v_1))] + (1-\xi)\,[(\lambda_1(u_1) \wedge \lambda_2(v_1))] \wedge \xi\,[\\
&\quad \times\ \lambda_1(u_2) \vee \lambda_2(v_2)] + (1-\xi)\,[\lambda_1(u_2) \wedge \lambda_2(v_2)]\\
&= \lambda(u_{1,}v_1) \wedge \lambda(u_{2,}v_2).
\end{aligned}
$$

ii. If $v_1 = v_2,\ u_1u_2 \in E_{1i}$

$$
\begin{aligned}
&\mu_i((u_1, v_1)(u_2, v_2))\\
&= \xi\,[\lambda_2(v_1) \vee \mu_{1i}(u_1u_2)] + (1-\xi)\,[\lambda_2(v_1) \wedge \mu_{1i}(u_1u_2)]\\
&= \xi\,[\lambda_2(v_1) \vee (\lambda_1(u_1) \wedge \lambda_1(u_2)] + (1-\xi)\,[\lambda_2(v_1) \wedge (\lambda_1(u_1\\
&\quad \times\ ) \wedge \lambda_1(u_2))]\\
&= \xi\,[(\lambda_2(v_1) \vee \lambda_1(u_1)) \wedge (\lambda_2(v_1) \vee \lambda_1(u_2))] + (1-\xi)\,[\lambda_2\\
&\quad \times\ (v_1) \wedge \lambda_1(u_1) \wedge \lambda_2(v_1) \wedge \lambda_2(u_2)]\\
&= \xi\,[(\lambda_1(u_1) \vee \lambda_2(v_1))] + (1-\xi)\,[(\lambda_1(u_1) \wedge \lambda_2(v_1))] \wedge \xi\,[\\
&\quad \times\ \lambda_1(u_2) \vee \lambda_2(v_1)] + (1-\xi)\,[\lambda_1(u_2) \wedge \lambda_2(v_2)]\\
&= \xi\,[(\lambda_1(u_1) \vee \lambda_2(v_1))] + (1-\xi)\,[(\lambda_1(u_1) \wedge \lambda_2(v_1))] \wedge \xi\,[\\
&\quad \times\ \lambda_2(u_2) \vee \lambda_2(v_2)] + (1-\xi)\,[\lambda_1(u_2) \wedge \lambda_2(v_2)]\\
&= \lambda(u_{1,}v_1) \wedge \lambda(u_{2,}v_2).
\end{aligned}
$$

This shows that $\mu_i((u_1, v_1)(u_2, v_2)) = \lambda(u_{1,}v_1) \wedge \lambda(u_{2,}v_2)$ for all $(u_{1,}v_1)(u_{2,}v_2) \in E_i$. Threrfore, $\widetilde{G} = \widetilde{G}_1 * \widetilde{G}_2 = (V, E_i, \lambda, \mu_i)$, $1 \leq i \leq n$ is *SFGS*. $\square$

Remark 1: For certain value of $\xi \in [0, 1]$, an FGS $\widetilde{G} = \widetilde{G}_1 * \widetilde{G}_2$ may be SFGS while $\widetilde{G}_1$ and $\widetilde{G}_2$ are not SFGSs.

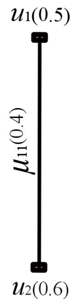
Example 4: Let us suppose, we have two FGSs $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$ and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$, $i = 1$, $j = 1, 2$, that do not carry strong conditions as given in following Figure (5). The weighted mean product of $\widetilde{G}_1$ and $\widetilde{G}_2$, for $\xi = 1$, can be constructed as shown in Figure (6). From the Figure (6), we can easily observe that:

$$
\begin{aligned}
\mu_1((u_1, v_1)(u_1, v_2)) &= \lambda(u_{1,}v_1) \wedge \lambda_2(u_1, v_2),\\
\mu_1((u_1, v_1)(u_2, v_1)) &= \lambda(u_{1,}v_1) \wedge \lambda_2(u_2, v_1),\\
\mu_1((u_1, v_2)(u_2, v_2)) &= \lambda(u_{1,}v_2) \wedge \lambda_2(u_2, v_2),\\
\mu_1((u_2, v_1)(u_2, v_2)) &= \lambda(u_{2,}v_1) \wedge \lambda_2(u_2, v_2),\\
\mu_1((u_1, v_3)(u_2, v_3)) &= \lambda(u_{1,}v_3) \wedge \lambda_2(u_2, v_3),\\
\mu_2((u_1, v_1)(u_1, v_3)) &= \lambda(u_{1,}v_1) \wedge \lambda_2(u_1, v_3),\\
\mu_2((u_2, v_1)(u_2, v_3)) &= \lambda(u_{2,}v_1) \wedge \lambda_2(u_2, v_3).
\end{aligned}
$$

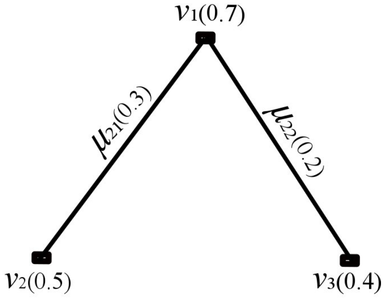
Therefore, $\widetilde{G} = (V, E_i, \lambda, \mu_i)$, $i = 1, 2$, is an SFGS.

Definition 5: If $\widetilde{G}$ is WMFGS of two $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$, $1 \leq i \leq n$, and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$, $1 \leq j \leq n$, then the vertex degree in $\widetilde{G}$ is defined as

$$
\begin{aligned}
&d_{\widetilde{G}}(u_i, v_j)\\
&\quad = \sum_{\substack{u_iu_k \in E_{1i},\\ v_j = v_l}} \xi(\lambda_2(v_j) \vee \mu_{1i}(u_iu_k)) + (1-\xi)(\lambda_2\\
&\qquad \times (v_j) \wedge \mu_{1i}(u_iu_k)) + \sum_{\substack{v_jv_l \in E_{2j},\\ u_i = u_k}} \xi(\lambda_1(u_i) \vee \mu_{2j}(v_jv_l))\\
&\qquad + (1-\xi)(\lambda_1(u_i) \wedge \mu_{2j}(v_jv_l))
\end{aligned}
$$



(a) $\widetilde{G}_1 = (V_1, E_{11}, \lambda_1, \mu_{11})$.



(b) $\widetilde{G}_2 = (V_1, E_{2j}, \lambda_2, \mu_{2j}), j = 1, 2$.

FIGURE 5. Two non *SFGSs* $\widetilde{G}_1$ and $\widetilde{G}$.

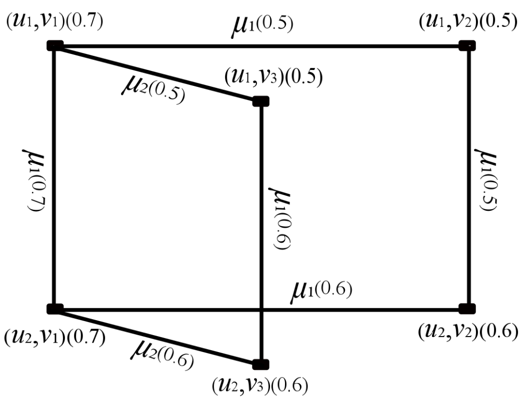


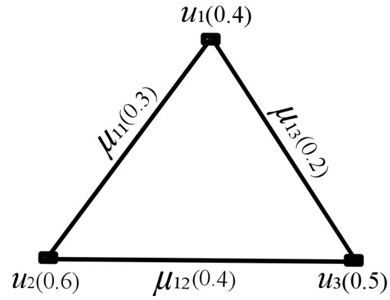
FIGURE 6. The *WMFGS* of $\widetilde{G}_1$ and $\widetilde{G}_2$ i.e., $\widetilde{G} = \widetilde{G}_1 * \widetilde{G}_2$.

while μ_i-degree of a vertex in $\widetilde{G}$ is defined as

$$\begin{aligned}
&\mu_i - d_{\widetilde{G}}(u_i, v_j) \\
&\quad = \sum_{\substack{u_iu_k \in E_{1i}, \\ v_j = v_l}} \xi(\lambda_2(v_j) \vee \mu_{1i}(u_iu_k)) + (1-\xi) \\
&\qquad \times (\lambda_2(v_j) \wedge \mu_{1i}(u_iu_k)) \\
&\qquad + \sum_{\substack{v_jv_l \in E_{2j}, \\ u_i = u_k}} \xi(\lambda_1(u_i) \vee \mu_{2i}(v_jv_l)) \\
&\qquad + (1-\xi)(\lambda_1(u_i) \wedge \mu_{2i}(v_jv_l)).
\end{aligned}$$

Example 5: If we have two FGSs $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$, $i = 1, 2, 3$, and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$, $j = 1$, as given in Figure (7)

The weighted mean product $\widetilde{G} = \widetilde{G}_1 * \widetilde{G}_2$, for $\xi = 0.5$, can be constructed as shown in Figure (8).



(a) $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$, $i = 1, 2, 3$.

(b) $\widetilde{G}_2 = (V_1, E_{21}, \lambda_2, \mu_{21})$.

FIGURE 7. Two *FGSs* $\widetilde{G}_1$ and $\widetilde{G}$.

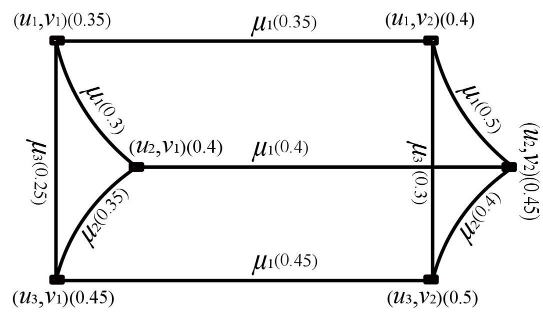


FIGURE 8. The *WMFGS* $\widetilde{G}$ of $\widetilde{G}_1$ and $\widetilde{G}_2$ i.e., $\widetilde{G} = \widetilde{G}_1 * \widetilde{G}_2$.

Now, we can calculate degrees of vertices in $\widetilde{G}$ using definition, as follows:

$$\begin{aligned}
d_{\widetilde{G}}(u_1, v_1) = {} & 0.5(\lambda_2(v_1) \vee \mu_{11}(u_1u_2)) + 0.5(\lambda_2(v_1) \\
& \wedge \mu_{11}(u_1u_2)) + 0.5(\lambda_2(v_1) \vee \mu_{13}(u_1u_3)) \\
& + 0.5(\lambda_2(v_1) \wedge \mu_{13}(u_1u_3)) + 0.5(\lambda_1(u_1) \\
& \vee \mu_{21}(v_1v_2)) + 0.5(\lambda_1(u_1) \wedge \mu_{21}(v_1v_2)) \\
= {} & 0.5(0.3 \vee 0.3) + 0.5(0.3 \wedge 0.3) \\
& + 0.5(0.3 \vee 0.2) + 0.5(0.3 \wedge 0.2) \\
& + 0.5(0.4 \vee 0.3) + 0.5(0.4 \wedge 0.3) = 0.9, \\
d_{\widetilde{G}}(u_1, v_2) = {} & 0.5(\lambda_2(v_2) \vee \mu_{11}(u_1u_2)) + 0.5(\lambda_2(v_2) \\
& \wedge \mu_{11}(u_1u_2)) + 0.5(\lambda_2(v_2) \vee \mu_{13}(u_1u_3)) \\
& + 0.5(\lambda_2(v_2) \wedge \mu_{13}(u_1u_3)) + 0.5(\lambda_1(u_1) \\
& \vee \mu_{21}(v_1v_2)) + 0.5(\lambda_1(u_1) \wedge \mu_{21}(v_1v_2))
\end{aligned}$$

$$
\begin{aligned}
&= 0.5(0.4 \vee 0.3) + 0.5(0.4 \wedge 0.3) \\
&\quad + 0.5(0.4 \vee 0.2) + 0.5(0.4 \wedge 0.2) \\
&\quad + 0.5(0.4 \vee 0.3) + 0.5(0.4 \wedge 0.3) = 1, \\
d_{\widetilde{G}}(u_2, v_1) &= 0.5(\lambda_2(v_1) \vee \mu_{11}(u_2u_1)) + 0.5(\lambda_2(v_1) \\
&\quad \wedge \mu_{11}(u_2u_1)) + 0.5(\lambda_2(v_1) \vee \mu_{12}(u_2u_3)) \\
&\quad + 0.5(\lambda_2(v_1) \wedge \mu_{12}(u_2u_3)) + 0.5(\lambda_1(u_2) \\
&\quad \vee \mu_{21}(v_1v_2)) + 0.5(\lambda_1(u_2) \wedge \mu_{21}(v_1v_2)) \\
&= 0.5(0.3 \vee 0.3) + 0.5(0.3 \wedge 0.3) \\
&\quad + 0.5(0.3 \vee 0.4) + 0.5(0.3 \wedge 0.4) \\
&\quad + 0.5(0.5 \vee 0.3) + 0.5(0.5 \wedge 0.3) = 1.05, \\
d_{\widetilde{G}}(u_2, v_2) = &= 0.5(\lambda_2(v_2) \vee \mu_{11}(u_2u_1)) + 0.5(\lambda_2(v_2) \\
&\quad \wedge \mu_{11}(u_2u_1)) + 0.5(\lambda_2(v_2) \vee \mu_{12}(u_2u_3)) \\
&\quad + 0.5(\lambda_2(v_2) \wedge \mu_{12}(u_2u_3)) + 0.5(\lambda_1(u_2) \\
&\quad \vee \mu_{21}(v_2v_1)) + 0.5(\lambda_1(u_2) \wedge \mu_{21}(v_2v_1)) \\
&= 0.5(0.4 \vee 0.3) + 0.5(0.4 \wedge 0.3) \\
&\quad + 0.5(0.4 \vee 0.4) \\
&\quad + 0.5(0.4 \wedge 0.4) + 0.5(0.5 \vee 0.3) \\
&\quad + 0.5(0.5 \wedge 0.3) = 1.15, \\
d_{\widetilde{G}}(u_3, v_1) &= 0.5(\lambda_2(v_1) \vee \mu_{13}(u_3u_1)) + 0.5(\lambda_2(v_1) \\
&\quad \wedge \mu_{13}(u_3u_1)) + 0.5(\lambda_2(v_1) \vee \mu_{12}(u_3u_2)) \\
&\quad + 0.5(\lambda_2(v_1) \wedge \mu_{12}(u_3u_2)) \\
&\quad + 0.5(\lambda_1(u_3) \vee \mu_{21}(v_1v_2)) \\
&\quad + 0.5(\lambda_1(u_3) \wedge \mu_{21}(v_1v_2)) \\
&= 0.5(0.3 \vee 0.2) + 0.5(0.3 \wedge 0.2) \\
&\quad + 0.5(0.3 \vee 0.4) + 0.5(0.3 \wedge 0.4) \\
&\quad + 0.5(0.6 \vee 0.3) + 0.5(0.6 \wedge 0.3) = 1.05, \\
d_{\widetilde{G}}(u_3, v_2) &= 0.5(\lambda_2(v_2) \vee \mu_{13}(u_3u_1)) + 0.5(\lambda_2(v_2) \\
&\quad \wedge \mu_{13}(u_3u_1)) + 0.5(\lambda_2(v_2) \vee \mu_{12}(u_3u_2)) \\
&\quad + 0.5(\lambda_2(v_2) \wedge \mu_{12}(u_3u_2)) + 0.5(\lambda_1(u_3) \\
&\quad \vee \mu_{21}(v_1v_2)) + 0.5(\lambda_1(u_3) \wedge \mu_{21}(v_1v_2)) \\
&= 0.5(0.4 \vee 0.2) + 0.5(0.4 \wedge 0.2) \\
&\quad + 0.5(0.4 \vee 0.4) + 0.5(0.4 \wedge 0.4) \\
&\quad + 0.5(0.6 \vee 0.3) + 0.5(0.6 \wedge 0.3) = 1.15.
\end{aligned}
$$

By direct evaluation, one can verifies the above results as follows:

$$
\begin{aligned}
d_{\widetilde{G}}(u_1, v_1) &= 0.35 + 0.3 + 0.25 = 0.9, \\
d_{\widetilde{G}}(u_1, v_2) &= 0.35 + 0.35 + 0.3 = 1.0, \\
d_{\widetilde{G}}(u_2, v_1) &= 0.4 + 0.35 + 0.3 = 1.05, \\
d_{\widetilde{G}}(u_2, v_2) &= 0.35 + 0.4 + 0.4 = 1.15, \\
d_{\widetilde{G}}(u_3, v_1) &= 0.35 + 0.45 + 0.25 = 1.05, \\
d_{\widetilde{G}}(u_3, v_2) &= 0.45 + 0.4 + 0.3 = 1.15.
\end{aligned}
$$

Now μ_i-degrees of vertices in *WMFGS* $\widetilde{G}$ are estimated using definition as follows:

$$
\begin{aligned}
\mu_1 - d_{\widetilde{G}}(u_1, v_1) &= 0.5(\lambda_2(v_1) \\
&\quad \vee \mu_{11}(u_1u_2)) + 0.5(\lambda_2(v_1) \wedge \mu_{11}(u_1u_2)) \\
&\quad + 0.5(\lambda_1(u_1) \vee \mu_{21}(v_1v_2)) + 0.5(\lambda_1(u_1) \\
&\quad \wedge \mu_{21}(v_1v_2)) \\
&= 0.5((0.3 \vee 0.3) + (0.3 \wedge 0.3) \\
&\quad + (0.4 \vee 0.3) + (0.4 \wedge 0.3)) = 0.65, \\
\mu_1 - d_{\widetilde{G}}(u_1, v_2) &= 0.5(\lambda_2(v_2) \vee \mu_{11}(u_1u_2)) + 0.5(\lambda_2(v_2) \\
&\quad \wedge \mu_{11}(u_1u_2)) + 0.5(\lambda_1(u_1) \vee \mu_{21}(v_2v_1)) \\
&\quad + 0.5(\lambda_1(u_1) \wedge \mu_{21}(v_2v_1)) \\
&= 0.5((0.4 \vee 0.3) + (0.4 \wedge 0.3) \\
&\quad + (0.4 \vee 0.3) + (0.4 \wedge 0.3)) = 0.7, \\
\mu_1 - d_{\widetilde{G}}(u_2, v_1) &= 0.5(\lambda_2(v_1) \vee \mu_{11}(u_2u_1)) \\
&\quad + 0.5(\lambda_2(v_1) \wedge \mu_{11}(u_2u_1)) + 0.5(\lambda_1(u_2) \\
&\quad \vee \mu_{21}(v_1v_2)) + 0.5(\lambda_1(u_2) \wedge \mu_{21}(v_1v_2)) \\
&= 0.5((0.3 \vee 0.3) + (0.3 \wedge 0.3) \\
&\quad + (0.5 \vee 0.3) + (0.5 \wedge 0.3)) = 0.7, \\
\mu_1 - d_{\widetilde{G}}(u_2, v_2) &= 0.5(\lambda_2(v_2) \vee \mu_{11}(u_2u_1)) + 0.5(\lambda_2(v_2) \\
&\quad \wedge \mu_{11}(u_2u_1)) + 0.5(\lambda_1(u_2) \vee \mu_{21}(v_2v_1)) \\
&\quad + 0.5(\lambda_1(u_2) \wedge \mu_{21}(v_2v_1)) \\
&= 0.5((0.4 \vee 0.3) + (0.4 \wedge 0.3) \\
&\quad + (0.5 \vee 0.3) + (0.5 \wedge 0.3)) = 0.75, \\
\mu_1 - d_{\widetilde{G}}(u_3, v_1) &= 0.5(\lambda_1(u_3) \vee \mu_{21}(v_1v_2)) + 0.5(\lambda_1(u_3) \\
&\quad \wedge \mu_{21}(v_1v_2)) \\
&= 0.5((0.6 \vee 0.3) + (0.6 \wedge 0.3) = 0.45, \\
\mu_1 - d_{\widetilde{G}}(u_3, v_2) &= 0.5(\lambda_1(u_3) \vee \mu_{21}(v_2v_1)) + 0.5(\lambda_1(u_3) \\
&\quad \wedge \mu_{21}(v_2v_1)) \\
&= 0.5((0.6 \vee 0.3) + (0.6 \wedge 0.3) = 0.45, \\
\mu_2 - d_{\widetilde{G}}(u_2, v_1) &= 0.5(\lambda_2(v_1) \vee \mu_{12}(u_2u_3)) + 0.5(\lambda_2(v_1) \\
&\quad \wedge \mu_{12}(u_2u_3)) \\
&= 0.5((0.3 \vee 0.4) + (0.3 \wedge 0.4) = 0.35, \\
\mu_2 - d_{\widetilde{G}}(u_2, v_2) &= 0.5(\lambda_2(v_2) \vee \mu_{12}(u_2u_3)) + 0.5(\lambda_2(v_2) \\
&\quad \wedge \mu_{12}(u_2u_3)) \\
&= 0.5((0.4 \vee 0.4) + (0.4 \wedge 0.4) = 0.4, \\
\mu_2 - d_{\widetilde{G}}(u_3, v_1) &= 0.5(\lambda_2(v_1) \vee \mu_{12}(u_3u_2)) + 0.5(\lambda_2(v_1) \\
&\quad \wedge \mu_{12}(u_3u_2)) \\
&= 0.5((0.3 \vee 0.4) + (0.3 \wedge 0.4) = 0.35, \\
\mu_2 - d_{\widetilde{G}}(u_3, v_2) &= 0.5(\lambda_2(v_2) \\
&\quad \vee \mu_{12}(u_3u_2)) + 0.5(\lambda_2(v_2) \\
&\quad \wedge \mu_{12}(u_3u_2)) \\
&= 0.5((0.4 \vee 0.4) + (0.4 \wedge 0.4) = 0.4, \\
\mu_3 - d_{\widetilde{G}}(u_1, v_1) &= 0.5(\lambda_2(v_1) \vee \mu_{13}(u_1u_3)) + 0.5(\lambda_2(v_1) \\
&\quad \wedge \mu_{13}(u_1u_3)) \\
&= 0.5((0.3 \vee 0.2) + (0.3 \wedge 0.2) = 0.25,
\end{aligned}
$$

$$\begin{aligned}\mu_3 - d_{\widetilde{G}}(u_1, v_2) &= 0.5(\lambda_2(v_2) \vee \mu_{13}(u_1u_3)) + 0.5(\lambda_2(v_2) \\ &\quad \wedge \mu_{13}(u_1u_3)) \\ &= 0.5((0.4 \vee 0.2) + (0.4 \wedge 0.2) = 0.3, \\ \mu_3 - d_{\widetilde{G}}(u_3, v_1) &= 0.5(\lambda_2(v_1) \vee \mu_{13}(u_3u_1)) + 0.5(\lambda_2(v_1) \\ &\quad \wedge \mu_{13}(u_3u_1)) \\ &= 0.5((0.3 \vee 0.2) + (0.3 \wedge 0.2) = 0.25, \\ \mu_3 - d_{\widetilde{G}}(u_3, v_2) &= 0.5(\lambda_2(v_2) \vee \mu_{13}(u_3u_1)) + 0.5(\lambda_2(v_2) \\ &\quad \wedge \mu_{13}(u_3u_1)) \\ &= 0.5((0.4 \vee 0.2) + (0.4 \wedge 0.2) = 0.3.\end{aligned}$$

Theorem 2: if $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$, $1 \leq i \leq n$, and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$, $1 \leq j \leq n$, are two FGSs such that $\lambda_1 \leq \mu_{2j}$, then a vertex degree in WMFGS $\widetilde{G} = \widetilde{G}_1 * \widetilde{G}_2$ is evaluated as

$$d_{\widetilde{G}}(u_i, v_j) = \xi(d_{G_1}(u_i)\lambda_2(v_j) + d_{\widetilde{G}_2}(v_j)) + (1-\xi)(d_{G_2}(v_j) \times \lambda_1(u_i) + d_{\widetilde{G}_1}(u_i)).$$

Proof: Suppose that $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$, $1 \leq i \leq n$, and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$, $1 \leq j \leq n$, are two *FGSs* such that $\lambda_1 \leq \mu_{2j}$, then $\mu_{1i} \leq \lambda_2$. Furthermore, by definition of vertex degree in *WMFGS* $\widetilde{G}$, we have

$$\begin{aligned}&d_{\widetilde{G}}(u_i, v_j) \\ &= \sum_{\substack{u_iu_k \in E_{1i},\\ v_j = v_l}} \xi(\lambda_2(v_j) \vee \mu_{1i}(u_iu_k)) + (1-\xi)(\lambda_2 \\ &\quad \times (v_j) \wedge \mu_{1i}(u_iu_k)) + \sum_{\substack{v_jv_l \in E_{2j},\\ u_i = u_k}} \xi(\lambda_1(u_i) \vee \mu_{2j}(v_jv_l)) \\ &\quad + (1-\xi)(\lambda_1(u_i) \wedge \mu_{2j}(v_jv_l)) \\ &= \sum_{\substack{u_iu_k \in E_{1i},\\ v_j = v_l}} \xi(\lambda_2(v_j)) + (1-\xi)(\mu_{1i}(u_iu_k)) + \sum_{\substack{v_jv_l \in E_{2j},\\ u_i = u_k}} \\ &\quad \times \xi(\mu_{2j}(v_jv_l)) + (1-\xi)(\lambda_1(u_i)) \\ &= \xi \sum_{\substack{u_iu_k \in E_{1i},\\ v_j = v_l}} (\lambda_2(v_j) + \mu_{2j}(v_jv_l)) + (1-\xi) \sum_{\substack{v_jv_l \in E_{2j},\\ u_i = u_k}} \\ &\quad \times (\mu_{1i}(u_iu_k) + \lambda_1(u_i)) \\ &= \xi(d_{G_1}(u_i)\lambda_2(v_j) + d_{\widetilde{G}_2}(v_j)) + (1-\xi)(d_{G_2}(v_j)\lambda_1(u_i) \\ &\quad + d_{\widetilde{G}_1}(u_i)).\end{aligned}$$

□

Theorem 3: if $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$, $1 \leq i \leq n$, and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$, $1 \leq j \leq n$, are two FGSs such that $\lambda_1 \leq \mu_{2j}$, and λ_2 remains constant as 'β', then a vertex degree in WMFGS $\widetilde{G} = \widetilde{G}_1 * \widetilde{G}_2$ is evaluated as

$$d_{\widetilde{G}}(u_i, v_j) = \xi(d_{G_1}(u_i)\beta + d_{\widetilde{G}_2}(v_j)) + (1-\xi)(d_{G_2}(v_j)\lambda_1(u_i) + d_{\widetilde{G}_1}(u_i)).$$

Proof: Suppose that $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$, $1 \leq i \leq n$, and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$, $1 \leq j \leq n$, are two *FGSs* such that $\lambda_1 \leq \mu_{2j}$ and λ_2 is a constant function denoted by β, furthermore, $\lambda_1 \leq \mu_{2j} \implies \mu_{1i} \leq \lambda_2$. Hence, by definition of vertex degree in *WMFGS* $\widetilde{G}$, we have

$$\begin{aligned}&d_{\widetilde{G}}(u_i, v_j) \\ &= \sum_{\substack{u_iu_k \in E_{1i},\\ v_j = v_l}} \xi(\lambda_2(v_j) \vee \mu_{1i}(u_iu_k)) + (1-\xi)(\lambda_2 \\ &\quad \times (v_j) \wedge \mu_{1i}(u_iu_k)) + \sum_{\substack{v_jv_l \in E_{2j},\\ u_i = u_k}} \xi(\lambda_1(u_i) \vee \mu_{2j}(v_jv_l)) \\ &\quad + (1-\xi)(\lambda_1(u_i) \wedge \mu_{2j}(v_jv_l)) \\ &= \sum_{\substack{u_iu_k \in E_{1i},\\ v_j = v_l}} \xi(\beta) + (1-\xi)(\mu_{1i}(u_iu_k)) + \sum_{\substack{v_jv_l \in E_{2j},\\ u_i = u_k}} \\ &\quad \times \xi(\mu_{2j}(v_jv_l)) + (1-\xi)(\lambda_1(u_i)) \\ &= \xi \sum_{\substack{u_iu_k \in E_{1i},\\ v_j = v_l}} (\beta + \mu_{2j}(v_jv_l)) + (1-\xi) \sum_{\substack{v_jv_l \in E_{2j},\\ u_i = u_k}} (\mu_{1i}(u_iu_k) \\ &\quad + \lambda_1(u_i)) = \xi(d_{G_1}(u_i)\beta + d_{\widetilde{G}_2}(v_j)) + (1-\xi)(d_{G_2}(v_j) \\ &\quad \times \lambda_1(u_i) + d_{\widetilde{G}_1}(u_i)).\end{aligned}$$

□

Theorem 4: if $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$, $1 \leq i \leq n$, and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$, $1 \leq j \leq n$, are two FGSs such that $\lambda_2 \leq \mu_{1i}$, then a vertex degree in WMFGS $\widetilde{G} = \widetilde{G}_1 * \widetilde{G}_2$ is evaluated as

$$d_{\widetilde{G}}(u_i, v_j) = \xi(d_{\widetilde{G}_1}(u_i) + d_{G_2}(v_j)\lambda_1(u_i)) + (1-\xi)(d_{\widetilde{G}_2}(v_j) + d_{G_1}(u_i)\lambda_2(v_j)).$$

Proof: Suppose that $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$, $1 \leq i \leq n$, and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$, $1 \leq j \leq n$, are two *FGSs* such that $\lambda_2 \leq \mu_{1i}$, then $\mu_{2j} \leq \lambda_1$. Furthermore, by definition of vertex degree in *WMFGS* $\widetilde{G}$, we have

$$\begin{aligned}&d_{\widetilde{G}}(u_i, v_j) \\ &= \sum_{\substack{u_iu_k \in E_{1i},\\ v_j = v_l}} \xi(\lambda_2(v_j) \vee \mu_{1i}(u_iu_k)) + (1-\xi)(\lambda_2 \\ &\quad \times (v_j) \wedge \mu_{1i}(u_iu_k)) + \sum_{\substack{v_jv_l \in E_{2j},\\ u_i = u_k}} \xi(\lambda_1(u_i) \vee \mu_{2j}(v_jv_l)) \\ &\quad + (1-\xi)(\lambda_1(u_i) \wedge \mu_{2j}(v_jv_l)) \\ &= \sum_{\substack{u_iu_k \in E_{1i},\\ v_j = v_l}} \xi(\mu_{1i}(u_iu_k)) + (1-\xi)(\lambda_2(v_j)) + \sum_{\substack{v_jv_l \in E_{2j},\\ u_i = u_k}} \\ &\quad \times \xi(\lambda_1(u_i)) + (1-\xi)(\mu_{2j}(v_jv_l)) \\ &= \xi \sum_{\substack{u_iu_k \in E_{1i},\\ v_j = v_l}} (\lambda_1(u_i) + \mu_{1i}(u_iu_k)) + (1-\xi) \sum_{\substack{v_jv_l \in E_{2j},\\ u_i = u_k}} \\ &\quad t(\lambda_2(v_j) + \mu_{2j}(v_jv_l)) = \xi(d_{\widetilde{G}_1}(u_i) + d_{G_2}(v_j)\lambda_1(u_i)) \\ &\quad + (1-\xi)(d_{\widetilde{G}_2}(v_j) + d_{G_1}(u_i)\lambda_2(v_j)).\end{aligned}$$

□

Theorem 5: if $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$, $1 \leq i \leq n$, and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$, $1 \leq j \leq n$, are two FGSs such that $\lambda_2 \leq \mu_{1i}$, and λ_1 remains constant, say 'β', then a vertex *degree in WMFGS $\widetilde{G} = \widetilde{G}_1 * \widetilde{G}_2$ is evaluated as*

$$d_{\widetilde{G}}(u_i, v_j) = \xi(d_{\widetilde{G}_1}(u_i) + d_{G_2}(v_j)\beta) + (1-\xi)(d_{\widetilde{G}_2}(v_j) + d_{G_1}(u_i)\lambda_2(v_j)).$$

Proof: Suppose that $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$, $1 \le i \le n$, and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$, $1 \le j \le n$, are two *FGSs* such that $\lambda_2 \le \mu_{1i}$ and λ_1 is a constant function denoted by β, furthermore, $\lambda_2 \le \mu_{2j} \implies \mu_{2j} \le \lambda_1$. Hence, by definition of vertex degree in *WMFGS* $\widetilde{G}$, we have

$$\begin{aligned}
d_{\widetilde{G}}(u_i, v_j) &= \sum_{\substack{u_iu_k \in E_{1i},\\ v_j = v_l}} \xi(\lambda_2(v_j) \vee \mu_{1i}(u_iu_k)) + (1-\xi)(\lambda_2 \\
&\quad \times (v_j) \wedge \mu_{1i}(u_iu_k)) + \sum_{\substack{v_jv_l \in E_{2j},\\ u_i = u_k}} \xi(\lambda_1(u_i) \vee \mu_{2j}(v_jv_l)) \\
&\quad + (1-\xi)(\lambda_1(u_i) \wedge \mu_{2j}(v_jv_l)) \\
&= \sum_{\substack{u_iu_k \in E_{1i},\\ v_j = v_l}} \xi(\mu_{1i}(u_iu_k)) + (1-\xi)(\lambda_2(v_j)) + \sum_{\substack{v_jv_l \in E_{2j},\\ u_i = u_k}} \\
&\quad \times \xi(\beta) + (1-\xi)(\mu_{2j}(v_jv_l)) \\
&= \xi \sum_{\substack{u_iu_k \in E_{1i},\\ v_j = v_l}} (\beta + \mu_{1i}(u_iu_k)) + (1-\xi) \sum_{\substack{v_jv_l \in E_{2j},\\ u_i = u_k}} (\lambda_2(v_j) \\
&\quad + \mu_{2j}(v_jv_l)) = \xi(d_{\widetilde{G}_1}(u_i) + d_{G_2}(v_j)\beta) + (1-\xi) \\
&\quad \times (d_{\widetilde{G}_2}(v_j) + d_{G_1}(u_i)\lambda_2(v_j)).
\end{aligned}$$

□

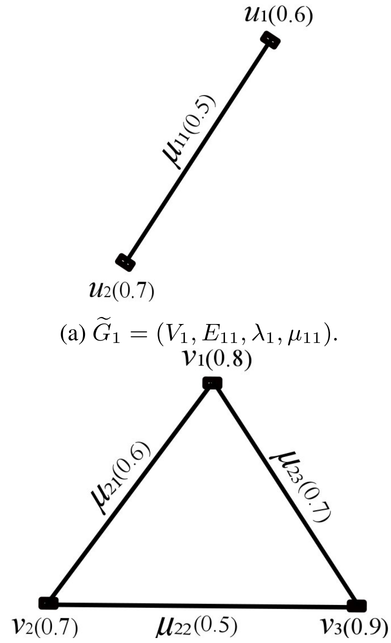
Theorem 6: if $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$, $1 \le i \le n$, and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$, $1 \le j \le n$, are two FGSs such that $mu_{2j} \le \lambda_1$ and $mu_{1i} \le \lambda_2$, then a vertex degree in WMFGS $\widetilde{G} = \widetilde{G}_1 * \widetilde{G}_2$ is evaluated as

$$d_{\widetilde{G}}(u_i, v_j) = \xi(d_{G_1}(u_i)\lambda_2(v_j) + d_{G_2}(v_j)\lambda_1(u_i)) + (1-\xi) \times (d_{\widetilde{G}_1}(u_i) + d_{\widetilde{G}_2}(v_j)).$$

Proof: Suppose that $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$, $1 \le i \le n$, and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$, $1 \le j \le n$, are two *FGSs* such that $mu_{2j} \le \lambda_1$ and $mu_{1i} \le \lambda_2$, therefore, by definition of vertex degree in *WMFGS* $\widetilde{G}$, we have

$$\begin{aligned}
d_{\widetilde{G}}(u_i, v_j) &= \sum_{\substack{u_iu_k \in E_{1i},\\ v_j = v_l}} \xi(\lambda_2(v_j) \vee \mu_{1i}(u_iu_k)) + (1-\xi)(\lambda_2 \\
&\quad \times (v_j) \wedge \mu_{1i}(u_iu_k)) + \sum_{\substack{v_jv_l \in E_{2j},\\ u_i = u_k}} \xi(\lambda_1(u_i) \vee \mu_{2j}(v_jv_l)) \\
&\quad + (1-\xi)(\lambda_1(u_i) \wedge \mu_{2j}(v_jv_l)) \\
&= \sum_{\substack{u_iu_k \in E_{1i},\\ v_j = v_l}} \xi(\lambda_2(v_j)) + (1-\xi)(\mu_{1i}(u_iu_k)) + \sum_{\substack{v_jv_l \in E_{2j},\\ u_i = u_k}} \\
&\quad \times \xi(\lambda_1(u_i)) + (1-\xi)(\mu_{2j}(v_jv_l)) \\
&= \sum_{\substack{u_iu_k \in E_{1i},\\ v_j = v_l}} \xi(\lambda_2(v_j)) + (1-\xi)(\mu_{2j}(v_jv_l)) + \sum_{\substack{v_jv_l \in E_{2j},\\ u_i = u_k}} \\
&\quad \times \xi(\lambda_1(u_i)) + (1-\xi)(\mu_{1i}(u_iu_k)) = \xi(d_{G_1}(u_i)\lambda_2(v_j) \\
&\quad + d_{G_2}(v_j)\lambda_1(u_i)) + (1-\xi)(d_{\widetilde{G}_1}(u_i) + d_{\widetilde{G}_2}(v_j)).
\end{aligned}$$

□



(a) $\widetilde{G}_1 = (V_1, E_{11}, \lambda_1, \mu_{11})$.

(b) $\widetilde{G}_2 = (V_1, E_{2j}, \lambda_2, \mu_{2j})$, $j = 1, 2, 3$.

FIGURE 9. Two *FGSs* $\widetilde{G}_1$ and $\widetilde{G}$.

Example 6: If we have two FGSs $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{11})$, and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$, $j = 1, 2, 3$, as given in Figure (9) The weighted mean product $\widetilde{G} = \widetilde{G}_1 * \widetilde{G}_2$ for $\xi = 0.5$, can be constructed as shown in Figure (10).

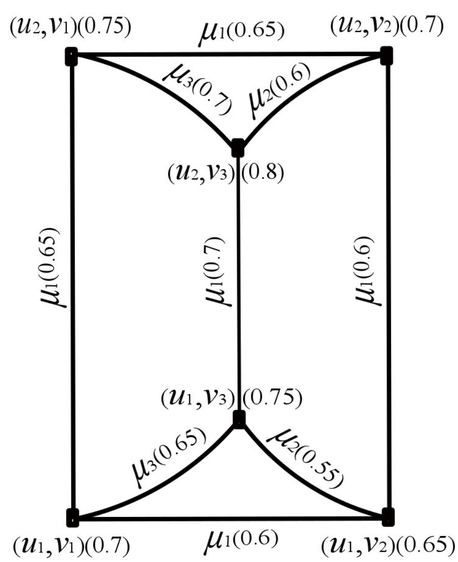


FIGURE 10. The *WMFGS* $\widetilde{G}$ of $\widetilde{G}_1$ and $\widetilde{G}_2$ i.e., $\widetilde{G} = \widetilde{G}_1 * \widetilde{G}_2$.

From Figure (9) we can easily observe that $\mu_{2j} \le \lambda_1$, $j = 1, 2, 3$, and $\mu_{11} < \lambda_2$. Therefore, the vertex degree in $\widetilde{G}$ can be evaluated under the use of following formula:

$$\begin{aligned}
d_{\widetilde{G}}(u_i,v_j) &= 0.5(d_{G_1}(u_i)\lambda_2(v_j)+d_{G_2}(v_j)\lambda_1(u_i))+0.5\\
&\quad\times(d_{\widetilde{G}_1}(u_i)+d_{\widetilde{G}_2}(v_j)).\\
d_{\widetilde{G}}(u_1,v_1) &= 0.5(d_{G_1}(u_1)\lambda_2(v_1)+d_{G_2}(v_1)\lambda_1(u_1))+0.5\\
&\quad\times(d_{\widetilde{G}_1}(u_1)+d_{\widetilde{G}_2}(v_1))\\
&= 0.5((1)(0.8)+(2)(0.6))+0.5(0.5+1.3)\\
&= 1.9,\\
d_{\widetilde{G}}(u_1,v_2) &= 0.5(d_{G_1}(u_1)\lambda_2(v_2)+d_{G_2}(v_2)\lambda_1(u_1))+0.5\\
&\quad\times(d_{\widetilde{G}_1}(u_1)+d_{\widetilde{G}_2}(v_2))\\
&= 0.5((1)(0.7)+(2)(0.6))+0.5(0.5+1.1)\\
&= 1.75,\\
d_{\widetilde{G}}(u_1,v_3) &= 0.5(d_{G_1}(u_1)\lambda_2(v_3)+d_{G_2}(v_3)\lambda_1(u_1))+0.5\\
&\quad\times(d_{\widetilde{G}_1}(u_1)+d_{\widetilde{G}_2}(v_3))\\
&= 0.5((1)(0.9)+(2)(0.6))+0.5(0.5+1.2)\\
&= 1.9,\\
d_{\widetilde{G}}(u_2,v_1) &= 0.5(d_{G_1}(u_2)\lambda_2(v_1)+d_{G_2}(v_1)\lambda_1(u_2))+0.5\\
&\quad\times(d_{\widetilde{G}_1}(u_2)+d_{\widetilde{G}_2}(v_1))\\
&= 0.5((1)(0.8)+(2)(0.7))+0.5(0.5+1.3)\\
&= 2,\\
d_{\widetilde{G}}(u_2,v_2) &= 0.5(d_{G_1}(u_2)\lambda_2(v_2)+d_{G_2}(v_2)\lambda_1(u_2))+0.5\\
&\quad\times(d_{\widetilde{G}_1}(u_2)+d_{\widetilde{G}_2}(v_2))\\
&= 0.5((1)(0.7)+(2)(0.7))+0.5(0.5+1.1)\\
&= 1.85,\\
d_{\widetilde{G}}(u_1,v_3) &= 0.5(d_{G_1}(u_2)\lambda_2(v_3)+d_{G_2}(v_3)\lambda_1(u_2))+0.5\\
&\quad\times(d_{\widetilde{G}_1}(u_2)+d_{\widetilde{G}_2}(v_3))\\
&= 0.5((1)(0.9)+(2)(0.7))+0.5(0.5+1.2)\\
&= 2.
\end{aligned}$$

The direct evaluation of the degrees of vertices in $\widetilde{G}$ provides following values:

$$\begin{aligned}
d_{\widetilde{G}}(u_1,v_1) &= 0.65+0.65+0.6=1.9,\\
d_{\widetilde{G}}(u_1,v_2) &= 0.6+0.6+0.55=1.75,\\
d_{\widetilde{G}}(u_1,v_3) &= 0.7+0.65+0.55=1.9,\\
d_{\widetilde{G}}(u_2,v_1) &= 0.7+0.65+0.65=2.0,\\
d_{\widetilde{G}}(u_2,v_2) &= 0.65+0.6+0.6=1.85,\\
d_{\widetilde{G}}(u_2,v_3)7 &= 0.7+0.7+0.6=2.0.
\end{aligned}$$

One can easily observe that the degrees of vertices evaluated from the formula, established in Theorem 6, and the direct calculations are exactly same.

Theorem 7: if $\widetilde{G}_1=(V_1,E_{1i},\lambda_1,\mu_{1i})$, $1\le i\le n$, and $\widetilde{G}_2=(V_2,E_{2j},\lambda_2,\mu_{2j})$, $1\le j\le n$, are two FGSs such that $\mu_{2j}\ge\lambda_1$, then a vertex total degree in WMFGS $\widetilde{G}=\widetilde{G}_1*\widetilde{G}_2$ is evaluated as

$$\begin{aligned}
d_{\widetilde{G}}(u_i,v_j) &= \xi(d_{G_1}(u_i)\lambda_2(v_j)+td_{\widetilde{G}_2}(v_j))+(1-\xi)(d_{G_2}(v_j)\\
&\qquad\times\lambda_1(u_i)+td_{\widetilde{G}_1}(u_i)).
\end{aligned}$$

Proof: Suppose that $\widetilde{G}_1=(V_1,E_{1i},\lambda_1,\mu_{1i})$, $1\le i\le n$, and $\widetilde{G}_2=(V_2,E_{2j},\lambda_2,\mu_{2j})$, $1\le j\le n$, are two *FGSs* such that $\mu_{2j}\ge\lambda_1$, then $\mu_{1i}\le\lambda_2$, $\lambda_1\le\lambda_2$. Therefore, the vertex total degree in *WMFGS* $\widetilde{G}$ is given as

$$\begin{aligned}
&td_{\widetilde{G}}(u_i,v_j)\\
&= \sum_{\substack{u_iu_k\in E_{1i},\\ v_j=v_l}}\xi(\lambda_2(v_j)\vee\mu_{1i}(u_iu_k))+(1-\xi)(\lambda_2\\
&\quad\times(v_j)\wedge\mu_{1i}(u_iu_k))+\sum_{\substack{v_jv_l\in E_{2j},\\ u_i=u_k}}\xi(\lambda_1(u_i)\vee\mu_{2j}(v_jv_l))\\
&\quad+(1-\xi)(\lambda_1(u_i)\wedge\mu_{2j}(v_jv_l))+\lambda(u_iv_j)\\
&= \sum_{\substack{u_iu_k\in E_{1i},\\ v_j=v_l}}\xi(\lambda_2(v_j))+(1-\xi)(\mu_{1i}(u_iu_k))+\sum_{\substack{v_jv_l\in E_{2j},\\ u_i=u_k}}\\
&\quad\times\xi(\mu_{2j}(v_jv_l))+(1-\xi)(\lambda_1(u_i))+\xi(\lambda_1(u_i)\vee\lambda_2(v_j))\\
&\quad+(1-\xi)(\lambda_1(u_i)\wedge\lambda_2(v_j))\\
&= \xi\sum_{\substack{u_iu_k\in E_{1i},\\ v_j=v_l}}(\lambda_2(v_j)+\mu_{2j}(v_jv_l))+(1-\xi)\sum_{\substack{v_jv_l\in E_{2j},\\ u_i=u_k}}\\
&\quad\times(\mu_{1i}(u_iu_k)+\lambda_1(u_i))+\xi(\lambda_2(v_j))+(1-\xi)(\lambda_1(u_i))\\
&= \xi(d_{G_1}(u_i)\lambda_2(v_j)+d_{\widetilde{G}_2}(v_j)+\lambda_2(v_j))+(1-\xi)(d_{G_2}(v_j)\\
&\quad\times\lambda_1(u_i)+d_{\widetilde{G}_1}(u_i)+\lambda_1(u_i))\\
&= \xi(d_{G_1}(u_i)\lambda_2(v_j)+td_{\widetilde{G}_2}(v_j))+(1-\xi)(d_{G_2}(v_j)\lambda_1(u_i)\\
&\quad+td_{\widetilde{G}_1}(u_i)).
\end{aligned}$$

□

Theorem 8: if $\widetilde{G}_1=(V_1,E_{1i},\lambda_1,\mu_{1i})$, $1\le i\le n$, and $\widetilde{G}_2=(V_2,E_{2j},\lambda_2,\mu_{2j})$, $1\le j\le n$, are two FGSs such that $\mu_{1i}\ge\lambda_2$, then a vertex total degree in WMFGS $\widetilde{G}=\widetilde{G}_1*\widetilde{G}_2$ is evaluated as

$$\begin{aligned}
td_{\widetilde{G}}(u_i,v_j) &= \xi(d_{G_2}(v_j)\lambda_1(u_i)+td_{\widetilde{G}_1}(u_i))+(1-\xi)(d_{G_1}\\
&\qquad\times(u_i)\lambda_2(v_j)+td_{\widetilde{G}_2}(v_j)).
\end{aligned}$$

Proof: Suppose that $\widetilde{G}_1=(V_1,E_{1i},\lambda_1,\mu_{1i})$, $1\le i\le n$, and $\widetilde{G}_2=(V_2,E_{2j},\lambda_2,\mu_{2j})$, $1\le j\le n$, are two *FGSs* such that $\lambda_2\le\mu_{1i}$, then $\mu_{2j}\le\lambda_1$, $\lambda_1\ge\lambda_2$. Thus, the vertex total degree in *WMFGS* $\widetilde{G}$ is given as

$$\begin{aligned}
&td_{\widetilde{G}}(u_i,v_j)\\
&= \sum_{\substack{u_iu_k\in E_{1i},\\ v_j=v_l}}\xi(\lambda_2(v_j)\vee\mu_{1i}(u_iu_k))+(1-\xi)(\lambda_2\\
&\quad\times(v_j)\wedge\mu_{1i}(u_iu_k))+\sum_{\substack{v_jv_l\in E_{2j},\\ u_i=u_k}}\xi(\lambda_1(u_i)\vee\mu_{2j}(v_jv_l))\\
&\quad+(1-\xi)(\lambda_1(u_i)\wedge\mu_{2j}(v_jv_l))+\lambda(u_iv_j)\\
&= \sum_{\substack{u_iu_k\in E_{1i},\\ v_j=v_l}}\xi(\mu_{1i}(u_iu_k))+(1-\xi)(\lambda_2(v_j))+\sum_{\substack{v_jv_l\in E_{2j},\\ u_i=u_k}}\\
&\quad\times\xi(\lambda_1(u_i))+(1-\xi)(\mu_{2j}(v_jv_l))+\xi(\lambda_1(u_i)\vee\lambda_2(v_j))\\
&\quad+(1-\xi)(\lambda_1(u_i)\wedge\lambda_2(v_j))
\end{aligned}$$

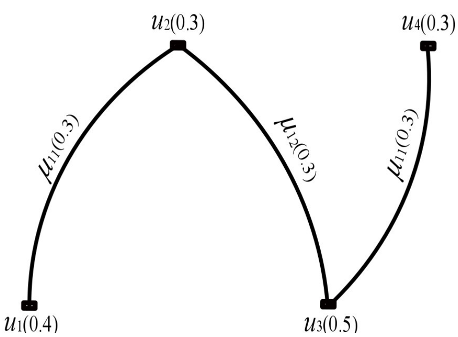
$$= \sum_{\substack{u_iu_k\in E_{1i},\\ v_j=v_l}} (\lambda_1(u_i)+\mu_{1i}(u_iu_k)) + (1-\xi)\sum_{\substack{v_jv_l\in E_{2j},\\ u_i=u_k}} (\lambda_2(v_j)$$
$$+\mu_{2j}(v_jv_l)) + \xi(\lambda_1(u_i)) + (1-\xi)(\lambda_2(v_j))$$
$$= \xi(d_{G_2}(v_j)\lambda_1(u_i) + d_{\widetilde{G}_1}(u_i) + \lambda_1(u_i)) + (1-\xi)(d_{G_1}(u_i)$$
$$\times \lambda_2(v_j) + d_{\widetilde{G}_2}(v_j) + \lambda_2(v_j))$$
$$= \xi(d_{G_2}(v_j)\lambda_1(u_i) + td_{\widetilde{G}_1}(u_i)) + (1-\xi)(d_{G_1}(u_i)\lambda_2(v_j)$$
$$+ td_{\widetilde{G}_2}(v_j)).$$

□

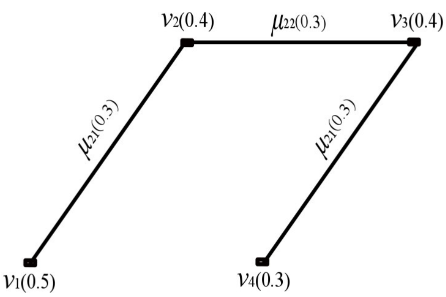
Definition 6: An FGS $\widetilde{G} = (V, E_i, \lambda, \mu_i)$, $1 \le i \le n$, is said to be regular fuzzy graph structure (RFGS), if each vertex of $\widetilde{G}$ carries same degree, and is $m\mu_k$-regular if all vertices of $\widetilde{G}$ have same degrees $m \in]0, 1]$ under membership function μ_k, $k \in \{1, 2, \ldots, n\}$.

Remark 2: The weighted mean product of two RFGSs could be an irregular FGS.

Example 7: Let $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$, $i = 1, 2$, and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$, $j = 1, 2$, be two FGSs given in Figure (11).



(a) $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i}), i = 1, 2.$



(b) $\widetilde{G}_2 = (V_1, E_{2j}, \lambda_2, \mu_{2j}), j = 1, 2.$

FIGURE 11. **μ_{11}-regular and μ_{21}-regular *FGSs*.**

After observing Figure (11), one can easily conclude that $\widetilde{G}_1$ and $\widetilde{G}_2$ are $0.3\mu_{11}$-regular and $0.3\mu_{21}$-regular *FGSs*, respectively. But, the Figure (12) shows that each vertex in *WMFGS* of $\widetilde{G}_1$ and $\widetilde{G}_2$ carries different number of edges with regard to membership value. Therefore, $\widetilde{G}_1 * \widetilde{G}_2$ is not an *RFGS*. Hence, it is concluded under this example that the *WMFGS* of two *RFGS* is not an *RFGS*.

Theorem 9: if $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$ is $m\mu_{1k}$-regular FGS and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$ is FGS such that $\mu_{2j} \ge \lambda_1$, $1 \le i \le n$, $1 \le i \le n$, $k \in \{1, 2, \ldots, n\}$, with a constant

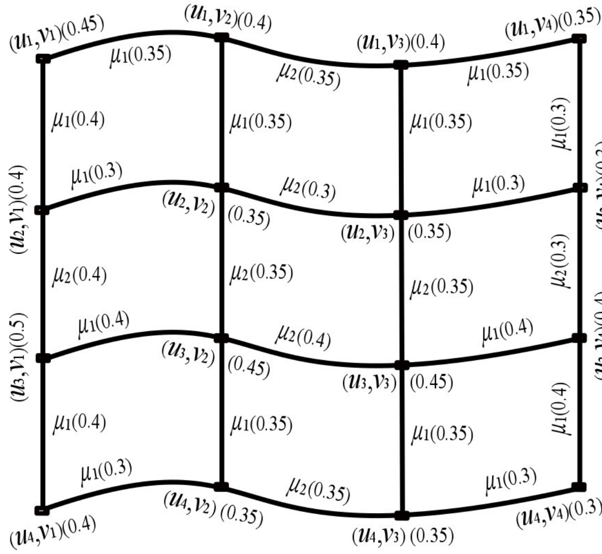


FIGURE 12. **The *WMFGS* of two *FGS* $\tilde{G}_1$ and $\tilde{G}$.**

mapping λ_2, say γ, then WMFGS of $\widetilde{G} = \widetilde{G}_1 * \widetilde{G}_2$ is regular iff $\widetilde{G}_2$ is regular.

Proof: Let $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$ be $m\mu_{1k}$-regular *FGS* with r_1-regular core *GS* G_1 and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$ be an *FGS* such that $\mu_{2j} \ge \lambda_1$, $1 \le i \le n$, $1 \le i \le n$, $k \in \{1, 2, \ldots, n\}$, with a constant mapping λ_2, say γ. Now suppose that $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$ is an α *RFGS* with r_2-regular core *GS* G_2, then we have

$$d_{\widetilde{G}}(u_i, v_j) = \xi(d_{G_1}(u_i)\lambda_2(v_j) + d_{\widetilde{G}_2}(v_j)) + (1-\xi)(d_{G_2}$$
$$\times (v_j)\lambda_1(u_i) + d_{\widetilde{G}_1}(u_i))$$
$$= \xi(\alpha + r_1\gamma) + (1-\xi)(m + r_2\lambda_1(u_i)).$$

This is valid for all vertices of $\widetilde{G}$, and is an *RFGS*. In converse, suppose that $\widetilde{G}$ is regular, then for any vertices (u_1, v_1) and (u_2, v_2) of $\widetilde{G}$ we have

$$d_{\widetilde{G}}(u_1, v_1)$$
$$= d_{\widetilde{G}}(u_2, v_2)$$
$$\implies \xi(d_{G_1}(u_1)\lambda_2(v_1) + d_{\widetilde{G}_2}(v_1)) + (1-\xi)(d_{G_2}(v_1)\lambda_1(u_1)$$
$$+ d_{\widetilde{G}_1}(u_1)) = \xi(d_{G_1}(u_2)\lambda_2(v_2) + d_{\widetilde{G}_2}(v_2))$$
$$+ (1-\xi)(d_{G_2}(v_2)\lambda_1(u_2) + d_{\widetilde{G}_1}(u_2)),$$
$$\implies \xi(d_{\widetilde{G}_2}(v_1) + r_1\gamma) + (1-\xi)(m + d_{G_2}(v_1)\lambda_1(u_1))$$
$$= \xi(d_{\widetilde{G}_2}(v_2) + r_1\gamma) + (1-\xi)(m + d_{G_2}(v_2)\lambda_1(u_2)),$$
$$\implies \xi(d_{\widetilde{G}_2}(v_1)) + (1-\xi)(d_{G_2}(v_1)\lambda_1(u_1)) = \xi(d_{\widetilde{G}_2}(v_2))$$
$$+ (1-\xi)(d_{G_2}(v_2)\lambda_1(u_2)).$$

It validates for all vertices of $\widetilde{G}_2$, and hence, $\widetilde{G}_2$ is an *RFGS*.

□

Theorem 10: if $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$ is $m\mu_{2k}$-regular FGS and $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$ is FGS such that $\mu_{1i} \ge \lambda_2$, $1 \le i \le n$, $1 \le i \le n$, $k \in \{1, 2, \ldots, n\}$, with a constant mapping λ_1, say β, then WMFGS of $\widetilde{G} = \widetilde{G}_1 * \widetilde{G}_2$ is regular iff $\widetilde{G}_1$ is regular.

TABLE 1. Development levels of five cities.

City	Development level
Islamabad	0.9
Karachi	0.8
Lahore	0.8
Faislabad	0.7
Multan	0.6

Proof: Let $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$ is $m\mu_{2k}$-regular *FGS* with r_2-regular core *GS* G_2 and $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$ is *FGS* such that $\mu_{1i} \geq \lambda_2$, $1 \leq i \leq n$, $1 \leq i \leq n$, $k \in \{1, 2, \ldots, n\}$, with a constant mapping λ_1, say β. Now suppose that $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$ is an α *RFGS* with r_1-regular core *GS* G_1, then we have

$$\begin{aligned} d_{\widetilde{G}}(u_i, v_j) &= \xi(d_{\widetilde{G}_1}(u_i) + d_{G_2}(v_j)\lambda_1(u_i)) + (1-\xi)(d_{\widetilde{G}_2} \\ &\quad \times (v_j) + d_{G_1}(u_i)\lambda_2(v_j)) \\ &= \xi(\alpha + r_2\beta) + (1-\xi)(m + r_1\lambda_2(v_j)). \end{aligned}$$

This is valid for all vertices of $\widetilde{G}$, and is an *RFGS*. In converse, suppose that $\widetilde{G}$ is regular, then for any vertices (u_1, v_1) and (u_2, v_2) of $\widetilde{G}$ we have

$$\begin{aligned} & d_{\widetilde{G}}(u_1, v_1) \\ = \ & d_{\widetilde{G}}(u_2, v_2) \\ \Longrightarrow \ & \xi(d_{\widetilde{G}_1}(u_1) + d_{G_2}(v_1)\lambda_1(u_1)) + (1-\xi)(d_{\widetilde{G}_2}(v_1) \\ & + d_{G_1}(u_1)\lambda_2(v_1)) = \xi(d_{\widetilde{G}_1}(u_2) + d_{G_2}(v_2)\lambda_1(u_2)) \\ & + (1-\xi)(d_{\widetilde{G}_2}(v_2) + d_{G_1}(u_2)\lambda_2(v_2)), \\ \Longrightarrow \ & \xi(d_{\widetilde{G}_1}(u_1) + r_2\beta) + (1-\xi)(m + d_{G_1}(u_1)\lambda_2(v_1)) \\ = \ & \xi(d_{\widetilde{G}_1}(u_2) + r_2\beta) + (1-\xi)(m + d_{G_1}(u_2)\lambda_2(v_2)), \\ \Longrightarrow \ & \xi(d_{\widetilde{G}_1}(u_1)) + (1-\xi)(d_{G_1}(u_1)\lambda_2(v_1)) \\ = \ & \xi(d_{\widetilde{G}_1}(u_2)) + (1-\xi)(d_{G_1}(u_2)\lambda_2(v_2)). \end{aligned}$$

It validates for all vertices of $\widetilde{G}_1$, and hence, $\widetilde{G}_1$ is an *RFGS*. □

Theorem 11: if $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$ and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$ are $m_1\mu_{1k}$-regular and $m_2\mu_{2l}$-regular FGSs, respectively, such that $\mu_{2j} \leq \lambda_1$, $\mu_{1i} \leq \lambda_2$, $1 \leq i \leq n$, $1 \leq i \leq n$, $k, l \in \{1, 2, \ldots, n\}$, with a constant mapping λ_2, say γ, then WMFGS of $\widetilde{G} = \widetilde{G}_1 * \widetilde{G}_2$ is regular iff λ_1 is constant function.

Proof: Let $\widetilde{G}_1 = (V_1, E_{1i}, \lambda_1, \mu_{1i})$ and $\widetilde{G}_2 = (V_2, E_{2j}, \lambda_2, \mu_{2j})$ be $m_1\mu_{1k}$-regular and $m_2\mu_{2l}$-regular *FGSs*, respectively, such that $\mu_{2j} \leq \lambda_1$, $\mu_{1i} \leq \lambda_2$, $1 \leq i \leq n$, $1 \leq i \leq n$, $k, l \in \{1, 2, \ldots, n\}$, with a constant mapping λ_2, say γ. Also, let G_1 and G_2 be the two core graph structures being r_1-regular and r_2-regular, respectively. Now suppose that λ_1 is a constant mapping of value, say β, then we have

$$\begin{aligned} d_{\widetilde{G}}(u_i, v_j) &= \xi(d_{G_1}(u_i)\lambda_2(v_j) + d_{G_2}(v_j)\lambda_1(u_i)) + (1-\xi) \\ &\quad \times (d_{\widetilde{G}_1}(u_i) + d_{\widetilde{G}_2}(v_j)) \\ &= \xi(r_1\gamma + r_2\beta) + (1-\xi)(k + l). \end{aligned}$$

his is valid for all vertices of $\widetilde{G}$, and is an *RFGS*. In converse, suppose that $\widetilde{G}$ is regular, then for any vertices (u_1, v_1) and (u_2, v_2) of $\widetilde{G}$ we have

$$\begin{aligned} & d_{\widetilde{G}}(u_1, v_1) \\ = \ & d_{\widetilde{G}}(u_2, v_2) \\ \Longrightarrow \ & \xi(d_{G_1}(u_1)\lambda_2(v_1) + d_{G_2}(v_1)\lambda_1(u_1)) + (1-\xi)(d_{\widetilde{G}_1}(u_1) \\ & + d_{\widetilde{G}_2}(v_1)) = \xi(d_{G_1}(u_2)\lambda_2(v_2) + d_{G_2}(v_2)\lambda_1(u_2)) \\ & + (1-\xi)(d_{\widetilde{G}_1}(u_2) + d_{\widetilde{G}_2}(v_2)), \\ \Longrightarrow \ & \xi(r_1\gamma + r_2\lambda_1(u_1)) + (1-\xi)(k+l) \\ = \ & \xi(r_1\gamma + r_2\lambda_1(u_2)) + (1-\xi)(k+l)), \\ \Longrightarrow \ & \lambda_1(u_1) = \lambda_1(u_2). \end{aligned}$$

It validates for all vertices of $\widetilde{G}_1$, and hence, λ_1 is a constant mapping. □

III. APPLICATION

Recognition of the Pakistan's most evolved and modern city. If we compare different cities of a country, they definitely

TABLE 2. Comparison of Islamabad with other cities.

Criteria	(Islamabad,Karachi)	(Islamabad,Lahore)	(Islamabad,Faisalabad)	(Islamabad,Multan)
Hospitals	0.8	0.8	0.7	0.6
Educational facilities	0.8	0.8	0.7	0.6
Industry development	0.7	0.7	0.6	0.5
Roads' network	0.8	0.8	0.7	0.6

TABLE 3. Comparison of Karachi with other cities.

Criteria	(Karachi,Islamabad)	(Karachi,Lahore)	(Karachi,Faisalabad)	(Karachi,Multan)
Hospitals	0.7	0.6	0.6	0.5
Educational facilities	0.6	0.7	0.5	0.4
Industry development	0.8	0.8	0.7	0.6
Roads' network	0.5	0.6	0.6	0.5

TABLE 4. Comparison of lahore with other cities.

Criteria	(Lahore,Islamabad)	(Lahore,Karachi)	(Lahore,Faisalabad)	(Lahore,Multan)
Hospitals	0.8	0.8	0.6	0.5
Educational facilities	0.7	0.7	0.7	0.6
Industry development	0.7	0.7	0.6	0.6
Roads' network	0.6	0.6	0.7	0.5

TABLE 5. Comparison of faisalabad with other cities.

Criteria	(Faisalabad,Islamabad)	(Faisalabad,Karachi)	(Faisalabad,Lahore)	(Faisalabad,Multan)
Hospitals	0.6	0.6	0.6	0.6
Educational facilities	0.6	0.5	0.6	0.5
Industry development	0.7	0.7	0.7	0.6
Roads' network	0.7	0.6	0.6	0.6

TABLE 6. Comparison of multan with other cities.

Criteria	(Multan,Islamabad)	(Multan,Karachi)	(Multan,Lahore)	(Multan,faisalabad)
Hospitals	0.4	0.6	0.5	0.5
Educational facilities	0.3	0.5	0.4	0.4
Industry development	0.4	0.5	0.4	0.3
Roads' network	0.5	0.4	0.5	0.4

have different characteristics and recognition level. Some cities are known for their constructed highways, while others have a variety of universities. For example, Islamabad and Multan are two significant and famous cities in Pakistan. Islamabad is the Pakistan's most advanced and modern city, but Multan appears to be underdeveloped. Everything that Multan would have in future, Islamabad already has. There are some characteristics that every city is famous for. For example, Faisalabad is famous for the Textile industry, Lahore is famous for its grand hospitals and Educational institutes, Karachi is the premier industrial and financial center of Pakistan.

We can use a fuzzy-graphic framework to show the most advanced and evolved city after comparing any two in a given time frame. We can also say the degree of the slow and under-developed phase at that time, with the assistance of the membership function. The fuzzy-graphic structure of the most developed and evolving cities can be very useful for a country to be concentrated on.

Let us have a set C of five cities of Pakistan: $C = \{$Islamabad, Karachi, Lahore, Faisalabad, Multan$\}$.

Let λ be a fuzzy mapping on C i.e., $\lambda : C \longleftrightarrow [0, 1]$ that evaluates the level of a developed city in Pakistan under four considered parameters, as shown in Table 1.

Tables 2–6 represent the development levels for each pair of cities using the law $\mu(v_1 v_2) = \sigma(v_1) \wedge \sigma(v_2)$, $\forall\, v_1, v_2 \in C$.

Now against set C, we can define several labeling functions, for instance, let us have as: $f_1 =$ Hospitals, $f_2 =$ Educational Facilities, $f_3 =$ Industry development, $f_4 =$ Roads' network, so that $G = (C, E_1, E_2, E_3, E_4)$ is a graph structure, where E_1, E_2, E_3 and E_4 are disjoint subsets of edges

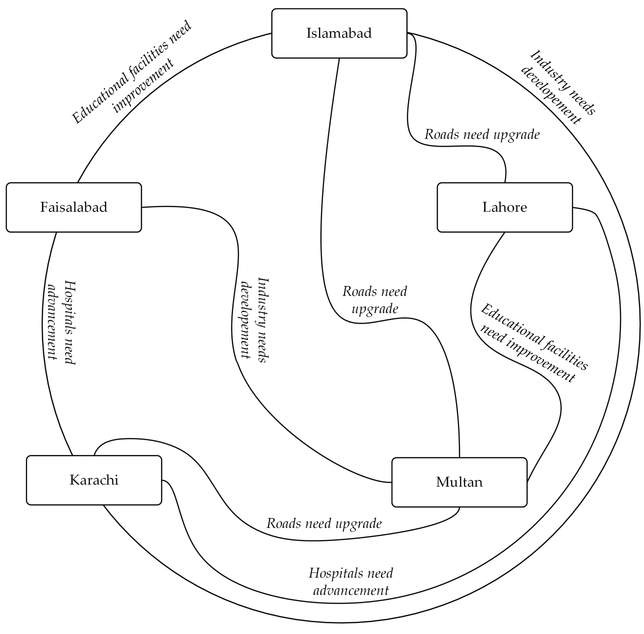


FIGURE 13. An *FGS* representing the parameter that needs improvement for each pair of cities.

regarding pairs of cities. Let us consider

$$E_1 = \{(\text{Islamabad,Karachi}), (\text{Islamabad,Multan}), (\text{Karachi,} \times \text{Faisalabad}), (\text{Lahore,Islamabad})\},$$
$$E_2 = \{(\text{Faisalabad,Islamabad}), (\text{Lahore,Multan})\},$$
$$E_3 = \{(\text{Karachi,Lahore}), (\text{Faisalabad,Multan})\},$$
$$E_4 = \{(\text{Multan,Lahore}), (\text{Lahore,Islamabad})\},$$

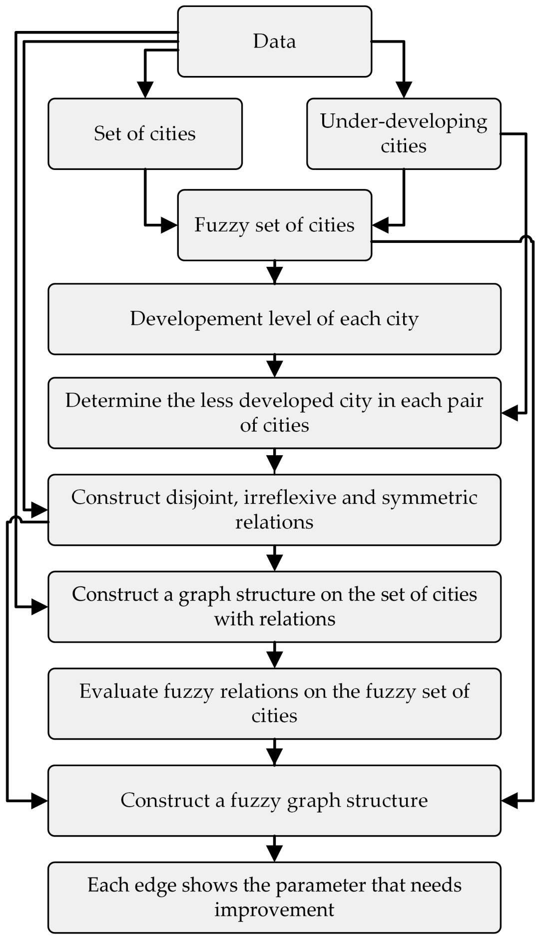


FIGURE 14. Flowchart of the procedure to determine the criteria which require enhancement.

and the corresponding fuzzy sets under membership functions μ_1, μ_2, μ_3 and μ_4, respectively, are:

$$\begin{aligned}
\mu_1 &= \{((\text{Islamabad,Karachi}), 0.8), \\
&\times ((\text{Islamabad,Multan}), 0.6), \\
&\times ((\text{Karachi,Faisalabad}), 0.7), \\
&\times ((\text{Lahore,Islamabad}), 0.8)\}, \\
\mu_2 &= \{((\text{Faisalabad,Islamabad}), 0.7), \\
&\times ((\text{Lahore,Multan}), 0.5)\}, \\
\mu_3 &= \{((\text{Karachi,Lahore}), 0.8), \\
&\times ((\text{Faisalabad,Multan}), 0.6)\}, \\
\mu_4 &= \{((\text{Multan,Lahore}), 0.5), \\
&\times ((\text{Lahore,Faisalabad}), 0.7)\}.
\end{aligned}$$

It is clear that $\widetilde{G} = (V, E_i, \lambda, \mu_i)$, $1 \leq i \leq 4$, is an *FGS* as shown in Figure (13).

In Figure (13), each edge of *FGS* is used to represent the parameter that needs to be improved in corresponding city. For instance, according to this *FGS*, Faisalabad and Multan have to improve their educational institutions as compare to Islamabad and Lahore, respectively. In the same way, Faisalabad needs to enhance hospitals in comparison with Karachi, while Karachi has to improve hospitals as compare to Lahore. An *FGS* of all cities can be very helpful for a country to maintain the assets in order to facilitate the people. It would highlight those cities which need some enhancement. The basic idea used in this application is illustrated by a flowchart on Figure (14).

IV. CONCLUSION

The graph theory has numerous implementations to address diverse problems in different fields, such as networking, connectivity, data analysis, cluster analysis, signal processing, image optimization, scheduling and planning. In order to deal with the uncertainty and vagueness of the graphical system, the use of fuzzy-graphical methods is very natural. Fuzzy Graph Theory has a wide range of uses in the simulation of various real-time processes. The level of work contained in the structure differs with divergent degrees of accuracy. In this paper, we presented a number of various concepts relating to fuzzy-graphic structures, such as the weighted mean product of two fuzzy-graphic structures and regular fuzzy-graphic structures, and examined several related attributes. The degree and total degree of a vertex in the weighted mean product of fuzzy-graphic structures were also described with the help of examples. In addition, we have suggested the execution of fuzzy-graphic frameworks to compare five major cities of Pakistan based on four criteria, such as Hospitals, Education, Industry and Roads. The procedure followed was visualized with the help of a flowchart presented in Figure (14). In the future, we aim to extend our work to (1) soft fuzzy-graph structures, (2) rough fuzzy graph structures and (3) rough fuzzy soft graph structures.

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