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Robust Asynchronous Switching Fault-Tolerant Control for Multi-Phase Batch Processes With Time-Varying Delay

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ABSTRACT A robust asynchronous switching fault-tolerant control method is proposed to solve the problems of uncertainties, unknown disturbances, time-varying delays and partial actuator failures in multi-phase batch processes. Firstly, an asynchronous system composed of subsystems with different dimensions including stable and unstable case is established to describe such multi-phase batch processes more accurately. Then introducing the output tracking error, the established switching model of different dimensions is extended. On this basis, a robust asynchronous switching fault-tolerant control law is designed, which improves the system's ability to cope with negative factors such as actuator failure and can obtain greater adjustment freedom. Secondly, by using relevant theories and methods, the sufficient conditions in the form of linear matrix inequality (LMI) are given to ensure the exponential stability of the system and the asymptotic stability at each phase. By solving these LMIs conditions, the shortest running time under stable case, the longest running time under unstable case and the control law gain of each phase are obtained. Finally, the effectiveness and feasibility of the proposed method are verified with injection molding process.

INDEX TERMS Asynchronous switching, fault-tolerant control, time-varying delays, multi-phase batch processes.

I. INTRODUCTION

With people's increasingly personalized and diversified needs, batch processes with high adjustability and small-scale are more popular. It has been widely used in many fields such as chemistry, pharmacy and injection molding. In view of the high accuracy and complexity of the control of batch process, a good deal of more mature control method applied to continuous processes are difficult to effectively control the batch process. How to make batch process run efficiently and stably has attracted extensive attention from scholars and researchers at home and abroad [1]–[5].

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The modern batch processes are becoming more and more complex. And the actual production processes are affected by many uncertain factors, such as equipment wear and tear, line aging and changes in production environment, etc. Therefore, uncertainty, unknown disturbances and time-varying delays are problem that have to be considered. Liu *et al.* [6] considered the influence of uncertainties, unknown disturbances and time-varying delays in the study of robust model predictive control based on matrix inequality. Shi *et al.* [7] considered the problems caused by uncertainties, unknown disturbances and time-varying delays in the robust model predictive control for industrial processes. However, compared with problems such as uncertainty, unknown disturbances and time-varying delays, the impact of failures on the system is more fatal. Considering the continuous high-intensity operation of production equipment and corresponding control systems. The control system will inevitably break down. Caccavale *et al.* [8] put forward a comprehensive judgment scheme using redundant temperature measurement combined with state observer. Kerkhof *et al.* [9] proposed a fault diagnosis scheme based on dynamic model for batch processes. Wang *et al.* [10] proposed a scheme based on dynamic event trigger fault estimation and sliding mode fault tolerance control. The above [8]–[10] on fault research are fault detection methods, and there is no in-depth study on how to deal with faults. Usually, the fault can be divided into sensor fault, system internal fault and actuator fault. The actuator is the part with the highest intensity and frequency in the whole system, so the failure of the actuator is also the most common. If the failure of the actuator cannot be dealt with effectively and promptly, it will degrade the performance of the control system. In the worst case, it will cause production stagnation and even casualties. Therefore, the study of fault-tolerant control has more practical significance [11], [12]. Wang *et al.* [13] adopted a fault tolerant control method of iterative learning in the batch process. Zhang *et al.* [14] proposed a minimum quadratic linear fault-tolerant tracking control scheme for batch process. Shi *et al.* [15] proposed a robust constraint model to predict fault-tolerant control for industrial processes with actuator fault. Shi *et al.* [16] proposed a fuzzy predictive fault-tolerant control method for nonlinear system with partial actuator failures. In [13]–[16], in view of the problem of partial actuator failure, the control method adopted is to introduce the failure factor into the controller design to overcome the effect of actuator failure on the system. Furthermore, time delay is also an important factor affecting system stability and control performance. It is common to adopt iterative learning fault-tolerant control method for batch processes with time delay and failure. Wang *et al.* [17] proposed a two-dimensional robust iterative learning fault-tolerant control (FTC) method for batch process with time delay and actuator failures. Wang *et al.* [18] put forward an iterative learning fault-tolerant control method considering uncertainty and cost function in controller design. Tao *et al.* [19] proposed a fault-tolerant iterative learning control method for batch processes with actuator failures and differential time-delay. Wang *et al.* [20] developed a fuzzy iterative learning fault-tolerant controller for batch processes with time delays and actuator failures. These studies [17]–[20] aim at the batch processes with time delay and actuator failures and obtain a better control effect. But their research focuses on the single-phase batch processes.

In the actual industrial production, the batch processes are mostly multi-phase. Wang *et al.* [21] proposed a robust iterative learning fault-tolerant control method for multi-phase batch processes (MPBP) with actuator failures and uncertainties under the framework of a two-dimensional theory. Shen *et al.* [22] proposed a two-dimensional iterative learning fault-tolerant control method for MPBP with time delays and failures. Wang *et al.* [23] proposed a two-dimensional robust mixed iterative learning method with guaranteed cost for MPBP with time delay, disturbances and actuator failures. In the above research on MPBP with failures [21]–[23], iterative learning control method is adopted. If there are non-repetitive disturbances in the system or batch length is inconsistent, the control effect of iterative learning control method will be greatly worsened. Yu *et al.* [24] proposed a robust predictive fault-tolerant control method for MPBP with time-varying delays, uncertainty and disturbances. This control method can effectively solve the problem of inconsistent batch length in the iterative learning control method. However, the above [21]–[24] adopt the control method of synchronous switching. Due to the influence of some uncontrollable factors in the actual industrial production process, the system state and the controller are difficult to keep in sync, i.e., the model and controller is mismatched. Therefore, Wang *et al.* [25] proposed an iterative learning control method for multi-phase batch processes with asynchronous switching. However, the control scheme in [25] uses the same control law from beginning to end. Hence it cannot to deal with the deviation of the system state.

To sum up, a robust model predictive fault-tolerant control method is proposed for MPBP with uncertainties, unknown disturbances, interval time-varying delays, and partial actuator failures. Based on model-dependent average dwell time method [26]–[29], Lyapunov stability theory and switching theory, the sufficient conditions with linear matrix inequality (LMI) form are given. By solving these LMI conditions, the control gain of the system, the shortest running time (SRT) for each phase of stable case and the longest running time (LRT) for each phase of unstable case are obtained. Depending on LRT of the unstable case, we put the switching signal in advance. It can ensure that the model matches the controller, and the stability is avoided. Compared with other studies, the contributions of this paper are as follows.

(1) A switching model of extended state space containing stable and unstable conditions at each phase is established to describe a class of MPBP with time-varying delays and partial actuator failures. Based on this model, a robust model predictive fault-tolerant control law is designed, which can effectively improve the control performance and improve the degree of freedom of the system.

(2) When the switching occurs, the model has completed the switching. However, it is difficult for the controller to switch in time due to some factors such as system identification speed, signal transmission blocked. This will lead to a mismatch between the model and the controller. Compared with literature [21]–[24], the asynchronous switching method proposed in this paper is more in line with the actual situation of industrial application. By calculating the LRT under unstable case, the switching signal is given in advance to match the model with the controller. The smooth operation of the system can be ensured.

(3) Compared with the asynchronous switched method in [25], the method proposed in this paper can effectively deal with the problems of non-repetitive disturbances and inconsistent batch length in the system. Furthermore, [25]

has cannot to deal with the deviation of the system state. The control scheme in [25] uses the same control law from beginning to end, and the deviation of the system state will become larger and larger as time goes by. The method proposed in this paper can solve the control gain in real time to avoid the increase of system deviation over time.

II. PROCESS FORMULATION

A. ESTABLISHMENT OF MODEL

For a MPBP with uncertainties, unknown disturbances and time-varying delays, a state space model is proposed to describe it as follows.

$$\begin{cases} x^{\Gamma(k)}(k+1) = A^{\Gamma(k)}(k)x(k) + A_d^{\Gamma(k)}(k)x^{\Gamma(k)}(k-d(k)) \\ \qquad\qquad + B^{\Gamma(k)}u^{\Gamma(k)}(k) + \omega^{\Gamma(k)}(k) \\ y^{\Gamma(k)}(k) = C^{\Gamma(k)}x^{\Gamma(k)}(k) \end{cases} \tag{1}$$

where $x^{\Gamma(k)}(k) \in R^{n_x^{\Gamma(k)}}$, $u^{\Gamma(k)}(k) \in R^{n_u^{\Gamma(k)}}$, $y^{\Gamma(k)}(k) \in R^{n_y^{\Gamma(k)}}$ and $\omega^{\Gamma(k)}(k)$ are the system state, control input, system output and unknown disturbances at the discrete-time k, respectively. $d(k)$ represents the time-varying delays based on discrete time k, which satisfies

$$d_m \leq d(k) \leq d_M \tag{2}$$

where d_M and d_m are the upper and lower bounds of the time-varying delays $d(k)$. $\Gamma(k)$ is the signal of whether the system has a switchover, which satisfies $\Gamma(k) : R^+ \to p := \{1, 2, 3 \cdots, P\}$, p is the phase of the system. In the pth phase, the sub-model of the system is $A^p(k) = A^p + \Delta_a^p(k)$ and $A_d^p(k) = A_d^p + \Delta_d^p(k)$, where A^p, A_d^p, B^p and C^p are the constant matrices matching the system dimension, respectively. $\Delta_a^p(k)$ and $\Delta_d^p(k)$ are the uncertain matrices matching the system dimension respectively, which can be expressed as follows.

$$\begin{bmatrix} \Delta_a^p(k) & \Delta_d^p(k) \end{bmatrix} = N^p\Delta^p(k)\begin{bmatrix} H^p & H_d^p \end{bmatrix} \tag{3}$$

$$\Delta^{p\mathrm{T}}(k)\Delta^p(k) \leq I^p \tag{4}$$

where N^p, H^p and H_d^p are constant matrices. $\Delta^p(k)$ is the uncertainties under the discrete time k.

When $\Gamma(k) = p$, Eq. (1) can be rewritten as Eq. (5). Considering the problem of asynchronous switching, we divide each phase into stable and unstable cases. Eq. (5a) is the model of the stable case at pth phase and Eq. (5b) is the model of the unstable case at pth phase. Therefore, Eq. (1) is transformed as follow form.

$$\begin{cases} x^p(k+1) = A^p(k)x^p(k) + A_d^p(k)x^p(k-d(k)) \\ \qquad\qquad + B^pu^p(k) + \omega^p(k) \\ y^p(k) = C^px^p(k) \end{cases} \tag{5a}$$

$$\begin{cases} x^p(k+1) = A^p(k)x^p(k) + A_d^p(k)x^p(k-d(k)) \\ \qquad\qquad + B^pu^{p-1}(k) + \omega^p(k) \\ y^p(k) = C^px^p(k) \end{cases} \tag{5b}$$

Furthermore, in the actual industrial production processes, controller failure is inevitable because of long-term continuous operation. Therefore, the actuator cannot get the correct calculated value and operate normally. In general, actuator failure includes three types, namely partial failure ($\alpha^p > 0$), complete failure ($\alpha^p = 0$) and stuck failure ($\alpha^p > u_\alpha^p$). Because of the latter two kinds of failures, the system cannot be operated, so the case of $\alpha^p > 0$ situation is studied emphatically. The following equation is defined.

$$u^{pF}(k) = \alpha^pu^p(k) \tag{6}$$

$$0 \leq \underline{\alpha}^p \leq \alpha^p \leq \bar{\alpha}^p \tag{7}$$

where $u^p(k)$ is the control input of the controller, $u^{pF}(k)$ is the control value in the case of partial failure of the actuator, $\underline{\alpha}^p \leq 1$ and $\bar{\alpha}^p \geq 1$ are known scalars.

To improve the control performance of the controller in case of failure, the following equation is introduced.

$$\beta^p = \frac{\bar{\alpha}^p + \underline{\alpha}^p}{2}, \quad \beta_0^p = \frac{\bar{\alpha}^p - \underline{\alpha}^p}{\bar{\alpha}^p + \underline{\alpha}^p} \tag{8}$$

Combining Eqs. 6-8, the following equation can be obtained.

$$\alpha^p = (I^p + \alpha_0^p)\beta^p \tag{9}$$

with

$$\left|\alpha_0^p\right| \leq \beta_0^p \leq I^p$$

After considering actuator failure, the state space model describing MPBP in the pth phase is rewritten as:

$$\begin{cases} x^p(k+1) = A^p(k)x^p(k) + A_d^p(k)x^p(k-d(k)) \\ \qquad\qquad + B^p\alpha^pu^p(k) + \omega^p(k) \\ y(k) = C^px^p(k) \end{cases} \tag{10a}$$

$$\begin{cases} x^p(k+1) = A^p(k)x^p(k) + A_d^p(k)x^p(k-d(k)) \\ \qquad\qquad + B^p\alpha^pu^{p-1}(k) + \omega^p(k) \\ y^p(k) = C^px^p(k) \end{cases} \tag{10b}$$

Due to each batch contains multi-phases, switching between phases is inevitable. Therefore, the system states of the two adjacent phases meet the following equation.

$$x^p(T^{p-1}) = \Re^px^{p-1}(T^{p-1}) \tag{11}$$

where $\Re^p$ represents a state-transition matrix between the p-1th phase and the pth phase.

If the switching occurs between adjacent phases, the switching conditions are as follows.

$$\Gamma(k+1) = \begin{cases} \Gamma(k) + 1 & if\ \gamma^{\Gamma(k)+1}(x(k)) < 0 \\ \Gamma(k) & other \end{cases} \tag{12}$$

where $\gamma^{\Gamma(k)+1}(x(k)) < 0$ is a switching condition that determines whether a switching has occurred.

Since the system must switch between two phases, and the switching time is also a key factor affecting the system, the relationship between the time of each switching point and the switching conditions can be expressed as the following equation.

$$T^p = \min\left\{k > T^{p-1} | \gamma^p(x(k)) < 0\right\}, \quad T^0 = 0 \tag{13}$$

In the pth phase, T^{pS} and T^{pU} are used to represent the switching time of stable case and unstable case in the same phase. In order to more clearly describe the multi-phase batch process with asynchronous switching, a time series is defined. It contains the switching point and the completion point of the adjacent phase in a batch.

$$\sum = \Big\{(T^{1s}, \Gamma(T^{1s})), (T^{2U}, \Gamma(T^{2U})), (T^{2S}, \Gamma(T^{2S})), (T^{3U}, \Gamma(T^{3U})), \cdots, (T^{pU}, \Gamma(T^{pU})), (T^{pS}, \Gamma(T^{pS}))\Big\} \quad (14)$$

where $(T^{pU}, \Gamma(T^{pU}))$ refers to the unstable switching point in the pth phase caused by the controller switching untimely. $(T^{pS}, \Gamma(T^{pS}))$ refers to the point where the controller has completed the switching.

And the switching time of the two phases are satisfied $T^{pS} - T^{pU} \geq \tau_S^p$, $T^{pU} - T^{(p-1)U} \leq \tau_U^p$, where τ_S^p and τ_U^p are the SRT of the stable case and the LRT of the unstable case, respectively.

B. EXTENDED STATE SPACE MODEL

To reduce the steady-state error of the system and improve the regulating ability of the controller, an incremental state is adopted. And output tracking error is introduced to establish a new state-space model as shown in the following equation.

$$\begin{cases} \bar{x}^{pS}(k+1) = A_k^p(k)\bar{x}^{pS}(k) + A_{dk}^p(k)\bar{x}^{pS}(k-d(k)) \\ \qquad\qquad + B_k^p\alpha^p\Delta u^p(k) + G_k^p\bar{\omega}_k^p(k) \\ \Delta y^p(k) = C_k^p\bar{x}^{pS}(k) \\ z^{pS}(k) = e^{pS}(k) = E_k^p\bar{x}^{pS}(k) \end{cases} \quad (15a)$$

$$\begin{cases} \bar{x}^{pU}(k+1) = A_k^p(k)\bar{x}^{pU}(k) + A_{dk}^p(k)\bar{x}^{pU}(k-d(k)) \\ \qquad\qquad + B_k^p\alpha^p\Delta u^{p-1}(k) + G_k^p\bar{\omega}_k^p(k) \\ \Delta y^p(k) = C_k^p\bar{x}^{pU}(k) \\ z^{pU}(k) = e^{pU}(k) = E_k^p\bar{x}^{pU}(k) \end{cases} \quad (15b)$$

where

$$\bar{x}^{pS}(k) = \begin{bmatrix} \Delta x^p(k) \\ e^{pS}(k) \end{bmatrix}, \quad \bar{x}^{pU}(k) = \begin{bmatrix} \Delta x^p(k) \\ e^{pU}(k) \end{bmatrix},$$

$$\bar{x}^{pS}(k-d(k)) = \begin{bmatrix} \Delta x^{pS}(k-d(k)) \\ e^{pS}(k-d(k)) \end{bmatrix},$$

$$\bar{x}^{pU}(k-d(k)) = \begin{bmatrix} \Delta x^{pU}(k-d(k)) \\ e^{pU}(k-d(k)) \end{bmatrix},$$

$$A_k^p(k) = A_k^p + \Delta_{ak}^p(k), \quad A_k^p = \begin{bmatrix} A^p & 0 \\ C^pA^p & I \end{bmatrix},$$

$$\Delta_{ak}^p(k) = N_k^p\Delta^p(k)H_k^p, \quad A_{dk}^p = \begin{bmatrix} A_d^p & 0 \\ C^pA_d^p & 0 \end{bmatrix},$$

$$\Delta_{dk}^p(k) = N_k^p\Delta^p(k)H_{dk}^p, \quad B_k^p = \begin{bmatrix} B^p \\ C^pB^p \end{bmatrix},$$

$$N_k^p = \begin{bmatrix} N^p \\ C^pN^p \end{bmatrix}, \quad G_k^p = \begin{bmatrix} I^p \\ C^p \end{bmatrix},$$

$$C_k^p = \begin{bmatrix} C^p & 0 \end{bmatrix}, \quad E_k^p = \begin{bmatrix} 0 & I^p \end{bmatrix},$$

$$H_k^p = \begin{bmatrix} H^p & 0 \end{bmatrix}, \quad H_{dk}^p = \begin{bmatrix} H_d^p & 0 \end{bmatrix},$$

$$\begin{aligned} \bar{\omega}^{pS}(k) &= (\Delta_a^p(k) - \Delta_a^p(k-1))x^p(k-1) \\ &\quad + (\Delta_d^p(k) - \Delta_d^p(k-1)) \\ &\quad \cdot x^p(k-1-d(k-1)) + \Delta\omega^p(k), \\ \bar{\omega}^{pU}(k) &= (\Delta_a^p(k) - \Delta_a^p(k-1))x^p(k-1) \\ &\quad + (\Delta_d^p(k) - \Delta_d^p(k-1))x^p(k-1 \\ &\quad - d(k-1)) + \Delta\omega^p(k), \\ \Delta\omega^p(k) &= \omega^p(k) - \omega^p(k-1). \end{aligned}$$

And $\vartheta^p(k)$ is the setpoint of the batch processes in the pth phase. The output tracking error of the batch processes are defined as $e^p(k) = y^p(k) - \vartheta^p(k)$. The following equation can be obtained.

$$\begin{aligned} e^{pS}(k+1) &= e^{pS}(k) + C^pA^p(k)\Delta x^p(k) + C^pB^p(k)\alpha^p\Delta u^p(k) \\ &\quad + C^p\bar{\omega}^{pS}(k) \\ e^{pU}(k+1) &= e^{pU}(k) + C^pA^p(k)\Delta x^p(k) \\ &\quad + C^pB^p(k)\alpha^p\Delta u^{p-1}(k) + C^p\bar{\omega}^{pU}(k) \end{aligned}$$

Hence, the model of the extended system in two contiguous phases meet the following equation:

$$\begin{aligned} \begin{bmatrix} \Delta x^{p+1}(T^p) \\ e^{p+1}(T^p) \end{bmatrix} &= \begin{bmatrix} \Re^p\Delta x^p(T^p) \\ C^p\Re^p(\Delta x^p(T^p) + x^p(T^p-1)) - \vartheta^{p+1} \end{bmatrix} \\ &= \begin{bmatrix} \Re^p \\ C^p\Re^p \end{bmatrix} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \Delta x^p(T^p) \\ e^p(T^p) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ C^p\Re^px^p(T^p-1) - \vartheta^{p+1} \end{bmatrix} \end{aligned} \quad (16)$$

where

$$\bar{\Re}^p = \begin{bmatrix} \Re^p \\ C^p\Re^p \end{bmatrix}\begin{bmatrix} I & 0 \end{bmatrix}, \quad \Upsilon^p = \begin{bmatrix} 0 \\ C^p\Re^px^p(T^p-1) - \vartheta^{p+1} \end{bmatrix},$$

then $\bar{x}^{p+1}(k) = \bar{\Re}^p\bar{x}^p(k) + \Upsilon^p$.

Remark 1: In order to clearly show the design idea of the controller in this paper, a block diagram of each batch is given. As shown in Figure 1.

III. DESIGN OF FAULT CONTROLLER

In this section, a robust asynchronous switch fault tolerant controller is designed to keep the MPBP stable during switching and occurring failure. Then the SRT of each stable case and the LRT of each unstable case can be solved by the following theorems. The proposed control method can not only guarantee the stability of the system in the event of failure, but also guarantee the smooth stability of the system when switching between two adjacent phases.

A. DESIGN OF THE FAULT-TOLERANT CONTROL LAW UNDER ASYNCHRONOUS CONDITION

The fault-tolerant control law based on the Eq. (15) in the stable and unstable cases are designed, the following equation can be obtained.

$$\Delta u^p(k) = K_k^p\bar{x}^{pS}(k) = K_k^p\begin{bmatrix} \Delta x^p(k) \\ e^{pS}(k) \end{bmatrix} \quad (17a)$$

$$\Delta u^{p-1}(k) = K_k^{p-1}\bar{x}^{(p-1)U}(k)$$

$$
\begin{aligned}
&= K_k^{p-1}\left(\left(\Re^{p-1}\right)^{-1}\bar{x}^p(k)+\left(\Re^{p-1}\right)^{-1}\Upsilon^{p-1}\right)\\
&= K_k^{p-1}\left(\Re^{p-1}\right)^{-1}\begin{bmatrix}\Delta x^p(k)\\ e^p(k)\end{bmatrix}\\
&\quad + K_k^{p-1}\left(\Re^{p-1}\right)^{-1}\Upsilon^{p-1}
\end{aligned} \tag{17b}
$$

where K_k^p and K_k^{p-1} are the control law gains of the system in the phase pth and phase p-1th, respectively.

Due to $\bar{x}^{p-1}(k) = (\bar{\Re}^{p-1})^{-1}\cdot\bar{x}^p(k) - (\bar{\Re}^{p-1})^{-1}\cdot\Upsilon^{p-1} \leq (\bar{\Re}^{p-1})^{-1}\cdot\bar{x}^p(k)$, $\Upsilon^{p-1}\geq 0$, the following equation can be obtained by scaling the Eq. (17b).

$$\Delta u^{p-1}(k) \leq K_k^{p-1}(\Re^{p-1})^{-1}\bar{x}^p(k) \tag{18}$$

Therefore, the Eq. (18) is substituted into the Eq. (15) to obtain the state space model of the closed-loop system in the stable case and unstable case. The following equations can be obtained.

$$
\begin{cases}
\bar{x}^{pS}(k+1) = A_k^p(k)\bar{x}^{pS}(k) + A_{dk}^p(k)\bar{x}^{pS}(k-d(k))\\
\qquad + B_k^p(k)\alpha^p\Delta u^p(k) + G_k^p\bar{\omega}_k^p(k)\\
\Delta y^p(k) = C_k^p\bar{x}^{pS}(k)\\
z^{pS}(k) = e^{pS}(k) = E_k^p\bar{x}^{pS}(k)
\end{cases} \tag{19a}
$$

$$
\begin{cases}
\bar{x}^{pU}(k+1) = A_k^p(k)\bar{x}^{pU}(k) + A_{dk}^p(k)\bar{x}^{pU}(k-d(k))\\
\qquad + B_k^p(k)\alpha^p K_k^{p-1}(\Re^{p-1})^{-1}\bar{x}_k^p(k)\\
\qquad + G_k^p\bar{\omega}_k^p(k)\\
\Delta y^p(k) = C_k^p\bar{x}^{pU}(k)\\
z^{pU}(k) = e^{pU}(k) = E_k^p\bar{x}^{pU}(k)
\end{cases} \tag{19b}
$$

B. DEFINITIONS AND LEMMAS

Due to design requirements, the following definitions and lemmas are given.

Definition 1 (Robust MPC Problem): In order to achieve the robust predictive control objective of batch processes with asynchronous condition, the following performance index of the robust predictive control are considered:

$$
\min_{\Delta u(k+i),i\geq 0}\ \max_{[A^p(k+i)\ A_d^p(k+i)\ B^p]\in\Omega,i\geq 0} J_\infty^p
$$

$$
\begin{aligned}
J_\infty^p(k) = \sum_{i=0}^{\infty}&[(x_1^p(k+i|k))^{\mathrm{T}}Q^p(x_1^p(k+i|k))\\
&+u^p(k+i|k)^{\mathrm{T}}R^p u^p(k+i|k)]
\end{aligned} \tag{20}
$$

where $Q^p > 0$ and $R^P > 0$ are the weighting matrices of the pth phase for system state variables and the control input, respectively. $x^p(k+i|k)$ and $u^p(k+i|k)$ are the state and input of the system at time the $k+i$ predicted by at the time k in the pth phase.

Definition 2 [26]: In a batch process, k satisfies $k = O \geq d = T_O^p$ for any discrete time. $\mathrm{N}_\Gamma^{pS}(d,O)$ denotes the number in the time interval $[d,O]$ for the switching signal Γ appearing in the pth phase of each batch. τ_S^p represents the running time of the system during a single stable case in a batch. In conclusion, we can get $\mathrm{N}_\Gamma^{pS}(d,O) \leq \mathrm{N}_0^S + \frac{T_S^p(d,O)}{\tau_S^p}$, where N_0^S is chatter bound and $\tau_S^p > 0$. Similarly, the switching number of the system in unstable case can be obtained as $\mathrm{N}_\Gamma^{pU}(d,O) \leq \mathrm{N}_0^U + \frac{T_U^p(d,O)}{\tau_U^p}$, where N_0^U is chatter bound and $\tau_U^p > 0$.

Definition 3 [27]: Given a scalar $r^p > 0$, the asymptotic stability of the system with unknown disturbance $\bar{\omega}^p(k)$ can be guaranteed, and the output $z^p(k)$ of the system can also satisfy $||z^p|| \leq r^p||\bar{\omega}^p||$. The discrete-time robust H_∞ performance is considered.

Lemma 1 [28]: Consider the matrices W, L and V of the appropriate dimensions, where W and L are real matrices, and the following formula can be obtained.

$$L^{\mathrm{T}}VL - W < 0 \tag{21}$$

if and only if

$$\begin{bmatrix}-W & L^{\mathrm{T}}\\ L & -V^{-1}\end{bmatrix} < 0 \quad\text{or}\quad \begin{bmatrix}-V^{-1} & L\\ L^{\mathrm{T}} & -W\end{bmatrix} < 0 \tag{22}$$

Lemma 2 [29]: For any vector $\bar{\delta}(k) \in R^n$, two positive integers κ_0, κ_1, and matrix $0 < \bar{R} \in R^{n\times n}$, the following inequality holds.

$$-(\kappa_1-\kappa_0+1)\sum_{t=\kappa_0}^{\kappa_1}\bar{\delta}^{\mathrm{T}}(k)\bar{R}\bar{\delta}(k) \leq -\sum_{t=\kappa_0}^{\kappa_1}\bar{\delta}^{\mathrm{T}}(k)\bar{R}\sum_{t=\kappa_0}^{\kappa_1}\bar{\delta}^{\mathrm{T}}(k) \tag{23}$$

Lemma 3 [30]: Let D, F, E and M be real matrices of appropriate dimensions with satisfying $M = M^{\mathrm{T}}$, then for all $F^{\mathrm{T}}F \leq I$

$$M + DFE + E^{\mathrm{T}}F^{\mathrm{T}}D^{\mathrm{T}} < 0 \tag{24}$$

if and only if there exists $\varepsilon > 0$ such that

$$M + \varepsilon^{-1}DD^{\mathrm{T}} + \varepsilon E^{\mathrm{T}}E < 0 \tag{25}$$

Lemma 4: For the pth phase, it will ensure the sub-system to be asymptotically stable, if the Lyapunov function $V^p(\bar{x}_1(k))$ can be found and satisfy the following conditions:

1) $V^p(\bar{x}_1(k)) \geq 0$ for $\bar{x}(k) \in R^{n_x+n_e}$, and $V^p(\bar{x}(k)) = 0 \leftrightarrow \bar{x}(k) = 0$;

2) $V^p(\bar{x}(k)) = \infty \leftrightarrow \|\bar{x}(k)\| = \infty$;

3) For any bounded condition, $0 < \Xi^p < 1$

$$V^p(\bar{x}(k+1)) \leq \Xi^p V^p(\bar{x}(k)) \tag{26}$$

C. MAIN THEOREM AND COROLLARY

In this section, the robust predict fault-tolerant control law is designed, including two theorems and the corresponding proof. Theorem 1 is a sufficient condition to ensure the stability of the MPBP with uncertainties, time-varying delays and partial actuator failure. Theorem 2 considers unknown disturbances on the basis of Theorem 1.

Theorem 1: The considered system (19) with $\bar{\omega} = 0$ is the asymptotical stability in each phase and exponentially stable in a batch, if there are some scalar $0 < \varsigma_p^S < 1$, $\varsigma_p^U > 1$, $\theta^p > 0$, $\theta^{p-1} > 0$, $0 \leq d_m \leq d_M$, some positive definite matrices

Q^{pS}, R^{pS}, $Q^{(p-1)U}$, $R^{(p-1)U}$, some positive definite symmetric matrices P_1^{pS}, T_1^{pS}, M_1^{pS}, G_1^{pS}, L_1^{pS}, S_1^{pS}, S_2^{pS}, M_3^{pS}, M_4^{pS}, X_1^{pS}, X_2^{pS}, β^{pS}, Y^{pS}, $P_1^{(p-1)U}$, $T_1^{(p-1)U}$, $M_1^{(p-1)U}$, $G_1^{(p-1)U}$, β^{pU} Y^{pU}, $L_1^{(p-1)U}$, $S_1^{(p-1)U}$, $S_2^{(p-1)U}$, $M_3^{(p-1)U}$, $M_4^{(p-1)U}$, $X_1^{(p-1)U}$, $X_2^{(p-1)U} \in R^{(n_x+n_e)}$ and some scalars ε_1^{pS}, ε_2^{pS}, ε_1^{pU}, ε_2^{pU}, $\theta^P > 0$, $\mu_p^S > 1$, $0 < \mu_p^U < 1$, so that the following LMIs hold.

$$\begin{bmatrix} \Lambda_{11}^{pS} & \Lambda_{12}^{pS} & \Lambda_{13}^{pS} & \Lambda_{14}^{pS} & \Lambda_{15}^{pS} \\ * & \Lambda_{22}^{pS} & 0 & 0 & 0 \\ * & * & \Lambda_{33}^{pS} & 0 & 0 \\ * & * & * & \Lambda_{44}^{pS} & 0 \\ * & * & * & * & \Lambda_{55}^{pS} \end{bmatrix} < 0 \tag{27}$$

$$\begin{bmatrix} \Lambda_{11}^{pU} & \Lambda_{12}^{pU} & \Lambda_{13}^{pU} & \Lambda_{14}^{pU} & \Lambda_{15}^{pU} \\ * & \Lambda_{22}^{pU} & 0 & 0 & 0 \\ * & * & \Lambda_{33}^{pU} & 0 & 0 \\ * & * & * & \Lambda_{44}^{pU} & 0 \\ * & * & * & * & \Lambda_{55}^{pU} \end{bmatrix} < 0 \tag{28}$$

$$\begin{cases} V_p^S(\bar{x}_1(k)) \le \mu_p^S V_{p-1}^S(\bar{x}_1(k)) \\ V_p^S(\bar{x}_1(k)) \le \mu_p^S V_p^U(\bar{x}_1(k)) \\ V_p^U(\bar{x}_1(k)) \le \mu_p^U V_{p-1}^S(\bar{x}_1(k)) \end{cases} \tag{29}$$

$$\begin{bmatrix} -1 & \bar{x}^{p\mathrm{T}}(k|k) \\ \bar{x}^p(k|k) & -P_1^p \end{bmatrix} \le 0 \tag{30}$$

$$\Lambda_{11}^{pS} = \begin{bmatrix} \phi_1^{pS} & 0 & (\varsigma_p^S)^{d_M} L_1^{pS} \\ * & -(\varsigma_p^S)^{d_M} S_1^{pS} & 0 \\ * & * & -(\varsigma_p^S)^{d_M}((M_4^{pS}) + (X_1^{pS})) \end{bmatrix},$$

$$\Lambda_{12}^{pS} = \begin{bmatrix} L_1^{pS} A_k^{p\mathrm{T}} + Y_1^{pS\mathrm{T}} \beta^{pS} B_k^{p\mathrm{T}} & L_1^{pS} A_k^{p\mathrm{T}} + Y_1^{pS\mathrm{T}} \beta^{pS} B_k^{p\mathrm{T}} - L_1^{pS} \\ S_1^{pS} A_{dk}^{p\mathrm{T}} & S_1^{pS} A_{dk}^{p\mathrm{T}} \\ 0 & 0 \end{bmatrix},$$

$$\Lambda_{13}^{pS} = \begin{bmatrix} L_1^{pS}(Q^{pS})^{\frac{1}{2}} & Y_1^{pS\mathrm{T}}(R^{pS})^{\frac{1}{2}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Lambda_{14}^{pS} = \Lambda_{15}^{pS} = \begin{bmatrix} L_1^{pS} H_k^{p\mathrm{T}} & Y_1^{pS\mathrm{T}} \beta^{pS} \\ S_1^{pS} H_{dk}^{p\mathrm{T}} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Lambda_{22}^{pS} = \begin{bmatrix} -L_1^{pS} + \varepsilon_1^{pS} N_k^p N_k^{p\mathrm{T}} + \varepsilon_2^{pS} B_k^p (\beta_0^{pS})^2 B_k^{p\mathrm{T}} & 0 \\ 0 & -X_1^{pS}(D_2^p)^{-1} + \varepsilon_1^{pS} N_k^p N_k^{p\mathrm{T}} + \varepsilon_2^{pS} B_k^p (\beta_0^{pS})^2 B_k^{p\mathrm{T}} \end{bmatrix},$$

$$\Lambda_{33}^{pS} = \begin{bmatrix} -\theta^p I^p & 0 \\ 0 & -\theta^p I^p \end{bmatrix}, \quad \Lambda_{44}^{pS} = \Lambda_{55}^{pS} = \begin{bmatrix} -\varepsilon_1^{pS} I^p & 0 \\ 0 & -\varepsilon_2^{pS} I^p \end{bmatrix},$$

$$\Lambda_{11}^{pU} = \begin{bmatrix} \phi_1^{pU} & 0 & (\varsigma_p^U)^{d_M} L_1^{(p-1)U} \\ * & -(\varsigma_p^{\mathrm{U}})^{d_M} S_1^{(p-1)U} & 0 \\ * & * & -(\varsigma_p^U)^{d_M}((M_4^{(p-1)U}) + (X_1^{(p-1)U})) \end{bmatrix},$$

$$\Lambda_{12}^{pU} = \begin{bmatrix} L_1^{(p-1)U} A_k^{p\mathrm{T}} + L_1^{(p-1)U} \left((\Re^{p-1})^{-1}\right)^{\mathrm{T}} \left(K^{p-1}\right)^{\mathrm{T}} \beta^{pU} B_k^{p\mathrm{T}} & L_1^{(p-1)U} A_k^{p\mathrm{T}} + L_1^{(p-1)U} \left((\Re^{p-1})^{-1}\right)^{\mathrm{T}} \left(K^{p-1}\right)^{\mathrm{T}} \beta^{pU} B_k^{p\mathrm{T}} - L_1^{(p-1)U} \\ S_1^{(p-1)U} A_{dk}^{p\mathrm{T}} & S_1^{(p-1)U} A_{dk}^{p\mathrm{T}} \\ 0 & 0 \end{bmatrix},$$

$$\Lambda_{13}^{pU} = \begin{bmatrix} L_1^{(p-1)U}(Q^{(p-1)U})^{\frac{1}{2}} & L_1^{(p-1)U} \left((\Re^{p-1})^{-1}\right)^{\mathrm{T}} \left(K^{p-1}\right)^{\mathrm{T}} (R^{(p-1)U})^{\frac{1}{2}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Lambda_{14}^{pU} = \Lambda_{15}^{pU} = \begin{bmatrix} L_1^{(p-1)U} H_k^{p\mathrm{T}} & L_1^{(p-1)U} \left((\Re^{p-1})^{-1}\right)^{\mathrm{T}} \left(K^{p-1}\right)^{\mathrm{T}} \beta^{pU} \\ S_1^{(p-1)U} H_{dk}^{p\mathrm{T}} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Lambda_{22}^{pU} = \begin{bmatrix} -L_1^{(p-1)U} + \varepsilon_1^{pU} N_k^p N_k^{p\mathrm{T}} + \varepsilon_2^{pU} B_k^p (\beta_0^{pU})^2 B_k^{p\mathrm{T}} & 0 \\ 0 & -X_1^{(p-1)U}(D_2^p)^{-1} + \varepsilon_1^{pU} N_k^p N_k^{p\mathrm{T}} + \varepsilon_2^{pU} B_k^p (\beta_0^{pU})^2 B_k^{p\mathrm{T}} \end{bmatrix},$$

$$\Lambda_{33}^{pU} = \begin{bmatrix} -\theta^{p-1} I^{p-1} & 0 \\ 0 & -\theta^{p-1} I^{p-1} \end{bmatrix}, \quad \Lambda_{44}^{pU} = \Lambda_{55}^{pU} = \begin{bmatrix} -\varepsilon_1^{pU} I^p & 0 \\ 0 & -\varepsilon_2^{pU} I^p \end{bmatrix},$$

$$\phi_1^{pS} = -\varsigma_p^S L_1^{pS} + M_3^{pS} + D_1^p S_2^{pS} + S_2^{pS} - (\varsigma_p^S)^{d_M} X_2^{pS},$$

$$\phi_1^{pU} = -\varsigma_p^U L_1^{(p-1)U} + M_3^{(p-1)U} + D_1^p S_2^{(p-1)U} + S_2^{(p-1)U} - (\varsigma_p^U)^{d_M} X_2^{(p-1)U}, \; D_1^p = (d_M - d_m + 1) I^p, \; D_2^p = (d_M)^2 I^p.$$

where $\Lambda_{11}^{pS}, \Lambda_{12}^{pS}, \Lambda_{13}^{pS}, \Lambda_{14}^{pS}, \Lambda_{22}^{pS}, \Lambda_{33}^{pS}, \Lambda_{11}^{pU}, \Lambda_{12}^{pU}, \Lambda_{13}^{pU}, \Lambda_{14}^{pU}, \Lambda_{22}^{pU}, \Lambda_{33}^{pU}$, and ϕ_1^{pS} as shown at the bottom of the previous page.

In Eq. (27) and Eq. (28), "$*$" is derived from symmetry and "0" is the zero matrix of the corresponding dimension.

The SRT of stable case and the LRT of unstable case in each phase are as follows.

$$\begin{cases} \tau_S^p \geq -\dfrac{\ln \mu_p^S}{\ln \varsigma_p^S} \\ \tau_U^p \leq -\dfrac{\ln \mu_p^U}{\ln \varsigma_p^U} \end{cases} \tag{31}$$

Proof 1: When $\bar{\omega}_k^p = 0$, the proof of the stable case and unstable case of the MPBP in the pth is as follows.

The stable case of the pth phase is proof:

Firstly, the asymptotic stability the uncertain discrete-time closed system (19) with w(k) = 0 are proved. When there is no disturbance in the system, according to Lemma 3, the Lyapunov function and robust performance indexes of the system meet the following equation:

$$\begin{aligned} &V(\bar{x}^{pS}(k+i+1|k)) - V(\bar{x}^{pS}(k+i|k)) \\ &\leq -[(\bar{x}^{pS}(k+i|k)^{\mathrm{T}} Q^{pS} (\bar{x}^{pS}(k+i|k) \\ &\quad + \Delta u^{pS}(k+i|k)^{\mathrm{T}} R^{pS} \Delta u^{pS}(k+i|k)] \end{aligned} \tag{32}$$

Accumulating left and right sides of Eq. (32) from $i = 0$ to ∞ and requiring that $V(\bar{x}_k^{pS}(\infty)) = 0$ or $\bar{x}_k^{pS}(\infty) = 0$, the following equation can be obtained:

$$J_\infty^{pS}(k) \leq V_p^S(\bar{x}_k^p(k)) \leq \theta^p \tag{33}$$

where θ^p is the upper bound of $J_\infty^{pS}(k)$.

The selection of Lyapunov-Krasovskii function can be expressed as follows.

$$V_p^S(x^p(k+i)) = \sum_{j=1}^{5} V_j^{pS}(x^p(k+i)) \tag{34}$$

where, to simplify, define

$$\begin{aligned} \bar{x}_d^p(k+i) &= \bar{x}^p(k+i-d(k+i)), \\ \bar{x}_{d_M}^p(k+i) &= \bar{x}^p(k+i-d_M), \\ \delta^p(k+i) &= \bar{x}^p(k+i+1) - \bar{x}^p(k+i), \\ \varphi^p(k+i) &= \left[\bar{x}^{p\mathrm{T}}(k+i) \quad \bar{x}_d^{p\mathrm{T}}(k+i) \quad \bar{x}_{d_M}^{p\mathrm{T}}(k+i)\right]^{\mathrm{T}}, \end{aligned}$$

$$\begin{aligned} V_1^{pS}(x_k^p(k+i)) &= \bar{x}^{p\mathrm{T}}(k+i) P_1^{pS} \bar{x}^p(k+i) \\ &= \bar{x}^{p\mathrm{T}}(k+i)\theta^p (L_1^{pS})^{-1} \bar{x}^p(k+i), \\ V_2^{pS}(\bar{x}^p(k+i)) &= \sum_{r=k-d(k)}^{k-1} \bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-1-r} T_1^{pS} \bar{x}^p(r+i), \\ V_3^{pS}(\bar{x}^p(k+i)) &= \sum_{r=k-d_M}^{k-1} \bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-1-r} M_1^{pS} \bar{x}^p(r+i) \\ &= \sum_{r=k-d_M}^{k-1} \bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-1-r} \theta^p (M_2^{pS})^{-1} \\ &\quad \times \bar{x}^p(r+i), \\ V_4^{pS}(\bar{x}^p(k+i)) &= \sum_{s=-d_M}^{-d_m} \sum_{r=k+s}^{k-1} \bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-1-r} T_1^{pS} \\ &\quad \times \bar{x}^p(r+i) \\ &= \sum_{s=-d_M}^{-d_m} \sum_{r=k+s}^{k-1} \bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-1-r} \theta^p \\ &\quad \times (S_1^{pS})^{-1} \bar{x}^p(r+i), \\ V_5^{pS}(\bar{x}^p(k+i)) &= d_M \sum_{s=-d_M}^{-1} \sum_{r=k+s}^{k-1} \delta^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-1-r} \\ &\quad \times G_1^{pS} \delta^p(r+i) \\ &= d_M \sum_{s=-d_M}^{-1} \sum_{r=k+s}^{k-1} \delta^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-1-r} \\ &\quad \times \theta^p (X_1^{pS})^{-1} \delta^p(r+i), \end{aligned}$$

$P_1^{pS}, T_1^{pS}, M_1^{pS}, M_2^{pS}$ and G_1^{pS} are positive definite matrices. Let

$$\begin{aligned} \xi^p(k+i) &= \Big[\bar{x}^p(k+i)^{\mathrm{T}} \quad \bar{x}^p(k+i-d(k))^{\mathrm{T}} \quad \cdots \\ &\qquad \bar{x}^p(k+i-d_M)^{\mathrm{T}} \quad \cdots \quad \delta^p(k+i-1)^{\mathrm{T}}\Big], \\ \psi_1^{pS} &= \mathrm{diag}\left[P_1^{pS} \quad T_1^{pS} \quad \cdots \quad M_1^{pS} \quad \cdots \quad d_M G_1^{pS}\right], \\ (\Pi^{pS})^{-1} &= \mathrm{diag}\Big[(L_1^{pS})^{-1} \quad (S_1^{pS})^{-1} \quad \cdots \quad (M_2^{pS})^{-1} \quad \cdots \\ &\qquad d_M (X_1^{pS})^{-1}\Big]. \end{aligned}$$

Therefore, the following Eq. (35) can be obtained.

$$\begin{aligned} V_p^S(\bar{x}^p(k+i)) &= \xi^{p\mathrm{T}}(k+i)\psi_1^p \xi^p(k+i) \\ &= \xi^{p\mathrm{T}}(k+i)\theta^p (\Pi^{pS})^{-1} \xi^p(k+i) \end{aligned} \tag{35}$$

By multiplying Eq. (35) with differential operator Δ, the following expression is designed.

$$\begin{aligned} \Delta V_p^S(\bar{x}^p(k+i)) &\leq V_p^S(\bar{x}^p(k+i+1)) - \varsigma_p^S V_p^S(\bar{x}^p(k+i)) \\ &= \sum_{j=1}^{5} \Delta V_j^{pS}(\bar{x}^p(k+i)) \end{aligned} \tag{36}$$

where

$$\begin{aligned} &\Delta V_1^{pS}(\bar{x}(k+i)) \\ &= \bar{x}^{p\mathrm{T}}(k+i+1)\theta^p (L_1^{pS})^{-1} \bar{x}^p(k+i+1) \\ &\quad - \varsigma_p^S \bar{x}^{p\mathrm{T}}(k+i)\theta^p (L_1^{pS})^{-1} \bar{x}^p(k+i), \\ &\Delta V_2^{pS}(\bar{x}^p(k+i)) \\ &= \sum_{r=k+1-d(k+1)}^{k} \bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-r} \theta^p (S_1^{pS})^{-1} \bar{x}^p(r+i) \\ &\quad - \sum_{r=k-d(k)}^{k-1} \varsigma_p^S \bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-1-r} \theta^p (S_1^{pS})^{-1} \bar{x}^p(r+i) \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{r=k+1-d_M}^{k-1} \bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-r}\theta^p(S_1^{pS})^{-1}\bar{x}^p(r+i) \\
&\quad + \bar{x}^{p\mathrm{T}}(k+i)\theta^p(S_1^{pS})^{-1}\bar{x}^p(k+i) \\
&\quad - \sum_{r=k-d_m}^{k-1} \varsigma_p^S\bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-1-r}\theta^p(S_1^{pS})^{-1}\bar{x}^p(r+i) \\
&= \bar{x}^{p\mathrm{T}}(k+i)\theta^p(S_1^{pS})^{-1}\bar{x}^p(k+i) \\
&\quad - \bar{x}_d^{p\mathrm{T}}(k+i)(\varsigma_p^S)^{d_M}\theta^p(S_1^{pS})^{-1}\bar{x}_d^p(k+i) \\
&\quad + \sum_{r=k-d_M+1}^{k-d_m} \bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-r}\theta^p(S_1^{pS})^{-1}\bar{x}^p(r+i), \\
&\Delta V_3^{pS}(\bar{x}^p(k+i)) \\
&= \sum_{r=k+1-d_M}^{k} \bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-r}\theta^p(M_2^{pS})^{-1}\bar{x}^p(r+i) \\
&\quad - \sum_{r=k-d_M}^{k-1} \varsigma_p^S\bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-1-r}\theta^p(M_2^{pS})^{-1}\bar{x}^p(r+i) \\
&= \sum_{r=k+1-d_M}^{k-1} \bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-r-1}\theta^p(M_2^{pS})^{-1}\bar{x}^p(r+i) \\
&\quad + \bar{x}^{p\mathrm{T}}(k+i)\theta^p(M_2^{pS})^{-1}\bar{x}^p(k+i) \\
&\quad - \bar{x}_{d_M}^{p\mathrm{T}}(k+i)(\varsigma_p^S)^{d_M}\theta^p(M_2^{pS})^{-1}\bar{x}_{d_M}^p(k+i), \\
&\quad - \sum_{r=k+1-d_M}^{k-1} \varsigma_p^S\bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-1-r}\theta^p(M_2^{pS})^{-1}\bar{x}^p(r+i) \\
&= \bar{x}^{p\mathrm{T}}(k+i)\theta^p(M_2^{pS})^{-1}\bar{x}^p(k+i) \\
&\quad - \bar{x}_{d_M}^{p\mathrm{T}}(k+i)(\varsigma_p^S)^{d_M}\theta^p(M_2^{pS})^{-1}\bar{x}_{d_M}^p(k+i), \\
&\Delta V_4^{pS}(\bar{x}^p(k+i)) \\
&= \sum_{s=-d_M}^{-d_m}\sum_{r=k+s+1}^{k} \bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-r}\theta^p(S_1^{pS})^{-1}\bar{x}^p(r+i) \\
&\quad - \sum_{s=-d_M}^{-d_m}\sum_{r=k+s}^{k-1} \varsigma_p^S x^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-1-r}\theta^p(S_1^{pS})^{-1}\bar{x}^p(r+i) \\
&< (d_M - d_m + 1)\bar{x}^{p\mathrm{T}}(k+i)\theta^p(S_1^{pS})^{-1}\bar{x}^p(k+i) \\
&\quad - \sum_{r=k-d_M+1}^{k-d_m} \bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-r}\theta^p(S_1^{pS})^{-1}\bar{x}^p(r+i), \\
&\Delta V_5^{pS}(\bar{x}^p(k+i)) \\
&= d_M \sum_{s=-d_M}^{-1}\sum_{r=k+s+1}^{k} \delta^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-r}\theta^p(X_1^{pS})^{-1} \\
&\quad \cdot \delta^p(r+i) - d_M \sum_{s=-d_M}^{-1}\sum_{r=k+s}^{k-1} \varsigma_p^S\delta^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-1-r} \\
&\quad \cdot \theta^p(X_1^{pS})^{-1}\delta^p(r+i) \\
&= d_M^2\delta^{p\mathrm{T}}(k+i)\theta^p(X_1^{pS})^{-1}\delta^p(k+i).
\end{aligned}
$$

According to Lemma 2, the following expression can be obtained.

$$
\begin{aligned}
&\Delta V_5^{pS}(\bar{x}^p(k+i)) \\
&\le d_M^2\delta^{p\mathrm{T}}(k+i)\theta^p(X_1^{pS})^{-1}\delta^p(k+i) \\
&\quad - \sum_{r=k-d_M}^{k-1} \delta^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-r}\theta^p(X_1^{pS})^{-1}\sum_{r=k-\tilde{d}_M}^{k-1}\delta^p(r+i) \\
&< d_M^2(\bar{x}^p(k+i+1) - \bar{x}^p(k+i))^{\mathrm{T}}\theta^p(X_1^{pS})^{-1}(\bar{x}^p(k+i+1) \\
&\quad - \bar{x}^p(k+i)) - (\bar{x}^p(k+i) - x_{d_M}^p(k+i))^{\mathrm{T}}(\varsigma_p^S)^{d_M}\theta^p(X_1^{pS})^{-1} \\
&\quad \cdot (\bar{x}^p(k+i) - \bar{x}_{d_M}^p(k+i))
\end{aligned} \tag{37}
$$

Based on the Eq. (32), it holds that

$$
(\theta^p)^{-1}\Delta V_p^S(\bar{x}^p(k+i|k)) + (\theta^p)^{-1}J^{pS}(k) \le 0, \tag{38}
$$

where

$$
\begin{aligned}
J^{pS}(k) = (\bar{x}^p(k+i|k))^{\mathrm{T}}Q_1^{pS}(\bar{x}^p(k+i|k)) \\
+ (\Delta u^{pS}(k+i|k))^{\mathrm{T}}R_1^{pS}
\end{aligned}
$$

$\Delta u^{pS}(k+i|k)$ is the optimal performance index.

Considering Eqs. (36)-(38), the following expression can be obtained.

$$
\begin{aligned}
&(\theta^p)^{-1}\Delta V_p^S(\bar{x}^p(k+i)) + (\theta^p)^{-1}J^{pS}(k) < \varphi^{p\mathrm{T}}(k)\Phi^{pS}\varphi^p(k) \\
&\Phi^{pS} = \begin{bmatrix} \phi_{1k}^{pS} & 0 & (\varsigma_p^S)^{\tilde{d}_M}(X_1^{pS})^{-1} \\ * & -(\varsigma_p^S)^{d_M}(S_1^{pS})^{-1} & 0 \\ * & 0 & -(\varsigma_p^S)^{\tilde{d}_M}((M_2^{pS})^{-1} + (X_1^{pS})^{-1}) \end{bmatrix} \\
&\quad + \Lambda_1^{p\mathrm{ST}}(L_1^{pS})^{-1}\Lambda_1^{pS} + \Lambda_2^{p\mathrm{ST}}(D_2^{pS})^2(X_1^{pS})^{-1}\Lambda_2^{pS} \\
&\quad + \lambda_1^{p\mathrm{ST}}(\theta^p)^{-1}\lambda_1^{pS} + \lambda_2^{pST}(\theta^p)^{-1}\lambda_2^{pS} \\
&\phi_{1k}^{pS} = -\varsigma_p^S(L_1^{pS})^{-1} + (S_1^{pS})^{-1} + (M_2^{pS})^{-1} + D_1^p(S_1^{pS})^{-1} \\
&\quad - (\varsigma_p^S)^{d_M}(X_1^{pS})^{-1}, \\
&\Lambda_1^{pS} = \begin{bmatrix} A_{kb}^{pS}(k) & A_{dk}^p(k) & 0 \end{bmatrix}, \\
&\Lambda_2^{pS} = \begin{bmatrix} A_{kb}^{pS}(k) - I & A_{dk}^p(k) & 0 \end{bmatrix}, \\
&\lambda_1^{pS} = \begin{bmatrix} (Q_1^{pS})^{\frac{1}{2}} & 0 & 0 \end{bmatrix}, \\
&\lambda_2^{pS} = \begin{bmatrix} (R_1^{pS})^{\frac{1}{2}}Y_1^{pS}(L_1^{pS})^{-1} & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{39}
$$

According to the Lemma 1, it can be expressed as the following LMI form (40), as shown at the bottom of the next page.

Pre- and post-multiplying LMI condition (43) by diag$[\,L_1^{pS}\ S_1^{pS}\ X_1^{pS}\ I^p\ I^p\ I^p\ I^p\,]$ and let $L_1^{pS}(M_2^{pS})^{-1}L_1^{pS} = M_3^{pS}, L_1^{pS}(S_1^{pS})^{-1}L_1^{pS} = S_2^{pS}, L_1^{pS}(X_1^{pS})^{-1}L_1^{pS} = X_2^{pS}, X_1^{pS}$ $(M_2^{pS})^{-1}X_1^{pS} = M_4^{pS}, K^{pS} = Y_1^{pS}(L_1^{pS})^{-1}$, the following equation can be obtained.

According to Eq. (41), as shown at the bottom of the next page, and Lemma 3, the Eq. (27) can be obtained.

The unstable case of the pth phase is proof:

When the MPBP under the asynchronous condition, the state of the system has been switched, but the controller of the system has not been switched. Therefore, the control law K_k^{p-1} is a known quantity. Similar to the stable case, the unstable phase proved by Lemma 4 satisfies $V_U^p(\bar{x}_k(k+1)) \leq \varsigma_p^U V_U^p(\bar{x}_k(k))$. Similar to the proof of steady case, the following Lyapunov-Krasovskii function can be obtained.

$$\begin{aligned} V_p^U(\bar{x}^p(k+i)) &= \xi^{p\mathrm{T}}(k+i)\psi_1^{pU}\xi^p(k+i) \\ &= \xi^{p\mathrm{T}}(k+i)\theta^p(\Pi^{pU})^{-1}\xi^p(k+i) \end{aligned} \tag{42}$$

where

$$\begin{aligned} \xi^p(k+i) &= \Big[x^p(k+i)^{\mathrm{T}} \quad x^p(k+i-d(k))^{\mathrm{T}} \quad \cdots \\ &\qquad x^p(k+i-d_M)^{\mathrm{T}} \quad \cdots \quad \delta^p(k+i-1)^{\mathrm{T}}\Big] \\ \psi_1^{pU} &= \mathrm{diag}\Big[P_1^{(p-1)U} \quad T_1^{(p-1)U} \quad \cdots \quad M_1^{(p-1)U} \\ &\qquad \cdots \quad d_M G_1^{(p-1)U}\Big] \\ (\Pi^{pU})^{-1} &= \mathrm{diag}\Big[(L_1^{(p-1)U})^{-1} \quad (S_1^{(p-1)U})^{-1} \quad \cdots \\ &\qquad (M_2^{(p-1)U})^{-1} \quad \cdots \quad d_M(X_1^{(p-1)U})^{-1}\Big] \end{aligned}$$

The incremental form of Eq. (42) can be obtained as follow.

$$\begin{aligned} \Delta V_p^U(\bar{x}^p(k+i)) &\leq V_p^U(\bar{x}^p(k+i+1)) - \varsigma_p^U V_p^U(\bar{x}^p(k+i)) \\ &= \sum_{j=1}^{5}\Delta V_j^{pU}(\bar{x}^p(k+i)) \end{aligned} \tag{43}$$

where

$$\begin{aligned} &\Delta V_1^{pU}(\bar{x}(k+i)) \\ &= \bar{x}^{p\mathrm{T}}(k+i+1)\theta^p(L_1^{(p-1)U})^{-1}\bar{x}^p(k+i+1) \\ &\quad - \varsigma_p^S\bar{x}^{p\mathrm{T}}(k+i)\theta^p(L_1^{pS})^{-1}\bar{x}^p(k+i) \\ &\Delta V_2^{pU}(\bar{x}^p(k+i)) \\ &= \sum_{r=k+1-d(k+1)}^{k}\bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-r}\theta^p(S_1^{pS})^{-1}\bar{x}^p(r+i) \\ &\quad - \sum_{r=k-d(k)}^{k-1}\varsigma_p^S\bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-1-r}\theta^p(S_1^{pS})^{-1}\bar{x}^p(r+i) \\ &\leq \bar{x}^{p\mathrm{T}}(k+i)\theta^p(S_1^{pS})^{-1}\bar{x}^p(k+i) \\ &\quad - \bar{x}^{p\mathrm{T}}(k+i)(\varsigma_p^S)^{d_M}\theta^p(S_1^{pS})^{-1}\bar{x}^p(k+i) \\ &\quad + \sum_{r=k-d_M+1}^{k-d_m}\bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-r}\theta^p(S_1^{pS})^{-1}\bar{x}^p(r+i) \\ &\Delta V_3^{pU}(\bar{x}^p(k+i)) \\ &= \sum_{r=k+1-d_M}^{k}\bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-r}\theta^p(M_2^{pS})^{-1}\bar{x}^p(r+i) \\ &\quad - \sum_{r=k-d_M}^{k-1}\varsigma_p^S\bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-1-r}\theta^p(M_2^{pS})^{-1}\bar{x}^p(r+i) \end{aligned}$$

$$\begin{bmatrix} \phi_k^{pS} & 0 & (\varsigma_p^S)^{d_M}(X_1^{pS})^{-1} & A_{kb}^{pS}(k) & (A_{kb}^{pS}(k)-I)^{\mathrm{T}} & (Q_1^{pS})^{\frac{1}{2}} & ((L_1^{pS})^{-1})^{\mathrm{T}}Y_1^{pS\mathrm{T}}((R_1^{pS})^{\frac{1}{2}})^{\mathrm{T}} \\ * & -(\varsigma_p^S)^{\tilde{d}_M}(S_1^{pS})^{-1} & 0 & A_{dk}^{p\mathrm{T}}(k) & A_{dk}^{p\mathrm{T}}(k) & 0 & 0 \\ * & * & -(\varsigma_p^S)^{d_M}((M_2^{pS})^{-1}+(X_1^{pS})^{-1}) & 0 & 0 & 0 & 0 \\ * & * & * & -L_1^{pS} & 0 & 0 & 0 \\ * & * & * & * & -(D_2^p)^{-1}X_1^{pS} & 0 & 0 \\ * & * & * & * & * & -\theta^p I^p & 0 \\ * & * & * & * & * & * & -\theta^p I^p \end{bmatrix} < 0 \tag{40}$$

$$\begin{bmatrix} \phi_1^{pS} & 0 & (\varsigma_p^S)^{d_M}L_1^{pS} & L_1^{pS}A_k^{p\mathrm{T}}(k)+Y_1^{pS\mathrm{T}}B_k^{p\mathrm{T}}(k) & L_1^{pS}A_k^{p\mathrm{T}}(k)+Y_1^{pS\mathrm{T}}B_k^{p\mathrm{T}}(k)-L_1^{pS} & L_1^{pS}(Q_1^{pS})^{\frac{1}{2}} & Y_1^{pS\mathrm{T}}(R_1^{pS})^{\frac{1}{2}} \\ * & -(\varsigma_p^S)^{d_M}S_1^{pS} & 0 & S_1^{pS}A_{dk}^{p\mathrm{T}}(k) & S_1^{pS}A_{dk}^{p\mathrm{T}}(k) & 0 & 0 \\ * & * & -(\varsigma_p^S)^{d_M}((M_4^{pS})+(X_1^{pS})) & 0 & 0 & 0 & 0 \\ * & * & * & -L_1^{pS} & 0 & 0 & 0 \\ * & * & * & * & -(D_2^p)^{-1}X_1^{pS} & 0 & 0 \\ * & * & * & * & * & -\theta^p I^p & 0 \\ * & * & * & * & * & * & -\theta^p I^p \end{bmatrix} < 0 \tag{41}$$

$$= \bar{x}^{p\mathrm{T}}(k+i)\theta^{p}(M_2^{pS})^{-1}\bar{x}^{p}(k+i) - \bar{x}^{p\mathrm{T}}(k+i)(\varsigma_p^S)^{d_M}\theta^{p}(M_2^{pS})^{-1}\bar{x}^{p}(k+i)$$

$$\Delta V_4^{pU}(\bar{x}^{p}(k+i)) = \sum_{s=-d_M}^{-d_m}\sum_{r=k+s+1}^{k}\bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-r}\theta^{p}(S_1^{pS})^{-1}\bar{x}^{p}(r+i) - \sum_{s=-d_M}^{-d_m}\sum_{r=k+s}^{k-1}\varsigma_p^S\bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-1-r}\theta^{p}(S_1^{pS})^{-1}\bar{x}^{p}(r+i)$$
$$< (d_M-d_m+1)\bar{x}^{p\mathrm{T}}(k+i)\theta^{p}(S_1^{pS})^{-1}\bar{x}^{p}(k+i) - \sum_{r=k-d_M+1}^{k-d_m}\bar{x}^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-r}\theta^{p}(S_1^{pS})^{-1}\bar{x}^{p}(r+i)$$

$$\Delta V_5^{pU}(\bar{x}^{p}(k+i)) = d_M\sum_{s=-d_M}^{-1}\sum_{r=k+s+1}^{k}\delta^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-r}\theta^{p}(X_1^{pS})^{-1}\cdot\delta^{p}(r+i) - d_M\sum_{s=-d_M}^{-1}\sum_{r=k+s}^{k-1}\varsigma_p^S\delta^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-1-r}\cdot\theta^{p}(X_1^{pS})^{-1}\delta^{p}(r+i)$$
$$= d_M^2\delta^{p\mathrm{T}}(k+i)\theta^{p}(X_1^{pS})^{-1}\delta^{p}(k+i) - d_M\sum_{r=k-d_M}^{k-1}\delta^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-r}\theta^{p}(X_1^{pS})^{-1}\delta^{p}(r+i)$$

According to the Lemma 2, the following expression can be obtained.

$$\Delta V_5^{pU}(\bar{x}^{p}(k+i)) \le d_M^2\delta^{p\mathrm{T}}(k+i)\theta^{p}(X_1^{pS})^{-1}\delta^{p}(k+i) - \sum_{r=k-d_M}^{k-1}\delta^{p\mathrm{T}}(r+i)(\varsigma_p^S)^{k-r}\theta^{p}(X_1^{pS})^{-1}\sum_{r=k-\tilde{d}_M}^{k-1}\delta^{p}(r+i)$$
$$< d_M^2(\bar{x}^{p}(k+i+1)-\bar{x}^{p}(k+i))^{\mathrm{T}}\theta^{p}(X_1^{pS})^{-1}(\bar{x}^{p}(k+i+1) - \bar{x}^{p}(k+i)) - (\bar{x}^{p}(k+i)-\bar{x}^{p}(k+i))^{\mathrm{T}}(\varsigma_p^S)^{d_M}\theta^{p}(X_1^{pS})^{-1}\cdot(\bar{x}^{p}(k+i)-\bar{x}^{p}(k+i)) \tag{44}$$

Similar to the stable case, the following equation can be expressed.

$$(\theta^{p})^{-1}\Delta V_p^U(\bar{x}^{p}(k+i|k)) + (\theta^{p})^{-1}J^{pU}(k) \le 0, \tag{45}$$

where

$$J^{pU}(k) = (\bar{x}^{p}(k+i|k))^{\mathrm{T}}Q_1^{pU}(\bar{x}^{p}(k+i|k)) + (\Delta u^{pU}(k+i|k))^{\mathrm{T}}R_1^{pU}\Delta u^{pU}(k+i|k).$$

The unstable case is similar to the stable case. Hence, the proof of the unstable case is ignored.

Furthermore, assuming that the discrete time is k, the asynchronous switching system operates in the steady case of the pth phase. According to the above equations (27)-(29), the following relationship can be obtained.

$$\begin{aligned}
&V_p^S(x_k(k)) \\
&< (\varsigma_p^S)^{O-T^{p-1/p}}V_p^S(x_k(T^{p-1/p})) \\
&\le \mu_p^S(\varsigma_p^S)^{O-T^{p-1/p}}V_p^U(x_k(T^{p-1/p})) \\
&\le (\varsigma_p^S)^{O-T^{p-1/p}}\mu_p^S(\varsigma_p^U)^{T^{p-1/p}-T^{p-1}}V_p^U(x_k(T^{p-1})) \\
&\le (\varsigma_p^S)^{O-T^{p-1/p}}\mu_p^S(\varsigma_p^U)^{T^{p-1/p}-T^{p-1}}\mu_p^U V_{p-1}^S(x_k(T^{p-1})) \\
&\vdots \\
&\le \prod_{p=1}^{P}(\mu_p^S)^{N_0^p+\left(T_S^p(d,O)/\tau_S^p\right)}\times\prod_{p=1}^{P}(\varsigma_p^S)^{T_p^S(d,O)} \\
&\quad\times\prod_{p=1}^{P}(\mu_p^U)^{N_0^U+\left(T_U^p(d,O)/\tau_U^p\right)}\times\prod_{p=1}^{P}(\varsigma_p^U)^{T_U^p(d,O)} \\
&\quad\times V_1^S(x_k(T^1)) \\
&= \exp\left(\sum_{p=1}^{P}N_0^S\ln\mu_p^S + \sum_{p=1}^{P}N_0^U\ln\mu_p^U\right) \\
&\quad\times\prod_{p=1}^{P}((\mu_p^S)^{1/\tau_S^p}(\varsigma_p^S))^{T_S^p(d,O)} \\
&\quad\times\prod_{p=1}^{P}((\mu_p^U)^{1/\tau_U^p}(\varsigma_p^U))^{T_U^p(d,O)}V_1^S(x_k(T^1))
\end{aligned} \tag{46}$$

Based on the Eq. (31), the following equation can be obtained.

$$\begin{cases}\tau_S^p + \dfrac{\ln\mu_p^S}{\ln\varsigma_p^S} \ge 0\\[2mm] \tau_U^p + \dfrac{\ln\mu_p^U}{\ln\varsigma_p^U} \le 0\end{cases} \tag{47}$$

Due to $0 < \varsigma_p^S < 1, \varsigma_p^U > 1, \mu_p^S > 1, 0 < \mu_p^U < 1$, it holds that

$$\begin{cases}\tau_S^p\ln\varsigma_p^S + \ln\mu_p^S \le 0\\ \tau_U^p\ln\varsigma_p^U + \ln\mu_p^U \le 0\end{cases} \tag{48}$$

yet

$$\begin{cases}(\mu_p^S)^{1/\tau_S^p}(\varsigma_p^S) = \exp\left(\ln\left[(\mu_p^S)^{1/\tau_S^p}(\varsigma_p^S)\right]\right)\\ \qquad = \exp\left(\left[1/\tau_S^p\ln\mu_p^S + \ln\varsigma_p^S\right]\right)\\ (\mu_p^U)^{1/\tau_U^p}(\varsigma_p^U) = \exp\left(\ln\left[(\mu_p^U)^{1/\tau_U^p}(\varsigma_p^U)\right]\right)\\ \qquad = \exp\left(\left[1/\tau_U^p\ln\mu_p^U + \ln\varsigma_p^U\right]\right)\end{cases} \tag{49}$$

Let $\eta = \max_{p\in P}((\mu_p^S)^{\frac{1}{\tau_S^p}}(\varsigma_p^S), (\mu_p^U)^{\frac{1}{\tau_U^p}}(\varsigma_p^U)), \kappa = \exp(\sum_{p-1}^{P}N_0^S\ln\mu_p^S + \sum_{p=1}^{P}N_0^U\ln\mu_p^U)$, the following equation can be obtained:

$$V_p^S(x_{1k}(k)) \le \eta\kappa^{O-f}V_1^S(x_{1k}(T^1)) \tag{50}$$

It can ensure the exponential stability of the system based on Eq. (29). Also, under the normal circumstance, $V_p^S(x(k)) \leq \varsigma^{O-f} V_1^S(x(T^1))$.

Moreover, to get the invariant set of the system, we can take the maximum value of

$$\bar{x}^p(k) = \max\ (x^p(\bar{r}) \quad \delta^p(\bar{r})), \quad \bar{r} \in (k - d_M, k),$$

it has

$$V^p(x_{1k}^p(k)) \leq \bar{x}_l^{p\mathrm{T}}(k)\bar{\psi}_{lk}^p \bar{x}_l^p(k) \leq \theta^p \tag{51}$$

where $\bar{\psi}_{lk}^p = P_1^p + d_M T_1^p + d_M M_1^p + \frac{d_m + d_M}{2}(d_M - d_m + 1)T_1^p + d_M^2 \cdot \frac{1+d_M}{2} G_1^p$. Letting $\bar{\varphi}_l^p = \theta^p(\bar{\psi}_{lk}^p)^{-1}$, we can use Lemma 1 to get Eq. (30).

Theorem 2: The considered system (19) with $\bar{\omega} \neq 0$ is the asymptotical stability in each phase and exponentially stable in a batch, if there are some scalar $0 < \varsigma_p^S < 1$, $\varsigma_p^U > 1$, $\theta^p > 0$, $\theta^{p-1} > 0$, $0 \leq d_m \leq d_M$,some positive definite matrices Q^{pS}, R^{pS}, $Q^{(p-1)U}$, $R^{(p-1)U}$, some positive definite symmetric matrices P_1^{pS}, T_1^{pS}, M_1^{pS}, G_1^{pS}, L_1^{pS}, S_1^{pS}, S_2^{pS}, M_3^{pS}, M_4^{pS}, X_1^{pS}, X_2^{pS},β^{pS}, Y^{pS}, $P_1^{(p-1)U}$, $T_1^{(p-1)U}$, $M_1^{(p-1)U}$, $G_1^{(p-1)U}$, β^{pU}, Y^{pU}, $L_1^{(p-1)U}$, $S_1^{(p-1)U}$, $S_2^{(p-1)U}$, $M_3^{(p-1)U}$, $M_4^{(p-1)U}$, $X_1^{(p-1)U}$, $X_2^{(p-1)U} \in R^{(n_x+n_e)}$ and some scalars r^p, ε_1^{pS}, ε_2^{pS}, ε_1^{pU}, ε_2^{pU}, $\theta^P > 0$, $\mu_p^S > 1$, $0 < \mu_p^U < 1$, so that the following LMIs hold.

$$\begin{bmatrix} \coprod_{11}^{pS} & \coprod_{12}^{pS} & \coprod_{13}^{pS} & \coprod_{14}^{pS} & \coprod_{15}^{pS} \\ * & \coprod_{22}^{pS} & 0 & 0 & 0 \\ * & * & \coprod_{33}^{pS} & 0 & 0 \\ * & * & * & \coprod_{44}^{pS} & 0 \\ * & * & * & * & \coprod_{55}^{pS} \end{bmatrix} < 0 \tag{52}$$

$$\begin{bmatrix} \coprod_{11}^{pU} & \coprod_{12}^{pU} & \coprod_{13}^{pU} & \coprod_{14}^{pU} & \coprod_{15}^{pU} \\ * & \coprod_{22}^{pU} & 0 & 0 & 0 \\ * & * & \coprod_{33}^{pU} & 0 & 0 \\ * & * & * & \coprod_{44}^{pU} & 0 \\ * & * & * & * & \coprod_{55}^{pU} \end{bmatrix} < 0 \tag{53}$$

$$\begin{cases} V_p^S(\bar{x}_1(k)) \leq \mu_p^S V_{p-1}^S(\bar{x}_1(k)) \\ V_p^S(\bar{x}_1(k)) \leq \mu_p^S V_p^U(\bar{x}_1(k)) \\ V_p^U(\bar{x}_1(k)) \leq \mu_p^U V_{p-1}^S(\bar{x}_1(k)) \end{cases} \tag{54}$$

$$\begin{bmatrix} -1 & \bar{x}_l^{p\mathrm{T}}(k|k) \\ \bar{x}_l^p(k|k) & -\bar{\varphi}_l^p \end{bmatrix} \leq 0 \tag{55}$$

where $\coprod_{11}^{pS}$, $\coprod_{12}^{pS}$, $\coprod_{13}^{pS}$, $\coprod_{22}^{pS}$, $\coprod_{33}^{pS}$, $\coprod_{11}^{pU}$, $\coprod_{12}^{pU}$, $\coprod_{13}^{pU}$, $\coprod_{14}^{pU}$, $\coprod_{22}^{pU}$, $\coprod_{33}^{pU}$, ϕ_{12}^{pS}, and ϕ_{12}^{pU} as shown at the bottom of the next page.

Proof 2: Under stable case at each phase, when there are disturbances in the system, i.e., $\bar{\omega}_k^p \neq 0$, the following H_∞ performance index is introduced to ensure the stability of the system.

$$J_1^p = \sum_{k=0}^{\infty} [\left(z^{pS}(k)\right)^{\mathrm{T}} z^{pS}(k) - (r^p)^2 \left(\bar{\omega}_k^p(k)\right)^{\mathrm{T}} \bar{\omega}_k^p(k)] \tag{56}$$

For any $\bar{\omega}(k) \in l_2[0, \infty]$ with nonzero, as shown in the following equation:

$$J_\omega^p \leq \sum_{k=0}^{\infty} [\left(z^{pS}(k)\right)^{\mathrm{T}} z^{pS}(k) - (r^p)^2 \left(\omega_k^p(k)\right)^{\mathrm{T}} \omega_k^p(k) + (\theta^p)^{-1} \Delta V^p(\bar{x}_k^p(k)) + (\theta^p)^{-1} \bar{J}_1^{pS}(k)] \tag{57}$$

Similar to Theorem 1, the following expression can be obtained (58), as shown at the bottom of page 13.

The unstable case is similar to stable case, the following expression can be obtained (59), as shown at the bottom of page 13.

Meanwhile, $J_\omega^p \leq 0$, $||z^p||_{l_2} \leq r^p ||\bar{\omega}^p||_{l_2}$ is guaranteed.

Remark 2: With the development of LMI technology and the maturity of MATLAB's LMI tools, the construction of Lyapunov functions and solving given LMI conditions are more convenient. In other words, the method of solving LMI can effectively reduce the computational difficulty, compared with some non-LMI methods such as the method proposed in [35].

IV. SIMULATION CASE

A. DESCRIPTION OF SIMULATION SYSTEM

The injection molding process has many superiorities, for instance fast production speed, high efficiency, high degree of automation and a wide range of types and styles. Therefore, injection molding has a wide range of applications in the field of large-scale and complex shape product production in plastic processing. The simulation in this paper takes the working process of the reciprocating screw injection molding machine as an example. Reciprocating screw injection molding machine is the most common mechanical equipment in the injection molding industry, and the injection molding process is a typical MPBP, taking injection molding as an example can more accurately verify the effectiveness and feasibility of the method proposed in this paper. The structure of the injection molding machine is shown in Fig 2.

The four main phases of the injection molding processes are shown in Fig 3. Fig. 3 (a) shows the injection phase, in which molten plastic is injected into the mold. Then, the system enters pressure holding phase. The main purpose of this phase is to keep the pressure in the cavity stable so as to ensure the quality of the product Fig. 3(b) shows the holding phase. Fig. 3 (c) and Fig. 3 (d) show the cooling phase and product removal, respectively. The simulation case in this paper takes the switch between injection phase and pressure holding phase as an example.

B. SIMULATION MODEL

The model established in injection molding machine is used for simulation with MATLAB software in order to verify

$$\coprod_{11}^{pS}=\begin{bmatrix}\phi_{1w}^{pS} & 0 & (\varsigma_p^S)^{d_M}L_1^{pS} & 0\\ * & -(\varsigma_p^S)^{d_M}S_1^{pS} & 0 & 0\\ * & * & -(\varsigma_p^S)^{d_M}((M_4^{pS})+(X_1^{pS})) & 0\\ * & * & * & -(r^p)^2I^p\end{bmatrix},$$

$$\coprod_{12}^{pS}=\begin{bmatrix}L_1^{pS}A_k^{p\mathrm{T}}+Y_1^{pS\mathrm{T}}\beta^{pS}B_k^{p\mathrm{T}} & L_1^{pS}A_k^{p\mathrm{T}}+Y_1^{pS\mathrm{T}}\beta^{pS}B_k^{p\mathrm{T}}-L_1^{pS}\\ S_1^{pS}A_{dk}^{p\mathrm{T}} & S_1^{pS}A_{dk}^{p\mathrm{T}}\\ 0 & 0\\ G_k^{p\mathrm{T}} & G_k^{p\mathrm{T}}\end{bmatrix},$$

$$\coprod_{13}^{pS}=\begin{bmatrix}L_1^{pS}E_k^{p\mathrm{T}} & L_1^{pS}(Q^{pS})^{\frac{1}{2}} & Y_1^{pS\mathrm{T}}(R^{pS})^{\frac{1}{2}}\\ 0&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad \coprod_{14}^{pS}=\coprod_{15}^{pS}=\begin{bmatrix}L_1^{pS}H_k^{p\mathrm{T}} & Y_1^{pS\mathrm{T}}\beta^{pS}\\ S_1^{pS}H_{dk}^{p\mathrm{T}} & 0\\ 0&0\\0&0\end{bmatrix},$$

$$\coprod_{22}^{pS}=\begin{bmatrix}-L_1^{pS}+\varepsilon_1^{pS}N_k^pN_k^{p\mathrm{T}}+\varepsilon_2^{pS}B_k^p(\beta_0^{pS})^2B_k^{p\mathrm{T}} & 0\\ 0 & -X_1^{pS}(D_2^p)^{-1}+\varepsilon_1^{pS}N_k^pN_k^{p\mathrm{T}}+\varepsilon_2^{pS}B_k^p(\beta_0^{pS})^2B_k^{p\mathrm{T}}\end{bmatrix},$$

$$\coprod_{33}^{pS}=\begin{bmatrix}-I^p & 0 & 0\\ 0 & -\theta^pI^p & 0\\ 0 & 0 & -\theta^pI^p\end{bmatrix},\quad \coprod_{44}^{pS}=\coprod_{55}^{pS}=\begin{bmatrix}-\varepsilon_1^{pS}I^p & 0\\ 0 & -\varepsilon_2^{pS}I^p\end{bmatrix},$$

$$\coprod_{11}^{pU}=\begin{bmatrix}\phi_{12}^{pU} & 0 & (\varsigma_p^U)^{d_M}L_1^{(p-1)U} & 0\\ * & -(\varsigma_p^{\mathrm{U}})^{d_M}S_1^{(p-1)U} & 0 & 0\\ * & * & -(\varsigma_p^U)^{d_M}((M_4^{(p-1)U})+(X_1^{(p-1)U})) & 0\\ * & * & * & -(\gamma^{pU})^2I^p\end{bmatrix},$$

$$\coprod_{12}^{pU}=\begin{bmatrix}L_1^{(p-1)U}A_k^{p\mathrm{T}}+L_1^{(p-1)U}\left((\Re^{p-1})^{-1}\right)^{\mathrm{T}}\left(K^{p-1}\right)^{\mathrm{T}}\beta^{pU}B_k^{p\mathrm{T}} & L_1^{(p-1)U}A_k^{p\mathrm{T}}+L_1^{(p-1)U}\left((\Re^{p-1})^{-1}\right)^{\mathrm{T}}\left(K^{p-1}\right)^{\mathrm{T}}\beta^{pU}B_k^{p\mathrm{T}}-L_1^{(p-1)U}\\ S_1^{(p-1)U}A_{dk}^{p\mathrm{T}} & S_1^{(p-1)U}A_{dk}^{p\mathrm{T}}\\ 0 & 0\\ G_k^{p\mathrm{T}} & G_k^{p\mathrm{T}}\end{bmatrix},$$

$$\coprod_{13}^{pU}=\begin{bmatrix}L_1^{(p-1)U}E_k^{pT} & L_1^{(p-1)U}(Q^{(p-1)U})^{\frac{1}{2}} & L_1^{(p-1)U}\left((\Re^{p-1})^{-1}\right)^{\mathrm{T}}\left(K^{p-1}\right)^{\mathrm{T}}(R^{(p-1)U})^{\frac{1}{2}}\\ 0&0&0\\0&0&0\\0&0&0\end{bmatrix},$$

$$\coprod_{14}^{pU}=\coprod_{15}^{pU}=\begin{bmatrix}L_1^{(p-1)U}H_k^{p\mathrm{T}} & L_1^{(p-1)U}\left((\Re^{p-1})^{-1}\right)^{\mathrm{T}}\left(K^{p-1}\right)^{\mathrm{T}}\beta^{pU}\\ S_1^{(p-1)U}H_{dk}^{p\mathrm{T}} & 0\\ 0&0\\0&0\end{bmatrix},$$

$$\coprod_{22}^{pU}=\begin{bmatrix}-L_1^{(p-1)U}+\varepsilon_1^{pU}N_k^pN_k^{p\mathrm{T}}+\varepsilon_2^{pU}B_k^p(\beta_0^{pU})^2B_k^{p\mathrm{T}} & 0\\ 0 & -X_1^{(p-1)U}(D_2^p)^{-1}+\varepsilon_1^{pU}N_k^pN_k^{p\mathrm{T}}+\varepsilon_2^{pU}B_k^p(\beta_0^{pU})^2B_k^{p\mathrm{T}}\end{bmatrix},$$

$$\coprod_{33}^{pU}=\begin{bmatrix}-I^p & 0 & 0\\ 0 & -\theta^{p-1}I^{p-1} & 0\\ 0 & 0 & -\theta^{p-1}I^{p-1}\end{bmatrix},\quad \coprod_{44}^{pU}=\coprod_{55}^{pU}=\begin{bmatrix}-\varepsilon_1^{pU}I^p & 0\\ 0 & -\varepsilon_2^{pU}I^p\end{bmatrix}$$

$$\phi_{12}^{pS}=-\varsigma_p^SL_1^{pS}+M_3^{pS}+D_1^pS_2^{pS}+S_2^{pS}-(\varsigma_p^S)^{d_M}X_2^{pS},$$

$$\phi_{12}^{pU}=-\varsigma_p^UL_1^{(p-1)U}+M_3^{(p-1)U}+D_1^pS_2^{(p-1)U}+S_2^{(p-1)U}-(\varsigma_p^U)^{d_M}X_2^{(p-1)U},\quad D_1^p=(d_M-d_m+1)I^p, D_2^p=(d_M)^2I^p.$$

the effectivity of the designed controller. Through repeated tests, the transfer function of injection velocity (IV) and valve opening (VO) of the system in the injection phase can be obtained as shown in Equation (60).

$$\frac{\mathrm{IV}}{\mathrm{VO}}=\frac{8.687z^{-1}-5.617z^{-2}}{1-0.9291z^{-1}-0.03191z^{-2}} \tag{60}$$

In the holding pressure phase, the transfer functions of nozzle pressure (NP) and injection velocity (IV) in the pressure holding phase can be obtained as shown in Equation (61).

$$\frac{\mathrm{NP}}{\mathrm{VO}}=\frac{171.8z^{-1}-156.8z^{-2}}{1-1.317z^{-1}-0.3259z^{-2}} \tag{61}$$

And injection velocity (IV) and nozzle pressure (NP) satisfy the transfer function as shown in Eq. (62).

$$\frac{\mathrm{NP}}{\mathrm{IV}}=\frac{0.1054z^{-1}}{1-z^{-1}} \tag{62}$$

Define

$$\begin{aligned}
x^1(k) &= [\mathrm{IV}(k) \quad 0.03191\mathrm{IV}(k-1)-5.617\mathrm{VO}(k-1) \\
&\qquad \mathrm{NP}(k)]^{\mathrm{T}}, \\
u^1(k) &= \mathrm{VO}(k), \quad y^1(k)=\mathrm{IV}(k), \\
x^2(k) &= \left[\mathrm{NP}(k) \quad -0.3259\mathrm{NP}(k-1)-156.8\mathrm{VO}(k-1)\right]^{\mathrm{T}}, \\
u^2(k) &= \mathrm{VO}(k), \quad y^2(k)=\mathrm{NP}(k).
\end{aligned}$$

The state space model of system is as follows:

$$\begin{cases} x^p(k+1)=A^p x^p(k)+B^p u^p(k)+\omega^p(k) \\ y^p(k)=C^p x^p(k) \quad p=1,2 \end{cases} \tag{63}$$

where

$$A^1=\begin{bmatrix} 0.9291 & 1 & 0 \\ 0.03191 & 0 & 0 \\ 0.1054 & 0 & 1 \end{bmatrix}, \quad B^1=\begin{bmatrix} 8.687 \\ -5.617 \\ 0 \end{bmatrix},$$

$$\begin{aligned}
&\left(z^{pS}(k)\right)^{\mathrm{T}} z^{pS}(k)-(r^p)^2\left(\bar{\omega}_k^p(k)\right)^{\mathrm{T}}\bar{\omega}_k^p(k)+(\theta^p)^{-1}\Delta V^p(x_{1k}^p(k))+(\theta^p)^{-1}J^{pS}(k)=\begin{bmatrix}\varphi_1^p(k)\\ \bar{\omega}_k^p(k)\end{bmatrix}^{\mathrm{T}} \\
&\left\{\begin{bmatrix} \phi_{12}^{pS} & 0 & (\varsigma_p^S)^{d_M}(X_1^{pS})^{-1} & 0 \\ * & -(\varsigma_p^S)^{d_M}(S_1^{pS})^{-1} & 0 & 0 \\ * & * & -(\varsigma_p^S)^{d_M}((M_2^{pS})^{-1}+(X_1^{pS})^{-1}) & 0 \\ * & * & * & -(r^p)^2 \end{bmatrix}+\begin{bmatrix}\left(\Lambda_1^{pS}\right)^{\mathrm{T}} \\ G_k^{p\mathrm{T}}\end{bmatrix}(L_1^{pS})^{-1}\begin{bmatrix}\Lambda_1^{pS} & G_k^p\end{bmatrix}\right. \\
&+\begin{bmatrix}\left(\Lambda_2^{pS}\right)^{\mathrm{T}} \\ G_k^{p\mathrm{T}}\end{bmatrix}(D_2^p)^{-1}(X_1^{pS})^{-1}\begin{bmatrix}\Lambda_2^{pS} & G_k^p\end{bmatrix} \\
&+\begin{bmatrix}E_k^{p\mathrm{T}} \\ 0 \\ 0 \\ 0\end{bmatrix}\begin{bmatrix}E_k^p & 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}\lambda_1^{pS} & 0\end{bmatrix}^{\mathrm{T}}(\theta^p)^{-1}\begin{bmatrix}\lambda_1^{pS} & 0\end{bmatrix} \\
&\left.+\begin{bmatrix}\lambda_2^{pS} & 0\end{bmatrix}^{\mathrm{T}}(\theta^p)^{-1}\begin{bmatrix}\lambda_2^{pS} & 0\end{bmatrix}\right\}\begin{bmatrix}\varphi_1^p(k)\\ \bar{\omega}_k^p(k)\end{bmatrix}
\end{aligned} \tag{58}$$

$$\begin{aligned}
&\begin{bmatrix} \phi_{12}^{pU} & 0 & (\varsigma_p^U)^{d_M}(X_1^{(p-1)U})^{-1} & 0 \\ * & -(\varsigma_p^U)^{d_M}(S_1^{(p-1)U})^{-1} & 0 & 0 \\ * & * & -(\varsigma_p^U)^{d_M}((M_2^{(p-1)U})^{-1}+(X_1^{(p-1)U})^{-1}) & 0 \\ * & * & * & -(r^p)^2 \end{bmatrix}+\begin{bmatrix}\left(\Lambda_1^{pU}\right)^{\mathrm{T}} \\ G_k^{p\mathrm{T}}\end{bmatrix}(L_1^{(p-1)U})^{-1}\begin{bmatrix}\Lambda_1^{pU} & G_k^p\end{bmatrix} \\
&+\begin{bmatrix}\left(\Lambda_2^{pU}\right)^{\mathrm{T}} \\ G_k^{p\mathrm{T}}\end{bmatrix}(D_2^p)^{-1}(X_1^{(p-1)U})^{-1}\begin{bmatrix}\Lambda_2^{pU} & G_k^p\end{bmatrix} \\
&+\begin{bmatrix}E_k^{p\mathrm{T}} \\ 0 \\ 0 \\ 0\end{bmatrix}\begin{bmatrix}E_k^p & 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}\lambda_1^{pU} & 0\end{bmatrix}^{\mathrm{T}}(\theta^p)^{-1}\begin{bmatrix}\lambda_1^{pU} & 0\end{bmatrix}+\begin{bmatrix}\lambda_2^{pU} & 0\end{bmatrix}^{\mathrm{T}}(\theta^p)^{-1}\begin{bmatrix}\lambda_2^{pU} & 0\end{bmatrix}<0
\end{aligned} \tag{59}$$

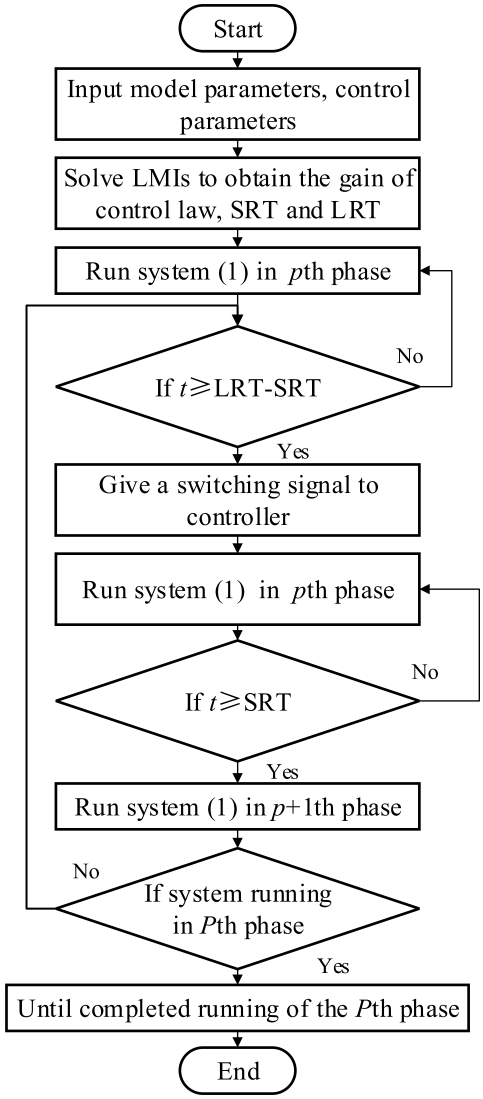

FIGURE 1. A block diagram of each batch.

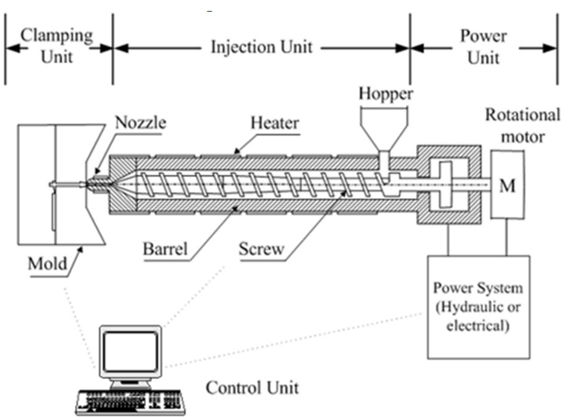

FIGURE 2. The structure of the injection molding machine.

$$C^1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 1.317 & 1 \\ -0.3259 & 0 \end{bmatrix},$$

$$B^2 = \begin{bmatrix} 171.8 \\ -156.8 \end{bmatrix}, \quad C^2 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

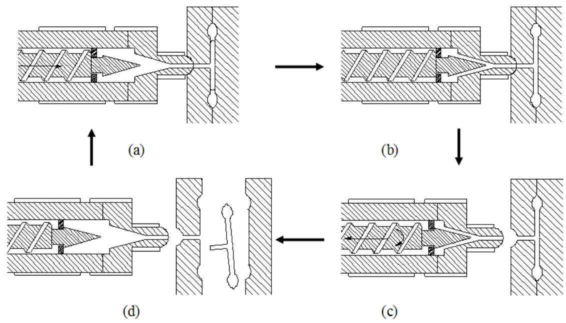

FIGURE 3. The main process of injection molding.

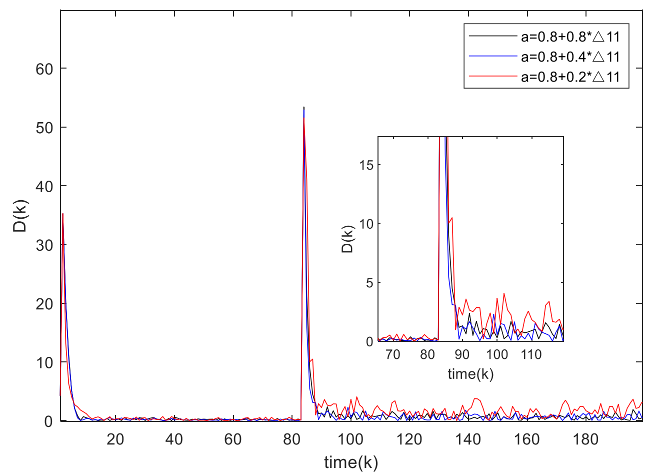

FIGURE 4. The tracking performance of the system under different fault conditions.

When $p = 1$, the system works in the injection phase. When $p = 2$, the system works in the pressure holding phase. The switching conditions of the two phases can be obtained in the following equation.

$$\gamma^1(x(k)) = 350 - \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x^1(k) < 0 \tag{64}$$

Considering the existence of time-varying delays, uncertainties, unknown disturbances and actuator fault in the system, the following equation can be obtained.

$$\begin{cases} x^p(k+1) = A^p(k)x^p(k) + A_d^p(k)x^p(k-d(k)) \\ \qquad\qquad + B^p\alpha^p u^p(k) + \omega^p(k) \\ y^p(k) = C^p x^p(k) \quad p = 1, 2 \end{cases} \tag{65}$$

where

$$1 \le d(k) \le 3, \quad N^1 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$H^1 = \begin{bmatrix} 0.104 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_d^1 = \begin{bmatrix} 0.004 & 0 & 0 \\ 0.05 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Delta^1(k) = \begin{bmatrix} \Delta_1 & 0 & 0 \\ 0 & \Delta_2 & 0 \\ 0 & 0 & \Delta_3 \end{bmatrix}, \quad N^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

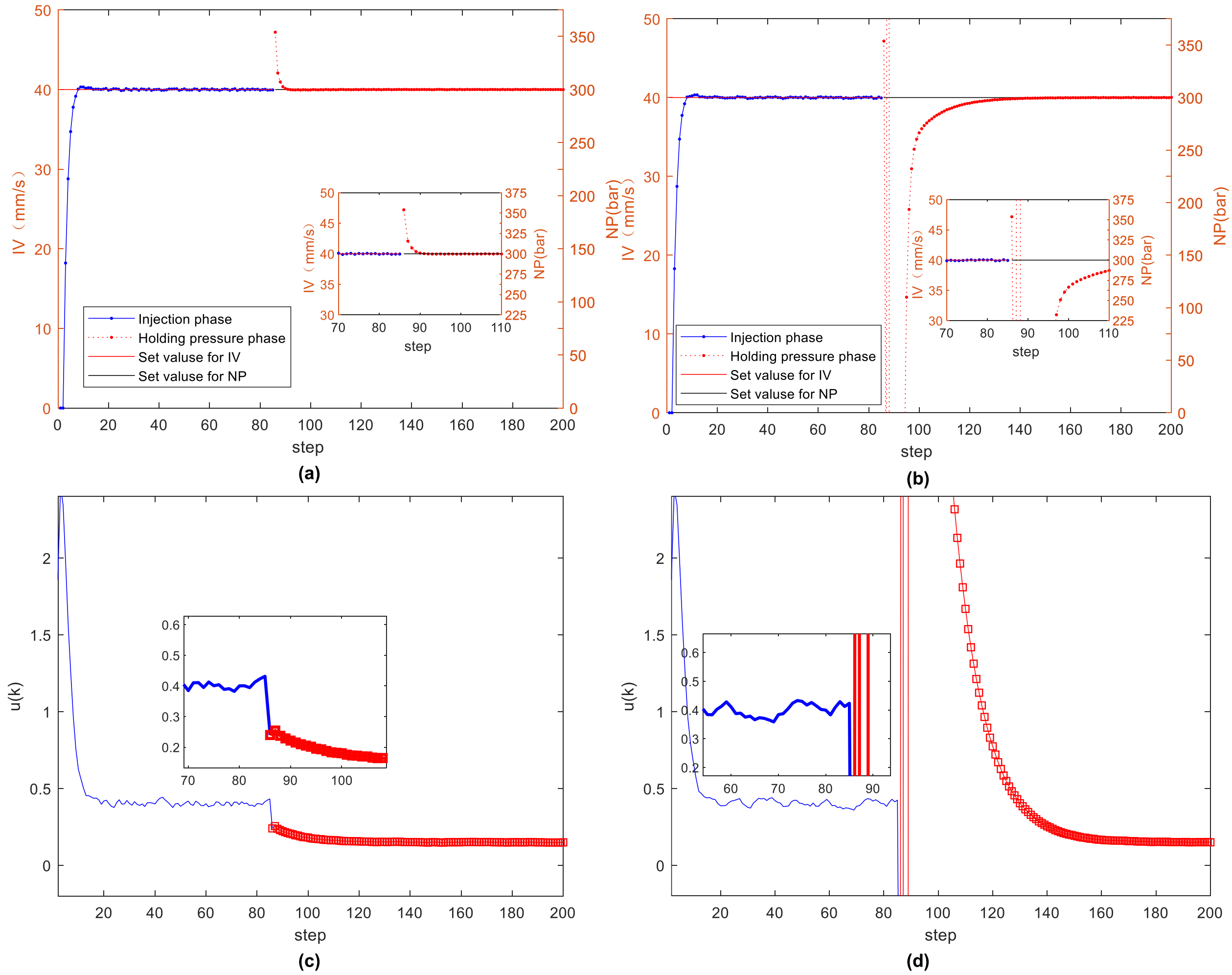

FIGURE 5. Comparison of system output response and control output in case 1.

$$H^2 = \begin{bmatrix} 0.0104 & 0 \\ -0.0304 & 0 \end{bmatrix}, \quad H_d^1 = \begin{bmatrix} 0.001 & 0 \\ -0.002 & 0 \end{bmatrix} \Delta^2(k)$$
$$= \begin{bmatrix} \Delta_4 & 0 \\ 0 & \Delta_5 \end{bmatrix}, \quad \omega^1(k) = 0.5 \times [\, \Delta_6 \;\; \Delta_7 \;\; \Delta_8 \,]^{\mathrm{T}},$$
$$\omega^2(k) = 0.5 \times [\, \Delta_9 \quad \Delta_{10} \,]^{\mathrm{T}}, \;\; |\Delta_{ii}| \le 1,$$

$ii = 1, 2, \cdots, 5$. Suppose there is an unknown fault α^p causing the actuator to fail. In this simulation, we know that $0.4 = \underline{\alpha}^p < \alpha^p < \bar{\alpha}^p = 1.2$. According to the Eq. (8), $\beta^p = 0.8$, $\beta_0^p = 0.5$ can be obtained. The setpoint values for the two phases can be obtained in the following equation:

$$\begin{cases} \vartheta^1(k) = 40, & 0 < k \le T^1 \\ \vartheta^2(k) = 300, & T^1 < k < T^1 + T^2 \end{cases} \tag{66}$$

To describe the tracking performance of the system more accurately, the following equation is defined:

$$D(k) = \begin{cases} \sqrt{e^{1\mathrm{T}}(k)e^1(k)} & 0 \le k < T_1 \\ \sqrt{e^{2\mathrm{T}}(k)e^2(k)} & T_1 \le k \le T_1 + T_2 \end{cases} \tag{67}$$

C. SIMULATION RESULTS

This section is mainly to verify the proposed method. One of the purposes is to verify whether the system can switch smoothly between two adjacent phases. The other is to verify whether the proposed method can effectively restrain the influence of factors such as uncertainty, unknown disturbance, time-varying delays and actuator fault in the system. In order to verify the effectiveness and feasibility of the proposed control method. In this paper, injection molding process is taken as an example for simulation verification. After repeated tests, the parameters of the controller in the injection phase and the pressure holding phase are chosen as $Q_1^1 = \mathrm{diag}\,[0.5,\, 1,\, 1,\, 1]$, $R_1^1 = 0.1$ and $Q_1^2 = \mathrm{diag}\,[10,\, 2.6,\, 9]\, R_1^2 = 0.1$, respectively. Then, the SRT of the injection phase is $T^1 = 86$s. And the SRT of the pressure holding phase is $T^2 = 113$s. The control law gain of the system after stabilization in the injection phase and the pressure holding phase are $K_1 = [-0.00874 - 0.0952\ 0 \ - 0.0476]$ and $K_2 = [-0.0084\ -0.0066\ -0.0032]$, respectively. The above parameters are obtained by solving LMIs in Theorem 2.

Figure 4 shows the tracking performance of the system of the proposed method under three different case. The three cases can be expression as following formula.

$$\begin{cases} Case\ 1: \alpha^p = 0.8 + 0.2\Delta_{11} \\ Case\ 2: \alpha^p = 0.8 + 0.4\Delta_{11} \\ Case\ 3: \alpha^p = 0.8 + 0.8\Delta_{11} \end{cases} \tag{68}$$

where Δ_{11} is a random number in $[-1, 1]$. Figure 4 gives the tracking performance of the system under three different random failure conditions. It can be seen that the tracking performance of the system deteriorates with the increase of the random fault range. But the proposed control method can still make the system converge rapidly. In order to intuitively reflect the superiority of the control method proposed in this paper, we selected a traditional control method [20] for comparison. Figure 5 (a) shows the system output response with asynchronous switching controller. Figure 5(b) shows the system output response with synchronous switching controller in [20]. Important purpose of the designed controller is that the system state and controller can be switched simultaneously. Avoid the "escape" of system state caused by the occurrence of asynchronous switching. It can be seen from the output response curve that the proposed asynchronous control method can realize the smooth switch from the injection phase to the pressure holding phase. However, when the synchronous switching control method is adopted, the state of the system fluctuates greatly.

Figure 5 (c) shows the system of control inputs with asynchronous switching controller. Figure 5 (d) shows the system of control inputs synchronous switching controller. The designed asynchronous switching controller can ensure the smooth and stable switching. After switching, the asynchronous switching controller can be stable and fast-tracking setpoint value. The synchronization switching controller has great fluctuation, and the response time of the controller is long, which has great influence on the actual production.

V. CONCLUSION

A robust asynchronous switching fault-tolerant controller is designed for MPBP with uncertainties, unknown disturbances, time-varying delays and partial actuator failures. In combination with the model-dependent average dwell time method, Lyapunov stability theory and switched system theory, sufficient conditions with LMIs form are given to ensure the exponential stability of the system and the asymptotic stability of each phase. Finally, the simulation results show that the proposed method has better tracking performance and anti-disturbance capability under the condition of time-varying delays and partial actuator failure.

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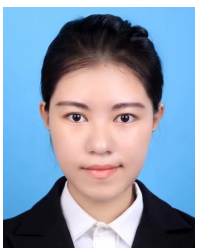

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