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Analysis of Third-Order Nonlinear Multi-Singular Emden–Fowler Equation by Using the LeNN-WOA-NM Algorithm

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ABSTRACT In this paper, a novel soft computing algorithm is designed for the numerical solution of third-order nonlinear multi-singular Emden–Fowler equation (TONMS-EFE) using the strength of universal approximation capabilities of Legendre polynomials based Legendre neural networks supported with optimization power of the Whale Optimization Algorithm (WOA) and Nelder-Mead (NM) algorithm. Unsupervised error functions are constructed in terms of mean square error for governing TONMS-EF equations of first and second order. Unknown designed parameters in LeNN structure are optimized initially by WOA for global search while NM algorithm further enhances the rapid local search convergence. The proposed algorithm's objective is to show the accuracy and robustness in solving challenging problems like TONMS-EFE. To study our designed scheme's performance and effectiveness, LeNN-WOA-NM is implemented on four cases of TONMS-EFE. The results obtained by the proposed algorithm are compared with the Particle Swarm Optimization (PSO) algorithm, Cuckoo search algorithm (CSA), and WOA. Extensive graphical and statistical analysis for fitness value, absolute errors, and performance indicators in terms of mean, median, and standard deviations show the proposed algorithm's efficiency and accuracy.

INDEX TERMS Singular Emden–Fowler equation, soft computing algorithm, weighted legendre neural networks, Nelder-Mead algorithm, whale optimization algorithm.

I. INTRODUCTION

Singular differential equations models various phenomenon's occurring in daily life. Therefore, they gain an immense importance specially in physics and applied mathematics. Singular non-linear model of famous Lane–Emden equations were introduced by astrophysicists Homer Lane [1] and Robert Emden [2] while working on thermal performance of gas and classical laws of heat and thermodynamics [3]. Singular systems of differential equations originates in field of numerical sciences and physical sciences [4], electromagnetic [5], catalytic diffusion and reactions [6], isothermal gas phenomenons [7], quantum mathematical model [8], classical and quantum mechanics [9], gaseous density [10], oscillating magnetic systems [11], isotropic mediums [12] and fluid mechanical systems [13].

Few techniques in the existing literature are used to solve non-linear singular models like TONMS-EFE. Shawagfeh presents Adomain decomposition method (ADM) [14], in 2001 Wazwaz [15] uses ADM to get over the difficulty of singularity, an analytical scheme for the solution of non-linear singular model was implemented by Liao [16], a numerical technique was established by He and Ji [17] using Taylor series and power series solutions are used by Nouh [18] along with the transformation of Euler-Abel. Kalabas and Bellman quasi-linearization scheme was developed by Mandelzweig, and Tabakin [19]. Variational iteration method (VIM) [20], Finite difference method (FDM) [21] and Optimal homotopy

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TABLE 1. Legendre polynomials.

n	$L_n(t)$
0	1
1	t
2	$\frac{1}{2}\left(3t^2-1\right)$
3	$\frac{1}{2}\left(5t^3-3t\right)$
4	$\frac{1}{8}\left(35t^4-30t^2+3\right)$
5	$\frac{1}{8}\left(63t^5-70t^3+15t\right)$
6	$\frac{1}{16}\left(231t^6-315t^4+105t^2-5\right)$
7	$\frac{1}{16}\left(429t^7-693t^5+315t^3-35t\right)$
8	$\frac{1}{128}\left(6435t^8-12012t^6+6930t^4-1260t^2+35\right)$
9	$\frac{1}{128}\left(12155t^9-25740t^7+18018t^5-4620t^3+315t\right)$
10	$\frac{1}{256}\left(46189t^{10}-109395t^8+90090t^6-30030t^4+3465t^2-63\right)$

TABLE 2. Parameter setting for WOA, NM, PSO and CSA.

Algorithm	Parameters	Settings	Parameters	Settings
WOA	Max. iterations	6,000	Limits	[-1,1]
	Selection of Candidate	Uniform	Search agents	50
NM Algorithm	function evaluations	200,000	Initial weights	Global best of WOA
	X-Tolerance 'TolX'	1.00E-20	Max. iterations	2,000
	Scaling	Objective and constraints	'TolFun'	1.00E-20
CSA	Max. iterations	8,000	Limits (lower, upper)	[-1,1]
	Search agents	50	Selection of Candidate	Uniform
PSO	Max. iterations	8,000	Limits (lower, upper)	[-1,1]
	Search agents	50	Selection of Candidate	Uniform

perturbation method (OHAM) [22], [23] are used to solve variety of ordinary and partial differential equation models. In terms of consistency, convergence, robustness, and applicability, the techniques mentioned above have advantages and limitations over each other. These techniques are based on well-established deterministic techniques. On the other hand, stochastic techniques based on artificial neural networks are less exploited and rapidly convergent.

In recent times, ANNs are used as universal function approximation procedures to develop stochastic numerical techniques. Due to their strength and stability, they are widely used for the solutions of variety of real world problems including multi-phase flow through porous media for imbibition phenomena [24], longitudinal heat transformation fins model [25], [26], Beam-Column designs [27], Optimal Model Selection for Regression [28], fractional models of damping material [26], nonlinear dusty plasma system [29], corneal Model for Eye Surgery [30] and temperature profile of porous fin model [31]. A plant propagation algorithm (PPA) and its modified version were developed to solve design engineering problems [32]–[35]. The above mentioned algorithms motivate authors to develop a soft computing technique based on artificial neural networks. The main features of this research work are summarized as

- This paper aims to establish a soft computing technique known as the LeNN-WOA-NM algorithm to solve non-linear multi-singular Emden-Folwer equations of a first and second type.
- LeNN-WOA-NM algorithm suggests series solutions for TONMS-EFE. Weighted Legendre polynomials are used for the approximation of our solutions. A fitness function is used to assess the unknown weights, and error is minimized by using the Nelder-Mead Algorithm.
- Results obtained by LeNN-WOA-NM algorithms are compared with exact solutions and other evolutionary algorithms, including Particle swarm optimization, Cuckoo search algorithm, and Whale optimization algorithm.

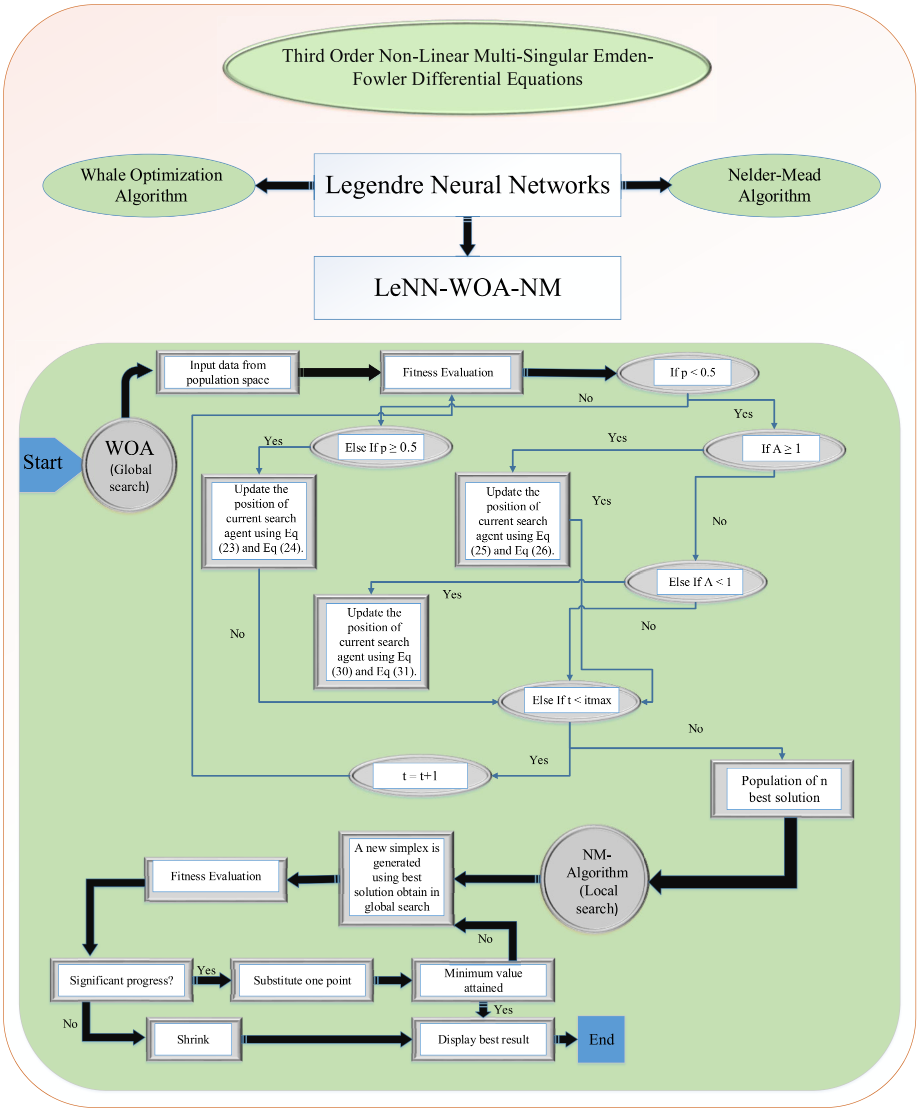


FIGURE 1. Flowchart for WOA-NM Algorithm.

- Mean absolute deviation (MAD), Theil's inequality coefficient(TIC) and Nash Sutcliffe efficiency (NSE), and Normal probability graphs are the performance indications that have been used for performance measurement of the proposed technique in providing the best possible solution for TONMS-EFE.
- The results for TONMS-EFE are shown through different graphs and tables, which show the dominance and robustness of the proposed (LeNN-WOA-NM) algorithm.

II. CONSTRUCTING EMDEN-FOWLER TYPE EQUATIONS OF THIRD-ORDER

To derive Emden-Fowler equation of third order we consider an equation of the form

$$x^{-\beta}\frac{d^m}{d\xi^m}\left(\xi^{\beta}\frac{d^n}{d\xi^n}\right)\phi + f(\xi)g(\phi) = 0, \tag{1}$$

where $f(\xi)$ and $g(\phi)$ are some functions of ξ and ϕ respectively. β is shape factor. Emden-Folwer equation given by Eq (1) represents multiple phenomenons in fluid mechanics, pattern formation, relativistic mechanics, pattern formation, relativistic mechanics and population evolution. To determine third order equations we select

$$m + n = 3, \qquad \text{and} \quad m, n \geq 1 \tag{2}$$

From Eq (2) we have following two choices

$$m = 2, \qquad n = 1, \tag{3}$$

and

$$m = 1, \qquad n = 2, \tag{4}$$

Substituting $m = 2$ and $n = 1$ in Eq (1). We get

$$\xi^{-\beta}\frac{d^2}{d\xi^2}\left(\xi^{\beta}\frac{d}{d\xi}\right)\phi + f(\xi)g(\phi) = 0, \tag{5}$$

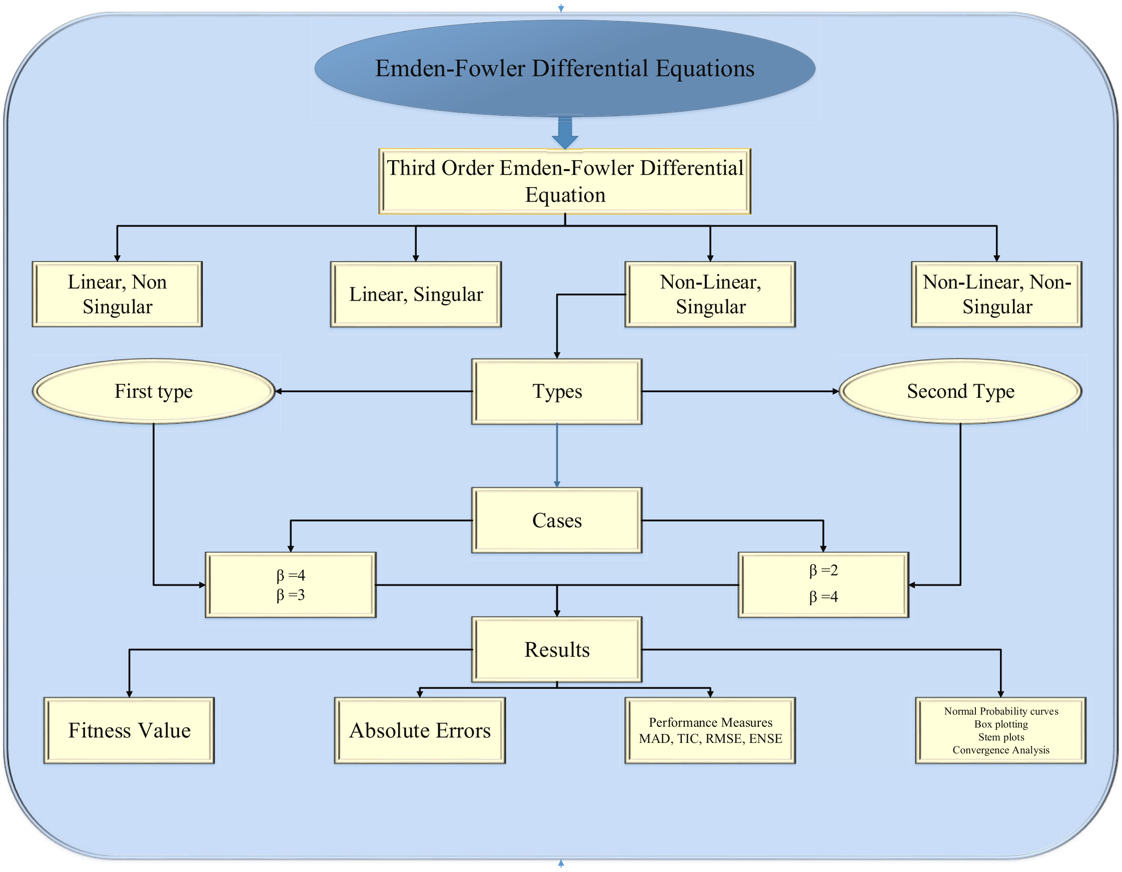


FIGURE 2. **Graphical overview of third order non-linear multi singular Emden-fowler differential equation with different cases depending on shape factor.**

with set of initial conditions given as

$$\phi(0) = A, \phi'(0) = 0 \quad \text{and} \quad \phi''(0) = 0,$$

Eq (5) in turn gives **First Emden-Folwer** type equation of order three as shown by Eq (6) along with initial conditions Eq (7).

$$\frac{d^3\phi}{d\xi^3} + \frac{2\beta}{\xi}\frac{d^2\phi}{d\xi^2} + \frac{\beta(\beta-1)}{\xi^2}\frac{d\phi}{d\xi} + f(\xi)g(\phi) = 0, \quad (6)$$

$$\phi(0) = A, \phi'(0) = 0 \quad \text{and} \quad \phi''(0) = 0, \quad (7)$$

Equivalently, Eq (6) can be written as

$$\phi''' + \frac{2\beta}{\xi}\phi'' + \frac{\beta(\beta-1)}{\xi^2}\phi' + f(\xi)g(\phi) = 0, \quad (8)$$

with

$$\phi(0) = A, \phi'(0) = 0 \quad \text{and} \quad \phi''(0) = 0,$$

It can be noticed that singularity lies at $\xi = 0$ and singular point appears twice as ξ and ξ^2 with shape factor β and $(\beta-1)$ respectively.

Now considering the case when $m = 1$ and $n = 2$. Substituting values of m and n in Eq (1). we have,

$$\xi^{-\beta}\frac{d}{d\xi}\left(\xi^{\beta}\frac{d^2}{d\xi^2}\right)\phi + f(\xi)g(\phi) = 0, \quad (9)$$

Eq (9) in turn gives **Second Emden-Folwer** type equation of order three as shown by Eq (10) along with initial conditions Eq (11).

$$\frac{d^3\phi}{d\xi^3} + \frac{\beta}{\xi}\frac{d^2\phi}{d\xi^2} + f(\xi)g(\phi) = 0, \quad (10)$$

$$\phi(0) = A, \phi'(0) = 0 \quad \text{and} \quad \phi''(0) = 0, \quad (11)$$

Equivalently, Eq (10) can be written as

$$\phi''' + \frac{\beta}{\xi}\phi'' + f(\xi)g(\phi) = 0, \quad (12)$$

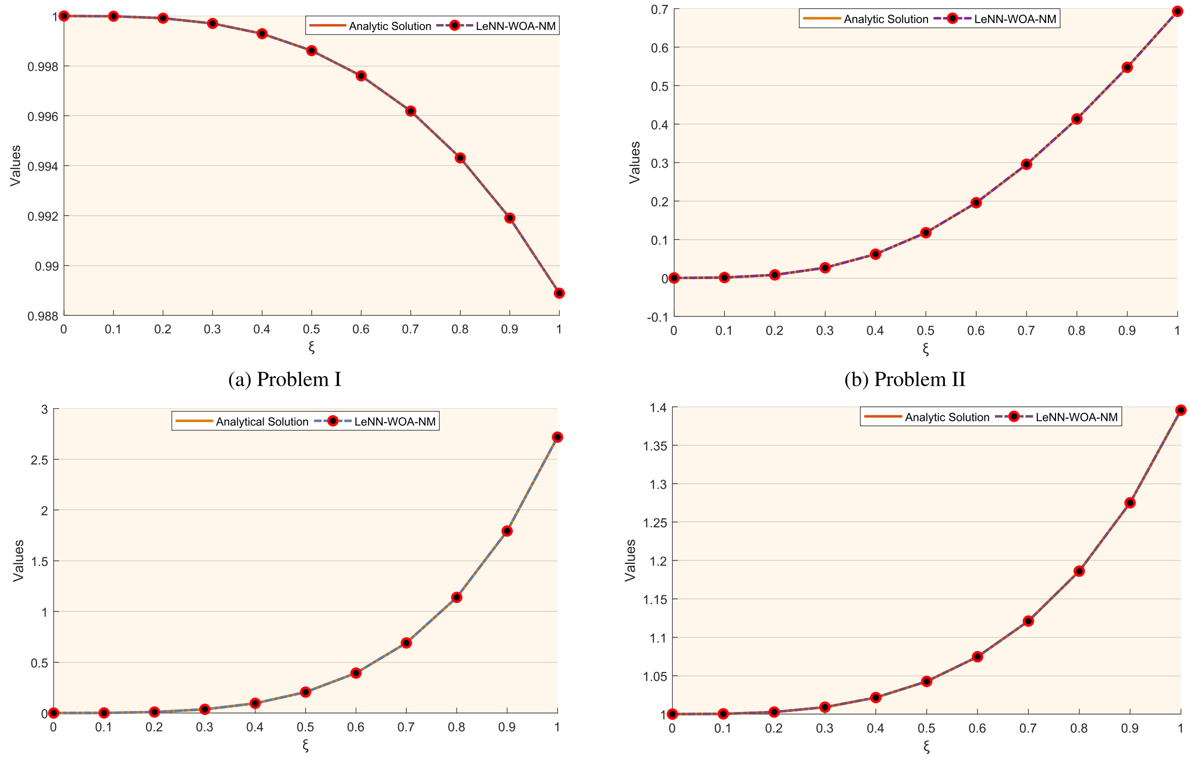
with

$$\phi(0) = A, \phi'(0) = 0 \quad \text{and} \quad \phi''(0) = 0.$$

Singular point is at $\xi = 0$ and appears with shape factor β once in second case.

III. SERIES SOLUTIONS USING WEIGHTED LEGENDRE POLYNOMIALS

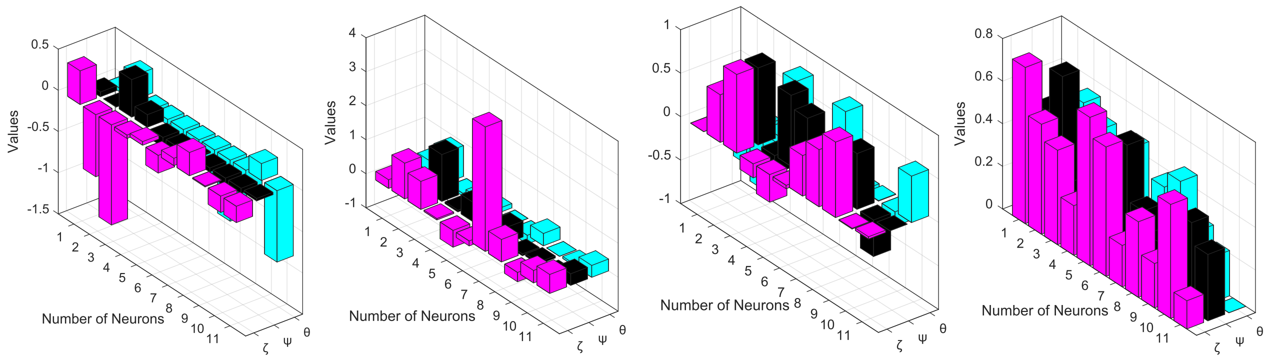
Legendre polynomials denoted by L_n are well known orthogonal polynomials that can be used to model approximate solutions. Table 1 represents first eleven ledendre polynomials.



(a) Problem I (b) Problem II

(c) Problem III (d) Problem IV

FIGURE 3. Solutions obtained by LeNN-WOA-NM approach for first and second type nonlinear singular Emden-Fowler differential equation.



(a) Problem I (b) Problem II (c) Problem III (d) Problem IV

FIGURE 4. Weights achieved by LeNN-WOA-NM algorithm for best solutions of Problem 1, 2, 3 and 4.

Polynomials of higher order are formulated by using Eq (13)

$$L_{n+1}(t) = \frac{1}{n+1}\left[(2n+1)tL_n(t) - nL_{n-1}(t)\right], \qquad (13)$$

trial solution or approximate series solution in term of weighted legendre polynomials for nonlinear Emden fowler is considered as

$$\phi_{appox}(\xi) = \sum_{n=0}^{N} \zeta_n L_n\left(\psi_n \xi + \theta_n\right), \qquad (14)$$

where, ζ_n, ψ_n and θ_n are unknown parameters.

Since, *nth* order continuous derivatives of Eq (14) exist. So first derivative $\phi'(\xi)$, second derivative $\phi''(\xi)$ and third derivative $\phi'''(\xi)$ of Eq (14) are represented by the following equations.

$$\phi'_{appox}(\xi) = \sum_{n=1}^{N} \zeta_n L'_n\left(\psi_n \xi + \theta_n\right), \qquad (15)$$

$$\phi''_{appox}(\xi) = \sum_{n=4}^{N} w_n L''_n\left(\psi_n \xi + \theta_n\right), \qquad (16)$$

$$\phi'''_{appox}(\xi) = \sum_{n=4}^{N} \zeta_n L'''_n\left(\psi_n \xi + \theta_n\right). \qquad (17)$$

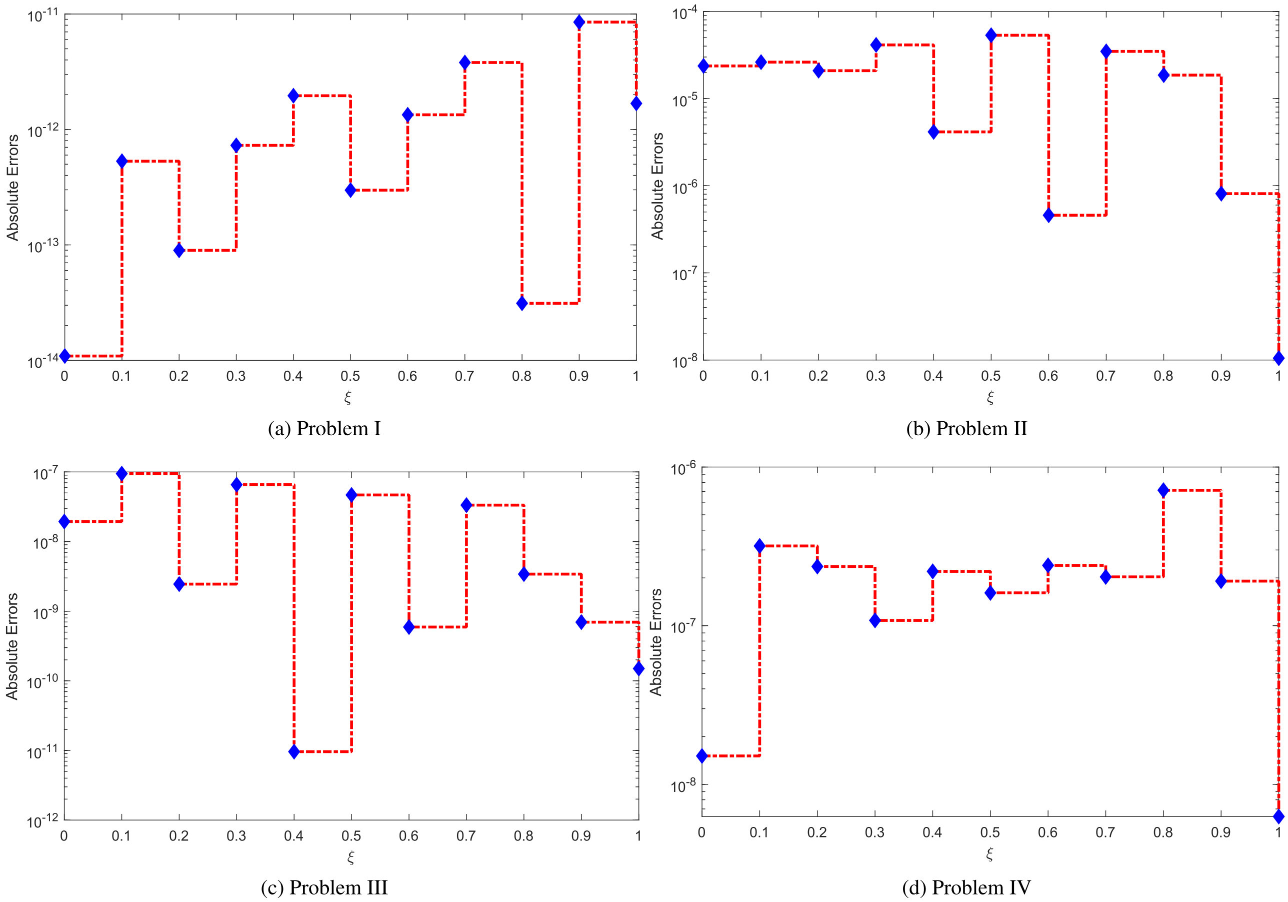


FIGURE 5. **Absolute errors in best solutions obtained by proposed algorithm for different problems.**

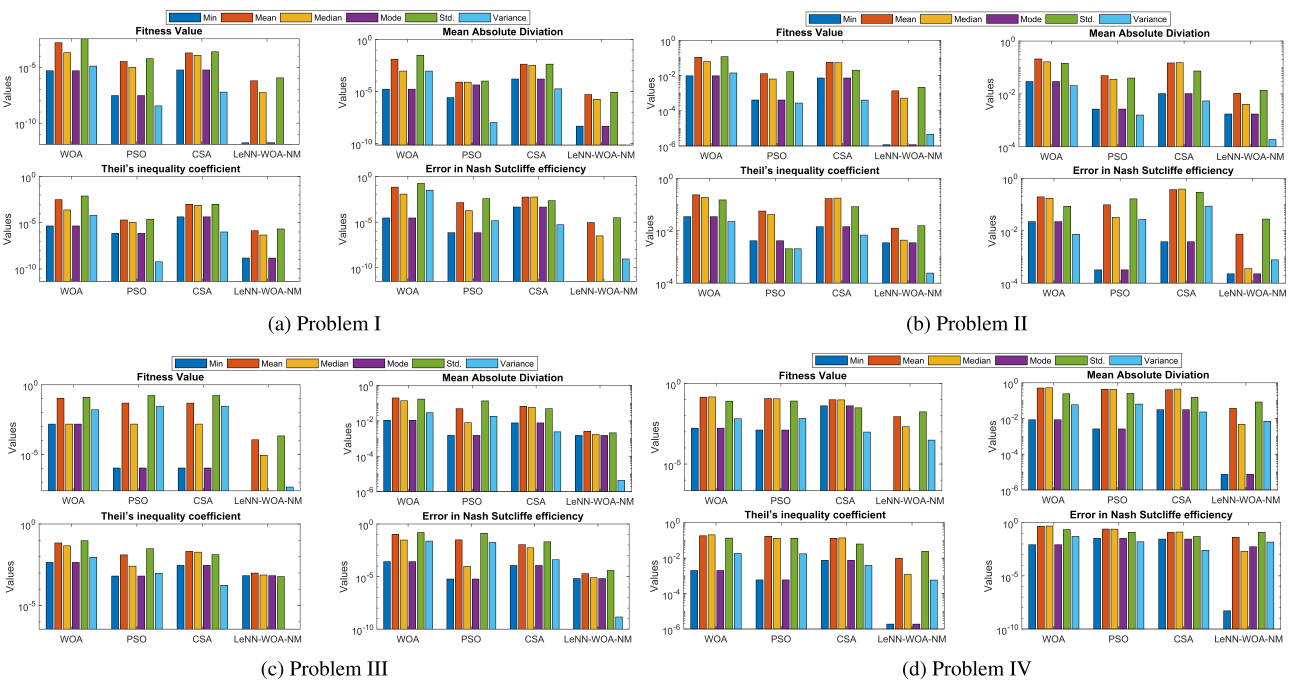


FIGURE 6. **Bar graphs of statistics representing attained values of LeNN-WOA-NM algorithm, PSO, CSA and WOA for performance indicators.**

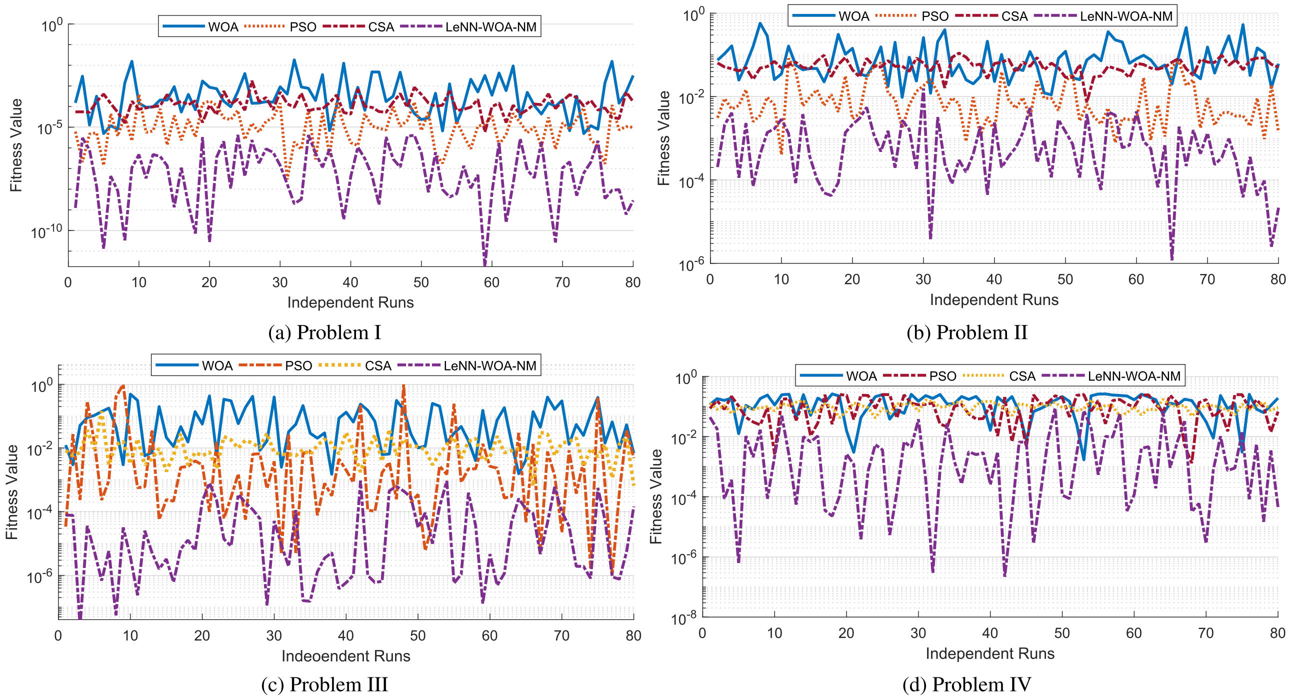


FIGURE 7. Fitness analysis for Problem 1, 2, 3 and 4 during 80 independent runs.

TABLE 3. Comparison of approximate solutions obtained by proposed algorithm with PSO, CSA, WOA and exact solution for Problem 1 and 2.

	Problem I					Problem II				
ξ	WOA	PSO	CSA	LeNN-WOA-NM	Exact	WOA	PSO	CSA	LeNN-WOA-NM	Exact
0	0.99939	1.000003	1.000287	1	1	0.048647	0.011755	0.032020	-3.06E-05	0
0.1	0.999393	0.999986	1.000293	0.999989	0.999989	0.050482	0.012868	0.025783	0.001000	0.001014
0.2	0.999259	0.999901	1.000226	0.999911	0.999911	0.057086	0.019442	0.026243	0.007968	0.007981
0.3	0.998965	0.999686	1.000022	0.999700	0.999700	0.073187	0.037255	0.039205	0.026642	0.026466
0.4	0.998473	0.999274	0.999618	0.999289	0.999289	0.103322	0.071394	0.069505	0.062035	0.061759
0.5	0.997736	0.998600	0.998947	0.998611	0.998611	0.151133	0.125451	0.120620	0.117783	0.117543
0.6	0.996693	0.997594	0.997940	0.997600	0.997600	0.218874	0.201158	0.194401	0.195567	0.195544
0.7	0.995279	0.996189	0.996531	0.996189	0.996189	0.307194	0.298401	0.290966	0.294906	0.295495
0.8	0.993421	0.994316	0.99465	0.994311	0.994311	0.415211	0.415537	0.408791	0.413433	0.413428
0.9	0.991043	0.991907	0.992231	0.991900	0.991900	0.540871	0.549907	0.545039	0.547543	0.547543
1	0.988066	0.988897	0.989207	0.988889	0.988889	0.681539	0.698434	0.696177	0.693147	0.693147

where ζ_n, ψ_n and θ_n are real valued unknown parameters.

IV. FITNESS FUNCTION FORMULATION

In this section, we formulate fitness/objective functions for first and second type nonlinear Emden-Fowler type equations. Fitness function is based on mean square error (MSE) in candidate solution that is used to train neurons (parameters) in LeNN. It is defined as

$$\text{Minimize} \quad \epsilon = \epsilon_1 + \epsilon_2, \tag{18}$$

where ϵ_1 is associated to first type nonlinear Emden-Fowler equation Eq (8) and ϵ_2 is associated to boundary conditions for Eq (8). Mathematically, ϵ_1 and ϵ_2 are given as

$$\epsilon_1 = \frac{1}{N}\sum_{\beta=1}^{N}\left(\frac{d^3\phi}{d\xi^3} + \frac{2\beta}{\xi}\frac{d^2\phi}{d\xi^2} + \frac{\beta(\beta-1)}{\xi^2}\frac{d\phi}{d\xi} + f(\xi)g(\phi)\right)^2, \tag{19}$$

$$\epsilon_2 = \frac{1}{3}\left((\phi(0)-A)^2 + \left(\frac{d\phi}{d\xi}(0)\right)^2 + \left(\frac{d^2\phi}{d\xi^2}(0)\right)^2\right), \tag{20}$$

For non-linear multi singular Emden-Fowler differential equation of type second, ϵ_1 and ϵ_2 can be mathematically

TABLE 4. Comparison of solutions obtained by proposed algorithm with WOA, PSO, CSA and exact solution for Problem 3 and Problem 4.

	Problem III					Problem IV				
ξ	WOA	PSO	CSA	Exact	LeNN-WOA-NM	WOA	PSO	CSA	Exact	LeNN-WOA-NM
0	-0.0291173	-5.580E-05	0.0064096	0	0	0.9909846	0.99863168	0.9312504	1	1
0.1	-0.0226260	0.0010363	0.0082296	0.0011051	0.0011051	0.9862820	0.9988289	0.9390017	1.0003333	1.0003333
0.2	-0.0086600	0.0097481	0.0175671	0.0097712	0.0097712	0.9835390	1.0009450	0.9489395	1.0026702	1.0026702
0.3	0.0233276	0.0366362	0.0448939	0.0364461	0.0364461	0.9847364	1.0070216	0.9632169	1.0090406	1.0090406
0.4	0.0876768	0.0961489	0.1046747	0.0954767	0.0954767	0.9919561	1.0192389	0.9838128	1.0215625	1.0215625
0.5	0.2034812	0.2075992	0.2162740	0.2060901	0.2060901	1.0074553	1.0399569	1.0127269	1.0425469	1.0425469
0.6	0.3957166	0.3962925	0.4050488	0.3935776	0.3935776	1.0337935	1.0718354	1.0522549	1.0746553	1.0746553
0.7	0.6965685	0.6948370	0.7036532	0.6907171	0.6907171	1.0740361	1.1180597	1.1053439	1.1211255	1.1211257
0.8	1.1469895	1.1446760	1.1535852	1.1394769	1.1394769	1.1320709	1.1826982	1.1760312	1.1860953	1.1860953
0.9	1.7985215	1.7978922	1.8070017	1.7930506	1.7930506	1.2130807	1.2712236	1.2699650	1.2750687	1.2750686
1	2.7154180	2.7193238	2.7288324	2.7182818	2.7182818	1.3242277	1.3912298	1.3950075	1.3956124	1.3956124

TABLE 5. Comparison between the absolute errors attained by proposed algorithm with WOA, PSO and CSA for Problem 1 and Problem 2.

	Problem I				Problem II			
ξ	WOA	PSO	CSA	LeNN-WOA-NM	WOA	PSO	CSA	LeNN-WOA-NM
0	6.61E-05	1.56E-08	7.17E-06	1.09E-14	0.003023	0.000224	0.004520	2.37E-05
0.1	7.33E-05	7.93E-07	1.16E-06	5.31E-13	0.000432	0.000431	0.006961	2.62E-05
0.2	0.000139	1.37E-07	5.54E-07	8.97E-14	0.003286	7.86E-05	0.004862	2.09E-05
0.3	5.30E-05	1.97E-07	6.02E-07	7.28E-13	0.008848	0.000497	0.003172	4.13E-05
0.4	3.56E-08	8.68E-07	3.05E-07	1.96E-12	0.006998	8.08E-06	0.003228	4.15E-06
0.5	4.88E-05	7.18E-07	5.95E-08	2.98E-13	0.000396	0.000999	0.003410	5.33E-05
0.6	0.000113	7.17E-08	2.58E-06	1.34E-12	0.005977	0.000161	0.001481	4.59E-07
0.7	9.69E-05	2.31E-07	1.01E-05	3.80E-12	0.013264	0.001392	4.41E-06	3.48E-05
0.8	2.20E-05	8.67E-07	1.79E-05	3.12E-14	9.32E-07	0.000610	0.000490	1.86E-05
0.9	1.15E-05	3.68E-07	1.32E-05	8.51E-12	0.028319	0.002989	0.000446	8.11E-07
1	0.000131	6.91E-07	2.13E-08	1.68E-12	0.005991	0.000438	0.003826	1.05E-08

expressed as

$$\epsilon_1 = \left(\frac{d^3\phi}{d\xi^3} + \frac{\beta}{\xi}\frac{d^2\phi}{d\xi^2} + f(\xi)g(\phi)\right)^2, \tag{21}$$

and

$$\epsilon_2 = \frac{1}{3}\left((\phi(0) - A)^2 + \left(\frac{d\phi}{d\xi}(0)\right)^2 + \left(\frac{d^2\phi}{d\xi^2}(0)\right)^2\right). \tag{22}$$

where $N = \frac{1}{h}$ and h is a step size.

A. WHALE OPTIMIZATION ALGORITHM

Whale Optimization Algorithm (WOA) is nature inspired technique given by Mirajlili and lewis [36] which imitate the social behaviour of whales. The algorithm is inspired by the bubble net hunting strategy.

Mathematical prescription for WOA is explained below:

1) ENCIRCLING PREY

Humpback whales encircles the recognized location of prey (small fishes). Initially, in candidate space the location of optimal design is not known. Position of encircled prey is modified by WOA towards the global optimal result with an increase in iterations. The hunting of prey is mathematically modeled as Eq (23) and Eq (24).

$$D = |C \cdot \overrightarrow{X^*}(t) - \vec{X}(t)|, \tag{23}$$

$$\vec{X}(t+1) = \overrightarrow{X^*}(t) - \vec{A}.D, \tag{24}$$

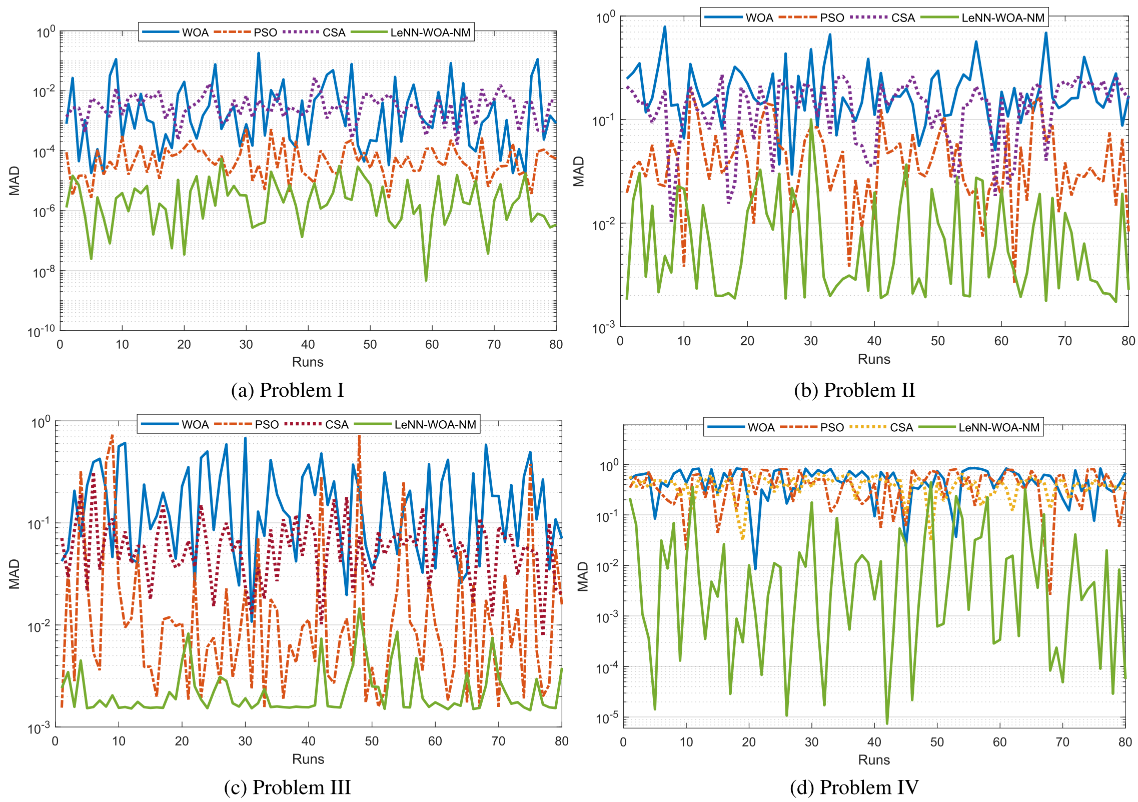
where "t" represents the current iterations, "X^*" indicates the best value obtained so far, "X" is a position vector, "||" gives the absolute value, "r" is a vector in interval [0,1], "." and "+" represents element wise multiplication and addition respectively. $\vec{A}$ and $\vec{C}$ are coefficient vectors and given as follows:

$$\vec{A} = 2\vec{a} \cdot \vec{r} - \vec{a}, \tag{25}$$

$$\vec{C} = 2 \cdot \vec{r}. \tag{26}$$

2) BUBBLE NET ATTACKING METHOD

To model mathematical equations for Bubble net attacking method two approaches are designed as follows:



(a) Problem I (b) Problem II

(c) Problem III (d) Problem IV

FIGURE 8. Convergence analysis of MAD during 80 independent runs for first and second type Emden-Fowler differential equations.

TABLE 6. Comparison between the absolute errors attained by proposed algorithm with WOA, PSO and CSA for Problem 3 and Problem 4.

	Problem III				Problem IV			
ξ	WOA	PSO	CSA	LeNN-WOA-NM	WOA	PSO	CSA	LeNN-WOA-NM
0	0.00251685	7.63E-05	0.00597363	1.94E-08	0.01390989	3.8100E-05	0.01144378	1.51E-08
0.1	0.00085218	9.73E-06	0.00016169	9.45E-08	0.00120851	0.00159428	0.00048074	3.18E-07
0.2	0.00014335	7.56E-05	0.00011854	2.45E-09	0.01035894	0.00026557	0.00194073	2.36E-07
0.3	0.00033636	3.21E-05	0.00014458	6.56E-08	0.00834433	0.00054039	0.00133826	1.08E-07
0.4	6.7800E-05	4.37E-06	0.00031391	9.61E-12	0.00059786	0.00147871	0.00150487	2.20E-07
0.5	0.00020100	5.67E-05	0.00020351	4.67E-08	0.00377830	0.00022670	0.00257252	1.61E-07
0.6	0.00036575	3.27E-05	0.00064464	5.92E-10	0.01339039	0.00085392	0.00179844	2.40E-07
0.7	0.00045345	3.69E-06	0.00014164	3.34E-08	0.00797716	0.00256672	0.00049265	2.03E-07
0.8	0.00024218	5.25E-05	0.00040872	3.41E-09	0.00089101	5.0800E-05	0.01962942	7.14E-07
0.9	2.4800E-06	4.29E-06	0.00013671	6.96E-10	0.02146690	0.00572957	0.03667775	1.91E-07
1	0.00017719	1.46E-05	0.00098814	1.50E-10	0.00559450	0.00098650	0.03621831	6.27E-09

1. **Shrinking encircling mechanism**: "a" is a randomly selected value and In the course of iterations, it linearly decreases from 2 to 0. Its value can be achieved by Eq (27).

$$a = 2\text{-t}\frac{2}{\text{MaxIter}}. \tag{27}$$

2. **Spiral updating position**: This approach evaluates the distance between the prey and the humpback whale. To mimic the helix-shaped movement a spiral equation is defined as follows:

$$\vec{X}(t+1) = \vec{D}' \cdot e^{bl} \cdot \cos(2\pi l) + \overrightarrow{X^*}(t), \tag{28}$$

where distance between the "ith" whale and the prey (best result attained so far) is represented by $\overrightarrow{D'} = |\overrightarrow{X^*}(t) - \vec{X}(t)|$, shape of the logarithmic spiral is denoted by constant b and l is a randomly selected number in $[-1,1]$.

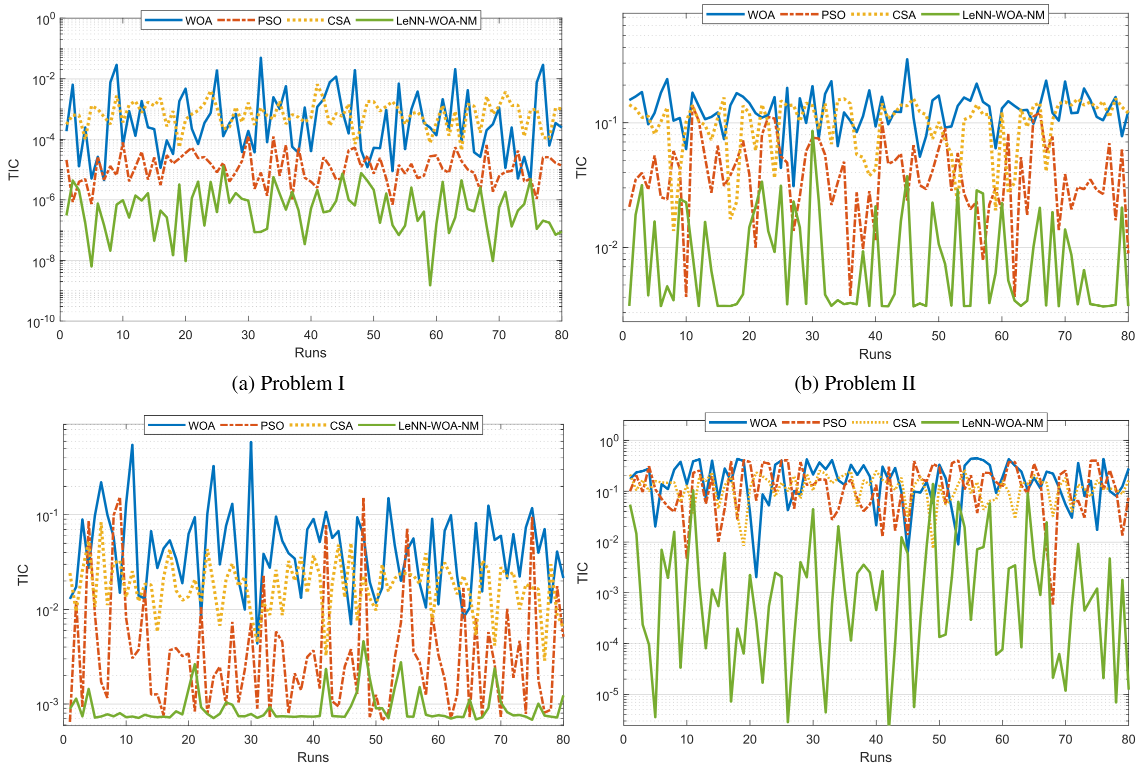


FIGURE 9. Convergence analysis of TIC during 80 independent runs for first and second type Emden-Fowler differential equations.

TABLE 7. Best weight achieved for optimization of Eq (38) and Eq (41) by proposed algorithm.

	Problem I			Problem II		
index	ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n
1	0.408826	0.057564	-0.044000	0.254318	0.114530	0.320272
2	-0.753480	-0.002450	0.295031	1.050468	0.041362	0.906695
3	-1.214790	0.454315	-0.167660	0.823686	1.375421	-0.080910
4	-0.041070	0.152337	-0.064490	0.041510	0.049329	0.169590
5	0.044146	-0.115580	-0.297110	-0.493580	0.571303	0.128204
6	-0.214880	-0.178810	-0.099960	-0.146110	-0.102080	-0.001930
7	0.088679	-0.016690	-0.300590	3.659748	0.467711	-0.253020
8	0.282416	-0.049280	-0.741580	0.659495	-0.054140	0.361584
9	-0.011820	-0.100280	-0.259620	-0.273810	0.061146	0.000882
10	-0.198010	-0.062350	0.204680	0.366455	-0.339450	0.058533
11	-0.198020	-0.029490	-0.866680	0.557791	0.340865	0.349444

We know that the humpback whale follows the spiral-shaped path and shrinking circle to hunt the prey. To model the simultaneous behaviour the probability is chosen to be 50% between the two paths so the position of the whales can be calculated by Eq (29).

$$\vec{X}(t+1) = \begin{cases} \vec{X}^{*}(t) - \vec{A}\cdot D, & \text{if } p < 0.5 \\ \vec{D'}\cdot e^{bl}\cdot\cos(2\pi l) + \vec{X^{*}}(t) & \text{if } p \geq 0.5, \end{cases} \tag{29}$$

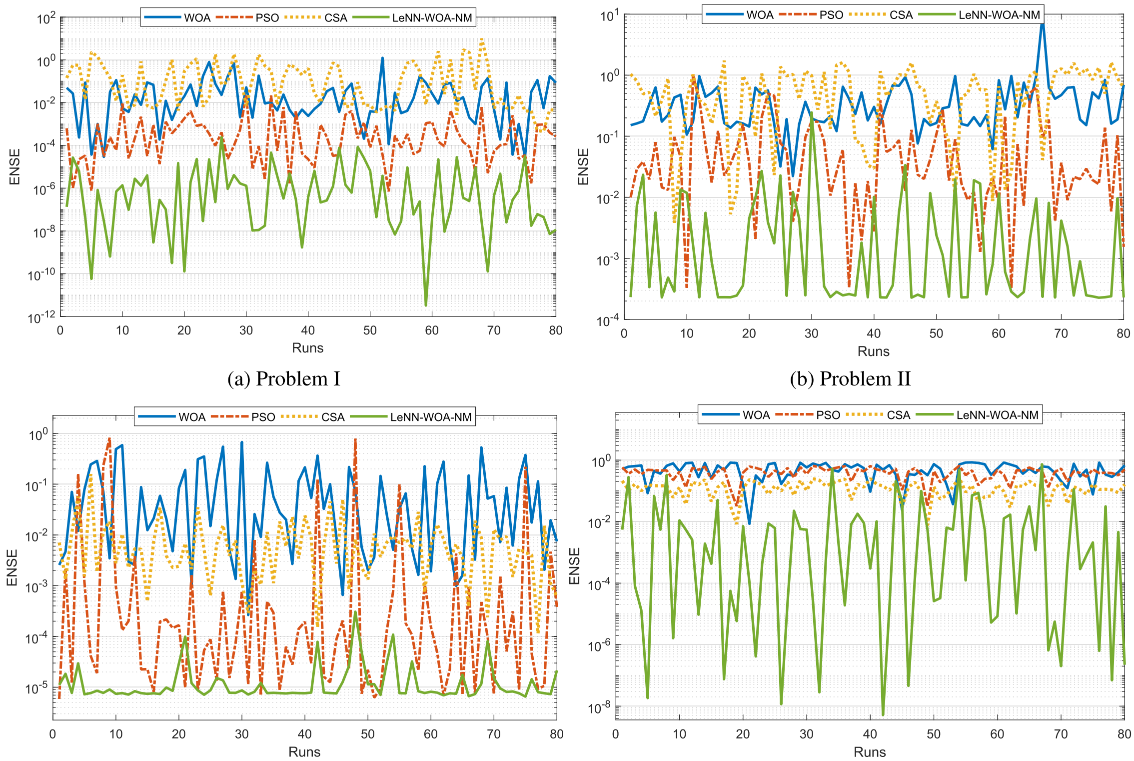
where "p" is a random value in interval [0,1].

3) SEARCH FOR PREY

A vector "A" with a random values less than 1 or greater then -1 is used to move a reference whale away from a whale. The mathematical model of this mechanism is given by Eq (30) and Eq (31).

$$\vec{D} = |\vec{C}\cdot\overrightarrow{X_{\text{rand}}} - \vec{X}|, \tag{30}$$

$$\vec{X}(t+1) = \overrightarrow{X_{\text{rand}}} - \vec{A}\cdot\vec{D}. \tag{31}$$



(a) Problem I (b) Problem II

(c) Problem III (d) Problem IV

FIGURE 10. Convergence analysis of ENSE during 80 independent runs for first and second type Emden-Fowler differential equations.

TABLE 8. Best weight achieved for optimization of Eq (44) and Eq (47) by proposed algorithm.

	Problem III			Problem IV		
index	ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n
1	-0.00025520	-0.07916260	-0.5659983	0.73308615	0.49157556	0.34432282
2	0.54710110	-0.09156670	-0.7318864	0.51580967	0.70179544	0.54020410
3	0.90428437	0.88504030	-0.0911262	0.44315273	0.20778173	0.46296807
4	-0.16085360	-0.20762780	0.7573477	0.23240265	0.48175825	0.22761992
5	-0.31751990	0.81663781	-0.2951611	0.69645753	0.19380327	0.47697631
6	-0.03909820	0.67666348	-0.0707813	0.60684850	0.57723262	0.16011539
7	0.46593053	0.28887830	0.7811133	0.19411520	0.31740822	0.39211912
8	0.65746316	0.43559105	-0.0846667	0.35456570	0.21980521	0.46826468
9	0.86795474	0.64078844	0.0057115	0.20798017	0.17682492	0.28322033
10	-0.00824190	-0.41933870	-0.1662330	0.53476583	0.41144797	0.21299134
11	0.02639607	0.00074241	0.5335582	0.13224697	0.31148234	0.00013506

where $\overrightarrow{X_{\text{rand}}}$ is an arbitrary whale taken from the current population.

When the process for optimization is started then WOA creates random population, initial population and calculate the fitness function. Flow chart of Whale optimization algorithm is given in Figure 1.

B. NELDER-MEAD ALGORITHM

The optimized weights obtained by WOA for solution for Eq (8) and Eq (11) are used as an initial guess or initial weights for Nelder-Mead algorithm. Hence, an effective local search mechanism is applied to furnish the approximate solution for the system. The detail procedure of Nelder-Mead algorithm is explained below.

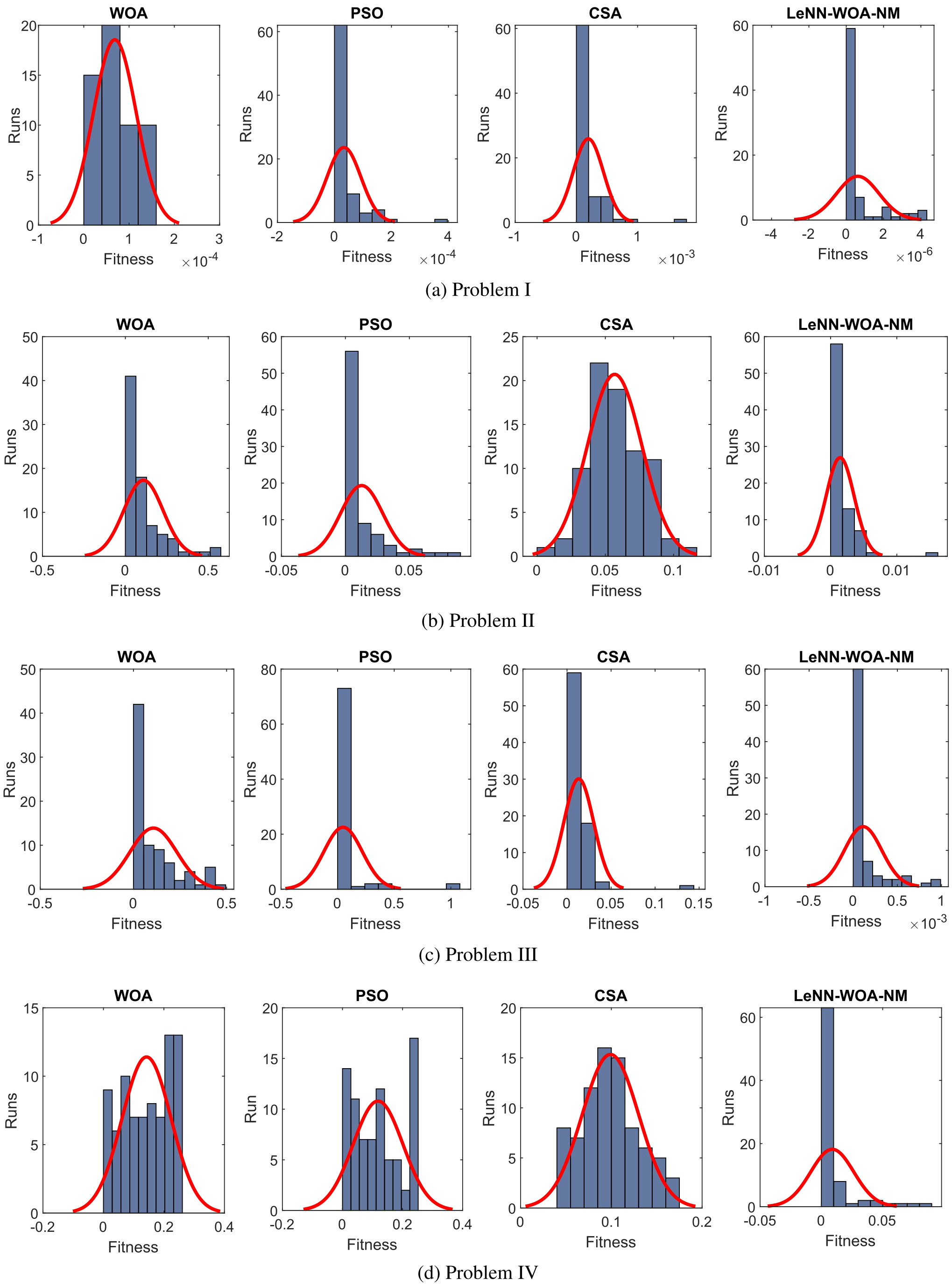


FIGURE 11. Analysis on normal probability curves for Fitness value attained by LeNN-WOA-NM algorithm in comparison with PSO, CSA and WOA for third order singular non-linear Emden-Fowler differential equation of type first (Problem 1, 2) and second (Problem 3, 4).

The Nelder–Mead (NM) simplex search method is a direct search method proposed by Nelder and Mead in 1965 [37]. It is a non-derivative search method that has widely been used to solve multidimensional constrained/unconstrained optimization problems [38], [39]. NM algorithm rescales the simplex of $n + 1$ points based on the local behavior of the function using four basic operations named as reflection, expansion, contraction and shrink [24]. The structure of Nelder-Mead algorithm described in flow chart given by Figure 1. Some recent application of NM algorithm includes numerical simulation of dynamical modeling of Li-ion batteries for electric vehicle [40], nonlinear Muskingum models [41], application to bankruptcy prediction in banks [42] and optimization of TIG welding

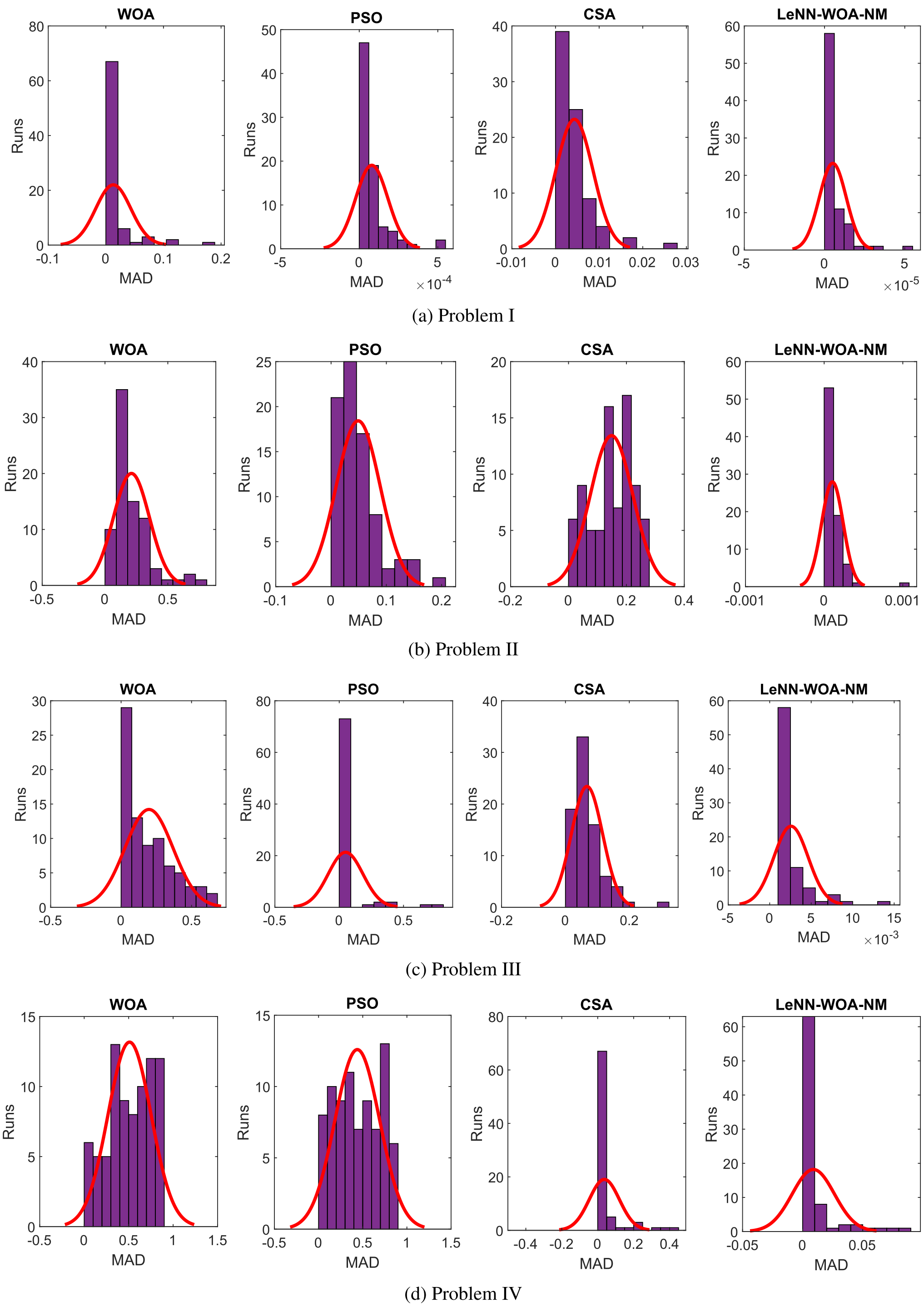


FIGURE 12. Analysis on normal probability curves for MAD attained by LeNN-WOA-NM algorithm in comparison with PSO, CSA and WOA for third order singular non-linear Emden-Fowler differential equation of type first (Problem 1, 2) and second (Problem 3, 4).

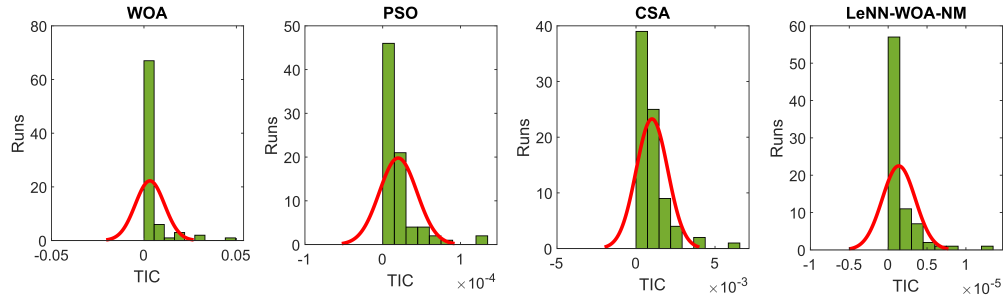
parameters [43]. Parameter setting for Nelder-Mead Algorithm is given in Table 38.

V. LeNN-WOA-NM ALGORITHM

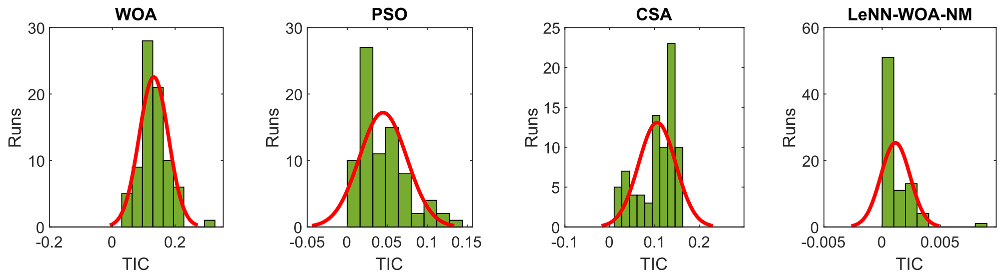
The steps for the proposed hybridized algorithm are summarized as:

Initialization: Approximate/trial solution is considered see Eq (14) and neurons in weighted Legendre polynomials are initialized with randomly generated real number form the candidate space.

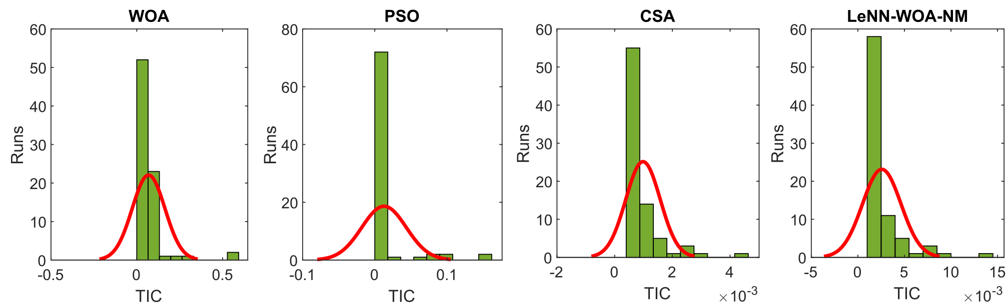
Fitness Calculation: Whale optimization algorithm is used to evaluate objective or fitness functions Eq (8)



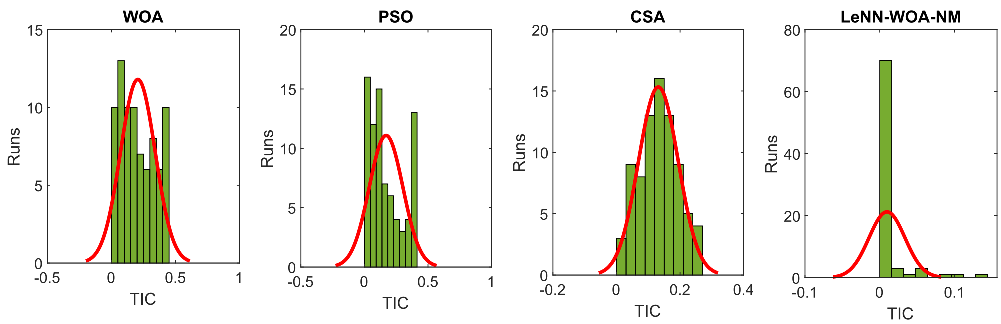
(a) Problem I



(b) Problem II



(c) Problem III



(d) Problem IV

FIGURE 13. Analysis on normal probability curves for TIC attained by LeNN-WOA-NM algorithm in comparison with PSO, CSA and WOA for third order singular non-linear Emden-Fowler differential equation of type first (Problem 1, 2) and second (Problem 3, 4).

and Eq (11) for first and second type non-linear Emden-Fowler equation to update the unknown neurons in LeNN structure until termination criteria is achieved.

Storage: Weights obtained by WOA for minimum value of fitness function are stored.

Initialize NM: Nelder-Mead algorithm starts the process of optimization by considering values of ζ_n, ψ_n and θ_n obtained by WOA as its initial guess.

Fitness Calculation: Fitness functions are evaluated with updated weights of WOA. The process stops when termination criteria is achieved.

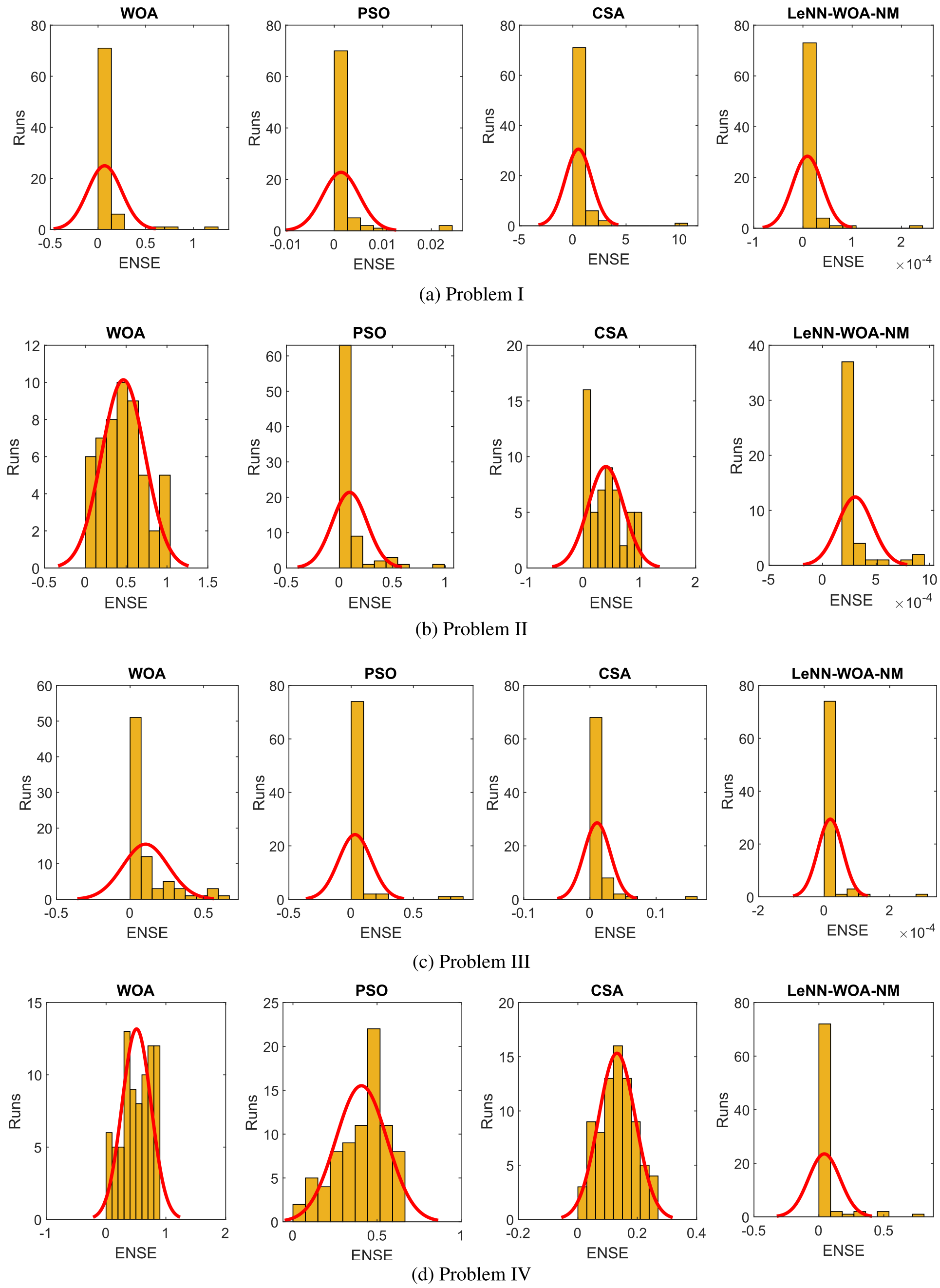


FIGURE 14. Analysis on normal probability curves for ENSE attained by LeNN-WOA-NM algorithm in comparison with PSO, CSA and WOA for third order singular non-linear Emden-Fowler differential equation of type first (Problem 1, 2) and second (Problem 3, 4).

Storage: Save the optimal weights or variables of the LeNN.

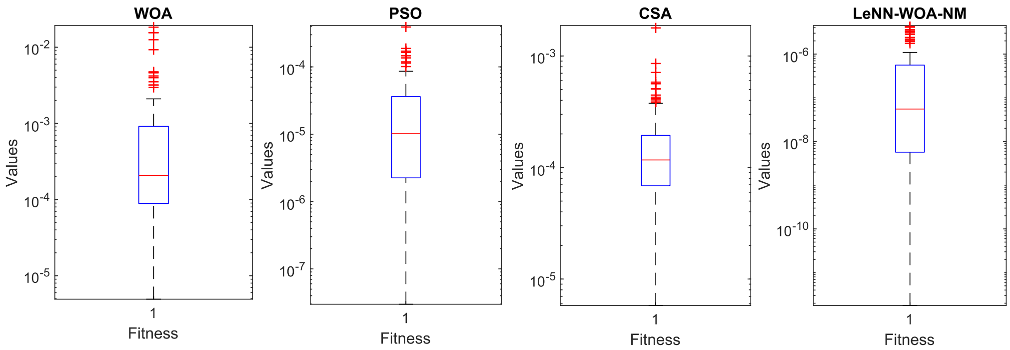
Flowchart of the proposed soft computing technique is shown in Figure 1.

VI. PERFORMANCE INDICES

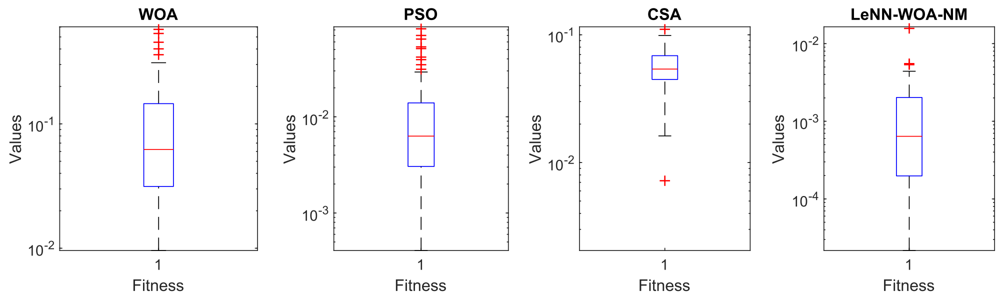
To check the efficiency of the designed technique in obtaining solution to non-linear Emden-Fowler differential equation of first and second order the statistical operators namely, mean absolute deviation (MAD), Theil's inequality coefficient (TIC) and Error in Nash Sutcliffe efficiency (ENSE) are defined [25]. The mathematical formulation of the operators is given as:

$$\mathrm{MAD} = \frac{1}{n}\sum_{m=1}^{n}\left|\phi(\xi) - \phi_{approx}(\xi)\right|, \tag{32}$$

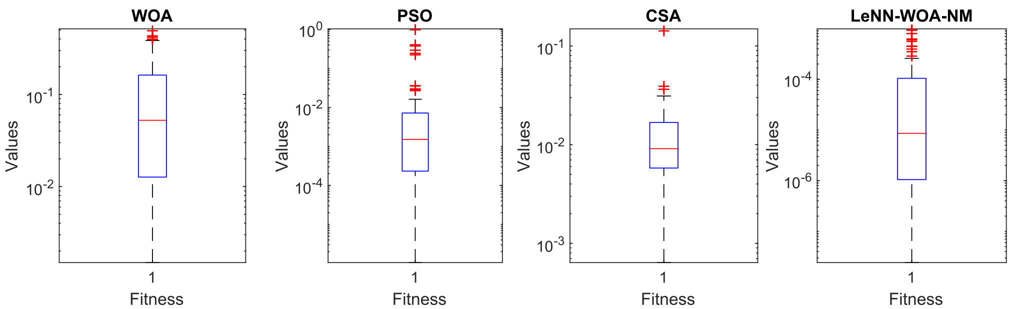
$$\mathrm{TIC} = \frac{\sqrt{\frac{1}{n}\sum_{n=1}^{n}\left(\phi(\xi) - \phi_{approx}(\xi)\right)^2}}{\left(\sqrt{\frac{1}{n}\sum_{n=1}^{n}(\phi(\xi))^2} + \sqrt{\frac{1}{n}\sum_{n=1}^{n}(\phi_{approx}(\xi))^2}\right)}, \tag{33}$$



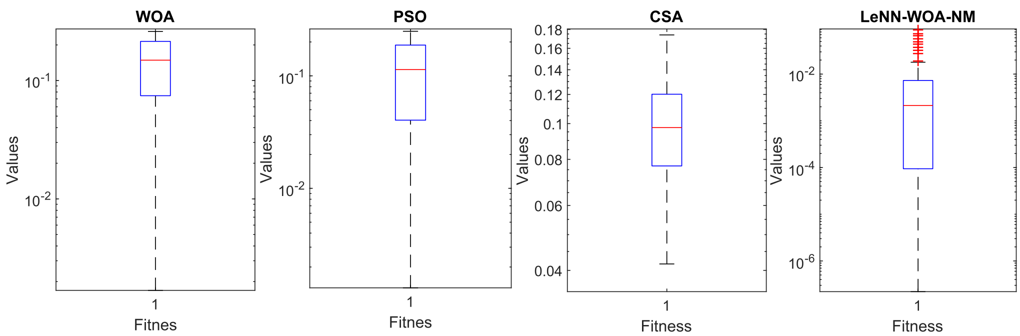
(a) Problem I



(b) Problem II



(c) Problem III



(d) Problem IV

FIGURE 15. Analysis of Boxplot for fitness value of Problem 1, 2, 3 and 4 attained by LeNN-WOA-NM algorithm in comparison with PSO, CSA and WOA.

TABLE 9. Statistical analysis on fitness analysis for Problem 1 during 80 independent runs by proposed algorithm, PSO, CSA and WOA.

Fitness Analysis						
	Min.	**Mean**	**Median**	**Mod.**	**Std.**	**Var.**
WOA	4.92E-06	1.60E-03	2.07E-04	4.92E-06	3.60E-03	1.29E-05
PSO	2.98E-08	3.33E-05	1.02E-05	2.98E-08	5.95E-05	3.54E-09
CSA	5.79E-06	1.95E-04	1.17E-04	5.79E-06	2.47E-04	6.08E-08
LeNN-WOA-NM	1.77E-12	6.20E-07	5.53E-08	1.77E-12	1.14E-06	1.29E-12

TABLE 10. Statistical analysis on mean absolute deviation for Problem 1 during 80 independent runs by proposed algorithm, PSO, CSA and WOA.

Mean Absolute Deviation						
	Min.	**Mean**	**Median**	**Mod.**	**Std.**	**Var.**
WOA	1.69E-05	0.0129	9.27E-04	1.69E-05	0.0305	9.28E-04
PSO	2.64E-06	7.98E-05	7.98E-05	4.31E-05	1.02E-04	1.04E-08
CSA	1.60E-04	0.0043	0.0033	1.60E-04	0.0043	1.81E-05
LeNN-WOA-NM	4.65E-09	5.15E-06	1.80E-06	4.65E-09	8.39E-06	7.04E-11

TABLE 11. Statistical analysis on Theil's inequality coefficient for Problem 1 during 80 independent runs by proposed algorithm, PSO, CSA and WOA.

Theil's inequality coefficient						
	Min.	**Mean**	**Median**	**Mod.**	**Std.**	**Var.**
WOA	4.48E-06	0.0032	2.43E-04	4.48E-06	0.0079	6.22E-05
PSO	6.99E-07	1.99E-05	1.12E-05	6.99E-07	2.42E-05	5.86E-10
CSA	4.34E-05	0.001	7.78E-04	4.34E-05	0.001	1.00E-06
LeNN-WOA-NM	1.51E-09	1.37E-06	4.67E-07	1.51E-09	2.13E-06	4.53E-12

TABLE 12. Statistical analysis on Error in Nash Sutcliffe Efficiency for Problem 1 during 80 independent runs by proposed algorithm, PSO, CSA and WOA.

Error in Nash Sutcliffe Efficiency						
	Min.	**Mean**	**Median**	**Mod.**	**Std.**	**Var.**
WOA	2.89E-05	0.0691	0.012	2.89E-05	0.179	0.0321
PSO	7.03E-07	0.0014	1.83E-04	7.03E-07	0.0038	1.44E-05
CSA	4.54E-04	0.0057	0.0057	4.54E-04	0.0023	5.20E-06
LeNN-WOA-NM	3.26E-12	9.13E-06	3.14E-07	3.26E-12	3.04E-05	9.22E-10

TABLE 13. Statistical analysis on Fitness Analysis for Problem 1 during 80 independent runs by proposed algorithm, PSO, CSA and WOA.

Fitness Analysis						
	Min.	**Mean**	**Median**	**Mod.**	**Std.**	**Var.**
WOA	0.00958519	0.10911432	0.06223133	0.00958519	0.1179656	0.01391588
PSO	0.000407	0.01272245	0.00629448	0.000407	0.01650894	0.000273
CSA	0.00720688	0.05682838	0.05390004	0.00720688	0.02003173	0.000401
LeNN-WOA-NM	1.18E-06	0.00134607	0.000524	1.18E-06	0.00212836	4.53E-06

$$\mathrm{NSE} = \left\{ 1 - \frac{\sum_{n=1}^{n}\left((\phi(\xi) - \phi_{approx}(\xi)\right)^2}{\sum_{n=1}^{n}\left((\phi(\xi) - \bar{\phi}(\xi)\right)^2}, \qquad \bar{\phi}(\xi) = \frac{1}{n}\sum_{m=1}^{n}\phi(\xi) \right. \tag{34}$$

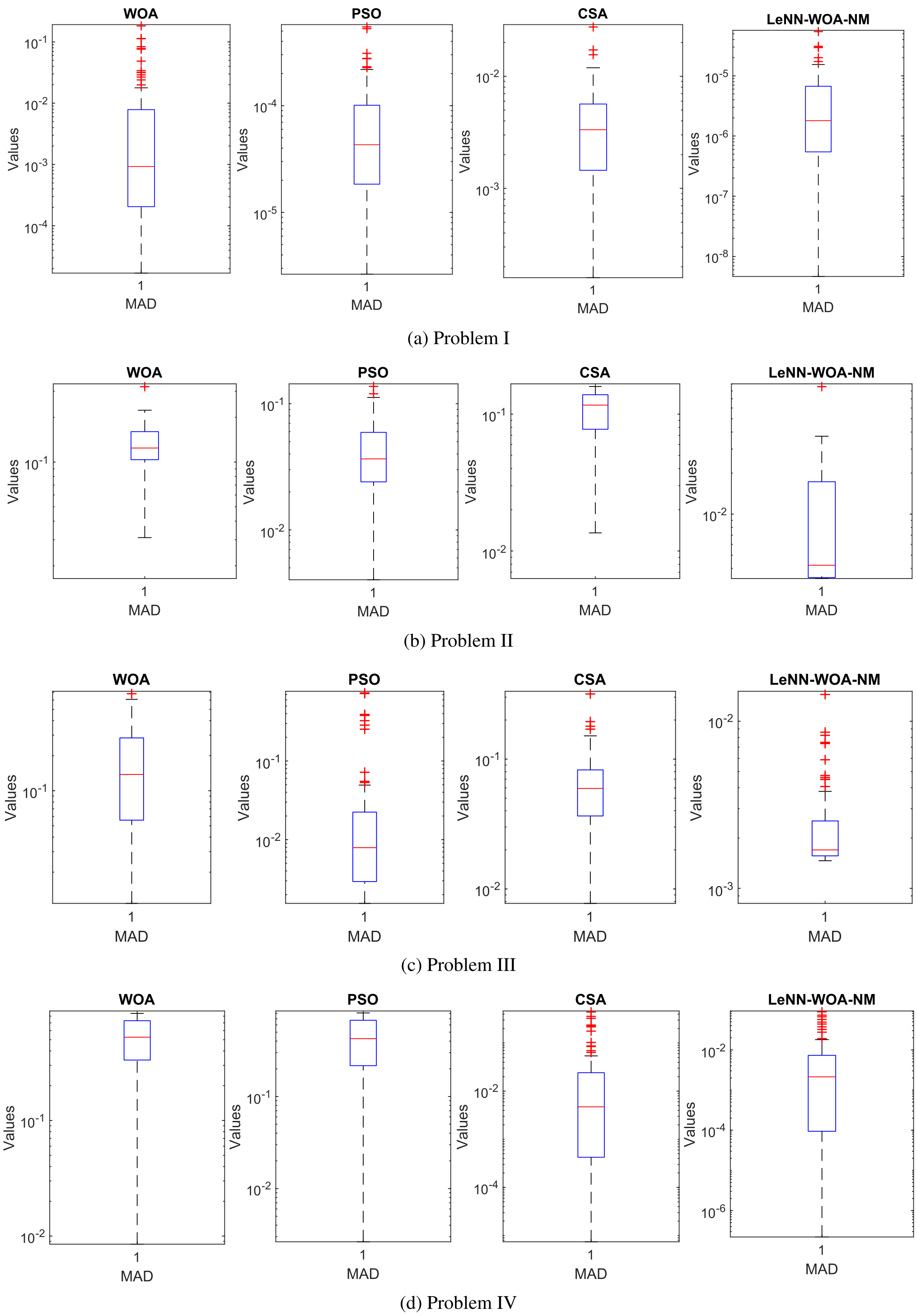


FIGURE 16. **Analysis of Boxplot for MAD of Problem 1, 2, 3 and 4 attained by LeNN-WOA-NM algorithm in comparison with PSO, CSA and WOA.**

$$ENSE = 1 - NSE, \tag{35}$$

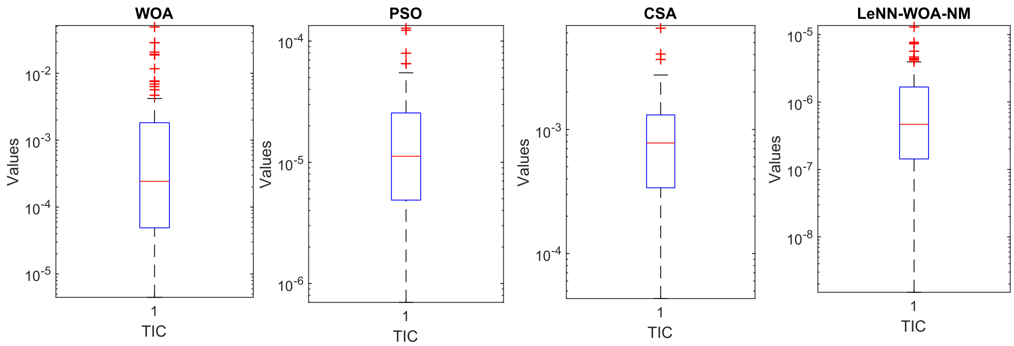
where n denotes the number of grid points.

VII. NUMERICAL EXPERIMENTATION

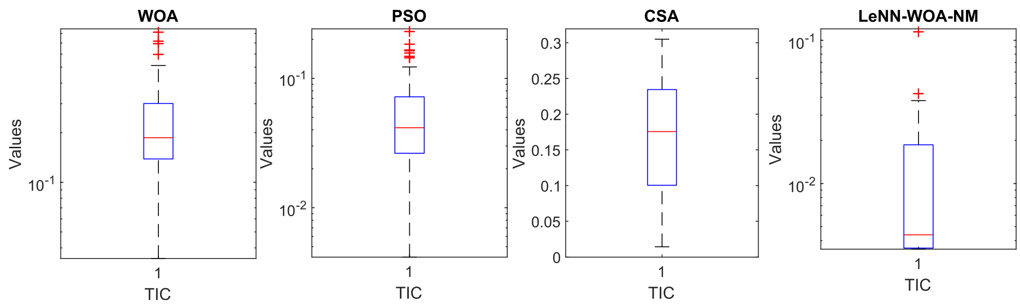
In this section, different problems are considered of first and second type multi singular non-linear third order Emden-Fowler differential equations. The detail explanation about problem is given below

Problem 1: Considering Non-Linear Emden-Fowler first type equation with shape factor $\beta = 4, f(\xi) = 1$ and $g(\phi) = \phi^m$ where $m = 0$.

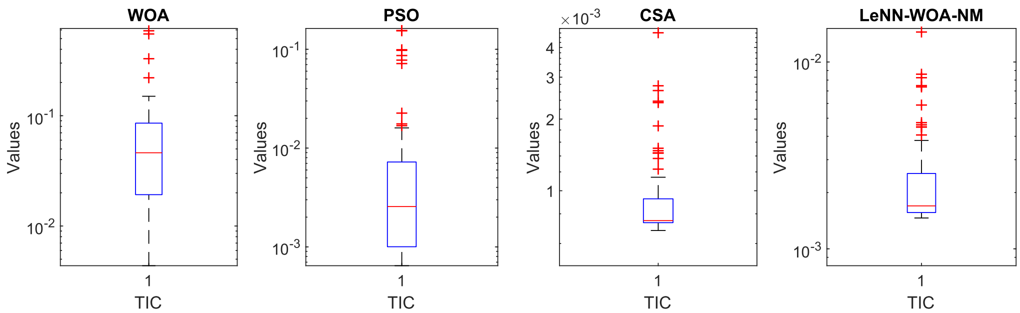
$$\frac{d^3\phi}{d\xi^3} + \frac{8}{\xi}\frac{d^2\phi}{d\xi^2} + \frac{12}{\xi^2}\frac{d\phi}{d\xi} + 1 = 0, \tag{36}$$



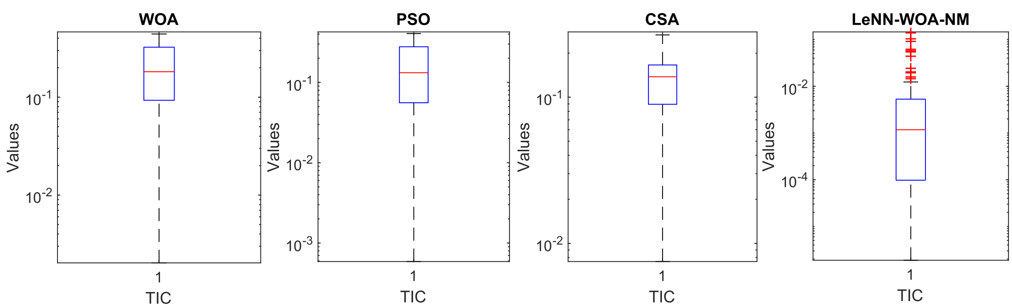
(a) Problem I



(b) Problem II



(c) Problem III



(d) Problem IV

FIGURE 17. Analysis of Boxplot for TIC of Problem 1, 2, 3 and 4 attained by LeNN-WOA-NM algorithm in comparison with PSO, CSA and WOA.

subjecting to initial conditions given as

$$\phi(0) = 1, \phi'(0) = 0 \quad \text{and} \quad \phi''(0) = 0,$$

exact solution for Eq (36) is given as $\phi(\xi) = 1 - \frac{1}{90}(\xi)^3$ [44].

Fitness function for Eq (36) is formulated as

$$\epsilon = \epsilon_1 + \epsilon_2, \tag{37}$$

equivalently,

$$\epsilon = \frac{1}{N}\sum_{m=1}^{N}\left(\xi_m^2\frac{d^3\phi}{d\xi_m^3} + 8\xi_m\frac{d^2\phi}{d\xi_m^2} + 12\frac{d\phi}{d\xi_m} + \xi_m^2\right)^2 + \frac{1}{3}\left((\phi_0 - 1)^2 + \left(\frac{d\phi(0)}{d\xi}\right)^2 + \left(\frac{d^2\phi(0)}{d\xi^2}\right)^2\right). \tag{38}$$

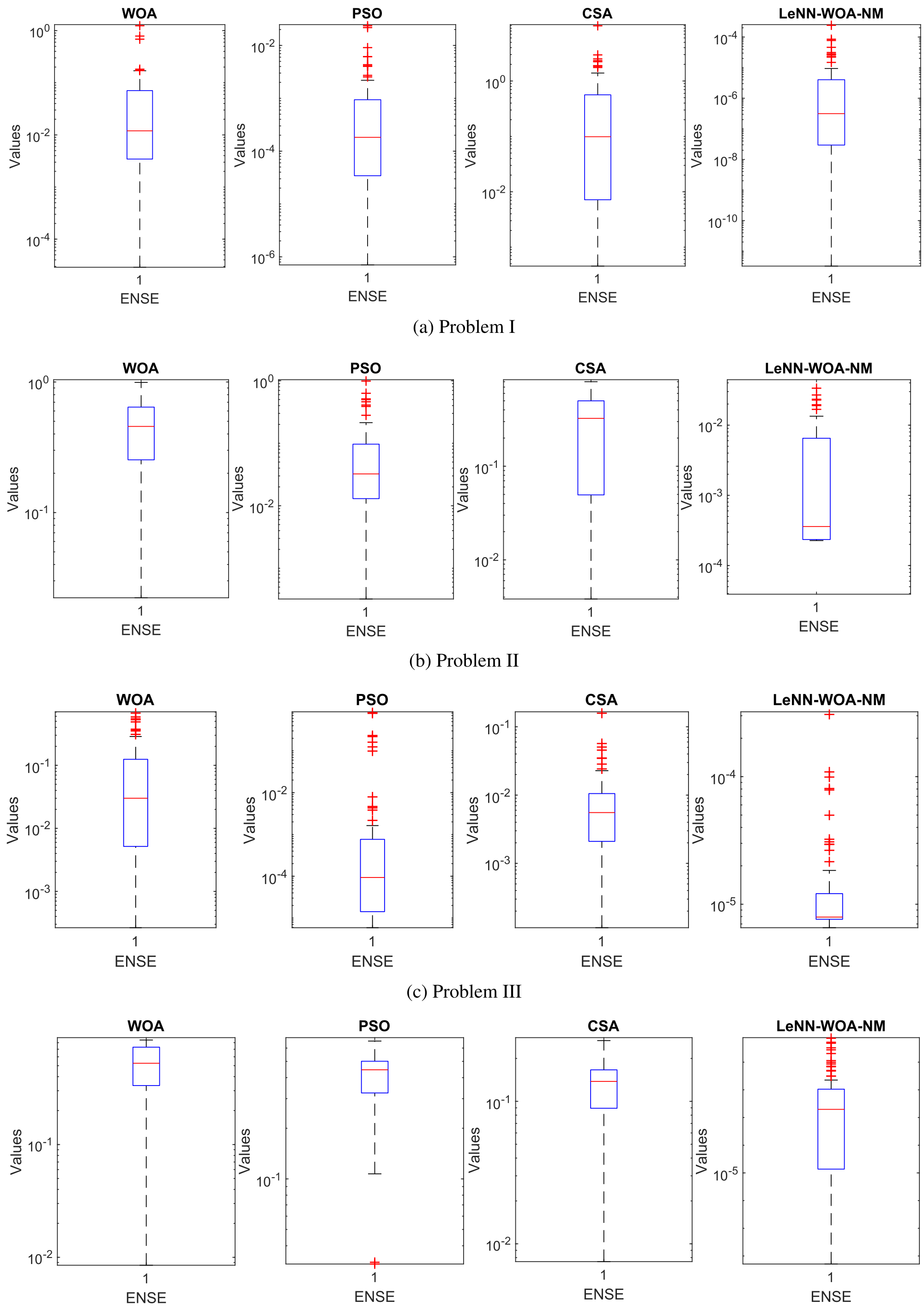


FIGURE 18. Analysis of Boxplot for ENSE of Problem 1, 2, 3 and 4 attained by LeNN-WOA-NM algorithm in comparison with PSO, CSA and WOA.

Problem 2: Let shape factor $\beta = 3, f(\xi) = -6(10+2\xi^3+6\xi^6)$ and $g(\phi) = e^{-3\phi}$

$$\frac{d^3\phi}{d\xi^3} + \frac{6}{\xi}\frac{d^2\phi}{d\xi^2} + \frac{6}{\xi^2}\frac{d\phi}{d\xi} - 6(10+2\xi^3+6\xi^6)e^{-3\phi} = 0, \tag{39}$$

with

$$\phi(0) = 0, \phi'(0) = 0 \quad \text{and} \quad \phi''(0) = 0,$$

exact solution for Eq (39) is given as $log(1+\xi^3)$ [44]. Fitness function for Eq (39) can be written as

$$\epsilon = \epsilon_1 + \epsilon_2, \tag{40}$$

TABLE 14. Statistical analysis on Mean Absolute Deviation for Problem 2 during 80 independent runs by proposed algorithm, PSO, CSA and WOA.

Mean Absolute Deviation						
	Min.	Mean	Median	Mod.	Std.	Var.
WOA	0.02946183	0.21032193	0.16364798	0.02946183	0.14350405	0.02059341
PSO	0.00266543	0.04903535	0.03593484	0.00266543	0.03983714	0.001587
CSA	0.01029353	0.14872052	0.15369413	0.01029353	0.07368662	0.00542972
LeNN-WOA-NM	0.00173657	0.01047509	0.00403578	0.00173657	0.01370237	1.88E-04

TABLE 15. Statistical analysis on Theil's inequality coefficient for Problem 2 during 80 independent runs by proposed algorithm, PSO, CSA and WOA.

Theil's inequality coefficient						
	Min.	Mean	Median	Mod.	Std.	Var.
WOA	0.03449301	0.23357385	0.18643734	0.03449301	0.14972866	0.02241867
PSO	0.0041522	0.0563613	0.04151384	0.0041522	0.00204901	0.00204901
CSA	0.01429876	0.16863342	0.17561954	0.01429876	0.08270124	0.0068395
LeNN-WOA-NM	0.00348245	0.01258072	0.00438835	0.00348245	0.01559042	2.43E-04

TABLE 16. Statistical analysis on Error in Nash Sutcliffe Efficiency for Problem 2 during 80 independent runs by proposed algorithm, PSO, CSA and WOA.

Error in Nash Sutcliffe Efficiency						
	Min.	Mean	Median	Mod.	Std.	Var.
WOA	0.022236	0.195314	0.173722	0.022236	0.085727	0.007349
PSO	3.22E-04	0.097186	0.032272	3.22E-04	0.164064	0.026917
CSA	0.003821	0.369791	0.388987	0.003821	0.294405	0.086674
LeNN-WOA-NM	2.27E-04	0.00744408	3.60E-04	2.27E-04	0.02785565	7.76E-04

TABLE 17. Statistical analysis on Fitness Analysis for Problem 2 during 80 independent runs by proposed algorithm, PSO, CSA and WOA.

Fitness Analysis						
	Min.	Mean	Median	Mod.	Std.	Var.
WOA	1.50E-03	0.107	1.50E-03	1.50E-03	0.1263	1.60E-02
PSO	1.05E-06	4.83E-02	1.50E-03	1.05E-06	1.70E-01	2.89E-02
CSA	1.05E-06	4.83E-02	1.50E-03	1.05E-06	1.70E-01	2.89E-02
LeNN-WOA-NM	2.43E-08	1.12E-04	8.58E-06	2.43E-08	2.12E-04	4.49E-08

TABLE 18. Statistical analysis on Mean Absolute Deviation for Problem 2 during 80 independent runs by proposed algorithm, PSO, CSA and WOA.

Mean Absolute Deviation						
	Min.	Mean	Median	Mod.	Std.	Var.
WOA	1.08E-02	0.1978	1.38E-01	1.08E-02	0.1706	2.91E-02
PSO	1.50E-03	4.94E-02	7.90E-03	1.50E-03	1.35E-01	1.82E-02
CSA	7.70E-03	0.0674	0.0594	7.70E-03	0.049	2.40E-03
LeNN-WOA-NM	1.50E-03	2.60E-03	1.70E-03	1.50E-03	2.10E-03	4.28E-06

$$\epsilon = \frac{1}{N}\sum_{m=1}^{N}\left(\frac{d^3\phi}{d\xi_m^3} + \frac{6}{\xi}\frac{d^2\phi}{d\xi^2} + \frac{6}{\xi_m^2}\frac{d\phi}{d\xi_m} - 6(10 + 2\xi_m^3 + 6\xi_m^6)e^{-3\phi}\right)^2 + \frac{1}{3}\left((\phi_0)^2 + \left(\frac{d\phi(0)}{d\xi}\right)^2 + \left(\frac{d^2\phi(0)}{d\xi^2}\right)^2\right). \quad (41)$$

Problem 3: Consider non liner Emden-Fowler second type equation with $\beta = 2$ and $f(\xi) = 6\,e^{\xi} - 6\xi e^{\xi} - 7\xi^2 e^{\xi} + \xi^6 e^{2\xi}$.

$$\frac{d^3\phi}{d\xi^3} - \frac{2}{\xi}\frac{d^2\phi}{d\xi^2} - \phi(\xi) - \phi^2(\xi) + 6e^{\xi} - 6\xi e^{\xi} - 7\xi^2 e^{\xi} + \xi^6 e^{2\xi} = 0, \quad (42)$$

TABLE 19. Statistical analysis on Theil's inequality coefficient for Problem 2 during 80 independent runs by proposed algorithm, PSO, CSA and WOA.

Theil's inequality coefficient						
	Min.	Mean	Median	Mod.	Std.	Var.
WOA	4.40E-03	0.0688	4.61E-02	4.40E-03	0.0955	9.10E-03
PSO	6.45E-04	1.28E-02	2.60E-03	6.45E-04	3.08E-02	9.50E-04
CSA	2.90E-03	0.021	1.88E-02	2.90E-03	0.0132	1.75E-04
LeNN-WOA-NM	6.81E-04	9.83E-04	7.49E-04	6.81E-04	5.96E-04	3.56E-07

TABLE 20. Statistical analysis on Error in Nash Sutcliffe Efficiency for Problem 2 during 80 independent runs by proposed algorithm, PSO, CSA and WOA.

Error in Nash Sutcliffe Efficiency						
	Min.	Mean	Median	Mod.	Std.	Var.
WOA	2.65E-04	0.1052	0.03	2.65E-04	0.1546	0.0239
PSO	5.87E-06	0.0311	9.35E-05	5.87E-06	0.1318	1.74E-02
CSA	1.14E-04	0.0108	0.0055	1.14E-04	0.0201	4.03E-04
LeNN-WOA-NM	6.54E-06	1.88E-05	7.93E-06	6.54E-06	3.80E-05	1.44E-09

TABLE 21. Statistical analysis on Fitness Analysis for Problem 4 during 80 independent runs by proposed algorithm, PSO, CSA and WOA.

Fitness Analysis						
	Min.	Mean	Median	Mod.	Std.	Var.
WOA	1.70E-03	0.1425	1.49E-01	1.70E-03	0.0811	6.60E-03
PSO	1.30E-03	1.18E-01	1.14E-01	1.30E-03	8.27E-02	6.80E-03
CSA	4.16E-02	9.90E-02	9.75E-02	4.16E-02	3.12E-02	9.73E-04
LeNN-WOA-NM	2.20E-07	8.90E-03	2.10E-03	2.20E-07	1.75E-02	3.08E-04

TABLE 22. Statistical analysis on Mean Absolute Deviation for Problem 4 during 80 independent runs by proposed algorithm, PSO, CSA and WOA.

Mean Absolute Deviation						
	Min.	Mean	Median	Mod.	Std.	Var.
WOA	8.50E-03	0.5099	5.26E-01	8.50E-03	0.2423	5.87E-02
PSO	2.60E-03	4.39E-01	4.25E-01	2.60E-03	2.53E-01	6.42E-02
CSA	3.12E-02	0.4072	0.4456	3.12E-02	0.1522	2.32E-02
LeNN-WOA-NM	7.43E-06	3.68E-02	4.70E-03	7.43E-06	8.37E-02	7.00E-03

TABLE 23. Statistical analysis on Theil's inequality coefficient for Problem 4 during 80 independent runs by proposed algorithm, PSO, CSA and WOA.

Theil's inequality coefficient						
	Min.	Mean	Median	Mod.	Std.	Var.
WOA	2.00E-03	0.1821	2.06E-01	2.00E-03	0.1352	1.83E-02
PSO	5.88E-04	1.70E-01	1.32E-01	5.88E-04	1.32E-01	1.75E-02
CSA	7.50E-03	0.1317	1.38E-01	7.50E-03	0.0625	3.90E-03
LeNN-WOA-NM	1.89E-06	9.70E-03	1.20E-03	1.89E-06	2.40E-02	5.78E-04

TABLE 24. Statistical analysis on Error in Nash Sutcliffe Efficiency for Problem 4 during 80 independent runs by proposed algorithm, PSO, CSA and WOA.

Error in Nash Sutcliffe Efficiency						
	Min.	Mean	Median	Mod.	Std.	Var.
WOA	8.50E-03	0.4649	0.4815	8.50E-03	0.2252	0.0507
PSO	3.34E-02	0.2512	2.47E-01	3.34E-02	0.1264	1.60E-02
CSA	2.86E-02	0.1216	0.1321	2.86E-02	0.0498	2.50E-03
LeNN-WOA-NM	5.12E-09	4.19E-02	2.00E-03	5.50E-03	1.22E-01	1.49E-02

with

$$\phi(0) = 0, \qquad \phi(1) = e \quad \text{and} \quad \phi'(0) = 0,$$

exact solution for Eq (42) is given as $\xi^3 e^\xi$ [45]. Fitness based error function for Eq (42) can be written as

$$\begin{aligned}\epsilon &= \epsilon_1 + \epsilon_2, \qquad (43)\\ \epsilon &= \frac{1}{N}\sum_{m=1}^{N}\left(\xi_m \frac{d^3\phi}{d\xi_m^3} - 2\frac{d^2\phi}{d\xi_m^2} - \xi_m\phi(\xi)\right.\\ &\quad \left. -\xi_m\phi^2(\xi) + 6\xi_m e^\xi - 6\xi e^\xi - 7\xi_m^3 e^\xi + \xi_m^7 e^{2\xi} + \frac{1}{2}\right)^2\\ &\quad +\frac{1}{3}\left((\phi_0)^2 + (\xi(1)-e)^2 + \left(\frac{d\phi(0)}{d\xi}\right)^2\right). \qquad (44)\end{aligned}$$

Problem 4: Let $\beta = 4, f(\xi) = -(10 + 10\xi^3 + \xi^6)$ and $g(\phi) = \phi$ then third order non-linear Emden-Folwer second type differential equation can be written as

$$\frac{d^3\phi}{d\xi^3} + \frac{4}{\xi}\frac{d^2\phi}{d\xi^2} - (10 + 10\xi^3 + \xi^6)\phi = 0, \qquad (45)$$

subjected to initial conditions

$$\phi(0) = 1, \phi'(0) = 0 \quad \text{and} \quad \phi''(0) = 0,$$

$$\begin{aligned}\phi_{approx} &= 0.408826 + (-0.75348\xi - 0.00245)(0.295031)\\ &+ \left(\frac{3(-1.21479\xi + 0.454315)^2 - 1}{2}\right)(-0.16766)\\ &+ \left(\frac{5(-0.04107\xi + 0.152337)^3 - 3(-0.04107\xi + 0.152337)}{2}\right)(-0.06449)\\ &+ \left(\frac{35(0.044146\xi - 0.11558)^4 - 30(0.044146\xi - 0.11558)^2}{8} + \frac{3}{8}\right)(-0.29711)\\ &+ \left(\frac{63(-0.21488\xi - 0.17881)^5 - 70(-0.21488\xi - 0.17881)^3}{8}\right.\\ &\left. + \frac{15(-0.21488\xi - 0.17881)}{8}\right)(-0.09996)\\ &+ \left(\frac{231(0.088679\xi - 0.01669)^6 - 315(0.088679\xi - 0.01669)^4}{16}\right.\\ &\left. + \frac{105(0.088679\xi - 0.01669)^2 - 5}{16}\right)(-0.30059)\\ &+ \left(\frac{429(0.282416\xi - 0.04928)^7 - 693(0.282416\xi - 0.04928)^5}{16}\right.\\ &\left. + \frac{315(0.282416\xi - 0.04928)^2 - 35(0.282416\xi - 0.04928)}{16}\right)(-0.74158)\\ &+ \left(\frac{6435(-0.01182\xi - 0.10028)^8 - 12012(-0.01182\xi - 0.10028)^6}{128}\right.\\ &\left. + \frac{6930(-0.01182\xi - 0.10028)^4 - 1260(-0.01182\xi - 0.10028)^2 + 35}{128}\right)(-0.25962)\\ &+ \left(\frac{12155(-0.19801\xi - 0.06235)^9 - 25740(-0.19801\xi - 0.06235)^7}{128}\right.\\ &+ \frac{18018(-0.19801\xi - 0.06235)^5 - 4620(-0.19801\xi - 0.06235)^3}{128}\\ &\left. + \frac{315(-0.19801\xi - 0.06235)}{128}\right)(0.20468)\\ &+ \left(\frac{46189(-0.19802\xi - 0.02949)^{10} - 109395(-0.19802\xi - 0.02949)^8}{256}\right.\\ &+ \frac{90090(-0.19802\xi - 0.02949)^6 - 30030(-0.19802\xi - 0.02949)^4}{256}\\ &\left. + \frac{3465(-0.19802\xi - 0.02949)^2 - 63}{256}\right)(-0.86668) \qquad (48)\end{aligned}$$

analytical solution obtained by [44] for Eq (45) is $e^{\frac{\xi^3}{3}}$. Fitness based error function for Eq (45) can be formulated as

$$\epsilon = \epsilon_1 + \epsilon_2, \tag{46}$$

equivalently,

$$\epsilon = \frac{1}{N}\sum_{m=1}^{N}\left(\frac{d^3\phi}{d\xi_m^3} + \frac{4}{\xi_m}\frac{d^2\phi}{d\xi_m^2} - (10 + 10\xi_m^3 + \xi_m^6)\phi\right)^2 + \frac{1}{3}\left((\phi_0 - 1)^2 + \left(\frac{d\phi(0)}{d\xi}\right)^2 + \left(\frac{d^2\phi(0)}{d\xi^2}\right)^2\right), \tag{47}$$

VIII. RESULTS AND DISCUSSION

This paper has presented the mathematical formulation and analysis of first and second-type third-order nonlinear multi singular Emden-Fowler equations (TONMS-EFE). Four problems are considered with different shape factor β, $f(\xi)$ and $g(\phi)$. Furthermore, an evolutionary soft computing technique is designed to solve the TONMS-EFE see Eq (8) and Eq (11). Approximate series solutions for different problems obtained by the LeNN-WOA-NM algorithm are compared with PSO, CSA, WOA, and exact solutions [44].

The optimization performance of the proposed technique for Eq (8) and Eq (11) is perform for 80 independent

$$\begin{aligned}
\phi_{approx} = {} & 0.254318 + (1.050468\xi + 0.041362)(0.906695) \\
& + \left(\frac{3(0.823686\xi + 1.375421)^2 - 1}{2}\right)(-0.08091) \\
& + \left(\frac{5(0.04151\xi + 0.049329)^3 - 3(0.04151\xi + 0.049329)}{2}\right)(0.16959) \\
& + \left(\frac{35(-0.49358\xi + 0.571303)^4 - 30(-0.49358\xi + 0.571303)^2}{8} + \frac{3}{8}\right)(0.128204) \\
& + \left(\frac{63(-0.14611\xi - 0.10208)^5 - 70(-0.14611\xi - 0.10208)^3}{8}\right. \\
& \left. + \frac{15(-0.14611\xi - 0.10208)}{8}\right)(-0.00193) \\
& + \left(\frac{231(3.659748\xi + 0.467711)^6 - 315(3.659748\xi + 0.467711)^4}{16}\right. \\
& \left. + \frac{105(3.659748\xi + 0.467711)^2 - 5}{16}\right)(-0.25302) \\
& + \left(\frac{429(0.659495\xi - 0.05414)^7 - 693(0.659495\xi - 0.05414)^5}{16}\right. \\
& \left. + \frac{315(0.659495\xi - 0.05414)^2 - 35(0.659495\xi - 0.05414)}{16}\right)(0.361584) \\
& + \left(\frac{6435(-0.27381\xi + 0.061146)^8 - 12012(-0.27381\xi + 0.061146)^6}{128}\right. \\
& \left. + \frac{6930(-0.27381\xi + 0.061146)^4 - 1260(-0.27381\xi + 0.061146)^2 + 35}{128}\right)(0.000882) \\
& + \left(\frac{12155(0.366455\xi - 0.33945)^9 - 25740(0.366455\xi - 0.33945)^7}{128}\right. \\
& + \frac{18018(0.366455\xi - 0.33945)^5 - 4620(0.366455\xi - 0.33945)^3}{128} \\
& \left. + \frac{315(0.366455\xi - 0.33945)}{128}\right)(0.058533) \\
& + \left(\frac{46189(0.557791\xi + 0.340865)^{10} - 109395(0.557791\xi + 0.340865)^8}{256}\right. \\
& + \frac{90090(0.557791\xi + 0.340865)^6 - 30030(0.557791\xi + 0.340865)^4}{256} \\
& \left. + \frac{3465(0.557791\xi + 0.340865)^2 - 63}{256}\right)(0.349444)
\end{aligned} \tag{49}$$

executions. The graphical performance of the design scheme for all four problems is illustrated in Figures 3-10. Approximate solutions obtained by the LeNN-WOA-NM algorithm for problem 1, 2, 3, and IV are demonstrated through Figures 3(a), 3(b), 3(c) and 3(d) respectively. The unknown weights in LeNN for calculation of best solutions are visualized in Figure 4. The absolute error graphs from the exact solution are demonstrated in Figure 5 for each problem. Figure 6 depicts the comparison of the minimum, mean, median, mode, standard deviation, and variance of fitness, MAD, TIC, and ENSE obtained by LeNN-WOA-NM algorithm with CSA, PSO, and WOA for the four problems.

Tables 3 and 4 represents the comparison of solutions at each step size. The values of absolute errors (AE) in Tables 5 and 6 lie around -10^{-12} to -10^{-14}, -10^{-5} to -10^{-8}, -10^{-8} to -10^{-10} and -10^{-7} to -10^{-9} for problem 1, 2, 3 and 4 respectively. Unknown weights obtained by proposed algorithm for optimization of fitness function Eqs (38), (41), (44) and (47) are dictated in Tables 7 and 8. It is clear

$$
\begin{aligned}
\phi_{approx} = {} & -0.0002552 + (0.5471011\xi - 0.0915667)(-0.7318864) \\
& + \left(\frac{3(0.90428437\xi + 0.8850403)^2 - 1}{2}\right)(-0.0911262) \\
& + \left(\frac{5(-0.1608536\xi - 0.2076278)^3 - 3(-0.1608536\xi - 0.2076278)}{2}\right)(0.75734771) \\
& + \left(\frac{35(-0.31751\xi + 0.816637)^4 - 30(-0.31751\xi + 0.816637)^2}{8} + \frac{3}{8}\right)(-0.295161) \\
& + \left(\frac{63(-0.0390982\xi + 0.67666348)^5 - 70(-0.0390982\xi + 0.67666348)^3}{8}\right. \\
& \left. + \frac{15(-0.0390982\xi + 0.67666348)}{8}\right)(-0.0707813) \\
& + \left(\frac{231(0.4659305\xi + 0.2888783)^6 - 315(0.4659305\xi + 0.2888783)^4}{16}\right. \\
& \left. + \frac{105(0.4659305\xi + 0.2888783)^2 - 5}{16}\right)(0.78111333) \\
& + \left(\frac{429(0.65746316\xi + 0.43559105)^7 - 693(0.65746316\xi + 0.43559105)^5}{16}\right. \\
& \left. + \frac{315(0.65746316\xi + 0.43559105)^2 - 35(0.65746316\xi + 0.43559105)}{16}\right)(-0.084667) \\
& + \left(\frac{6435(0.867974\xi + 0.640784)^8 - 12012(0.867974\xi + 0.640784)^6}{128}\right. \\
& \left. + \frac{6930(0.867974\xi + 0.640784)^4 - 1260(0.867974\xi + 0.640784)^2 + 35}{128}\right)(0.00571) \\
& + \left(\frac{12155(-0.0082419\xi - 0.4193387)^9 - 25740(-0.0082419\xi - 0.4193387)^7}{128}\right. \\
& + \frac{18018(-0.0082419\xi - 0.4193387)^5 - 4620(-0.0082419\xi - 0.4193387)^3}{128} \\
& \left. + \frac{315(-0.0082419\xi - 0.4193387)}{128}\right)(-0.1662331) \\
& + \left(\frac{46189(0.02639607\xi + 0.00074241)^{10} - 109395(0.02639607\xi + 0.00074241)^8}{256}\right. \\
& + \frac{90090(0.02639607\xi + 0.00074241)^6 - 30030(0.02639607\xi + 0.00074241)^4}{256} \\
& \left. + \frac{3465(0.02639607\xi + 0.00074241)^2 - 63}{256}\right)(0.5335582)
\end{aligned}
\tag{50}
$$

from Tables 9, 13, 17 and 21 the objective values (fitness values) lie round 10^{-12}, 10^{-6}, 10^{-8} and 10^{-7} for problem 1 to 4 respectively. It is clear from Tables 10, 14, 18 and 22 that values of mean absolute deviation (MAD) lie round 10^{-9}, 10^{-3}, 10^{-3} and 10^{-6} for problem 1 to 4 respectively. It is clear from Tables 11, 16, 19 and 23 that values of Theil's inequality coefficient (TIC) lie round 10^{-9}, 10^{-3}, 10^{-4} and 10^{-6} for problem 1 to 4 respectively. It is clear from Tables 12, 17, 20 and 24 that values of Error in Nash Sutcliffe efficiency (ENSE) lie round 10^{-12}, 10^{-4}, 10^{-6} and 10^{-9} for problem 1 to 4 respectively.

Bar graphs given in Figure 6 demonstrates the comparison of values of fitness, MAD, TIC, and ENSE obtained by the LeNN-WOA-NM algorithm with PSO, CSA, and WOA for each problem. The convergence of fitness value, MAD, TIC, and ENSE during 80 independent runs are shown through Figure 7-10. Normal probability curves and boxplots for performance indicators are shown in Figures 11-18. Extensive statistical and graphical analysis illustrates the effectiveness of the proposed algorithm in solving nonlinear-multi singular differential equations.

$$
\begin{aligned}
\phi_{approx} = {} & 0.73308615 + (0.51580967\xi + 0.70179544)(0.5402041) \\
& + \left(\frac{3(0.44315273\xi + 0.20778173)^2 - 1}{2}\right)(0.46296807) \\
& + \left(\frac{5(0.23240265\xi + 0.48175825)^3 - 3(0.23240265\xi + 0.48175825)}{2}\right)(0.22761992) \\
& + \left(\frac{35(0.696457\xi + 0.193803)^4 - 30(0.696457\xi + 0.193803)^2}{8} + \frac{3}{8}\right)(0.476976) \\
& + \left(\frac{63(0.6068485\xi + 0.57723262)^5 - 70(0.6068485\xi + 0.57723262)^3}{8}\right. \\
& \left. + \frac{15(0.6068485\xi + 0.57723262)}{8}\right)(0.16011539) \\
& + \left(\frac{231(0.1941152\xi + 0.31740822)^6 - 315(0.1941152\xi + 0.31740822)^4}{16}\right. \\
& \left. + \frac{105(0.1941152\xi + 0.31740822)^2 - 5}{16}\right)(0.39211912) \\
& + \left(\frac{429(0.3545657\xi + 0.21980521)^7 - 693(0.3545657\xi + 0.21980521)^5}{16}\right. \\
& \left. + \frac{315(0.3545657\xi + 0.21980521)^2 - 35(0.3545657\xi + 0.21980521)}{16}\right)(0.46826468) \\
& + \left(\frac{6435(0.20798017\xi + 0.17682492)^8 - 12012(0.20798017\xi + 0.17682492)^6}{128}\right. \\
& \left. + \frac{6930(0.20798017\xi + 0.17682492)^4 - 1260(0.20798017\xi + 0.17682492)^2 + 35}{128}\right)(0.28322033) \\
& + \left(\frac{12155(0.534765583\xi + 0.41144797)^9 - 25740(0.534765583\xi + 0.41144797)^7}{128}\right. \\
& + \frac{18018(0.534765583\xi + 0.41144797)^5 - 4620(0.534765583\xi + 0.41144797)^3}{128} \\
& \left. + \frac{315(0.534765583\xi + 0.41144797)}{128}\right)(0.21299134) \\
& + \left(\frac{46189(0.13224697\xi + 0.31148234)^{10} - 109395(0.13224697\xi + 0.31148234)^8}{256}\right. \\
& + \frac{90090(0.13224697\xi + 0.31148234)^6 - 30030(0.13224697\xi + 0.31148234)^4}{256} \\
& \left. + \frac{3465(0.13224697\xi + 0.31148234)^2 - 63}{256}\right)(0.00013506) \qquad (51)
\end{aligned}
$$

Nomenclature:

Abreviation	Discriptions
LeNN	Legendre Neural Networks
NM	Nelder-Mead
MAD	Mean Absolute Diviation
TIC	Theil's inequality coefficient
NSE	Nash Sutcliffe efficiency
ENSE	Error in Nash Sutcliffe efficiency
PSO	Particle Swarm Optimization
CSA	Cuckoo search Algorithm
WOA	Whale Optimization Algorithm
β	Shape factor
α	Reflection Coefficient
δ	Shrink
γ	Contraction
t	Current iteration in WOA
X^*	Best value obtained so far
$\vec{A}, \vec{C}$	Coefficient Vectors
$\vec{X}_{rand}$	Random whale

IX. CONCLUSION

In this work, we have formulated third-order nonlinear multi singular Emden-Fowler equations. Furthermore, we have designed novel soft computing that hybridized global search exploitation of WOA with local search exploration of the NM algorithm. The combination is named as LeNN-WOA-NM algorithm. Weighted Legendre polynomials are used to model approximate series solutions for third-order nonlinear multi-singular Emden-Fowler differential equations, and fitness functions are constructed to evaluate the candidate solutions. Some significant findings of the study are summarized below as:

- The design of a soft computing paradigm, the LeNN-WOA-NM algorithm, is effectively applied to solve nonlinear multi-singular third-order Emden–Fowler models of the first and second type.
- The accuracy and robustness of the present scheme are proven by comparing the proposed results with the exact solutions, PSO, CSA, and WOA for different Emden–Fowler equation problems.
- The statistical analysis and assessments based on 80 independent executions of the LeNN-WOA-NM algorithm establish the accuracy and convergence of the proposed algorithm for solving real-world problems.

Approximate solution for Eq (36) is given (48), as shown at the bottom of the 23rd page.

Approximate solution for Eq (39) is given (49), as shown at the bottom of the 24th page.

Approximate solution for Eq (42) is given (50), as shown at the bottom of the 25th page.

Approximate solution for Eq (45) is given (51), as shown at the bottom of the 26th page.

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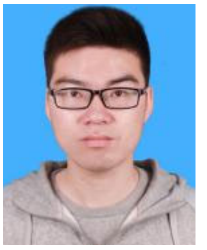

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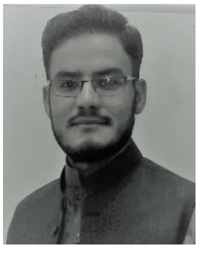

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