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Dynamic Preservation for a Class of Semi-Discrete Recurrent Neural Networks

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ABSTRACT The dynamic preservation in discrete simulations of the recurrent neural networks (RNNs) with discrete and infinite distributive asynchronous time delays is addressed. Firstly, we formulate the corresponding discrete-time model by semi-discrete technique. Secondly, we derive several mild algebra conditions to guarantee that the discussed discrete-time system is the global exponential stability in Lyapunov sense. It is shown analytically that the discrete-time technique is able to maintain the uniqueness of equilibrium point and its dynamic behavior of the continuous-time model under the same conditions. Meanwhile, we also show that there exists some degree of deviation in the spatial position among the equilibrium points of the continuous-time model and its discrete-time analogue due to the rounding error. Finally, we verify the validity of the main obtained results by comparing one continuous-time numerical example with its discrete-time counterpart.

INDEX TERMS Discrete-time network, dynamic preservation, mixed delays, semi-discrete technique.

I. INTRODUCTION

As we all know, RNN is a fundamental dynamic system, which is widely applied in the fields of optimal control, pattern recognition, prediction, and associative memory. Many studies have investigated its applications [1]–[4] and dynamics [5]–[16].

For a neural network, it is essential to introduce a time delay in its model due to the limitations of switching amplifier and propagation speed. Furthermore, one kind of delay is often not enough to describe the transmission process's signal propagation among neurons. For instance, a driver, his hands, feet, and eyes all exist delays during operation. Thus, it is entirely appropriate to take into account the mixed delays, such as the literature with mixed time delays [5]–[7]. Besides, the reaction delays in a moving vehicular system generally vary along with the time and the drivers. It follows that the asynchrony of time delays also needs to be considered in the model to make it more general [10]–[12], [14], [15]. Zhou *et al.* investigated the stability and periodicity for cellular neural networks with mixed asynchronous delays [10]. The paper [11] studied the dynamical behaviors of complex-valued RNNs with variable and infinite distributive asynchronous delays.

In the study of models of complex ordinary differential equations, we are familiar with approximating them by difference equations and expect their solutions to be similar to those of the corresponding of differential equations. But as pointed out in [16]–[18], the dynamic characteristics of the discretized equations are generally not identical with those of the original differential ones. Hence it is vital to find a suitable discrete method to preserve the dynamical behaviors of the continuous-time networks. Many scholars have studied the discrete methods of differential equations, which are generally divided into full-discretization [19]–[21] and semi-discretization [22]–[24]. In contrast to the full discretization in which all actual time-domain states are discretized, the semi-discretization only discretizes the delay states and the periodic coefficients. In [25], Ludovic *et al.* pointed out that the first semi-discrete method may have been proposed by Beverton Holt in [26]. Recently, Mohamad *et al.* proposed a semi-discrete technique in [27], which was shown in [28] with more advantages than the Euler method to retain the dynamics of the continuous system. Afterward, many scholars investigated the dynamic preservation in discrete simulations of continuous-time networks by semi-discrete technique [29]–[41]. For instance, Sun and Feng [32] showed

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that the discrete analogues keep the periodicity of a class of integro-differential equations. The paper [38] illustrated the multi-almost periodicity of a semi-discrete neural network. The stability preservation of a semi-discrete Cohen-Grossberg impulsive neural network was presented [40].

According to the previous studies, there is no literature on the stability preservation of the semi-discrete RNNs with discrete and infinite distributive asynchronous time delays. In this paper, we illustrate that the discrete model completely remain the dynamics of the corresponding continuous one. Meanwhile, we also show some degree of deviation in the spatial position among the equilibrium points of continuous-time model and its discrete-time analogue due to the rounding error. The main contributions of this paper is that the semi-discrete analogues are effective to describe the discrete networks and simulate the continuous networks. Therefore, the obtained results are significant for the dynamical study of the discrete and continuous networks.

Other parts are arranged as follows. In Section 2, the semi-discrete recurrent neural networks with some reasonable assumptions are listed. Some simple sufficient conditions for stability of the discussed networks are given in Section 3. In Section 4, some corollaries and comparisons are proposed. Three numerical examples and their simulations are illustrated in Section 5. Finally conclusions will be drawn.

Notations: Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all nature numbers and the set of all real numbers, respectively. In addition, The superscript T denotes matrix transposition. $[\cdot]$ is a ceiling function on $\mathbb{R}$, and $[x]$ denotes a maximum integer not exceed real number x. The other notations are standard.

II. PRELIMINARIES

The following neural networks are considered:

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -a_ix_i(t) + \sum_{j=1}^{n} b_{ij}f_j(x_j(t))\\ &\quad + \sum_{j=1}^{n} c_{ij}f_j(x_j(t-\tau_{ij}))\\ &\quad + \sum_{j=1}^{n} d_{ij}\int_0^{\infty} K_{ij}(s)f_j(x_j(t-s))ds + u_i, \qquad (1)\end{aligned}$$

where $i = 1, 2, \cdots, n$; $t \geq 0$; $x_i(t)$ denotes the state variable; a_i stands for the positive behaved number; $f_j(\cdot)$ represents the differentiable nonlinear activation function; b_{ij}, c_{ij} and d_{ij} stands for the connection weights; τ_{ij} stands for the asynchronous delay; $K_{ij}(\cdot)$ corresponds to the delay kernel of the distributive asynchronous delay; u_i represents the external input. Let $x_i(s) = \varphi_i(s), s \in (-\infty, 0]$ be a bounded and continuous initial condition, $i = 1, 2, \cdots, n$. In addition, we make the following hypotheses:

(1) $f_j(\cdot)$ satisfies that

$$|f_j(u) - f_j(v)| \leq l_j|u - v|, \quad l_j \geq 0, \text{ for all } u,\ v \in \mathbb{R}, \qquad (2)$$

$$|f_j(\cdot)| \leq M_j, \quad M_j \geq 0, \qquad (3)$$

for $j = 1, 2, \cdots, n$.

(2) $K_{ij}(\cdot)$ satisfies that

$$\begin{cases} K_{ij}(\cdot) \text{ is nonnegative continuous bounded on } [0, +\infty);\\ \int_0^{\infty} K_{ij}(s)ds = 1;\\ \int_0^{\infty} K_{ij}(s)e^{\mu s}ds < \infty \quad \text{for } \mu > 0,\end{cases} \qquad (4)$$

for $i, j = 1, 2, \cdots, n$.

Next, we use the semi-discrete method to formulate a discrete difference equation of (1). Let h be a positive uniform step-size of discretization and $[\cdot]$ be the ceiling function. Then the network (1) will be reformulated:

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -a_ix_i(t) + \sum_{j=1}^{n} b_{ij}f_j(x_j([\frac{t}{h}]h))\\ &\quad + \sum_{j=1}^{n} c_{ij}f_j(x_j([\frac{t}{h}]h - [\frac{\tau_{ij}}{h}]h))\\ &\quad + \sum_{j=1}^{n} d_{ij}\left[\sum_{[\frac{s}{h}]=1}^{\infty} \mathcal{K}_{ij}([\frac{s}{h}]h)f_j(x_j([\frac{t}{h}]h - [\frac{s}{h}]h))\right] + u_i,\end{aligned} \qquad (5)$$

where $\mathcal{K}_{ij}([\frac{s}{h}]h) = \omega_{ij}(h)K_{ij}([\frac{s}{h}]h)$, and $\omega_{ij}(h)$ denotes the positive weight such that $\mathcal{K}_{ij}([\frac{s}{h}]h)$ satisfies that

$$\begin{cases} \mathcal{K}_{ij}(\cdot) \text{ is a nonnegative bounded on } \mathbb{N};\\ \sum_{p=1}^{\infty} \mathcal{K}_{ij}(p) = 1;\\ \text{there exists a positive number} \quad \omega > 1\\ \text{so that } \sum_{p=1}^{\infty} \mathcal{K}_{ij}(p)\omega^p < \infty,\end{cases} \qquad (6)$$

for $i, j = 1, 2, \cdots, n$.

For convenience, let $m = [\frac{t}{h}]$, $p = [\frac{s}{h}]$ and define that

$$x_i(mh) \triangleq x_i(m), \qquad [\frac{\tau_{ij}}{h}]h \triangleq \nu_{ij}. \qquad (7)$$

Rewrite (5) as

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -a_ix_i(t) + \sum_{j=1}^{n} b_{ij}f_j(x_j(m))\\ &\quad + \sum_{j=1}^{n} c_{ij}f_j(x_j(m-\nu_{ij})) + u_i\\ &\quad + \sum_{j=1}^{n} d_{ij}\left[\sum_{p=1}^{\infty} \mathcal{K}_{ij}(p)f_j(x_j(m-p))\right].\end{aligned} \qquad (8)$$

Multiply (8) with e^{a_it}, and then integrate it over $[mh, t)$, $t < (m+1)h$. After taking the limit $t \to (m+1)h$ on the result, one can get the discrete-time analogue of network (1):

$$\begin{aligned}x_i(m+1) &= e^{-a_ih}x_i(m) + \psi_i(h)\left[\sum_{j=1}^{n} b_{ij}f_j(x_j(m))\right.\\ &\quad + \sum_{j=1}^{n} c_{ij}f_j(x_j(m-\nu_{ij})) + u_i\end{aligned}$$

$$+\sum_{j=1}^{n} d_{ij}\left[\sum_{p=1}^{\infty}\mathcal{K}_{ij}(p)f_j(x_j(m-p))\right]\Bigg], \tag{9}$$

where $\psi_i(h) = \frac{1-e^{-a_i h}}{a_i}$, and $\psi_i(h) \approx h + o(h^2)$ when h is enough small.

Because of the initial conditions of (1), the initial conditions of (9) can be written as:

$$x_i(s) = \varphi_i(s), \quad s \in \Gamma = \{\cdots, -2, -1, 0\},\ i = 1, \cdots, n. \tag{10}$$

Remark 1: Model (9) is a class of discrete RNNs with mixed delays. So it is more general than the models in [27], [29], [32], which only consider one delay.

Remark 2: It is different from those models in [36], [39] with the discrete time-variable delays. This paper considers the delays like [29], which will bring some convenience for illustrating the dynamics preservation in discrete simulations by Lyapunov function.

The following lemma and definition are extracted from the reference [27] and are required later.

Lemma 1 [27]: Let ω be a constant bigger than one so that $\sum_{p=1}^{\infty}\mathcal{K}_{ij}(p)\omega^p < \infty$. Then for any number $\lambda \in [0, \omega)$, one has $\sum_{p=1}^{\infty}\lambda^p p\mathcal{K}_{ij}(p) < \infty$. Especially, $\sum_{p=1}^{\infty}\lambda^p\mathcal{K}_{ij}(p) < \infty$.

Assume that $x^\star = (x_1^\star, x_2^\star, \ldots, x_n^\star)^T$ is an equilibrium point of the discrete-time analogue (9). Then by (6) and (9), one can obtain that

$$\begin{cases} a_i x_i^\star - \sum_{j=1}^{n} b_{ij}f_j(x_j^\star) - \sum_{j=1}^{n} c_{ij}f_j(x_j^\star) \\ -\sum_{j=1}^{n} d_{ij}f_j(x_j^\star) - u_i = 0, \quad i = 1, 2, 3, \cdots. \end{cases} \tag{11}$$

Definition 1 [27]: Let $0 < M < 1$ and $\gamma \geq 1$ be two constants. Then we say $x^\star$ is globally exponentially stable, if all solutions of (9) with the initial conditions (10) meet

$$\begin{cases} \sum_{i=1}^{n}\dfrac{|x_i(m) - x_i^\star|}{\psi_i(h)} \leq M^m\gamma\sum_{i=1}^{n}\sup_{s\in\Gamma}\dfrac{|\varphi_i(s) - x_i^\star|}{\psi_i(h)}, \\ m = 1, 2, 3, \cdots. \end{cases} \tag{12}$$

III. MAIN RESULTS

Theorem 1: Assume that (2), (3) and (6) hold. Then equation (9) has a unique equilibrium point if

$$a_i > l_i\sum_{j=1}^{n}(|b_{ji}| + |c_{ji}| + |d_{ji}|) \tag{13}$$

for all $i = 1, 2, \cdots, n$.

Proof: Let $x(t) = \big(x_1(t), x_2(t), \cdots, x_n(t)\big)^T$ and

$$\Theta = \{x(t)\Big||x_i(t) - \frac{u_i}{a_i}| \leq \frac{1}{a_i}\left[\sum_{j=1}^{n}(|b_{ij}| + |c_{ij}| + |d_{ij}|)M_j\right] \quad i = 1, 2, \cdots, n\}, \tag{14}$$

and so Θ is a closed set.

Let $g(x) = \big(g_1(x), g_2(x), \cdots, g_n(x)\big)^T$, where

$$\begin{cases} g_i(x) = g_i(x_1, \cdots, x_n) \\ = \dfrac{1}{a_i}\Big[\sum_{j=1}^{n} b_{ij}f_j(x_j) \\ \quad + \sum_{j=1}^{n} c_{ij}f_j(x_j) + \sum_{j=1}^{n} d_{ij}f_j(x_j) + u_i\Big], \\ i = 1, 2, 3, \cdots. \end{cases} \tag{15}$$

Assume that $x^\star$ is an equilibrium point of (9). Then by (3) and (11), for $\forall i \in \mathbb{N}$, we have

$$\begin{aligned} \left|g_i(x^\star) - \frac{x_i}{a_i}\right| &\leq \frac{1}{a_i}\left[\sum_{j=1}^{n}|b_{ij}|M_j + \sum_{j=1}^{n}|c_{ij}|M_j + \sum_{j=1}^{n}|d_{ij}|M_j\right] \\ &\leq \frac{1}{a_i}\left[\sum_{j=1}^{n}(|b_{ij}| + |c_{ij}| + |d_{ij}|)M_j\right]. \end{aligned} \tag{16}$$

Therefore, each equilibrium point of model (9) is in Θ.

Obviously, $g(x)$ is continuous from Θ to Θ. Therefore, it follows that we have one point $x^\star = (x_1^\star, x_2^\star, \ldots, x_n^\star)^T \in \Theta$ such that

$$g(x^\star) = \big(g_1(x^\star), \cdots, g_n(x^\star)\big)^T = \big(x_1^\star, \cdots, x_n^\star\big)^T = x^\star,$$

and then $x_i^\star = g_i(x^\star)$ for all $i = 1, 2, \cdots, n$. From (11), $x^\star$ is an equilibrium point of discrete-time system (9).

Let $v^\star = (v_1^\star, v_2^\star, \cdots, v_n^\star)^T$ be also an equilibrium point of (9). Then by (2) and (11),

$$\begin{aligned} a_i|x_i^\star - v_i^\star| &= \left|\sum_{j=1}^{n}(b_{ij} + c_{ij} + d_{ij})(f_j(x_j^\star) - f_j(v_j^\star))\right| \\ &\leq \sum_{j=1}^{n} l_j(|b_{ij}| + |c_{ij}| + |d_{ij}|)|x_j^\star - v_j^\star|, \end{aligned} \tag{17}$$

for all $i = 1, 2, \cdots, n$. Sum up both sides of all inequalities in (17), we can get that

$$\begin{aligned} &\sum_{i=1}^{n} a_i|x_i^\star - v_i^\star| \\ &\leq \sum_{i=1}^{n}\sum_{j=1}^{n} l_j(|b_{ij}| + |c_{ij}| + |d_{ij}|)|x_j^\star - v_j^\star| \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} l_i(|b_{ji}| + |c_{ji}| + |d_{ji}|)|x_i^\star - v_i^\star|, \end{aligned} \tag{18}$$

and then

$$\sum_{i=1}^{n}\left[a_i - l_i\sum_{j=1}^{n}(|b_{ji}| + |c_{ji}| + |d_{ji}|)\right]|x_i^\star - v_i^\star| \leq 0. \tag{19}$$

By (13), one can get $x_i^\star = v_i^\star$ for all $i = 1, 2, \cdots, n$, which imply that the equilibrium point of (9) is unique. ■

Remark 3: By comparing (1) and (9), it is found that there exist some errors between τ_{ij} and ν_{ij} as well as $K_{ij}(p)$

and $\mathcal{K}_{ij}(p)$. So there is some deviation in the spatial position of the state curve among model (1) and model (9), but the uniqueness of their equilibrium points are consistent by Ref. [7], [10], [11], [14]. In other words, equation (9) preserves the equilibrium point of its continuous counterpart (1).

The following will illustrate that the stability is also preserved in the discrete-time system (9).

Theorem 2: Suppose that (2), (3), (6) and (13) hold. Then the discrete model (9) with the initial conditions (10) is the global exponential stability.

Proof: Following from (2) and (9), one gets

$$\begin{aligned}
&|x_i(m+1)-x_i^\star| \\
&\le e^{-a_ih}|x_i(m)-x_i^\star| + \psi_i(h)\sum_{j=1}^n l_j|b_{ij}||x_j(m)-x_j^\star| \\
&\quad + \psi_i(h)\sum_{j=1}^n l_j|c_{ij}||x_j(m-\nu_{ij})-x_j^\star| \\
&\quad + \psi_i(h)\sum_{j=1}^n l_j|d_{ij}|\sum_{p=1}^\infty \mathcal{K}_{ij}(p)|x_j(m-p)-x_j^\star|. \qquad (20)
\end{aligned}$$

Let $z_i(m) = \lambda^m \frac{|x_i(m)-x_i^\star|}{\psi_i(h)}$, $m\in\mathbb{Z}$ for all $i=1,2,\cdots,n$. Then by (20)

$$\begin{aligned}
z_i(m+1) &= \lambda^{m+1}\frac{|x_i(m+1)-x_i^\star|}{\psi_i(h)} \\
&\le \lambda e^{-a_ih}z_i(m) + \sum_{j=1}^n l_j\psi_j(h)\lambda|b_{ij}|z_j(m) \\
&\quad + \sum_{j=1}^n l_j\psi_j(h)\lambda^{\nu_{ij}+1}|c_{ij}|z_j(m-\nu_{ij}) \\
&\quad + \sum_{j=1}^n l_j\psi_j(h)\lambda^{p+1}|d_{ij}|\sum_{p=1}^\infty \mathcal{K}_{ij}(p)z_j(m-p). \qquad (21)
\end{aligned}$$

In the following, we introduce a Lyapunov functional $V(m) = V(z_1,z_2,\ldots,z_n)(m)$ defined by

$$\begin{aligned}
V(m) = \sum_{i=1}^n \Big\{ z_i(m) &+ \sum_{j=1}^n l_j\psi_j(h)\lambda^{\nu_{ij}+1}|c_{ij}|\sum_{l=m-\nu_{ij}}^{m-1} z_j(l) \\
&+ \sum_{j=1}^n l_j\psi_j(h)|d_{ij}|\sum_{p=1}^\infty \lambda^{p+1}\mathcal{K}_{ij}(p)\sum_{k=m-p}^{m-1} z_j(k)\Big\}. \qquad (22)
\end{aligned}$$

Then,

$$\begin{aligned}
\Delta V(m) &= V(m+1)-V(m) \\
&\le \sum_{i=1}^n\Big\{(\lambda e^{-a_ih}-1)z_i(m) + \sum_{j=1}^n \lambda l_j\psi_j(h)|b_{ij}|z_j(m) \\
&\quad + \sum_{j=1}^n l_j\psi_j(h)\lambda^{\nu_{ij}+1}|c_{ij}|z_j(m) \\
&\quad + \sum_{j=1}^n l_j\psi_j(h)|d_{ij}|\sum_{p=1}^\infty \lambda^{p+1}\mathcal{K}_{ij}(p)z_j(m)\Big\} \\
&= -\sum_{i=1}^n\Big\{1-\lambda e^{-a_ih}-\lambda l_i\psi_i(h)\sum_{j=1}^n|b_{ji}| \\
&\quad - l_i\psi_i(h)\sum_{j=1}^n|c_{ji}|\lambda^{\nu_{ji}+1} \\
&\quad - l_i\psi_i(h)\sum_{j=1}^n|d_{ji}|\sum_{p=1}^\infty\lambda^{p+1}\mathcal{K}_{ji}(p)\Big\}z_i(m). \qquad (23)
\end{aligned}$$

From (6), we know that $\omega > 1$ is a constant and satisfies $\sum_{p=1}^\infty \mathcal{K}_{ij}(p)\omega^p < \infty$. And by Lemma 1, a continuous auxiliary function over an interval $[0,\omega)$ is considered and defined by

$$\begin{aligned}
G_i(\lambda_i) = 1-\lambda_ie^{-a_ih}-\lambda_il_i\psi_i(h)\sum_{j=1}^n|b_{ji}| &- l_i\psi_i(h)\sum_{j=1}^n|c_{ji}|\lambda_i^{\nu_{ji}+1} \\
&- l_i\psi_i(h)\sum_{j=1}^n|d_{ji}|\sum_{p=1}^\infty\lambda_i^{p+1}\mathcal{K}_{ji}(p), \qquad (24)
\end{aligned}$$

for all $i=1,2,\cdots,n$.

By (13), we can get

$$\begin{aligned}
G_i(1) &= 1-e^{-a_ih}-l_i\psi_i(h)\sum_{j=1}^n|b_{ji}| \\
&\quad - l_i\psi_i(h)\sum_{j=1}^n|c_{ji}| - l_i\psi_i(h)\sum_{j=1}^n|d_{ji}|\Big(\sum_{p=1}^\infty\mathcal{K}_{ji}(p)\Big) \\
&= \psi_i(h)\Big\{a_i - l_i\sum_{j=1}^n|b_{ji}| - l_i\sum_{j=1}^n|c_{ji}| - l_i\sum_{j=1}^n|d_{ji}|\Big\} \\
&> 0. \qquad (25)
\end{aligned}$$

By the continuity of G_i, there is a number $\lambda_i^\star$, $1<\lambda_i^\star<\omega$ so that $G_i(\lambda_i^\star)>0$ for $i=1,2,\cdots,n$. Given that $\lambda=\max\{\lambda_1^\star,\lambda_2^\star,\cdots,\lambda_n^\star\}>1$, we have

$$\begin{aligned}
G_i(\lambda) = 1-\lambda e^{-a_ih}-\lambda l_i\psi_i(h)\sum_{j=1}^n|b_{ji}| &- l_i\psi_i(h)\sum_{j=1}^n|c_{ji}|\lambda^{\nu_j+1} \\
&- l_i\psi_i(h)\sum_{j=1}^n|d_{ji}|\sum_{p=1}^\infty\lambda^{p+1}\mathcal{K}_{ji}(p) > 0 \qquad (26)
\end{aligned}$$

for all $i = 1, 2, \cdots, n$. From (22), (23) and (26), one has that $\Delta V(m) < 0$ for $m \in \mathbb{N}$, and further,

$$\sum_{i=1}^{n} z_i(m) \le V(m) \le V(0), \quad m = 1, 2, \cdots. \tag{27}$$

By Lemma 1, one has $\sum_{p=1}^{\infty} \lambda^{p+1} p\mathcal{K}_{ji}(p) < \infty$. Let

$$\gamma = \max_{1\le i\le n} \left\{ 1 + l_i\psi_i(h) \sum_{j=1}^{n} |c_{ji}| \lambda^{\nu_{ji}+1}\nu_{ji} + l_i\psi_i(h) \sum_{j=1}^{n} |d_{ji}| \sum_{p=1}^{\infty} \lambda^{p+1} p\mathcal{K}_{ji}(p) \right\}.$$

Then,

$$\begin{aligned} V(0) &= \sum_{i=1}^{n} \left\{ z_i(0) + \sum_{j=1}^{n} l_j\psi_j(h)\lambda^{\nu_{ij}+1}|c_{ij}| \sum_{l=-\nu_{ij}}^{-1} z_j(l) + \sum_{j=1}^{n} l_j\psi_j(h)|d_{ij}| \sum_{p=1}^{\infty} \lambda^{p+1}\mathcal{K}_{ij}(p) \sum_{k=p}^{-1} z_j(k) \right\} \\ &= \sum_{i=1}^{n} \left\{ z_i(0) + l_i\psi_i(h) \sum_{j=1}^{n} |c_{ji}|\lambda^{\nu_{ji}+1} \sum_{l=-\nu_{ji}}^{0} z_i(l) + l_i\psi_i(h) \sum_{j=1}^{n} |d_{ji}| \sum_{p=1}^{\infty} \lambda^{p+1}\mathcal{K}_{ji}(p) \sum_{k=p}^{-1} z_j(k) \right\} \\ &\le \sum_{i=1}^{n} \left\{ 1 + l_i\psi_i(h) \sum_{j=1}^{n} |c_{ji}|\lambda^{\nu_{ji}+1}\nu_{ji} + l_i\psi_i(h) \times \sum_{j=1}^{n} |d_{ji}| \sum_{p=1}^{\infty} \lambda^{p+1} p\mathcal{K}_{ji}(p) \right\} \sup_{s\in\Gamma} \frac{|\varphi_i(s) - x_i^\star|}{\psi_i(h)} \\ &\le \gamma \sum_{i=1}^{n} \sup_{s\in\Gamma} \frac{|\varphi_i(s) - x_i^\star|}{\psi_i(h)}. \end{aligned} \tag{28}$$

Combining (27) and (28), we have that

$$\sum_{i=1}^{n} \frac{|x_i(m) - x_i^\star|}{\psi_i(h)} \le \frac{1}{\lambda^m}\gamma \sum_{i=1}^{n} \sup_{s\in\Gamma} \frac{|\varphi_i(s) - x_i^\star|}{\psi_i(h)},$$

where $m = 1, 2, 3, \cdots$. On account of Definition 1, the discrete model (9) with the initial conditions (10) is the global exponential stability. ■

Remark 4: In [10], the authors have given some sufficient conditions to assure the global exponential stability of model (1). Under those conditions, Theorem 2 in this paper show that equation (9) remains the dynamical characteristics of its original equation (1).

IV. DISCUSSIONS

From the main results, one can get the following corollaries, some of which had been proved in the known literature.

If $b_{ij} = d_{ij} = 0$ for $i, j = 1, 2, \ldots n$, (9) turns into:

$$x_i(m+1) = e^{-a_i h} x_i(m) + \psi_i(h) \left[\sum_{j=1}^{n} c_{ij} f_j(x_j(m - \nu_{ij})) + u_i \right], \tag{29}$$

and the corresponding initial conditions can be assumed that

$$x_i(s) = \varphi_i(s), \quad s \in \{-\nu, \cdots, -1, 0\},\ i = 1, 2, \cdots, n, \tag{30}$$

where $\nu = \max_{1\le i,j\le n}\{\nu_{ij}\}$ and $\psi_i(h) = \frac{1-e^{-a_i h}}{a_i} \approx h + o(h^2)$ when h is enough small.

Corollary 1: Assume that (2) and (3) hold. If

$$a_i > l_i \sum_{j=1}^{n} |c_{ji}|, \quad i = 1, 2, \cdots, n, \tag{31}$$

then the discrete network (29) with the initial conditions (30) is globally exponentially stable.

If $b_{ij} = c_{ij} = 0$ for $i, j = 1, 2, \ldots n$, (9) becomes into

$$x_i(m+1) = e^{-a_i h} x_i(m) + \psi_i(h) \left[\sum_{j=1}^{n} d_{ij} \left[\sum_{p=1}^{\infty} \mathcal{K}_{ij}(p) f_j(x_j(m-p)) \right] + u_i \right], \tag{32}$$

and (10) is the corresponding initial condition.

Corollary 2: Assume that (2), (3) and (6) hold. If

$$a_i > l_i \sum_{j=1}^{n} |d_{ji}|, \quad i = 1, 2, \cdots, n, \tag{33}$$

then the discrete network (32) with the initial conditions (10) is globally exponentially stable.

Remark 5: Corollary 1 and Corollary 2 are the same as Theorem 4.2 and Theorem 4.3 in [27], respectively. Therefore, the obtained results are more general than that in [27].

When $d_{ij} = 0$ for $i, j = 1, 2, \ldots n$, (9) becomes into

$$x_i(m+1) = e^{-a_i h} x_i(m) + \psi_i(h) \left[\sum_{j=1}^{n} b_{ij} f_j(x_j(m)) + \sum_{j=1}^{n} c_{ij} f_j(x_j(m - \nu_{ij})) + u_i \right], \tag{34}$$

and (30)is the corresponding initial condition.

Corollary 3: Assume that (2) and (3) hold. If

$$a_i > l_i \sum_{j=1}^{n} (|b_{ji}| + |c_{ji}|), \quad i = 1, 2, \cdots, n, \tag{35}$$

then the discrete network (34) with the initial conditions (30) is globally exponentially stable.

Remark 6: Compared with the results in [29], we see that Corollary 3 is just Theorem 3.1 of [29]. Especially, if $c_{ij} = 0$ in model (34) for $i, j = 1, 2, \ldots n$, Corollary 3 is Theorem 4.1 of [27].

When $c_{ij} = 0$ for $i, j = 1, 2, \ldots n$, model (9) changes into

$$x_i(m+1) = e^{-a_i h} x_i(m) + \psi_i(h) \left[\sum_{j=1}^{n} b_{ij} f_j(x_j(m)) + \sum_{j=1}^{n} d_{ij} \left[\sum_{p=1}^{\infty} \mathcal{K}_{ij}(p) f_j(x_j(m-p)) \right] + u_i \right], \quad (36)$$

and (10) is the corresponding initial condition.

Corollary 4: Assume that (2), (3) and (6) hold. If

$$a_i > l_i \sum_{j=1}^{n} (|b_{ji}| + |d_{ji}|), \quad i = 1, 2, \cdots, n, \quad (37)$$

then the discrete network (36) with the initial conditions (10) is globally exponentially stable.

V. EXAMPLES

In this section, three numerical examples are provided to verify the validity of the obtained results.

Example 1: Consider the following continuous-time RNNs with mixed delays.

$$\begin{cases} \frac{dx_1(t)}{dt} = -6x_1(t) + 0.5f_1(x_1(t)) + 0.4f_2(x_2(t)) \\ +0.4f_1(x_1(t-\tau_{11})) + 0.6f_2(x_2(t-\tau_{12})) \\ +0.4\int_0^{\infty} K_{11}(s) f_1(x_1(t-s))ds \\ +0.15\int_0^{\infty} K_{12}(s) f_2(x_2(t-s))ds + 3, \\ \frac{dx_2(t)}{dt} = -7x_2(t) + 0.2f_1(x_1(t)) + 0.5f_2(x_2(t)) \\ -0.3f_1(x_1(t-\tau_{21})) + 0.2f_2(x_2(t-\tau_{22})) \\ -0.2\int_0^{\infty} K_{21}(s) f_1(x_1(t-s))ds \\ +0.7\int_0^{\infty} K_{22}(s) f_2(x_2(t-s))ds + 2, \end{cases} \quad (38)$$

where $f_i(x_i(t)) = \tanh(x_i(t))$, $\tau_{11} = \tau_{21} = 2.1$, $\tau_{12} = \tau_{22} = 3.2$, $K_{11}(t) = K_{21}(t) = (2e^{-2t})/(1-e^{-2t})$, $K_{12}(t) = K_{22}(t) = (3e^{-3t})/(1-e^{-3t})$.

After simple verification, we know that model (38) satisfies the assumed conditions. The state trajectories of x_1, x_2 of the continuous-time model (38) by three different initial conditions: (1) $x_1(s) = 0.1, x_2(s) = 0.9$; (2) $x_1(s) = -1.1, x_2(s) = -0.2$; (3) $x_1(s) = -1.5 + e^s, x_2(s) = 0.1sin(s) - 1, s \in [-3.2, 0]$, are illustrated in Fig. 1, which show that all the dynamical behaviors are converging toward the same equilibrium point.

Example 2: Consider the following discrete-time RNNs with mixed delays, which are the discretization for (38) by semi-discrete method.

$$\begin{cases} x_1(m+1) = e^{-6h}x_1(m) + \psi_1(h)[0.5f_1(x_1(m)) \\ +0.4f_2(x_2(m)) + 0.4f_1(x_1(t-\nu_{11}(t))) \\ +0.6f_2(x_2(t-\nu_{12}(t))) \\ +0.4\left[\sum_{p=1}^{\infty} \mathcal{K}_{11}(p) f_1(x_1(m-p))\right] \\ +0.15\left[\sum_{p=1}^{\infty} \mathcal{K}_{12}(p) f_2(x_2(m-p))\right] + 3, \\ x_2(m+1) = e^{-7h}x_2(m) + \psi_2(h)[0.2f_1(x_1(m)) \\ +0.5f_2(x_2(m)) - 0.3f_1(x_1(t-\nu_{11}(t))) \\ +0.2f_2(x_2(t-\nu_{12}(t))) \\ -0.2\left[\sum_{p=1}^{\infty} \mathcal{K}_{21}(p) f_1(x_1(m-p))\right] \\ +0.7\left[\sum_{p=1}^{\infty} \mathcal{K}_{22}(p) f_2(x_2(m-p))\right] + 2, \end{cases} \quad (39)$$

where $\psi_1(h) = \frac{1-e^{-6h}}{6}$, $\psi_2(h) = \frac{1-e^{-7h}}{7}$, $[\frac{t}{h}] = m$, $x_i(mh) = x_i(m)$, $[\frac{\tau_{ij}([\frac{t}{h}]h))}{h}]h = [\frac{\tau_{ij}(mh))}{h}]h \triangleq \nu_{ij}(m)$, $[\frac{K_{ij}([\frac{t}{h}]h))}{h}]h = [\frac{K_{ij}(m))}{h}]h \triangleq \mathcal{K}_{ij}(m)$, $\overline{\nu_i} = \max\{\nu_i, \mathcal{K}_i\}$, $\nu_i = \max_{1\le j\le n} \sup_{s\ge 0} \nu_{ij}(m)$, $\mathcal{K}_i = \max_{1\le j\le n} \sup_{s\ge 0} \mathcal{K}_{ij}(m)$.

Under the same conditions, the discrete-time model's state trajectories (39) are shown in Fig. 2 by three different initial conditions. It is illustrated that model (39) also converges to a unique equilibrium point. From Fig. 1 and Fig. 2, we know that model (39) preserves the dynamic characteristics of the corresponding continuous-time model (38). But the point is that the convergence process of model (39) has some deviation from that of the corresponding continuous model (38) because of the rounding error of $K_{ij}(t)$ in the process of discretization, which is shown in Fig. 3.

Example 3: Consider the following semi-discrete RNNs with infinite distributive asynchronous delays.

$$\begin{cases} x_1(m+1) = e^{-6.6h}x_1(m) + \psi_1(h)[0.8f_1(x_1(m)) \\ +0.3f_2(x_2(m)) + 1.2f_3(x_3(m))] \\ +0.4\left[\sum_{p=1}^{\infty} \mathcal{K}_{11}(p) f_1(x_1(m-p))\right] \\ +0.2\left[\sum_{p=1}^{\infty} \mathcal{K}_{12}(p) f_2(x_2(m-p))\right] \\ +0.15\left[\sum_{p=1}^{\infty} \mathcal{K}_{13}(p) f_3(x_3(m-p))\right] + 2, \\ x_2(m+1) = e^{-7.5h}x_2(m) + \psi_2(h)[0.2f_1(x_1(m)) \\ +0.6f_2(x_2(m)) + 0.5f_3(x_3(m))] \\ -0.3\left[\sum_{p=1}^{\infty} \mathcal{K}_{21}(p) f_1(x_1(m-p))\right] \\ +0.85\left[\sum_{p=1}^{\infty} \mathcal{K}_{22}(p) f_2(x_2(m-p))\right] \\ +0.7\left[\sum_{p=1}^{\infty} \mathcal{K}_{23}(p) f_3(x_3(m-p))\right] + 1.5, \\ x_3(m+1) = e^{-6.9h}x_3(m) + \psi_2(h)[0.2f_1(x_1(m)) \\ +0.4f_2(x_2(m)) + 0.5f_3(x_3(m))] \\ -0.1\left[\sum_{p=1}^{\infty} \mathcal{K}_{31}(p) f_1(x_1(m-p))\right] \\ +0.9\left[\sum_{p=1}^{\infty} \mathcal{K}_{32}(p) f_2(x_2(m-p))\right] \\ +0.7\left[\sum_{p=1}^{\infty} \mathcal{K}_{33}(p) f_3(x_3(m-p))\right] + 3, \end{cases} \quad (40)$$

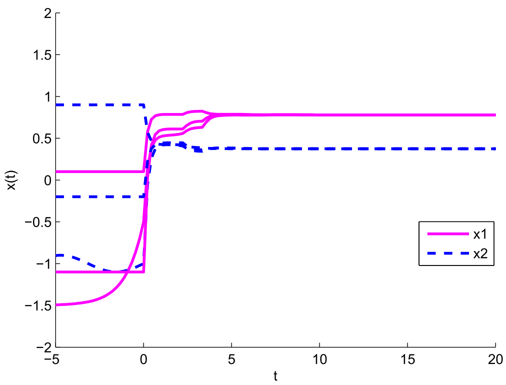

FIGURE 1. The state trajectories of the continuous-time model (38).

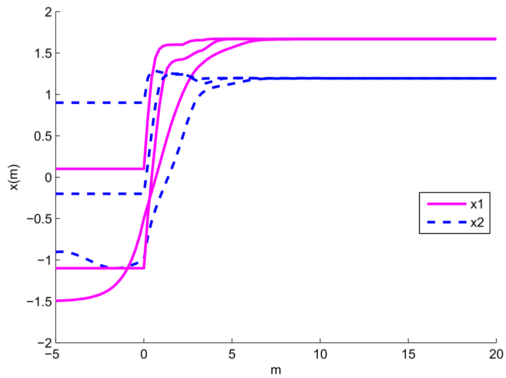

FIGURE 2. The state trajectories of discrete-time model (39).

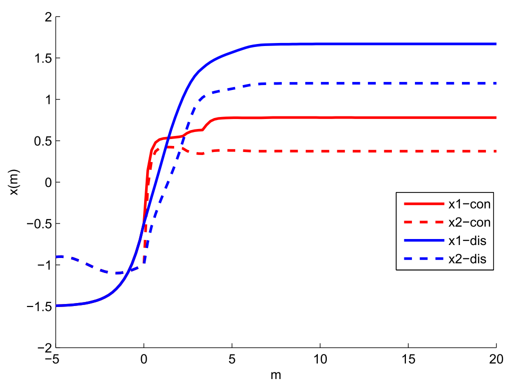

FIGURE 3. The state trajectories of model (39) and its continuous-time counterpart.

where $f_i(x_i(t)) = \sin(x_i(t))$, $\psi_1(h) = \frac{1-e^{-6.6h}}{6.6}$, $\psi_2(h) = \frac{1-e^{-7.5h}}{7.5}$, $\psi_3(h) = \frac{1-e^{-6.9h}}{6.9}$, $[\frac{t}{h}] = m$, $x_i(mh) = x_i(m)$, $[\frac{K_{ij}([\frac{t}{h}]h))}{h}]h = [\frac{K_{ij}(m))}{h}]h \triangleq \mathcal{K}_{ij}(m)$, $K_{11}(t) = K_{21}(t) = K_{31}(t) = (2e^{-2t})/(1-e^{-2t})$, $K_{12}(t) = K_{22}(t) = K_{32}(t) = (2.5e^{-2.5t})/(1-e^{-2.5t})$, $K_{13}(t) = K_{23}(t) = K_{33}(t) = (3e^{-3t})/(1-e^{-3t})$. $\mathcal{K}_i = \max_{1\leq j\leq n} \sup_{s\geq 0} \mathcal{K}_{ij}(m)$.

The state trajectories for the semi-discrete system (40) by three different initial conditions: (1) $x_1(s) = 2.5, x_2(s) = 2, x_3(s) = 3$; (2) $x_1(s) = -1.9, x_2(s) = -0.5, x_3(s) = -0.1$; (3) $x_1(s) = -1 + e^s, x_2(s) = 0.3e^s - 1, x_3(s) = 0.5e^s, s \in [-3, 0]$, are rendered in Fig. 4, which illustrate the effectiveness of the results in Corollary 4.

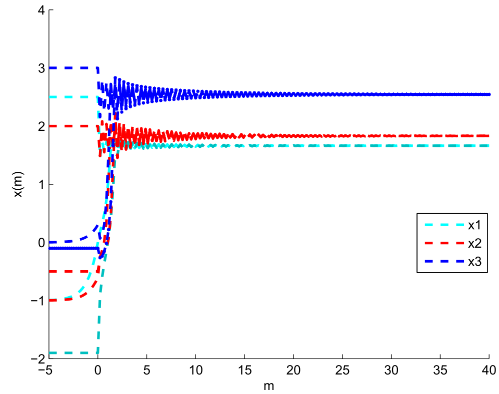

FIGURE 4. The state trajectories of discrete-time model (40).

VI. CONCLUSION

The paper gets a discrete analogue for a class of RNNs with discrete and infinite distributive asynchronous delays by semi-discrete technique, and investigate its preservation of dynamic characteristics. Some mild algebraic conditions insure the uniqueness and global exponential stability of the discussed discrete system's equilibrium point. It is shown analytically that the applied discrete technique is able to remain the dynamic behavior of its continuous counterpart. Meanwhile, we also show some degree of deviation in the spatial position among the equilibrium points of the continuous model and its discrete analogue due to the rounding error. Besides, the method and technique in this paper can be further applied to investigate other discrete neural networks, such as multiple neural networks and impulsive neural networks. The dynamical preserving problems for these discrete neural networks are not still fully studied, so we will focus on this topic in the future.

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