IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received April 24, 2021, accepted May 11, 2021, date of publication May 17, 2021, date of current version May 27, 2021.

Digital Object Identifier 10.1109/ACCESS.2021.3081570

A Spectral RMIL+ Conjugate Gradient Method for Unconstrained Optimization With Applications in Portfolio Selection and Motion Control

ALIYU MUHAMMED AWWAL [1,2], IBRAHIM MOHAMMED SULAIMAN [3], MAULANA MALIK [4], MUSTAFA MAMAT [3,5], POOM KUMAM [1,6], (Member, IEEE), AND KANOKWAN SITTHITHAKERNGKIET [7]

[1]Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangkok 10140, Thailand
[2]Department of Mathematics, Faculty of Science, Gombe State University, Gombe 760214, Nigeria
[3]Faculty of Informatics and Computing, Universiti Sultan Zainal Abidin (UniSZA), Kuala Terengganu 22200, Malaysia
[4]Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Indonesia (UI), Depok 16424, Indonesia
[5]Institute of Engineering Mathematics, Universiti Malaysia Perlis (UniMAP), Arau 02600, Malaysia
[6]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
[7]Intelligent and Nonlinear Dynamic Innovations Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok (KMUTNB), Bangkok 10800, Thailand

Corresponding authors: Poom Kumam (poom.kum@kmutt.ac.th), Ibrahim Mohammed Sulaiman (sulaimanib@unisza.edu.my), and Aliyu Muhammed Awwal (aliyumagsu@gmail.com)

This work was supported in part by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), King Mongkut's University of Technology Thonburi (KMUTT) and the Center through the Computational and Applied Science for Smart Innovation Research Cluster (CLASSIC), Faculty of Science, KMUTT, in part by the Thailand Science Research and Innovation (TSRI) Basic Research Fund Fiscal year 2021 under Project 64A306000005, in part by the Thailand Science Research and Innovation Fund, and in part by the King Mongkut's University of Technology North Bangkok under Contract KMUTNB-BasicR-64-22.

ABSTRACT The Spectral conjugate gradient (SCG) methods are among the efficient variants of CG algorithms which are obtained by combining the spectral gradient parameter and CG parameter. The success of SCG methods relies on effective choices of the step-size α_k and the search direction d_k. This paper presents an SCG method for unconstrained optimization models. The search directions generated by the new method possess sufficient descent property without the restart condition and independent of the line search procedure used. The global convergence of the new method is proved under the weak Wolfe line search. Preliminary numerical results are presented which show that the method is efficient and promising, particularly for large-scale problems. Also, the method was applied to solve the robotic motion control problem and portfolio selection problem.

INDEX TERMS Spectral algorithm, conjugate gradient algorithms, unconstrained optimization models, motion control, line search procedure, portfolio selection.

I. INTRODUCTION

In this paper, we consider the optimization model:

$$\min f(x), \quad x \in \mathbb{R}^n. \tag{1}$$

The function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ has continuous partial derivatives, whose gradient $\nabla f(x) = g(x)$ is available. This type of problems often arise in economics, management science, engineering and other industrial applications [1]–[4]. The spectral conjugate algorithms are widely used to solve problem (1) by

$$x_{k+1} = x_k + t_k d_k, \quad k \geq 0, \tag{2}$$

where x_0 is the initial guess. The step-size $t_k > 0$ is usually calculated along d_k. For $k = 0$, we have $d_0 = -g_0$ which is known as the Steepest descent direction. However, the subsequent directions of search are recursively computed as

$$d_k = -\theta_k g_k + \beta_k d_{k-1}, \quad k \geq 1. \tag{3}$$

Here θ_k is the spectral parameter and β_k is the conjugate gradient parameter that differentiate the types of spectral CG methods. Some of the known CG parameters are given by Hestenes-Stiefel (HS) [5], Polak-Ribiere-Polyak (PRP) [6],

The associate editor coordinating the review of this manuscript and approving it for publication was Guido Lombardi.

[7], Liu-Storey (LS) [8], Fletcher-Reeves (FR) [9], Conjugate Descent (CD) [10], Dai-Yuan (DY) [11] whose formulas are given as follow:

$$\beta_k^{HS} = \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}}, \quad \beta_k^{PRP} = \frac{g_k^T y_{k-1}}{\|g_{k-1}\|^2},$$
$$\beta_k^{LS} = \frac{g_k^T y_{k-1}}{-g_{k-1}^T d_{k-1}}, \quad \beta_k^{FR} = \frac{\|g_k\|^2}{\|g_{k-1}\|^2},$$
$$\beta_k^{CD} = \frac{\|g_k\|^2}{-d_{k-1}^T g_{k-1}}, \quad \beta_k^{DY} = \frac{\|g_k\|^2}{d_{k-1}^T y_{k-1}}.$$

In theory, when exact minimization rule is used, then, all these choices of β_k are equivalent for strongly convex quadratic function f. However, for non-quadratic objective functions, every β_k chosen leads to different numerical performances [12].

The convergence properties of these methods have been studied by various scholars. From the computational point of view, the method of PRP performed better than the FR method. Also, when f is a convex objective function, then, under the exact line search, the PRP algorithm would converge globally [6]. For the exact line search, t_k is computed to satisfy:

$$f(x_k + t_k d_k) := \min_{t \geq 0} f(x_k + t d_k). \tag{4}$$

However, for certain non-convex functions, Powell [13] shows that the PRP algorithm would not converge globally.

Other convergence results require the step-size t_k to satisfy the weak or strong Wolfe line search. The weak Wolfe search technique expect t_k to satisfy the inequalities.

$$f(x_k + t_k d_k) \leq f(x_k) + \delta t_k g_k^T d_k. \tag{5}$$
$$g(x_k + \alpha_k d_k)^T d_k \geq \sigma g_k^T d_k. \tag{6}$$

On the other hand, the strong Wolfe (SWP) search technique except t_k to satisfy (5) and

$$g(x_k + t_k d_k)^T d_k \leq -\sigma |g_k^T d_k|, \tag{7}$$

where $0 < \delta < \sigma < 1$. Al-Baali [14] showed that FR algorithm satisfies the sufficient descent condition:

$$g_k^T d_k \leq -C\|g_k\|^2, \quad C > 0, \tag{8}$$

and established the convergence proof for general function using SWP line search. For recent studies on conjugate gradient algorithms, please refer to [6], [10], [12], [13], [15]–[19].

Recently, Rivaie *et al.* [20] construct a variant of PRP method and give the formula as:

$$\beta_k^{RMIL} = \frac{g_k^T (g_k - g_{k-1})}{\|d_{k-1}\|^2}, \tag{9}$$

where the authors replaced $\|g_{k-1}\|^2$ with $\|d_{k-1}\|^2$ in the denominator of classical PRP CG parameter and showed that the method satisfies (8) and further proved its global convergence under exact minimization condition. However, Dai [21] pointed out the use of a wrong inequality in establishing the convergence proof of the result in [20]. As a result, Dai [21] presented a modification as:

$$\beta_k^{RMIL+} = \begin{cases} \frac{g_k^T (g_k - g_{k-1})}{\|d_{k-1}\|^2}, & \text{if } 0 \leq g_k^T g_{k-1} \leq \|g_k\|^2, \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

and studied the convergence of *RMIL+* using exact minimization rule. Numerical result obtained showed that the *RMIL+* is promising. More recently, Yousif [22], studied the convergence of *RMIL+* method under strong Wolfe line search.

However, some of the best performing conjugate gradients algorithms, developed recently, are those that incorporated the spectral parameters. An interesting feature of the spectral CG algorithm is that only gradient directions are incorporated at every line search while a non monotone strategy usually guarantees the global convergence [23]. The first known spectral CG algorithms was defined by Birgin and Martínez [24] with the parameters defined as follows:

$$\beta_k^1 = \frac{(\theta_k y_{k-1} - s_{k-1})^T g_k}{s_{k-1}^T y_{k-1}}, \quad \beta_k^2 = \frac{\theta_k g_k^T y_{k-1}}{t_{k-1}\theta_{k-1} g_{k-1}^T g_{k-1}},$$
$$\beta_k^3 = \frac{\theta_k g_k^T g_k}{t_{k-1}\theta_{k-1} g_{k-1}^T g_{k-1}},$$

where $y_{k-1} = g_k - g_{k-1}$ and $s_{k-1} = x_k - x_{k-1}$. The method β_k^1 reduces to a modified CG algorithm presented by Perry [25] if $\theta \equiv 1$, for all k. Moreover, if $\theta_k \equiv 1$ holds, for all k, and β_k^2 satisfies the exact minimization condition, then, the classical PRP CG method by Polak-Ribiere-Polyak [6] is obtained. However, if the successive gradients are orthogonal and the condition $\theta \equiv 1$ holds, for all k, then β_k^3 would reduce to the classical Fletcher-Reeves (FR) [9] algorithm. One of the drawbacks of this method is that d_k may not be a descent during the iteration process. Hence, restart procedure was employed in [26] to guarantee that the method has a descent direction in all iterations. The convergence analysis of this method was studied under the standard Wolfe line search. Birgin and Martínez [26] further extended their work and introduced an alternative spectral parameter:

$$\theta_k = \frac{s_{k-1}^T s_{k-1}}{s_{k-1}^T y_{k-1}}. \tag{11}$$

Computing the conjugate gradient coefficients β_k (11) with the spectral parameter in (11) leads to the spectral conjugate gradient (SCG) method. Numerical results obtained under various line search procedures illustrate that the SCG is more efficient compared to the classical methods of PRP [6], FR [9], Perry [25] and the spectral gradient method [27]. More work have been done to improve these methods. Recent research focuses on memoryless BFGS updates for unconstrained optimization [15], [28]–[31]. Another efficient spectral FR (sFR) conjugate gradient method was introduced by Zhang *et al.*, [32] with the spectral parameter θ_k and conjugate gradient parameter β_k defined as:

$$\beta_k = \beta_k^{FR} = \frac{\|g_k\|^2}{\|g_{k-1}\|}, \quad \theta_k = \frac{d_{k-1}^T y_{k-1}}{\|g_{k-1}\|^2}. \tag{12}$$

An interesting feature of sFR parameter is that, independent of line search procedures employed, d_k satisfies $g_k^T d_k = -\|g_k\|^2, \quad \forall\, k \geq 0$. Also, the parameter reduces to classical FR parameter provided the exact minimization condition is satisfied [29]. The convergence analysis was discussed under a modified Armijo line search and results obtained from numerical computations showed that sFR method is efficient and promising compare to PRP method. Recently, Liu *et al.* [33] extended the work of Birgin and Martinez, [24] and Zhang *et al.*, [32] to proposed a general spectral parameter that will reduce to the main CG algorithm if an exact line search is used. Under some mild conditions, the authors established the global convergence of the method. For more references on recent spectral conjugate gradient method, please refer to [33]–[39].

Motivated by the above contributions, a new spectral conjugate gradient method is developed in this paper. Some of the contributions of this paper are highlighted as follows:

- A new spectral conjugate gradient algorithm, based on RMIL+, for solving unconstrained optimization is developed.
- The search direction generated by the new algorithm satisfies the sufficient descent property without the restart condition and independent of any line search.
- The global convergence of the new method is proved under the weak Wolfe line search.
- The efficiency of the new algorithm is demonstrated on some standard large-scale problems.
- The new algorithm is applied to solve problems arising from portfolio selection.
- Lastly, the new algorithm is successfully applied to deal with robotic motion control problem.

The rest of the paper is designed as follows. In Section II, a new spectral parameter is presented and the corresponding algorithm is given. The global convergence results of the new formula under the weak Wolfe line search is presented in Section III. Experimental numerical results are presented in Section IV. Applications of the new algorithm in portfolio selection and motion control are discussed in Sections V and VI, respectively, where the finally conclusion in section VII.

II. NEW ALGORITHM

In this section, inspired by the idea of Birgin and Martinez, [24] and Zhang *et al.*, [40], we propose an efficient spectral parameter as follows.

Consider the sequence $\{x_k\}$ computed using the spectral CG parameter (2) and (3). Multiplying (3) by g_k^T gives

$$g_k^T d_k = -\theta_k g_k + \beta_k g_k^T d_{k-1} = \frac{g_{k-1}^T d_{k-1}}{\|d_{k-1}\|^2}\|g_k\|^2 \psi_k, \tag{13}$$

where,

$$\psi_k = -\frac{\|d_{k-1}\|^2}{g_{k-1}^T d_{k-1}}\theta_k + \beta_k \frac{\|d_{k-1}\|^2}{g_{k-1}^T d_{k-1}} \frac{g_k^T d_{k-1}}{\|g_k\|^2}.$$

This implies,

$$\frac{g_k^T d_k}{\|g_k\|^2} = \frac{g_{k-1}^T d_{k-1}}{\|d_{k-1}\|^2}\psi_k. \tag{14}$$

Suppose for every $k \geq 1$, we choose the parameter $\psi_k = 1$, then from (3) and (14), we get

$$\frac{g_k^T d_k}{\|g_k\|^2} = \frac{g_{k-1}^T d_{k-1}}{\|d_{k-1}\|^2} = \ldots = \frac{g_0^T d_0}{\|d_0\|^2} = -1. \tag{15}$$

From (15), we have

$$g_k^T d_k = -\|g_k\|^2, \quad \forall k \geq 1. \tag{16}$$

Thus, if we choose the spectral parameter θ_k to satisfy $\psi_k \equiv 1$, then, the direction of search will always satisfy the descent condition regardless of the search procedure used. This analysis motivated us on defining a new parameter θ_k as follows.

$$\theta_k = -\frac{g_{k-1}^T d_{k-1}}{\|d_{k-1}\|^2} + \beta_k \frac{g_k^T d_{k-1}}{\|g_k\|^2}, \tag{17}$$

where $k \geq 1$ and $\psi \equiv 1$.

The algorithm of our spectral conjugate method is as follows.

Algorithm 1 sRMIL+ Method

Step 1. Initialization: given $x_0 \in \mathbb{R}^n$, $\sigma > 0$, $\psi_k > 0$, set $k := 0$. If $\|g_k\| \leq \epsilon$, stop. Otherwise,
Step 2. Compute β_k by (10).
Step 3. Compute d_k using (3) where θ_k is defined in (17).
Step 4. Determine t_k based on (5) and (6).
Step 5. Update new iterate based on (2).
Step 6. Check if $\|g_k\| = 0$, terminate. Else, go to step 2 with $k = k + 1$.

To analyze the convergence of the conjugate gradient method, the assumptions below are often needed.

Assumption A:

1) $f(x)$ is bounded from below on the level set $\Omega = \{x \in \mathbb{R}^n | f(x) \leq f(x_0)\}$.
2) In some neighborhood N of Ω, f is smooth and $g(x)$ is Lipchitz continuous on an open convex set N that contains Ω, such that, there exist $L > 0$ (constant) satisfying;

$$\|g(x) - g(y)\| \leq L\|x - y\|, \quad \forall x, y \in N. \tag{18}$$

Remark 1: From the analysis above, we have shown that the proposed sRMIL+ satisfies the descent condition (16) *regardless of the line search. This condition would play a significant part in the convergence of the proposed sRMIL+ algorithm.*

III. CONVERGENCE ANALYSIS

The global convergence properties of sRMIL+ under weak Wolfe line search will be studied in this section. Consider the CG parameter defined by (10). If the condition $0 \leq g_k^T g_{k-1} \leq \|g_k\|^2$, holds then

$$\beta_k^{RMIL+} = \frac{\|g_k\|^2 - g_k^T g_{k-1}}{\|d_{k-1}\|^2} \leq \frac{\|g_k\|^2}{\|d_{k-1}\|^2},$$

where the inequality is obtained by dropping the second term since $g_k^T g_{k-1} \geq 0$. On the other hand, it also holds that the condition $0 \leq g_k^T g_{k-1} \leq \|g_k\|^2$ is equivalent to $-\|g_k\|^2 \geq -g_k^T g_{k-1}$. Therefore

$$\beta_k^{RMIL+} = \frac{\|g_k\|^2 - g_k^T g_{k-1}}{\|d_{k-1}\|^2} \geq \frac{\|g_k\|^2 - \|g_k\|^2}{\|d_{k-1}\|^2} = 0.$$

Hence, it holds that

$$0 \leq \beta_k^{RMIL+} \leq \frac{\|g_k\|^2}{\|d_{k-1}\|^2}, \quad \forall k \geq 1. \tag{19}$$

The proofs would be based on the above inequality (19).

Consider the Assumptions A, then, there exists a constant $\gamma > 0$, in such a way that we have

$$\|g_k\| \leq \gamma, \quad \forall k \in \mathbb{N}. \tag{20}$$

The lemma that follows is based on Zoutendijk [18] condition and has been extensively used in the global convergence analysis of different CG methods.

Lemma 1: Let's suppose Assumption A is true. For CG iterative method defined by (2) *and* (3), *where the search direction d_k satisfies*

$$g_k^T d_k < 0,$$

for $k \in \mathbb{N}$ and step-size t_k follows from the weak Wolfe line search, then,

$$\sum_{k=0}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < +\infty. \tag{21}$$

The proof of this Lemma follows from [17]. From (19) and using Lemma 1, we have the following convergence theorem.

Theorem 1: Suppose Assumption A holds. Consider the sequence $\{g_k\}$ and $\{d_k\}$ follows from the proposed algorithm, where β_k is given by (19), *and t_k satisfies the weak Wolfe line search, then,*

$$\frac{\|d_k\|^2}{\|g_k\|^4} \leq \sum_{i=0}^{k} \frac{1}{\|g_i\|}. \tag{22}$$

Proof: Applying Cauchy-Schwarz inequality on the descent condition (16), we deduce

$$\|d_k\| \geq \|g_k\| \implies \frac{1}{\|d_k\|} \leq \frac{1}{\|g_k\|}. \tag{23}$$

Now, from (16), (3) and (17), we have

$$\begin{aligned} d_k &= \frac{g_{k-1}^T d_{k-1}}{\|d_{k-1}\|^2} g_k - \beta_k \frac{g_k^T d_{k-1}}{\|g_k\|^2} g_k + \beta_k d_{k-1} \\ &= -\frac{\|g_{k-1}\|^2}{\|d_{k-1}\|^2} g_k - \beta_k \frac{g_k^T d_{k-1}}{\|g_k\|^2} g_k + \beta_k d_{k-1} \\ &= -\frac{\|g_{k-1}\|^2}{\|d_{k-1}\|^2} g_k + \beta_k \left(I - \frac{g_k g_k^T}{\|g_k\|^2} \right) d_{k-1}. \end{aligned} \tag{24}$$

Next, applying norm on both sides of (24) and using (23) gives

$$\begin{aligned} \|d_k\| &\leq \frac{\|g_{k-1}\|^2}{\|d_{k-1}\|^2} \|g_k\| + \beta_k \left\| I - \frac{g_k g_k^T}{\|g_k\|^2} \right\| \|d_{k-1}\| \\ &\leq \frac{\|g_{k-1}\|^2}{\|g_{k-1}\|^2} \|g_k\| + \beta_k \left\| I - \frac{g_k g_k^T}{\|g_k\|^2} \right\| \|d_{k-1}\| \\ &\leq \|g_k\| + \beta_k \left\| I - \frac{g_k g_k^T}{\|g_k\|^2} \right\| \|d_{k-1}\| \\ &= \|g_k\| + \beta_k \|d_{k-1}\|. \end{aligned} \tag{25}$$

Squaring both sides and using (19) and (23) yields

$$\begin{aligned} \|d_k\|^2 &\leq (\|g_k\| + \beta_k \|d_{k-1}\|)^2 \\ &= \|g_k\|^2 + 2\beta_k \|g_k\| \|d_{k-1}\| + \beta_k^2 \|d_{k-1}\|^2 \\ &\leq \|g_k\|^2 + 2\frac{\|g_k\|^2}{\|d_{k-1}\|^2} \|g_k\| \|d_{k-1}\| + \frac{\|g_k\|^4}{\|d_{k-1}\|^4} \|d_{k-1}\|^2 \\ &= \|g_k\|^2 + 2\frac{\|g_k\|^3}{\|d_{k-1}\|} + \frac{\|g_k\|^4}{\|d_{k-1}\|^2} \\ &\leq \|g_k\|^2 + 2\frac{\|g_k\|^3}{\|g_{k-1}\|} + \frac{\|g_k\|^4}{\|g_{k-1}\|^2} \\ &\leq \frac{\|g_k\|^4}{\|g_k\|^2} + 2\frac{\|g_k\|^4}{\|g_k\| \|g_{k-1}\|} + \frac{\|g_k\|^4}{\|g_{k-1}\|^2}. \end{aligned} \tag{26}$$

Rearranging gives

$$\begin{aligned} \frac{\|d_k\|^2}{\|g_k\|^4} &\leq \frac{1}{\|g_k\|^2} + 2\frac{1}{\|g_k\| \|g_{k-1}\|} + \frac{1}{\|g_{k-1}\|^2} \\ &= \left(\frac{1}{\|g_k\|} + \frac{1}{\|g_{k-1}\|} \right)^2. \end{aligned} \tag{27}$$

$$\frac{\|d_k\|^2}{\|g_k\|^4} \leq \left(\frac{1}{\|g_k\|} + \frac{1}{\|g_{k-1}\|} \right)^2 \leq \sum_{i=0}^{k} \frac{1}{\|g_i\|}. \tag{28}$$

The proof is completed. □

Theorem 2: Suppose Assumption A holds true. Consider the sequence $\{g_k\}$ and $\{d_k\}$ generated by the proposed method, where β_k is given by (19), *and t_k satisfies the weak Wolfe line search, then,*

$$\lim_{k \to \infty} \inf \|g_k\| = 0. \tag{29}$$

Proof: The proof of Theorem 2 would be done by contradiction. That is, if the conclusion of Theorem 2 is not true, then, there exist a constant $c > 0$ in such a way that

$$\|g_k\|^2 \geq c, \quad \forall k \geq 1. \tag{30}$$

TABLE 1. List of test functions, dimensions, and initial points.

Number	Functions	Dimension	Initial Points
1	Extended Rosenbrock	1,000	(-1.2, 1,...,-1.2,1)
2	Extended Rosenbrock	1,000	(10,...,10)
3	Extended Rosenbrock	10,000	(-1.2, 1,...,-1.2,1)
4	Extended Rosenbrock	10,000	(5,...,5)
5	Diagonal 4	500	(1,..., 1)
6	Diagonal 4	500	(-20,..., -20)
7	Diagonal 4	1,000	(1,..., 1)
8	Diagonal 4	1,000	(-30,..., -30)
9	Hager	10	(1,..., 1)
10	Hager	10	(-10,..., -10)
11	Trecanni	2	(-1, 0.5)
12	Trecanni	2	(-5, 10)
13	Shallow	1,000	(0,..., 0)
14	Shallow	1,000	(10,..., 10)
15	Shallow	10,000	(-1,..., -1)
16	Shallow	10,000	(-10,..., -10)
17	Leon	2	(2,2)
18	Leon	2	(8,8)
19	Extended Powel	100	(3,-1,0,1,...)
20	Extended Powel	100	(5,...,5)
21	Extended Beale	1,000	(1, 0.8,...,1,0.8)
22	Extended Beale	1,000	(0.5,...,0.5)
23	Extended Beale	10,000	(-1,..., -1)
24	Extended Beale	10,000	(0.5,...,0.5)
25	Six Hump Camel	2	(-1,2)
26	Six Hump Camel	2	(-5, 10)
27	Three Hump Camel	2	(-1,2)
28	Three Hump Camel	2	(2,-1)
29	POWER	10	(1,..., 1)
30	POWER	10	(10,..., 10)
31	Colville	4	(2,2,2,2)
32	Colville	4	(10,10,10,10)
33	Dixon and Price	3	(1, 1, 1)
34	Dixon and Price	3	(10, 10, 10)
35	Sphere	5,000	(1,...,1)
36	Sphere	5,000	(10,...,10)
37	Sum Squares	50	(0,1,...,0,1)
38	Sum Squares	50	(10,..., 10)
39	NONSCOMP	2	(3,3)
40	NONSCOMP	2	(10, 10)
41	Extended DENSCHNB	10	(1,..., 1)
42	Extended DENSCHNB	10	(10,..., 10)
43	Extended DENSCHNB	100	(10,..., 10)
44	Extended DENSCHNB	100	(-50,..., -50)
45	Extended Penalty	10	(1, 2,..., 10)
46	Extended Penalty	10	(-10,..., -10)
47	Extended Penalty	100	(1,...,1)
48	Extended Penalty	100	(-2,..., -2)
49	ENGVAL1	50	(2,...,2)
50	ENGVAL1	100	(2,...,2)
51	ENGVAL8	50	(0,...,0)
52	ENGVAL8	100	(0,...,0)

TABLE 1. *(Continued.)* List of test functions, dimensions, and initial points.

53	Extended White & Holst	1,000	(-1.2, 1,...,-1.2,1)
54	Extended White & Holst	1,000	(10,...,10)
55	Extended White & Holst	10,000	(-1.2, 1,...,-1.2,1)
56	Extended White & Holst	10,000	(5,...,5)
57	Extended Tridiagonal 1	500	(2,..., 2)
58	Extended Tridiagonal 1	500	(10,..., 10)
59	Extended Tridiagonal 1	1,000	(1,..., 1)
60	Extended Tridiagonal 1	1,000	(-10,..., -10)
61	FLETCHCR	10	(0,..., 0)
62	FLETCHCR	10	(10,..., 10)
63	Zettl	2	(-1, 2)
64	Zettl	2	(10, 10)
65	Quartic	4	(10,10,10,10)
66	Quartic	4	(15,15,15,15)
67	Generalized Tridiagonal 1	10	(2,..., 2)
68	Generalized Tridiagonal 1	10	(10,..., 10)
69	Ext Freudenstein & Roth	10,000	(-5,...,-5)
70	Ext Freudenstein & Roth	50,000	(-5,...,-5)
71	Extended Himmelblau	1,000	(1,..., 1)
72	Extended Himmelblau	1,000	(20,..., 20)
73	Extended Himmelblau	10,000	(-1,..., -1)
74	Extended Himmelblau	10,000	(50,..., 50)
75	Extended Maratos	10	(1.1, 0.1,..., 1.1,)
76	Extended Maratos	10	(-1,..., -1)
77	Booth	2	(5, 5)
78	Booth	2	(10, 10)
79	Quadratic QF2	50	(0.5,..., 0.5)
80	Quadratic QF2	50	(30,..., 30)
81	Matyas	2	(1, 1)
82	Matyas	2	(20, 20)
83	Quadratic QF1	50	(1,..., 1)
84	Quadratic QF1	50	(10,..., 10)
85	Quadratic QF1	500	(1,..., 1)
86	Quadratic QF1	500	(-5,..., -5)
87	Generalized Tridiagonal 2	4	(1,1,1,1)
88	Generalized Tridiagonal 2	4	(10,10,10,10)
89	Raydan 1	10	(1,..., 1)
90	Raydan 1	10	(-10,..., -10)
91	Raydan 1	100	(-1,..., -1)
92	Raydan 1	100	(-10,..., -10)
93	Generalized Quartic	1,000	(5,...,5)
94	Generalized Quartic	1,000	(20,...,20)
95	Extended Quadratic Penalty QP1	4	(1,1,1,1)
96	Extended Quadratic Penalty QP1	4	(10,10,10,10)
97	Extended Quadratic Penalty QP2	100	(1,..., 1)
98	Extended Quadratic Penalty QP2	100	(10,..., 10)
99	Extended Quadratic Penalty QP2	500	(10,...,10)
100	Extended Quadratic Penalty QP2	500	(20,..., 20)
101	QUARTICM	5,000	(2,...,2)
102	QUARTICM	10,000	(2,...,2)
103	QUARTICM	15,000	(2,...,2)
104	QUARTICM	20,000	(2,...,2)
105	DENSCHNA	10,000	(-1,...,-1)

TABLE 1. *(Continued.)* **List of test functions, dimensions, and initial points.**

106	DENSCHNA	100,000	(-1,...,-1)
107	DENSCHNC	5,000	(100,...,100)
108	DENSCHNC	50,000	(100,...,100)
109	Extended Block-Diagonal BD1	5,000	(1.02,...,1.02)
110	Extended Block-Diagonal BD1	50,000	(1.02,...,1.02)
111	Extended Block-Diagonal BD1	100,000	(1.02,...,1.02)
112	HIMMELBH	50	(0.2,...,0.2)
113	HIMMELBH	100	(0.2,...,0.2)
114	HIMMELBH	1,000	(0.2,...,0.2)
115	Extended Hiebert	10,000	(1.04,...,1.04)
116	Extended Hiebert	50,000	(1.04,...,1.04)
117	Extended Hiebert	100,000	(1.04,...,1.04)
118	Price 4	2	(-2,-2)
119	Price 4	2	(3,-2)
120	Rotated Ellipse	2	(-2,-2)
121	Rotated Ellipse	2	(0.2,0.2)
122	Zirilli or Aluffi-Pentini's	2	(2,2)
123	Zirilli or Aluffi-Pentini's	2	(-2,-2)
124	Diagonal Double Border Arrow Up	500	(1.005,...,1.005)
125	Diagonal Double Border Arrow Up	5,000	(1.005,...,1.005)
126	Diagonal Double Border Arrow Up	50,000	(1.005,...,1.005)
127	HARKERP	1,000	(1,...,1000)
128	HARKERP	50,000	(1,2,...,50000)
129	Extended Quadratic Penalty QP3	10	(15,15,...,15)
130	Extended Quadratic Penalty QP3	100	(15,15,...,15)

From Theorem 1, we have

$$\frac{\|d_k\|^2}{\|g_k\|^4} \leq \frac{k+1}{c}, \tag{31}$$

which implies

$$\sum_{k=0}^{\infty} \frac{\|g_k\|^4}{\|d_k\|^2} \geq \sum_{k=0}^{\infty} \frac{c}{k+1} = \infty. \tag{32}$$

However, since (16) hold true, then, from (21), we obtain

$$\sum_{k=0}^{\infty} \frac{\|g_k\|^4}{\|d_k\|^2} = \sum_{k=0}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < \infty. \tag{33}$$

It is clear that (32) and (33) yield contradiction. This implies that (29) holds and thus completes the proof. □

IV. PRELIMINARY RESULTS

This section presents the numerical results on 130 benchmark test problems to illustrate the efficiency of the proposed sRMIL+ method and compare the performance with the RMIL+ method [21], PRP method [6], [7], sFR method [40], and sPRP method [33]. These algorithms were coded in MATLAB R2019a and compiled with personal laptop; Intel Core i7 processor, 16 GB RAM, 64bit Windows 10 Pro operating system. A large number of the benchmark functions and initial points are considered by Andrei [41] and Jamil-Yang [42] as in Table 1. The dimensions of our test problems ranging from 2 to 100,000. All the methods are implemented using the weak Wolfe line search with $\delta = 0.0001$, $\sigma = 0.001$ and the termination criteria was set as $\|g_k\| \leq 10^{-6}$. We use "F" to denote when the iteration is bigger than 10,000 or never reach the optimal point.

All numerical results are presented in Table 2 and Table 3, where NOI represents the number of iterations, NOF represents the number of function evaluations, and CPU represents the central processing unit time. Tables 2 and 3 show that all methods failed to successfully solve the ENGVAL8 function with dimension 100, and for all, we get that the RMIL+ method solves 73% (95 out of 130), the PRP method 71% (93 out of 130), the sFR method 85% (111 out of 130), sPRP, and sRMIL+ methods 96% (125 out of 130). So, this is indicating that the sRMIL+ method is superior at solving the test problems considered compared with RMIL, PRP, and sFR methods. However, the sRMIL+ method is competitive with the sPRP method.

On the other hand, to compare and visually characterize the numerical results in Tables 2 and 3, we use the performance profile tool of Dolan and Moré [43] to describe the performance of the RMIL+, PRP, sFR, sPRP, and sRMIL+ methods based on the NOI, NOF, and CPU time, respectively. Let S is set of solvers, P is test problems, for n_s solvers and n_p problems, the performance profile $\xi : \mathbb{R} \to [0, 1]$ is formulated as follows:

$$\xi(\tau) := \frac{1}{n_p} size\{p \in P_l | \log_2(r_{p,s}) \leq \tau\}, \quad \forall \tau \in \mathbb{R}^+,$$

TABLE 2. Numerical results of RMIL+, PRP, and sFR methods.

Number	RMIL+			PRP			sFR		
	NOI	NOF	CPU	NOI	NOF	CPU	NOI	NOF	CPU
1	27	176	0.0488	19	123	0.0377	179	293	0.1955
2	40	243	0.0667	F	F	F	163	1181	0.2378
3	32	192	0.3874	19	123	0.231	200	986	1.6746
4	40	195	0.3768	20	136	0.4796	831	5101	8.4652
5	F	F	F	F	F	F	31	90	0.0254
6	F	F	F	F	F	F	56	165	0.0392
7	F	F	F	F	F	F	39	114	0.053
8	F	F	F	F	F	F	64	189	0.0549
9	F	F	F	F	F	F	34	105	0.0048
10	F	F	F	F	F	F	89	309	0.0091
11	1	3	0.0013	1	3	1.95E-04	1	3	1.88E-04
12	5	23	0.007	5	23	6.89E-04	7	30	7.20E-04
13	11	39	0.0154	F	F	F	55	171	0.0483
14	14	59	0.0303	13	51	0.018	96	389	0.0928
15	F	F	F	F	F	F	47	149	0.3167
16	F	F	F	F	F	F	372	1298	2.4379
17	31	179	0.0023	17	136	0.0012	66	363	0.0035
18	35	265	0.0033	28	243	0.0032	709	4235	0.0522
19	70	863	0.0716	3337	10084	0.7111	5589	16877	1.0729
20	39	225	0.0443	2312	7053	0.4623	6019	18187	1.2789
21	52	191	0.0992	15	69	0.0479	75	249	0.1303
22	F	F	F	9	44	0.0367	81	267	0.1455
23	11	48	0.2153	F	F	F	87	288	1.1917
24	F	F	F	10	47	0.222	87	285	1.2099
25	8	36	0.0053	6	30	0.007	27	96	0.0103
26	11	66	0.0026	F	F	F	536	2070	0.0438
27	F	F	F	F	F	F	F	F	F
28	15	400	0.0108	F	F	F	F	F	F
29	123	369	0.0102	10	30	7.66E-04	10	30	9.52E-04
30	139	417	0.0123	10	30	8.78E-04	10	30	9.73E-04
31	1032	4339	0.0726	148	818	0.2155	F	F	F
32	669	2819	0.0324	86	372	0.0167	33	169	0.0029
33	1	3	0.0083	1	3	0.0167	1	3	0.0071
34	1	3	0.0056	1	3	0.0071	1	3	0.0057
35	F	F	F	F	F	F	16	60	2.10E-03
36	46	194	0.0083	56	266	0.0063	25	122	0.0013
37	46	138	0.0152	25	75	0.0057	25	75	0.0043
38	81	243	0.2223	41	123	0.0097	41	123	0.0181
39	54	193	0.0183	15	76	0.0048	156	637	0.0136
40	17	94	0.2085	F	F	F	93	405	0.0115
41	6	22	8.45E-04	5	19	4.20E-03	9	31	0.0012
42	8	37	0.0022	8	37	0.0023	11	49	0.0017
43	8	37	0.0093	8	37	0.0044	11	49	0.01
44	9	43	0.0181	7	37	0.0032	63	223	0.0257
45	27	112	0.0038	31	117	0.0017	20	76	0.0027
46	26	103	0.0021	9	46	6.12E-04	19	74	0.0018
47	19	87	0.0056	12	82	0.006	F	F	F
48	19	89	0.0124	13	87	0.0077	F	F	F
49	47	817	0.0301	22	409	0.0147	23	431	0.019
50	F	F	F	22	416	0.0251	22	386	0.0273
51	14	63	0.2976	14	78	0.2305	F	F	F
52	F	F	F	F	F	F	F	F	F

TABLE 2. *(Continued.)* **Numerical results of RMIL+, PRP, and sFR methods.**

53	16	102	0.0588	15	104	0.0525	49	214	0.1055
54	F	F	F	21	181	0.0898	850	4788	1.763
55	16	102	0.3907	15	104	0.3841	50	127	0.8213
56	38	260	0.9512	22	203	0.7392	130	1308	4.4304
57	6	37	0.0144	F	F	F	26	97	0.0399
58	5	26	0.0145	5	26	0.0139	9	50	0.0182
59	7	40	0.0276	F	F	F	35	127	0.0716
60	9	55	0.0425	13	68	0.0458	29	121	0.0718
61	72	311	0.0084	56	263	0.0055	1208	5994	0.0791
62	111	548	0.0183	71	376	0.0078	299	1489	0.0333
63	16	69	0.0028	10	45	0.0011	26	95	2.50E-03
64	F	F	F	12	59	0.0012	20	82	0.0027
65	F	F	F	F	F	F	537	14628	2.2454
66	F	F	F	F	F	F	44	1330	0.5172
67	22	74	0.0058	23	77	0.0057	27	90	0.004
68	28	120	0.003	27	117	0.0037	43	173	0.0038
69	11	63	0.1356	8	54	0.1141	17	80	0.1977
70	11	63	0.4922	8	54	0.3937	17	80	0.5808
71	11	44	0.0215	8	34	0.0327	15	56	0.0243
72	7	34	0.0165	6	31	0.0127	9	40	0.0185
73	9	42	0.0952	8	45	0.1075	17	115	0.2521
74	11	50	0.137	8	44	0.1027	13	55	0.1317
75	207	923	0.0331	F	F	F	F	F	F
76	31	195	0.0134	25	188	0.004	182	1750	0.0338
77	2	6	2.58E-04	2	6	2.34E-04	2	6	2.41E-04
78	2	6	2.84E-04	2	6	4.06E-04	2	6	2.96E-04
79	78	280	0.0225	70	250	0.0241	116	407	0.2404
80	77	334	0.0327	58	275	0.0322	F	F	F
81	F	F	F	F	F	F	9	63	0.0102
82	F	F	F	F	F	F	11	77	0.0034
83	69	207	0.0115	38	114	0.0049	38	114	0.0061
84	78	234	0.0104	40	120	0.0073	40	120	0.0068
85	447	1341	0.1716	131	393	0.0719	131	393	0.063
86	500	1500	0.2046	137	411	0.072	137	411	0.0683
87	7	21	0.0027	F	F	F	34	101	2.18E-01
88	F	F	F	11	59	0.0019	F	F	F
89	27	105	0.0026	22	87	0.0021	21	84	0.0071
90	37	170	0.0062	37	157	0.0036	197	635	0.0325
91	109	505	0.0394	74	409	0.032	95	475	0.2468
92	180	841	0.0609	F	F	F	F	F	F
93	802	2788	0.0517	163	696	0.0138	272	1075	0.0248
94	806	2811	0.0454	113	495	0.0133	273	1096	0.0253
95	14	53	0.0012	6	28	6.26E-04	20	72	0.0041
96	15	68	0.0013	9	49	9.52E-04	51	194	0.0042
97	34	313	0.0254	22	235	0.0161	388	4075	0.1787
98	30	296	0.0252	27	296	0.0233	490	5020	0.2081
99	57	620	0.1127	39	493	0.087	1217	21367	3.459
100	69	743	0.1158	43	528	0.082	1436	25608	4.0323
101	F	F	F	F	F	F	3	27	0.0847
102	F	F	F	F	F	F	3	27	0.1332
103	F	F	F	F	F	F	3	27	0.17
104	F	F	F	F	F	F	3	27	0.217
105	F	F	F	15	44	0.1867	23	66	0.265
106	F	F	F	15	44	1.4056	27	78	2.4636

TABLE 2. *(Continued.)* **Numerical results of RMIL+, PRP, and sFR methods.**

107	10	86	0.1111	11	91	0.0745	50	222	0.2247
108	10	86	0.6165	11	91	0.7072	52	228	1.8732
109	12	269	0.2379	11	256	0.236	10	194	0.1955
110	13	246	1.8967	11	199	1.6463	9	131	1.0683
111	12	286	4.412	11	257	4.0298	12	255	4.0832
112	7	22	0.0076	5	15	0.0011	10	30	0.005
113	7	30	0.003	5	15	0.0031	10	30	0.006
114	7	48	0.0292	5	15	0.0101	F	F	F
115	F	F	F	26	220	0.4321	60	281	0.5859
116	F	F	F	26	220	1.5976	66	299	2.5239
117	F	F	F	26	220	3.6466	66	299	4.8565
118	20	127	0.0025	16	91	0.0018	4013	12128	0.1617
119	39	171	0.0172	24	150	0.0036	4796	14474	0.2015
120	42	1261	0.0185	48	1484	0.012	F	F	F
121	31	946	0.0086	42	1304	0.0124	F	F	F
122	F	F	F	F	F	F	33	97	0.0032
123	F	F	F	F	F	F	29	62	0.006
124	14	421	0.0595	F	F	F	F	F	F
125	36	1067	0.7225	F	F	F	F	F	F
126	F	F	F	F	F	F	F	F	F
127	F	F	F	F	F	F	16	114	0.0388
128	F	F	F	F	F	F	F	F	F
129	10	51	0.0018	13	77	0.0016	15	66	0.0022
130	27	159	0.0213	16	108	0.0089	F	F	F

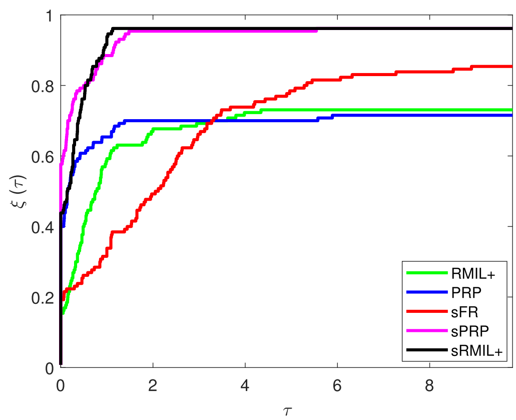

FIGURE 1. Performance profiles based on NOI.

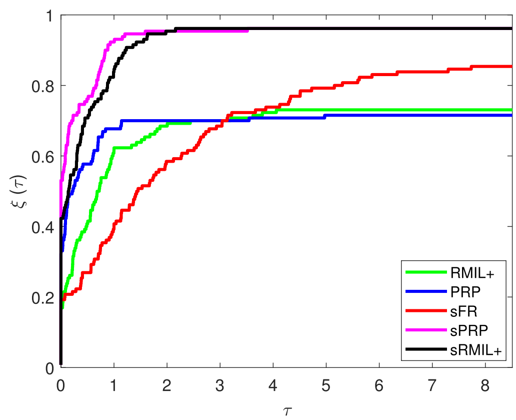

FIGURE 2. Performance profiles based on NOF.

where $1 \leq s \leq n_s$, $r_{p,s}$ is the performance ratio which is defined by: $r_{p,s} := t_{p,s}/\min\{t_{p,s}\}$ and for each solver $s \in S_l$ and each problem $p \in P_l$, they define $t_{p,s}$ is NOI or NOF or CPU time required to solve problem p by solver s. According to their rules, the method with the curve at the top is the best.

From Figures 1, 2 and 3, we can see that the sRMIL+ curve is mostly at the top of the RMIL+, PRP, and sFR curves, so it is indicating that the sRMIL+ algorithm outperforms the RMIL+, PRP, and sFR methods based on NOI, NOF, and CPU time. Meanwhile, when we compare the sRMIL+ method with the sPRP method, both methods are competitive based on NOI, NOF, and CPU time. Finally, by combining Tables 2, 3 and Figures 1, 2, 3, we can take conclusion that sRMIL+ method perform better than RMIL+, PRP and SFR, on other hand the sPRP and sRMIL+ methods are almost the same performance. This demonstrates that the methods proposed in this paper have good numerical performance.

V. APPLICATION IN PORTFOLIO SELECTION

In this section, we present the application of the proposed method for solving portfolio selection problem. Theory of portfolio selection was first proposed by Harry Markowitz in his paper "Portfolio Selection" [44]. In this paper, we only consider the stock portfolio. Stock portfolio can be defined as a collection of stocks owned by investors. Portfolio theory is based on the fact that investors will invest their money in

TABLE 3. Numerical results of sPRP, and sRMIL+ methods.

Number	sPRP			sRMIL+		
	NOI	NOF	CPU	NOI	NOF	CPU
1	18	120	0.0318	23	147	0.0365
2	20	144	0.0438	26	295	0.0676
3	18	120	0.2253	23	147	0.2603
4	19	129	0.243	36	508	0.7877
5	5	13	0.0051	6	16	0.0051
6	5	13	0.0058	6	16	0.0048
7	5	13	0.0062	8	22	0.0086
8	5	13	0.0094	8	22	0.0063
9	36	94	0.007	16	53	0.0016
10	31	95	0.0032	19	93	0.0023
11	1	3	2.22E-04	1	3	1.59E-04
12	5	23	5.43E-04	5	23	5.33E-04
13	8	32	0.019	7	30	0.0145
14	14	63	0.0287	17	74	0.026
15	12	43	0.0952	14	48	0.1018
16	11	57	0.1207	13	73	0.1853
17	14	108	0.002	19	197	0.0017
18	29	308	0.0039	34	303	0.0037
19	3261	9856	0.7004	70	863	0.0693
20	101	401	0.0382	39	225	0.0245
21	13	64	0.0349	16	121	0.0865
22	10	47	0.0371	10	43	0.0332
23	10	47	0.201	9	49	0.2068
24	10	47	0.2192	10	43	0.2101
25	6	30	8.17E-04	7	33	7.39E-04
26	8	42	1.10E-03	12	57	1.20E-03
27	16	430	0.2081	11	214	0.0044
28	9	240	0.0072	13	367	0.0112
29	10	30	1.10E-03	10	30	9.30E-04
30	10	30	9.12E-04	10	30	9.30E-04
31	137	801	0.0083	134	1471	0.0107
32	87	377	0.0043	67	433	0.0036
33	1	3	0.0041	1	3	0.0036
34	1	3	0.0074	1	3	0.0064
35	27	92	0.002	19	78	9.20E-04
36	60	281	0.0036	30	144	0.0027
37	25	75	0.004	50	150	0.0097
38	41	123	0.008	88	273	0.0108
39	15	79	0.009	15	72	0.0041
40	13	73	0.0041	16	69	0.003
41	5	19	8.60E-03	5	19	6.72E-04
42	8	37	0.0046	11	49	0.0013
43	8	37	0.0035	11	49	0.0035
44	8	41	0.0039	8	41	0.0034
45	33	125	0.1931	19	90	0.0095
46	7	36	0.0036	9	44	0.0026
47	13	85	0.0092	23	142	0.0171
48	F	F	F	21	141	0.0163
49	22	419	0.0229	48	813	0.0465
50	24	489	0.0323	43	719	0.0434

TABLE 3. ***(Continued.)*** **Numerical results of sPRP, and sRMIL+ methods.**

51	F	F	F	F	F	F
52	F	F	F	F	F	F
53	14	98	0.0547	17	111	0.0562
54	23	229	0.1045	30	408	0.1625
55	14	98	0.362	17	111	0.4085
56	31	308	1.0713	33	220	0.7971
57	13	63	0.0287	7	46	0.025
58	5	26	0.0128	5	26	0.0138
59	13	63	0.044	12	94	0.053
60	7	45	0.0338	13	104	0.059
61	56	263	0.0077	58	279	0.0108
62	53	285	0.006	68	524	0.0164
63	9	43	1.90E-03	12	57	8.27E-04
64	11	49	1.30E-03	15	67	0.0014
65	18	434	0.1061	18	411	0.1426
66	17	477	0.1394	14	340	0.1119
67	23	77	0.0019	24	83	0.0016
68	27	117	0.003	31	144	0.0026
69	F	F	F	F	F	F
70	F	F	F	F	F	F
71	7	31	0.0126	12	45	0.0155
72	6	31	0.0135	6	31	0.0228
73	8	45	0.1121	8	44	0.0999
74	6	33	0.0867	7	36	0.0909
75	34	263	0.2415	F	F	F
76	27	208	0.0048	30	257	0.0087
77	2	6	4.12E-04	2	6	2.43E-04
78	2	6	2.42E-04	2	6	2.34E-04
79	70	250	0.0065	101	497	0.0143
80	58	275	0.0069	78	392	0.0152
81	9	63	0.0079	9	63	7.60E-03
82	11	77	0.0033	11	77	0.0024
83	38	114	0.0059	76	353	0.0149
84	40	120	0.0091	71	321	0.0148
85	131	393	0.0661	262	1755	0.2009
86	137	411	0.0699	260	1211	0.22
87	16	40	7.40E-03	9	26	7.03E-04
88	14	60	7.10E-03	11	54	1.30E-03
89	20	80	0.0094	22	90	0.0024
90	32	146	0.0086	46	308	0.01
91	76	446	0.0363	117	1259	0.0537
92	169	900	0.0576	152	1789	0.0812
93	163	695	0.0131	88	1403	0.0176
94	113	495	0.0136	51	305	0.0091
95	6	28	5.27E-04	6	28	5.97E-04
96	9	50	9.32E-04	9	49	9.12E-04
97	21	228	0.0178	22	203	0.0189
98	27	296	0.0193	27	253	0.0158
99	41	508	0.0881	41	442	0.0828
100	37	440	0.0806	53	568	0.1064
101	3	27	0.0751	3	27	0.0709
102	3	27	0.1336	3	27	0.1228
103	3	27	0.1772	3	27	0.1687

TABLE 3. ***(Continued.)*** **Numerical results of sPRP, and sRMIL+ methods.**

104	3	27	0.2193	3	27	0.2145
105	7	27	0.1166	11	42	0.1878
106	7	27	1.1403	9	35	1.2819
107	11	91	0.086	12	92	0.0861
108	11	91	0.7223	11	89	0.6681
109	12	263	0.2703	11	149	0.1724
110	12	242	2.0159	12	195	1.5232
111	13	275	4.2859	11	149	2.7943
112	5	15	0.0012	5	15	0.0011
113	5	15	0.0024	5	15	0.0015
114	5	15	0.0111	5	15	0.0107
115	25	217	0.4152	23	144	0.2744
116	25	217	1.8073	21	136	0.969
117	25	217	3.3591	21	136	2.1777
118	18	110	0.0023	11	77	0.0022
119	19	119	0.0023	10	68	0.0018
120	50	1387	0.0151	40	1183	0.0108
121	43	1196	0.01	28	878	0.0083
122	7	20	7.89E-04	6	22	7.17E-04
123	22	55	0.0019	10	33	9.82E-04
124	39	1267	0.1274	14	421	0.0614
125	61	1825	1.2331	36	1066	0.7244
126	72	2154	14.1106	41	1322	10.273
127	8	68	0.0238	8	68	0.0218
128	10	170	1.2856	8	101	0.7042
129	8	43	0.001	8	43	9.96E-04
130	16	113	0.012	17	111	0.0126

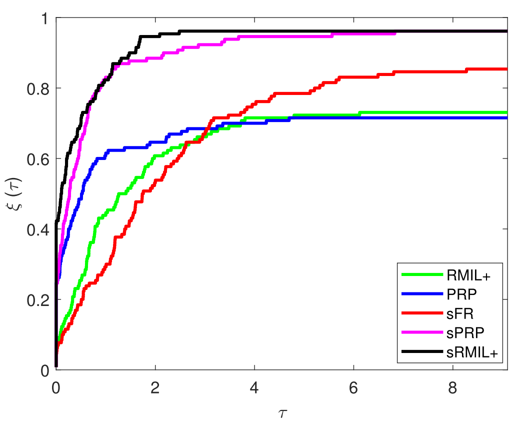

FIGURE 3. **Performance profiles based on CPU time.**

various types of stocks, where the main objective is to reduce risk. According to [45], the return R_i on stock s_i is formulated by

$$R_i = \frac{P_t - P_{t-1}}{P_{t-1}},$$

where P_t is the price of the stock at time t and P_{t-1} is the price of stock at time $t-1$. The expected return of the portfolio's return is defined as

$$\mu = \mathrm{E}\left(\sum_{i=1}^{n} w_i R_i\right) = \sum_{i=1}^{n} w_i \mu_i, \quad (34)$$

and variance of the portfolio's return is defined as

$$\sigma^2 = \mathrm{V}ar\left(\sum_{i=1}^{n} w_i R_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \mathrm{Cov}(R_i, R_j), \quad (35)$$

where n is number of stocks, w_i is the percentage of the value of the stock contained in the portfolio and $\mathrm{Cov}(R_i, R_j)$ is the covariance of R_i and R_j.

One way to optimize a portfolio is to minimize risk. Risk here is defined as the variance of the portfolio's return σ^2. So that the problem of portfolio selection can be written in the following model

$$\begin{cases} \text{minimize} : \sigma^2 = \mathrm{V}ar\left(\sum_{i=1}^{n} w_i R_i\right) \\ \text{subject to} : \sum_{i=1}^{n} w_i = 1. \end{cases} \quad (36)$$

In this research, the stock price used is the weekly closing price of 9 stocks and the stocks being considered are PT Bank Central Asia Tbk (BBCA), PT Bank Rakyat Indonesia (Persero) Tbk (BBRI), PT Unilever Indonesia Tbk (UNVR), PT Telekomunikasi Indonesia Tbk (TLKM), PT Indofood CBP Sukses Makmur Tbk (ICBP), PT Bank Mandiri (Persero) Tbk (BMRI), PT Perusahaan Gas Negara Tbk (PGAS), PT Astra International Tbk (ASII) and PT Bank Negara Indonesia Tbk (BBNI) where the stock closing price is obtained from the data http://finance.yahoo.com/., over a period of 3 years (Jan 1, 2018 - Dec 31, 2020). Based on this

TABLE 4. Mean and Variance of return of all stocks.

Stocks	BBCA	BBRI	UNVR	TLKM	ICBP	BMRI	PGAS	ASII	BBNI
Mean	-0.0020	0.0003	0.0031	0.0025	0.0005	0.0028	0.0036	0.0032	0.0037
Variance	0.0013	0.0027	0.0013	0.0017	0.0014	0.0031	0.0067	0.0024	0.0035

TABLE 5. Covariance of return of all stocks.

Stocks	BBCA	BBRI	UNVR	TLKM	ICBP	BMRI	PGAS	ASII	BBNI
BBCA	0.0013	0.0013	0.0005	0.0006	0.0005	0.0014	0.0017	0.0012	0.0015
BBRI	0.0013	0.0027	0.0006	0.0009	0.0006	0.0024	0.0023	0.0018	0.0025
UNVR	0.0005	0.0006	0.0013	0.0005	0.0006	0.0009	0.0010	0.0007	0.0007
TLKM	0.0006	0.0009	0.0005	0.0017	0.0005	0.0011	0.0016	0.0009	0.0012
ICBP	0.0005	0.0006	0.0006	0.0005	0.0014	0.0008	0.0009	0.0005	0.0008
BMRI	0.0014	0.0024	0.0009	0.0011	0.0008	0.0031	0.0028	0.0019	0.0028
PGAS	0.0017	0.0023	0.0010	0.0016	0.0009	0.0028	0.0067	0.0023	0.0031
ASII	0.0012	0.0018	0.0007	0.0009	0.0005	0.0019	0.0023	0.0024	0.0019
BBNI	0.0015	0.0025	0.0007	0.0012	0.0008	0.0028	0.0031	0.0019	0.0035

data, we have return of each weekly closing stock price and also obtained the mean, variance, and covariance values of return in Tables 4 and 5, respectively.

Let $w_1, w_2, \dots, w_9$ be the proportions allocated to BBCA, BBRI, UNVR, TLKM, ICBP, BMRI, PGAS, ASII and BBNI, respectively. By setting $w_9 = 1 - w_2 - w_2 - w_3 - w_4 - w_5 - w_6 - w_7 - w_8$ and using the data in Tables 4 and 5, we can form problem (36) into an unconstrained optimization problem as follows:

$$
\begin{aligned}
\min_{\mathbf{w}\in\mathbb{R}^8} \Big\{ & (-0.2e-3w_1 - 0.2e-3w_2 - 0.10e-2w_3 \\
& -0.9e-3w_4 - 0.10e-2w_5 - 0.1e-3w_6 \\
& +0.2e-3w_7 - 0.3e-3w_8 + 0.15e-2)w_1 + \\
& (-0.12e-2w_1 + 0.2e-3w_2 - 0.19e-2w_3 \\
& -0.16e-2w_4 - 0.19e-2w_5 - 0.1e-3w_6 \\
& -0.2e-3w_7 - 0.7e-3w_8 + 0.25e-2)w_2 \\
& +(-0.2e-3w_1 - 0.1e-3w_2 + 0.6e-3w_3 \\
& -0.2e-3w_4 - 0.1e-3w_5 + 0.2e-3w_6 \\
& +0.3e-3w_7 + 0.7e-3)w_3 + (-0.6e-3w_1 \\
& -0.3e-3w_2 - 0.7e-3w_3 + 0.5e-3w_4 \\
& -0.7e-3w_5 - 0.1e-3w_6 + 0.4e-3w_7 \\
& -0.3e-3w_8 + 0.12e-2)w_4 + (-0.3e-3w_1 \\
& -0.2e-3w_2 - 0.2e-3w_3 - 0.3e-3w_4 \\
& +0.6e-3w_5 + 0.8e-3 + 0.1e-3w_7 \\
& -0.3e-3w_8)w_5 + (-0.14e-2w_1 \\
& -0.4e-3w_2 - 0.19e-2w_3 - 0.17e-2w_4 \\
& -0.20e-2w_5 + 0.3e-3w_6 + 0.28e-2 \\
& -0.9e-3w_8)w_6 + (-0.14e-2w_1 - 0.8 \\
& e-3w_2 - 0.21e-2w_3 - 0.15e-2w_4 \\
& -0.22e-2w_5 - 0.3e-3w_6 + 0.36e-2w_7 \\
& -0.8e-3w_8 + 0.31e-2)w_7 + (-0.7e-3w_1 \\
& -0.1e-3w_2 - 0.12e-2w_3 - 0.10e-2w_4 \\
& -0.14e-2w_5 + 0.19e-2 + 0.4e-3w_7 \\
& +0.5e-3w_8)w_8 + (-0.20e-2w_1 \\
& -0.10e-2w_2 - 0.28e-2w_3 - 0.23e-2w_4 \\
& -0.27e-2w_5 - 0.7e-3w_6 - 0.4e-3w_7 \\
& -0.16e-2w_8 + 0.35e-2)(1 - w_1 - w_2 \\
& -w_3 - w_4 - w_5 - w_6 - w_7 - w_8) \Big\}.
\end{aligned}
$$

By running Algorithm 1 with an initial point $(0.25, \dots, 0.25)$, then the problem above has a solution $w_1 = 0.4322$, $w_2 = 0.1201$, $w_3 = 0.2892$, $w_4 = 0.2464$, $w_5 = 0.2333$, $w_6 = -0.1818$, $w_7 = -0.0854$, $w_8 = 0.0187$, and also we obtained $w_9 = 0.0727$. Furthermore, based on (34) and (35), we have $\mu = 0.0003$ and $\sigma^2 = 0.0006$, respectively. Therefore, we can take the proportion of each stock with minimal risk, i.e, 43.22% BBCA, 12.01% BBRI, 28.92% UNVR, 24.64% TLKM, 23.33% ICBP, −18.18%, BMRI, −8.54% PGAS, 1.87% ASII and 7.27% BBNI. Because there is a minus sign in the proportion of ICBP and BMRI stocks, it indicates that investor can do short shelling. As a final conclusion here, investors can consider this portfolio with a minimal risk is 0.0006 and an expected portfolio return value is 0.0003.

VI. APPLICATION IN TWO-JOINT PLANAR ROBOTIC MOTION CONTROL

Additional efficiency test for the good performance of the sRMIL+ method is demonstrated by implementing it to solve a two-joint planar robotic motion control problem. To begin with, a brief description of discrete-time kinematics equation of two-joint planar robot manipulator will be given as presented in [46]. Let $u_k \in \mathbb{R}^2$ denotes the joint angle vector and $v_k \in \mathbb{R}^2$ be end effector position vector. A discrete-time kinematics equation of two-joint planar robot manipulator at a position level is described by the following model

$$\Omega(u_k) = v_k. \tag{37}$$

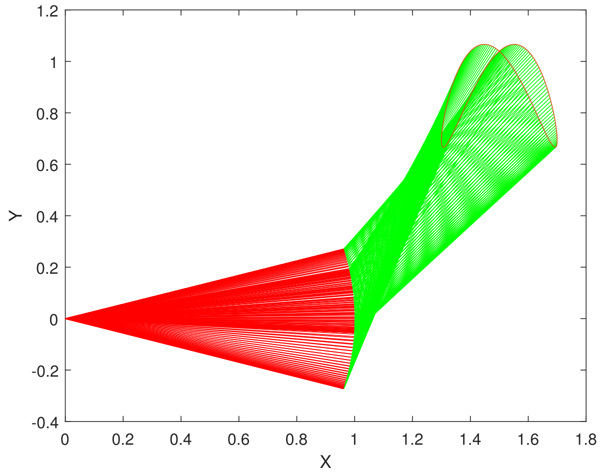

FIGURE 4. Manipulator trajectories.

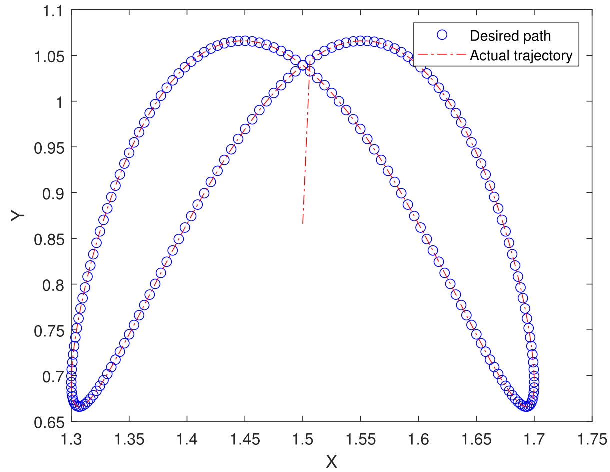

FIGURE 5. End effector trajectory and desired path.

Let ℓ_1 and ℓ_2, respectively, denote the lengths of the first and second rod. The mapping $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is the kinematics mapping where its structure is given as follows

$$\Omega(u) = \Big[\ell_1 \cos(u_1) + \ell_2 \cos(u_1 + u_2), \quad \ell_2 \sin(u_1) + \ell_2 \sin(u_1 + u_2)\Big]^T.$$

In motion control problem, at each instant time, say $t_k \in [0, t_{\text{final}}]$ where t_{final} is the end of task duration, a series nonlinear least squares problems which are a special case of the unconstrained optimization problem needed to be solved as follows

$$\min_{v_k \in \mathbb{R}^2} \frac{1}{2} \|v_k - \widehat{v}_k\|^2, \tag{38}$$

where $\widehat{v}_k$ represents the end effector controlled track.

Following similar approach presented in [1], [47], the end effector, that is $\widehat{v}_k$, used in this experiment, is controlled to track a Lissajous curve given as

$$\widehat{v}_k = \left[\frac{3}{2} + \frac{1}{5}\sin\left(\frac{\pi t_k}{5}\right), \quad \frac{\sqrt{3}}{2} + \frac{1}{5}\sin\left(\frac{2\pi t_k}{5} + \frac{\pi}{3}\right)\right]^T. \tag{39}$$

The implementation of the sRMIL+ algorithm with regards to the motion control experiment was performed using MATLAB R2019b and run on a PC with intel Core(TM) i5-8250u processor with 4 GB of RAM and CPU 1.60 GHZ. The initial point used is $u_0 = [u_1, \ u_2] = [0, \ \frac{\pi}{3}]^T$ with the task duration $[0, \ t_{\text{final}}]$ being divided into 200 equal parts, where $t_{\text{final}} = 10$ seconds and $\ell_1 = \ell_2 = 1$.

The motion control experimental results are presented in Figures 4–7, where Figure 4 depicts the robot trajectories synthesized by the sRMIL+ algorithm and Figure 5 plots the end effector trajectory and desired path. The errors of the sRMIL+ algorithm are reported in Figures 6–7, where Figure 6 shows the error recorded on horizontal axis and Figure 7 shows the error recorded on the vertical axis. It is apparent from Figures 4 and 5 that the sRMIL+ algorithm successfully executed the task given to it. The error recorded during the execution of the task is as low as 10^{-5}. This is evident from Figures 6 and 7. This confirms the efficiency and applicability of the proposed sRMIL+ algorithm.

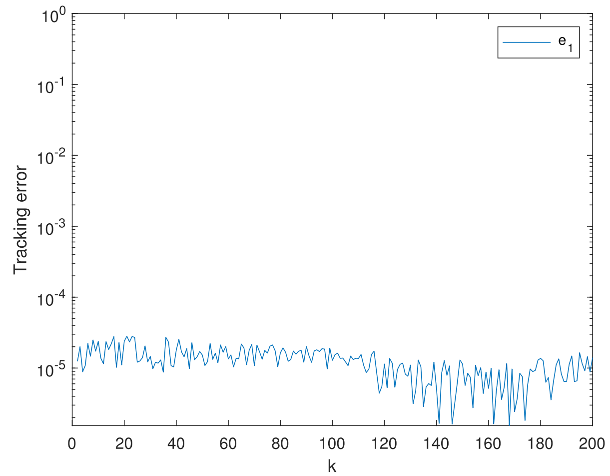

FIGURE 6. Tracking errors on the horizontal x-axis.

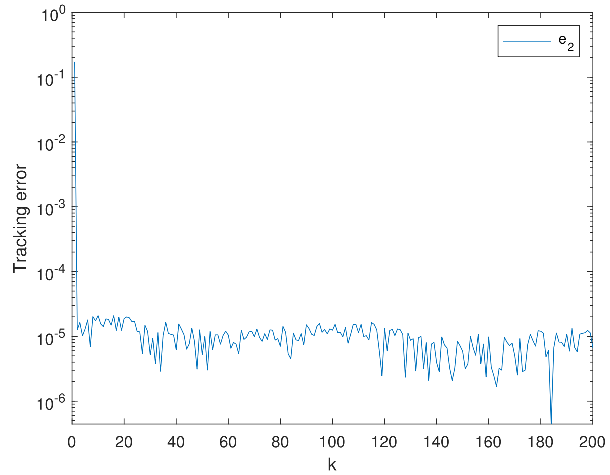

FIGURE 7. Tracking errors on the vertical y-axis.

VII. CONCLUSION

In this paper, we presented a new spectral conjugate gradient direction based on the idea of recent RMIL+ CG coefficient. For the proposed method, the sufficient descent condition always holds regardless of the line search procedure used. The global convergence proof was established under some standard assumptions. Preliminary experiment was conducted to check the performance of the proposed algorithm. The numerical results obtained showed that the new algorithm is not only efficient but also promising in practice when compared with some existing CG algorithms. Furthermore, the proposed spectral method was extended to solve problems of portfolio selection and robotic motion control to demonstrate its applicability to real-world problems.

ACKNOWLEDGMENT

The author Aliyu Muhammed Awwal would like to thank the Postdoctoral Fellowship from the King Mongkut's University of Technology Thonburi (KMUTT), Thailand. The author Ibrahim Mohammed Sulaiman would like to thank the Posdoctoral Fellowship from the Universiti Sultan Zainal Abidin, Kuala Terengganu, Malaysia.

REFERENCES

[1] A. M. Awwal, P. Kumam, L. Wang, S. Huang, and W. Kumam, "Inertial-based derivative-free method for system of monotone nonlinear equations and application," *IEEE Access*, vol. 8, pp. 226921–226930, 2020.

[2] I. Sulaiman and M. Mamat, "A new conjugate gradient method with descent properties and its application to regression analysis," *J. Numer. Anal., Ind. Appl. Math.*, vol. 12, nos. 1–2, pp. 25–39, 2020.

[3] Z. Dai and J. Kang, "Some new efficient mean-variance portfolio selection models," *Int. J. Finance Econ.*, vol. 2021, pp. 1–13, 2021.

[4] Z. Dai and H. Zhu, "Stock return predictability from a mixed model perspective," *Pacific-Basin Finance J.*, vol. 60, Apr. 2020, Art. no. 101267.

[5] M. R. Hestenes and E. Stiefel, "Methods of conjugate gradients for solving linear systems," *J. Res. Nat. Bur. Standards*, vol. 49, no. 6, p. 409, Dec. 1952, doi: 10.6028/jres.049.044.

[6] E. Polak and G. Ribiere, "Note sur la convergence de méthodes de directions conjuguées," *ESAIM, Math. Model. Numer. Anal.-Modélisation Mathématique et Analyse Numérique*, vol. 3, no. R1, pp. 35–43, 1969.

[7] B. T. Polyak, "The conjugate gradient method in extremal problems," *USSR Comput. Math. Math. Phys.*, vol. 9, no. 4, pp. 94–112, Jan. 1969.

[8] Y. Liu and C. Storey, "Efficient generalized conjugate gradient algorithms, Part 1: Theory," *J. Optim. Theory Appl.*, vol. 69, no. 1, pp. 129–137, Apr. 1991.

[9] R. Fletcher and M. J. D. Powell, "A rapidly convergent descent method for minimization," *Comput. J.*, vol. 6, no. 2, pp. 163–168, Aug. 1963.

[10] R. Fletcher, *Practical Methods of Optimization*. Hoboken, NJ, USA: Wiley, 2013.

[11] Y. Dai, J. Han, G. Liu, D. Sun, H. Yin, and Y.-X. Yuan, "Convergence properties of nonlinear conjugate gradient methods," *SIAM J. Optim.*, vol. 10, no. 2, pp. 345–358, Jan. 2000.

[12] W. W. Hager and H. Zhang, "A survey of nonlinear conjugate gradient methods," *Pacific J. Optim.*, vol. 2, no. 1, pp. 35–58, Jan. 2006.

[13] M. J. Powell, "Nonconvex minimization calculations and the conjugate gradient method," in *Numerical Analysis*. Berlin, Germany: Springer, 1984, pp. 122–141.

[14] M. Al-Baali, "Descent property and global convergence of the Fletcher–Reeves method with inexact line search," *IMA J. Numer. Anal.*, vol. 5, no. 1, pp. 121–124, 1985.

[15] Y. H. Dai and Y. Yuan, "A nonlinear conjugate gradient method with a strong global convergence property," *SIAM J. Optim.*, vol. 10, no. 1, pp. 177–182, Jan. 1999.

[16] L. Guanghui, H. Jiye, and Y. Hongxia, "Global convergence of the Fletcher–Reeves algorithm with inexact linesearch," *Appl. Math., J. Chin. Univ.*, vol. 10, no. 1, pp. 75–82, Mar. 1995.

[17] Y. Dai and Y.-X. Yuan, "Convergence properties of the Fletcher–Reeves method," *IMA J. Numer. Anal.*, vol. 16, no. 2, pp. 155–164, Apr. 1996.

[18] G. Zoutendijk, "Nonlinear programming, computational methods," in *Integer and Nonlinear Programming*, J. Abadie, Ed. Amsterdam, The Netherlands: North-Holland, 1970, pp. 37–86. [Online]. Available: https://ci.nii.ac.jp/naid/10030666308/en/

[19] M. Malik, M. Mamat, S. S. Abas, and I. M. Sulaiman, "A new coefficient of the conjugate gradient method with the sufficient descent condition and global convergence properties," *Eng. Lett.*, vol. 28, no. 3, pp. 704–714, 2020.

[20] M. Rivaie, M. Mamat, L. W. June, and I. Mohd, "A new class of nonlinear conjugate gradient coefficients with global convergence properties," *Appl. Math. Comput.*, vol. 218, no. 22, pp. 11323–11332, Jul. 2012.

[21] Z. Dai, "Comments on a new class of nonlinear conjugate gradient coefficients with global convergence properties," *Appl. Math. Comput.*, vol. 276, pp. 297–300, Mar. 2016.

[22] O. O. O. Yousif, "The convergence properties of RMIL+ conjugate gradient method under the strong Wolfe line search," *Appl. Math. Comput.*, vol. 367, Feb. 2020, Art. no. 124777.

[23] J. Liu and Y. Jiang, "Global convergence of a spectral conjugate gradient method for unconstrained optimization," *Abstract Appl. Anal.*, vol. 2012, pp. 1–12, Jan. 2012.

[24] E. G. Birgin, J. M. Martínez, and M. Raydan, "Nonmonotone spectral projected gradient methods on convex sets," *SIAM J. Optim.*, vol. 10, no. 4, pp. 1196–1211, Jan. 2000.

[25] A. Perry, "A modified conjugate gradient algorithm," *Oper. Res.*, vol. 26, no. 6, pp. 1073–1078, 1978.

[26] E. G. Birgin and J. M. Martínez, "A spectral conjugate gradient method for unconstrained optimization," *Appl. Math. Optim.*, vol. 43, no. 2, pp. 117–128, Jan. 2001.

[27] M. Raydan, "The Barzilai and Borwein gradient method for the large scale unconstrained minimization problem," *SIAM J. Optim.*, vol. 7, no. 1, pp. 26–33, Feb. 1997.

[28] X. Wu, "A new spectral Polak-Ribière-Polak conjugate gradient method," *ScienceAsia*, vol. 41, pp. 345–349, Oct. 2015.

[29] U. A. Yakubu, M. Mamat, M. A. Mohamed, and M. Rivaie, "Modification on spectral conjugate gradient method for unconstrained optimization," *Int. J. Eng. Technol., UAE*, vol. 7, no. 3.28, pp. 307–311, 2018.

[30] N. Andrei, "A scaled BFGS preconditioned conjugate gradient algorithm for unconstrained optimization," *Appl. Math. Lett.*, vol. 20, no. 6, pp. 645–650, Jun. 2007.

[31] S. Babaie-Kafaki, "Two modified scaled nonlinear conjugate gradient methods," *J. Comput. Appl. Math.*, vol. 261, pp. 172–182, May 2014.

[32] L. Zhang, W. Zhou, and D.-H. Li, "A descent modified Polak–Ribière–Polyak conjugate gradient method and its global convergence," *IMA J. Numer. Anal.*, vol. 26, no. 4, pp. 629–640, Oct. 2006.

[33] J. K. Liu, Y. M. Feng, and L. M. Zou, "A spectral conjugate gradient method for solving large-scale unconstrained optimization," *Comput. Math. with Appl.*, vol. 77, no. 3, pp. 731–739, Feb. 2019.

[34] P. Faramarzi and K. Amini, "A modified spectral conjugate gradient method with global convergence," *J. Optim. Theory Appl.*, vol. 182, no. 2, pp. 667–690, Aug. 2019.

[35] J. Jian, Q. Chen, X. Jiang, Y. Zeng, and J. Yin, "A new spectral conjugate gradient method for large-scale unconstrained optimization," *Optim. Methods Softw.*, vol. 32, no. 3, pp. 503–515, May 2017.

[36] J. Jian, L. Yang, X. Jiang, P. Liu, and M. Liu, "A spectral conjugate gradient method with descent property," *Mathematics*, vol. 8, no. 2, p. 280, Feb. 2020.

[37] M. Hamoda, M. Mamat, M. Rivaie, and Z. Salleh, "A conjugate gradient method with strong Wolfe–Powell line search for unconstrained optimization," *Appl. Math. Sci.*, vol. 10, pp. 721–734, 2016.

[38] S.-Y. Liu and Y.-Y. Huang, "Several guaranteed descent conjugate gradient methods for unconstrained optimization," *J. Appl. Math.*, vol. 2014, pp. 1–14, Jan. 2014.

[39] M. K. Riahi and I. A. Qattan, "Linearly convergent nonlinear conjugate gradient methods for a parameter identification problems," 2018, *arXiv:1806.10197*. [Online]. Available: http://arxiv.org/abs/1806.10197

[40] L. Zhang, W. Zhou, and D. Li, "Global convergence of a modified Fletcher–Reeves conjugate gradient method with Armijo-type line search," *Numerische Math.*, vol. 104, no. 4, pp. 561–572, Sep. 2006.

[41] N. Andrei, *Nonlinear Conjugate Gradient Methods for Unconstrained Optimization*. Cham, Switzerland: Springer, 2020.

[42] M. Jamil and X.-S. Yang, "A literature survey of benchmark functions for global optimisation problems," *Int. J. Math. Modeling Numer. Optim.*, vol. 4, no. 2, pp. 150–194, 2013.

[43] E. D. Dolan and J. J. Moré, "Benchmarking optimization software with performance profiles," *Math. Program.*, vol. 91, no. 2, pp. 201–213, Jan. 2002.
[44] H. Markowitz, "Portfolio selection," *J. Finance*, vol. 7, no. 1, pp. 77–91, 1952.
[45] S. Roman, *Introduction to the Mathematics of Finance: From Risk Management to Options Pricing*. Berlin, Germany: Springer, 2004.
[46] Y. Zhang, L. He, C. Hu, J. Guo, J. Li, and Y. Shi, "General four-step discrete-time zeroing and derivative dynamics applied to time-varying nonlinear optimization," *J. Comput. Appl. Math.*, vol. 347, pp. 314–329, Feb. 2019.
[47] S. Aji, P. Kumam, A. M. Awwal, M. M. Yahaya, and W. Kumam, "Two hybrid spectral methods with inertial effect for solving system of nonlinear monotone equations with application in robotics," *IEEE Access*, vol. 9, pp. 30918–30928, 2021.

ALIYU MUHAMMED AWWAL received the B.Sc. degree from Gombe State University, the M.Sc. degree from Bayero University Kano, and the Ph.D. degree in applied mathematics from the King Mongkut's University of Technology Thonburi (KMUTT). He has authored or coauthored a number of research articles in high impact journals. His research interest includes iterative algorithms for solving nonlinear problems, such as numerical optimization problems, nonlinear least squares problems and system of nonlinear equations with applications in signal recovery, image deblurring, and motion control.

IBRAHIM MOHAMMED SULAIMAN received the Ph.D. degree in fuzzy systems from the Universiti Sultan Zainal Abidin (UniSZA), Malaysia, in 2018. Since 2019, he has been a Postdoctoral Researcher with the Faculty of Informatics and Computing, UniSZA. He has published research articles in various international journals and attended international conferences. His research interests include numerical research, fuzzy nonlinear systems, and unconstrained optimization.

MAULANA MALIK received the B.Sc. and M.Sc. degrees in mathematics from the Universitas Indonesia (UI), Indonesia. He is currently pursuing the Ph.D. degree with the Universiti Sultan Zainal Abidin (UniSZA), Kuala Terengganu, Malaysia. Since 2016, he has been a Lecturer with the Department of Mathematics, Faculty of Mathematics and Natural Sciences, UI. His current research focuses on optimization includes the conjugate gradient (CG) method and its application in financial mathematics.

MUSTAFA MAMAT received the Ph.D. degree in optimization from the UMT, in 2007. He was appointed as a Senior Lecturer, in 2008, and an Associate Professor with UMT, in 2010. Since 2013, he has been a Professor with the Faculty of Informatics and Computing, Universiti Sultan Zainal Abidin. He has published more than 411 research articles in various international journals and conferences. His research interest includes applied mathematics, with a field of concentration of optimization include conjugate gradient, steepest descent methods, Broyden's family, and quasi Newton methods.

POOM KUMAM (Member, IEEE) received the Ph.D. degree in mathematics from Naresuan University, Thailand. He is currently a Full Professor with the Department of Mathematics, King Mongkut's University of Technology Thonburi (KMUTT). He is also the Head of the Fixed Point Theory and Applications Research Group, KMUTT, and also with the Theoretical and Computational Science Center (TaCS-Center), KMUTT. He is also the Director of the Computational and Applied Science for Smart Innovation Cluster (CLASSIC Research Cluster), KMUTT. He has provided and developed many mathematical tools in his fields productively over the past years. He has over 800 scientific articles and projects either presented or published. He is editorial board journals more than 50 journals and also he delivers many invited talks on different international conferences every year all around the world. His research interests include fixed point theory, variational analysis, random operator theory, optimization theory, approximation theory, fractional differential equations, differential game, entropy and quantum operators, fuzzy soft set, mathematical modeling for fluid dynamics, inverse problems, dynamic games in economics, traffic network equilibria, bandwidth allocation problem, wireless sensor networks, image restoration, signal and image processing, game theory, and cryptology.

KANOKWAN SITTHITHAKERNGKIET received the B.Sc., M.Sc., and Ph.D. degrees. She is currently a Lecturer with the Department of Mathematics, King Mongkut's University of Technology North Bangkok (KMUTNB). Her research interests include fuzzy optimization, fuzzy regression, fuzzy nonlinear mappings, least squares method, optimization problems, and image processing.