IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received March 9, 2021, accepted April 21, 2021, date of publication May 4, 2021, date of current version May 17, 2021.

Digital Object Identifier 10.1109/ACCESS.2021.3077513

Stability Analysis of Car-Following Systems With Uniformly Distributed Delays Using Frequency-Sweeping Approach

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This work was supported in part by the National Key Research and Development Program of China under Grant 2020AAA0109200 and Grant 2020AAA0109203, and in part by the Fundamental Research Funds for the Central Universities under Grant N2104007.

ABSTRACT In this paper, the stability problem of a class of deterministic car-following systems with uniformly distributed delays is addressed with a particular focus on the effects induced by the delay parameters on the stability. To perform such an analysis, the frequency-sweeping approach introduced recently by the authors will be adopted. The corresponding results are easy to derive and to implement. As a byproduct of the analysis, the complete characterization of the delay intervals guaranteeing stability is explicitly derived. Two case studies with different network topology types (linear and ring configurations) are studied in detail. The stability problem of car-following systems under consideration is a specific consensus problem of multi-agent systems. Such examples allow illustrating the approach as well as the positive effect of using uniformly distributed delay in the modeling (compared with the commonly-used pointwise delay). The derived results show that the approach is effective and may be adopted to more general consensus problems for multi-agent systems.

INDEX TERMS Car-following systems, uniformly distributed delay systems (UDDSs), complete stability problem, frequency-sweeping approach, consensus of multi-agent systems.

I. INTRODUCTION

For studying the traffic flow dynamics, various mathematical models have been proposed (see e.g., [3], [9], [11], and [30]). In this paper, we focus on the car-following models, which represent a type of microscopic traffic flow models and are usually described by delay-differential equations (see e.g., [6], [23], and [24]). Time delays in the car-following systems mainly arise from driver reactions and vehicle mechanical response (see [2] and [32]).

Compared with the pointwise-delay models, the distributed-delay ones may better reflect the inherent memory effects of drivers in car-following systems (a detailed analysis can be found in [28] and [29]).

In this paper, we are concerned with the model containing a uniformly distributed delay. The corresponding system is called a uniformly distributed delay system (UDDS). Mathematically speaking, the UDDS model covers the pointwise-delay model [29]. In addition, the UDDS model may lead to larger stability regions in the corresponding parameter-space [1]. Whereas, the stability analysis for a UDDS is much more complicated than that for a pointwise-delay system.

The objective of this paper is to study the complete stability problem w.r.t. the delay parameter τ for the car-following systems with uniformly distributed delays. The problem involves technical difficulties from two aspects: (1) The stability problem for car-following systems is a type of consensus problem for multi-agent systems. (2) The overall system is a time-delay one (more precisely, a UDDS).

For the stability analysis of time-delay systems, the existing methods can be mainly divided into the τ-decomposition method [10] (the delay is treated as the only free parameter) and the D-decomposition method [20] (the delay is fixed

The associate editor coordinating the review of this manuscript and approving it for publication was Xiwang Dong.

while some system and/or controller parameters are free parameters). The approach in our current paper falls in the former class.

To appropriately address the car-following systems with uniformly distributed delays, we adopt the frequency-sweeping framework which was proposed in [14] and discussed in [13] in the case of dynamical systems including uniformly distributed delays. However, such ideas cannot be adopted straightforwardly to higher dimension cases that represents an important inconvenience if the methodology is used in the analysis of traffic flow models. In order to make our frequency-sweeping approach more practically effective, two technical novelties are proposed and explicitly discussed.

First, the characteristic equation of the car-following model has a characteristic root $\lambda = 0$ independent of τ, as the configuration matrix (related with the network topology) has a simple zero eigenvalue. In the field of time-delay systems, the case with characteristic root $\lambda = 0$ generally does not need to be addressed since the associated time-delay systems cannot be asymptotically stable for any τ value. However, for multi-agent systems, such a characteristic root $\lambda = 0$ generally does not affect the consensus (see e.g., [8], [21], [22], [31], and [33]). In this paper, we will show that the frequency-sweeping approach can be directly used to study the stability of car-following systems. More precisely, the characteristic root $\lambda = 0$ does not correspond to a valid frequency-sweeping curve (FSC). We only need to analyze all the FSCs in a standard manner.

Second, the configuration matrix for a car-following system is an $n \times n$ one, where n is the number of vehicles. As n increases, the exponent of $e^{-\tau\lambda}$ for the corresponding characteristic function $f(\lambda, \tau)$ may be very large. It is not trivial to obtain the expansion of $f(\lambda, \tau)$, which is required in the earlier literature to generate the FSCs. In this paper, we generate the FSCs through computing the associated generalized eigenvalues, without needing to calculate the scalar form of $f(\lambda, \tau)$.

The above novelties considerably facilitate the analysis and will motivate us to further investigate more general consensus problems of multi-agent systems by using the frequency-sweeping approach in the future.

This paper is organized as follows. In Section II, some preliminaries and prerequisites are given. The frequency-sweeping approach for UDDSs is reviewed in Section III. In Section IV, the implementation of frequency-sweeping approach to car-following systems, two case studies, and some discussions are given. Finally, the paper concludes in Section V.

Notations: In this paper, $\mathbb{R}$ ($\mathbb{R}_+$) denotes the set of (positive) real numbers; $\mathbb{C}_-$ ($\mathbb{C}_+$) denotes the left-half (right-half) plane; $\mathbb{C}_0$ is the imaginary axis; $\partial\mathbb{D}$ is the unit circle; $\mathbb{N}$ is the set of non-negative integers. ε is a sufficiently small positive real number. I is the identity matrix of appropriate dimensions. For $\gamma \in \mathbb{R}$, $\lceil\gamma\rceil$ denotes the smallest integer greater than or equal to γ. $rank(\cdot)$ denotes the rank of a matrix. Finally, $\det(\cdot)$ denotes the determinant of its argument.

II. PRELIMINARIES AND PREREQUISITES

A. CAR-FOLLOWING SYSTEMS WITH UNIFORMLY DISTRIBUTED DELAYS

Drivers make decision based on the information continuously observed during a memory window (see [25] and [32]). That is why we adopt the distributed-delay model. In this paper, we adopt the uniform distribution function as the kernel of distributed-delay model, which represents the average of the information available in the memory window [16].

Consider a car-following system with n number of vehicles. The car-following dynamics can be described by

$$\dot{v}_i(t) = \alpha_i \int_0^{\infty} \kappa(\theta)(v_{i+1}(t-\theta) - v_i(t-\theta))d\theta,\quad i = 1, \ldots, n, \tag{1}$$

where $v_i(t)$ is the velocity of the i-th vehicle at time t, $\alpha_i > 0$ can be considered as the sensitivity of the i-th driver to the velocity difference between his vehicle and the one in front, and $\kappa(\theta)$ is the uniformly distributed delay kernel

$$\kappa(\theta) = \begin{cases} \dfrac{1}{d_1 + d_2}, & \textit{if } \tau - d_1 < \theta < \tau + d_2, \\ 0, & \textit{otherwise}, \end{cases} \tag{2}$$

where $\tau \geq d_1 \geq 0$, $d_2 \geq 0$, and $d_1 + d_2 > 0$.

All the n vehicles' dynamics can be expressed in a augmented form:

$$\dot{V}(t) = \int_0^{\infty} JV(t-\theta)\kappa(\theta)d\theta, \tag{3}$$

where $V(t) = (v_1(t), \ldots, v_n(t))^T$ and $J \in \mathbb{R}^{n\times n}$ is the configuration matrix weighted by $\alpha_i > 0$. The structure of J is determined by the network topology (two case studies will be given in Subsection IV-B).

B. STABILITY CONDITION OF CAR-FOLLOWING SYSTEMS

A car-following system is said to be stable if all the vehicles reach a constant speed. The associated characteristic function is (see e.g., [17] and [29]):

$$f(\lambda, \tau) = \det(\lambda I - J\mu(\lambda)e^{-\tau\lambda}), \tag{4}$$

where λ is the Laplace variable and $\mu(\lambda) = \frac{e^{d_1\lambda} - e^{-d_2\lambda}}{(d_1+d_2)\lambda}$.

We now summarize the stability condition for the car-following systems.

As J has a simple zero eigenvalue, the characteristic equation $f(\lambda, \tau) = 0$ has a characteristic root $\lambda = 0$ independent of τ, i.e., a fixed zero characteristic root. The stability of the car-following systems under consideration in this paper is determined by all the remaining characteristic roots.

In other words, a car-following system is asymptotically stable if and only if all the infinitely many characteristic roots (except the fixed zero characteristic root) are located in the open left half-plane $\mathbb{C}_-$.

C. COMPLETE STABILITY PROBLEM W.R.T. DELAY PARAMETER τ

Our objective in this paper is to study the complete stability problem w.r.t. the delay parameter τ. To be more precise, our objective is to find the whole stability τ-set along the semi-infinite interval $\tau \in [d_1, \infty)$.

As commonly adopted in the literature, the notation $NU(\tau) \in \mathbb{N}$ denotes the number of characteristic roots located in the open right half-plane $\mathbb{C}_+$, in the presence of delay τ. We will inspect $NU(\tau)$ as τ increases from the minimum value d_1 to ∞.

For the most common pointwise-delay system $\dot{x}(t) = Ax(t) + Bx(t-\tau)$, the characteristic function is a quasipolynomial $f(\lambda, \tau) = \det(\lambda I - A - Be^{-\tau\lambda})$ with infinitely many characteristic roots. One may refer to some monographs on stability of time-delay systems e.g., [7] and [18]. Generally speaking, it is non-trivial to study the spectrum and the complete stability problem has long been open (see [12] and [14] for some dedicated introduction and study).

Compared with the pointwise-delay systems, the UDDSs exhibit more complicated characteristic functions (belonging to general quasipolynomias [14]). In the next section, we will systematically introduce how to solve the complete stability problem for UDDSs.

III. FREQUENCY-SWEEPING APPROACH

The frequency-sweeping approach can be applied to study the stability of a broad class of time-delay systems [14], including retarded- and neutral-type time-delay systems, distributed time-delay systems, fractional time-delay systems, and multiple time-delay systems.

In this section, we review the results for UDDSs recently proposed in [13].

A. BASIC NOTIONS AND NOTATIONS

Consider the following general distributed delay system

$$\dot{x}(t) = Ax(t) + B\int_{-\infty}^{t} \kappa(t-\theta)x(\theta)d\theta, \tag{5}$$

where A and B are constant matrices and $\kappa(\theta) : [0, \infty) \mapsto [0, \infty)$ is a scalar kernel function.

Throughout this paper, we consider the case where $\kappa(\theta)$ is the uniform distribution (2).

The system described by (5) and (2) is called a uniformly distributed delay system (UDDS), and the characteristic function is

$$f(\lambda, \tau) = \det(\lambda I - A - B\frac{e^{-(\tau-d_1)\lambda} - e^{-(\tau+d_2)\lambda}}{(d_1+d_2)\lambda}), \quad \lambda \neq 0. \tag{6}$$

For the application of the UDDS model, one may refer to e.g., [4], [5], [15], and [29].

Remark 1: The UDDS model *includes* the pointwise-delay one. When $\kappa(\theta) = \delta(\theta - \tau)$ ($\delta(\theta)$ is the Dirac delta function), system (5) reduces to the pointwise-delay system $\dot{x}(t) = Ax(t) + Bx(t-\tau)$.

Although the characteristic function (6) is not a standard quasipolynomial, it falls in the class of general quasipolynomials (see [14]). The analysis for a general quasipolynomial is much more involved than that for a standard quasipolynomial.

Letting $z = e^{-\tau\lambda}$, we can rewrite the characteristic function $f(\lambda, \tau)$ (6) as

$$p(\lambda, z) = \det(\lambda I - A - B\mu(\lambda)z), \quad \lambda \neq 0. \tag{7}$$

It can be expanded as a scalar form:

$$p(\lambda, z) = a_0(\lambda) + a_1(\lambda)\mu(\lambda)z + \cdots + a_q(\lambda)\mu^q(\lambda)z^q, \tag{8}$$

where $q = rank(B)$.

Without loss of generality, suppose there are u critical pairs (λ, z) ($\lambda \in \mathbb{C}_0$ and $z \in \partial\mathbb{D}$) for $p(\lambda, z) = 0$. They are denoted by $(\lambda_0 = j\omega_0, z_0), \ldots, (\lambda_{u-1} = j\omega_{u-1}, z_{u-1})$ with $0 < \omega_0 \leq \cdots \leq \omega_{u-1}$.[1]

Once all the critical pairs $(\lambda_\alpha, z_\alpha)$ are found, all the critical pairs (λ, τ) ($\lambda \in \mathbb{C}_0$ and $\tau \in \mathbb{R}_+\backslash(0, d_1)$) for $f(\lambda, \tau) = 0$ can be obtained. The corresponding λ and τ are called a critical imaginary root (CIR) and a critical delay (CD), respectively.

For each CIR λ_α, the corresponding (infinitely many) CDs are given by $\tau_{\alpha,k} \triangleq \tau_{\alpha,0} + \frac{2k\pi}{\omega_\alpha}$, $k \in \mathbb{N}$, $\tau_{\alpha,0} \triangleq \min\{\tau \geq d_1 : e^{-\tau\lambda_\alpha} = z_\alpha\}$. The pairs $(\lambda_\alpha, \tau_{\alpha,k})$, $k \in \mathbb{N}$, define a set of critical pairs associated with $(\lambda_\alpha, z_\alpha)$.

For a UDDS, the frequency-sweeping curves (FSCs) can be generated by the following procedure.

Frequency-Sweeping Curves (FSCs): Sweep $\omega > 0$ and for each $\lambda = j\omega$ we have q solutions of z such that $p(j\omega, z) = 0$ (denoted by $z_1(j\omega), \ldots, z_q(j\omega)$). In this way, we obtain q FSCs $\Gamma_i(\omega)$: $|z_i(j\omega)|$ vs. ω, $i = 1, \ldots, q$. We denote by $\Im_1$ the line parallel to the abscissa axis with ordinate equal to 1. If $(\lambda_\alpha, \tau_{\alpha,k})$ is a critical pair, then some FSCs intersect $\Im_1$ at $\omega = \omega_\alpha$.

It is easy to see that all the CIRs and CDs can be detected from the FSCs.

For a set of critical pairs $(\lambda_\alpha, \tau_{\alpha,k})$, there must exist some FSCs such that $z_i(j\omega_\alpha) = z_\alpha = e^{-\tau_{\alpha,0}\lambda_\alpha}$ intersecting $\Im_1$ when $\omega = \omega_\alpha$. Among these FSCs, we denote the number of those when $\omega = \omega_\alpha + \varepsilon$ ($\omega = \omega_\alpha - \varepsilon$) above $\Im_1$ by $NF_{z_\alpha}(\omega_\alpha + \varepsilon)$ ($NF_{z_\alpha}(\omega_\alpha - \varepsilon)$). We introduce a notation $\Delta NF_{z_\alpha}(\omega_\alpha)$ to describe the asymptotic behavior of the FSCs, defined as

$$\Delta NF_{z_\alpha}(\omega_\alpha) = NF_{z_\alpha}(\omega_\alpha + \varepsilon) - NF_{z_\alpha}(\omega_\alpha - \varepsilon). \tag{9}$$

In order to solve the complete stability problem for UDDSs, some technical issues need to be appropriately addressed. See the subsequent subsections.

B. ON CHARACTERISTIC ROOT $\lambda = 0$

Although $f(\lambda, \tau)$ is not defined at $\lambda = 0$, $\lambda = 0$ may be a potential characteristic root. We can check it by L'Hôpital's rule (more details can be found in Subsection 3.1 of [13]).

[1] Due to the conjugate symmetry of the spectrum, it suffices to consider only the roots with non-negative imaginary parts.

For the car-following systems considered in this paper, there is always a simple characteristic root $\lambda = 0$ independent of τ. This is due to the simple zero eigenvalue of the configuration matrix J. Thus, $a_0(\lambda), \ldots, a_q(\lambda)$ in (8) have a common zero $\lambda = 0$ (i.e., the function $p(\lambda, z)$ (8) has a factor λ).

Remark 2: In the field of time-delay systems, the case that $a_0(\lambda), \ldots, a_q(\lambda)$ have a common zero not in $\mathbb{C}_-$ is generally treated as a trivial case, since the associated time-delay system cannot be asymptotically stable for any τ value. It is not hard to see that such a common zero does not have an effect on the FSCs.

C. SPECTRAL PROPERTY AT MINIMUM VALUE OF τ

To solve the complete stability problem, it is needed to analyze the spectrum at the minimum value of τ, i.e., $\tau = d_1$, where the system still has an infinite number of characteristic roots (unlike the pointwise-delay system $\dot{x}(t) = Ax(t) + Bx(t-\tau)$, which becomes a finite-dimensional system $\dot{x}(t) = (A+B)x(t)$ at the minimum value of τ, i.e., $\tau = 0$).

We can adopt the argument principle-based method to compute $NU(d_1)$ or $NU(d_1+\varepsilon)$ (more details can be found in Subsection 3.2 of [13]).

D. INVARIANCE PROPERTY

As a key step of the stability analysis, we need to analyze the asymptotic behavior of the CIRs at the corresponding CDs. Since a CIR has *infinitely many* CDs, it is impossible to analyze the asymptotic behavior at all the infinitely many CDs one by one.

Theorem 1 ([13]): For a critical imaginary root λ_α of the uniformly distributed delay system described by (5) and (2), $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is a constant $\Delta NF_{z_\alpha}(\omega_\alpha)$ for all $\tau_{\alpha,k} > d_1$.

The contribution of Theorem 1 is twofold:

First, the invariance property is confirmed for the UDDSs, with which we will be able to systematically study the complete stability problem.

Second, a graphical criterion is obtained to determine $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$, since the constant value of $\Delta NF_{z_\alpha}(\omega_\alpha)$ can be easily observed from the FSCs.

For each α, the constant value of $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is denoted by U_{λ_α}.

E. SOLUTION FOR COMPLETE STABILITY PROBLEM

Combining the results in the previous subsections, we are able to obtain the explicit expression of $NU(\tau)$ for a UDDS.

Theorem 2 ([13]): For any $\tau > d_1$ which is not a critical delay, $NU(\tau)$ for the uniformly distributed delay system described by (5) and (2) can be explicitly expressed as

$$NU(\tau) = NU(d_1+\varepsilon) + \sum_{\alpha=0}^{u-1} NU_\alpha(\tau), \tag{10}$$

where

$$NU_\alpha(\tau) = \begin{cases} 0, & \tau < \tau_{\alpha,0}, \\ 2U_{\lambda_\alpha}\left\lceil \dfrac{\tau - \tau_{\alpha,0}}{2\pi/\omega_\alpha} \right\rceil, & \tau > \tau_{\alpha,0}, \end{cases} \quad \text{if } \tau_{\alpha,0} \neq d_1,$$

$$NU_\alpha(\tau) = \begin{cases} 0, & \tau < \tau_{\alpha,1}, \\ 2U_{\lambda_\alpha}\left\lceil \dfrac{\tau - \tau_{\alpha,1}}{2\pi/\omega_\alpha} \right\rceil, & \tau > \tau_{\alpha,1}, \end{cases} \quad \text{if } \tau_{\alpha,0} = d_1,$$

The UDDS is asymptotically stable if and only if τ lies in the interval(s) with $NU(\tau) = 0$ excluding the CDs.

As a result, the whole stability τ-set may be found and the complete stability problem can be solved.

IV. IMPLEMENTATION OF FREQUENCY-SWEEPING APPROACH TO CAR-FOLLOWING SYSTEMS

In this section, we apply the frequency-sweeping approach reviewed in Section III to solve the complete stability problem for the car-following systems with uniformly distributed delays.

Compared with the work in [13], this paper contains some technical novelties as introduced in the next subsection.

A. NOVELTIES OF THE PAPER

For the car-following systems under consideration, the following properties hold:

(1) $\lambda = 0$ is a fixed characteristic root independent of τ.

(2) $rank(J) = n-1$ and the index q in (8) equals to $n-1$.

Taking into account the above properties, we present some novelties concerning the frequency-sweeping approach.

As mentioned, the case with characteristic root $\lambda = 0$ is a trivial case in the field of time-delay systems, since the associated time-delay systems cannot be asymptotically stable for any τ value. However, when studying the consensus of multi-agent systems, the characteristic root $\lambda = 0$ does not have an explicit affect, as the consensus is determined by all the other characteristic roots. In this paper, we will directly use the frequency-sweeping approach to address the car-following systems. More precisely, the characteristic root $\lambda = 0$ does not correspond to a valid FSC. There are $q = n-1$ number of valid FSCs. We only need to analyze all the $n-1$ FSCs in a standard manner. Moreover, this illustrates that the frequency-sweeping approach is applicable to some more general consensus problems of multi-agent systems.

Another novelty is that the FSCs can be generated through computing the generalized eigenvalue of $(\lambda I, J\mu(\lambda))$. To be more precise, for each ω value, the corresponding q solutions $z_1(j\omega), \ldots, z_q(j\omega)$ for $p(j\omega, z) = 0$ can be calculated by means of the generalized eigenvalues of $(j\omega I, J\mu(j\omega))$. We do not need to expand the function $p(\lambda, z)$ into the scalar form (8). As the configuration matrix of a car-following system (the Laplacian matrix of a general multi-agent system, see e.g., [19], [26], [27], and [34]) is in general of high dimension. This novelty considerably reduces the complexity of the analysis procedure and hence is of practical significance.

B. CASE STUDIES

In the sequel, we address a car-following system with 20 vehicles (borrowed from Subsection 5.1 of [29]) to illustrate the frequency-sweeping approach.

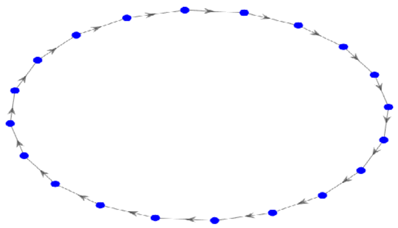
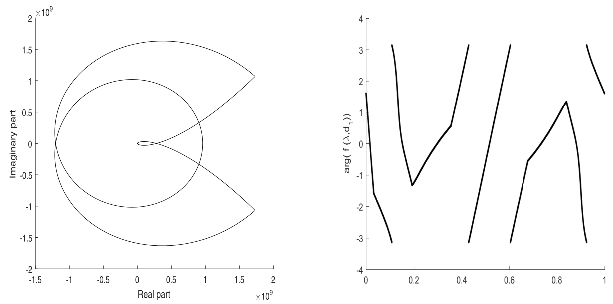


FIGURE 1. Ring configuration for example 1.



(a) Image (b) Argument

FIGURE 2. Image and argument of $f(\lambda, d_1)$ when $d_1 = d_2 = 0.1$ for example 1.

We study the complete stability problem under different values of d_1 and d_2. In particular, when $d_1 = d_2 = 0$, the system under consideration reduces to a pointwise-delay system. Therefore, we may compare the delay effect on stability between the uniformly distributed delay model and the commonly-used pointwise-delay model.

Two widely utilized configurations are considered in Example 1 and Example 2, respectively.

Example 1: Consider a car-following system with 20 vehicles (i.e., $n = 20$). With the ring configuration as shown in Fig. 1.

The configuration matrix J is the following 20×20 matrix

$$J = \begin{pmatrix} -2 & 0 & \cdots & 0 & 2 \\ 2 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2 & -2 \end{pmatrix}.$$

We here study the complete stability problem w.r.t. the delay parameter τ under different values of d_1 and d_2.

We first address the case $d_1 = d_2 = 0$, which exactly corresponds to the pointwise-delay model. We may adopt some existing approaches and have that the stability set of τ is $[0, 0.2510)$.

In the sequel, we study in detail the case $d_1 = d_2 = 0.1$.

As mentioned in Subsection III-B, there is a fixed simple characteristic root $\lambda = 0$. The reason for characteristic root $\lambda = 0$ is different with the one for the general UDDSs discussed in [13]. For the characteristic function $f(\lambda, \tau)$ (4) considered in this paper, λ is a factor and hence the characteristic root $\lambda = 0$ does not affect the FSCs.

By using the method mentioned in Subsection III-C, we have that $NU(0.1) = NU(0.1 + \varepsilon) = 0$. As λ varies along a proper Jordan curve (see [13] for details), the image of $f(\lambda, d_1)$ is given in Fig. 2(a) and the argument of $f(\lambda, d_1)$ (denoted by $\arg(f(\lambda, d_1)) \in (-\pi, \pi]$) is given in Fig. 2(b), where the abscissa axis denotes the proportion of the completion of λ's track along the Jordan curve.

The remaining analysis is as follows.

The FSCs for $d_1 = d_2 = 0.1$ are shown in Fig. 3(a) and Fig. 3(b). According to the discussions in Subsection IV-A, there are totally 19 FSCs. Among them, there are 9 pairs of coinciding FSCs.

From the FSCs, we can detect all CIRs λ_α and the corresponding CDs $\tau_{\alpha,k}$. They are listed in Table 1.

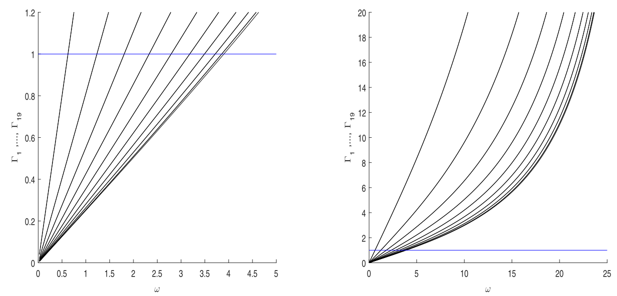
According to Theorem 1, it follows that $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = +1$ for all $\alpha = 0, \ldots, 18$ and $k \in \mathbb{N}$. Hence, $U_{\lambda_\alpha} = +1$ for all $\alpha = 0, \ldots, 18$.

In light of Theorem 2, for any $\tau > d_1$ which is not a critical delay, $NU(\tau)$ for the car-following system can be explicitly expressed as

$$NU(\tau) = 0 + \sum_{\alpha=0}^{18} NU_\alpha(\tau),$$

TABLE 1. CIRs and CDs for example 1.

CIRs	CDs ($k \in \mathbb{N}$)
$\lambda_{0,1}$=0.6253j	$\tau_{0,k} = 4.7727 + 10.0478k$
	$\tau_{1,k} = 0.2512 + 10.0478k$
$\lambda_{2,3}$=1.2329j	$\tau_{2,k} = 2.2932 + 5.0961k$
	$\tau_{3,k} = 0.2548 + 5.0961k$
$\lambda_{4,5}$=1.8061j	$\tau_{4,k} = 1.4785 + 3.4789k$
	$\tau_{5,k} = 0.2609 + 3.4789k$
$\lambda_{6,7}$=2.3299j	$\tau_{6,k} = 1.0787 + 2.6967k$
	$\tau_{7,k} = 0.2697 + 2.6967k$
$\lambda_{8,9}$=2.7918j	$\tau_{8,k} = 0.8440 + 2.2506k$
	$\tau_{9,k} = 0.2813 + 2.2506k$
$\lambda_{10,11}$=3.1817j	$\tau_{10,k} = 0.6912 + 1.9748k$
	$\tau_{11,k} = 0.2962 + 1.9748k$
$\lambda_{12,13}$=3.4920j	$\tau_{12,k} = 0.5848 + 1.7993k$
	$\tau_{13,k} = 0.3149 + 1.7993k$
$\lambda_{14,15}$=3.7172j	$\tau_{14,k} = 0.5071 + 1.6903k$
	$\tau_{15,k} = 0.3381 + 1.6903k$
$\lambda_{16,17}$=3.8537j	$\tau_{16,k} = 0.4484 + 1.6304k$
	$\tau_{17,k} = 0.3668 + 1.6304k$
λ_{18}=3.8994j	$\tau_{18,k} = 0.4028 + 1.6113k$



(a) For $\omega \in [0, 5]$ (b) For $\omega \in [0, 25]$

FIGURE 3. FSCs when $d_1 = d_2 = 0.1$ for example 1.

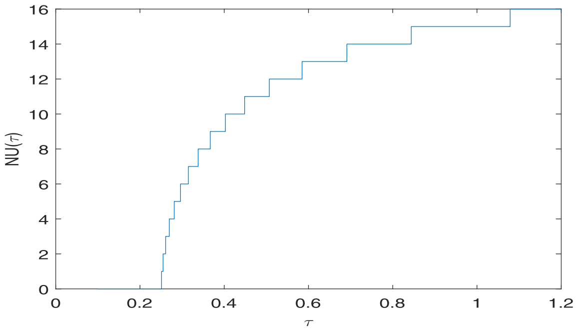


FIGURE 4. $NU(\tau)$ **when** $d_1 = d_2 = 0.1$ **for example 1.**

TABLE 2. Stability results for example 1.

$d_1 = d_2 =$	Stability τ-set	Range of θ with $\kappa(\theta) > 0$
0	$[0, 0.2510)$	
0.01	$[0.01, 0.2510)$	$[0, 0.02]$ to $(0.2410, 0.2610)$
0.05	$[0.05, 0.2511)$	$[0, 0.1]$ to $(0.2011, 0.3011)$
0.1	$[0.1, 0.2512)$	$[0, 0.2]$ to $(0.1512, 0.3512)$
0.15	$[0.15, 0.2514)$	$[0, 0.3]$ to $(0.1014, 0.4014)$
0.2	$[0.2, 0.2517)$	$[0, 0.4]$ to $(0.0517, 0.4517)$
0.25	$[0.25, 0.2520)$	$[0, 0.5]$ to $(0.0020, 0.5020)$
0.3	$\emptyset$	

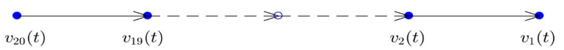


FIGURE 5. Linear configuration for example 2.

where

$$NU_\alpha(\tau) = \begin{cases} 0, & \tau < \tau_{\alpha,0}, \\ 2\left\lceil \dfrac{\tau - \tau_{\alpha,0}}{2\pi/\omega_\alpha} \right\rceil, & \tau > \tau_{\alpha,0}, \end{cases} \quad \alpha = 0, \ldots, 18.$$

Therefore, there is one and only one stability τ-interval: $[0.1, 0.2512)$. The $NU(\tau)$ distribution is plotted in Fig. 4.

For other values of d_1 and d_2, the complete stability problem w.r.t. τ can be solved analogously. We directly list the stability results in Table 2.

In Table 2, we also give the range of θ with $\kappa(\theta) > 0$ under which the stability may be retained. This range reflects the allowable range of delay effect. ■

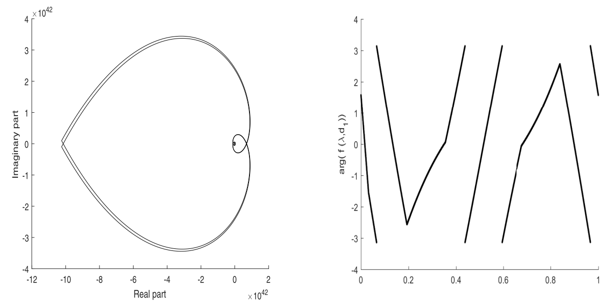
Example 2: Consider a car-following system with 20 vehicles (i.e., $n = 20$). With the linear configuration as shown in Fig. 5.

The configuration matrix J is the following 20×20 matrix

$$J = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 2 & -2 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2 & -2 \end{pmatrix}.$$

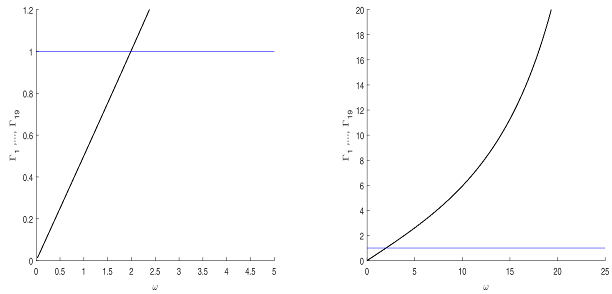
We here study the complete stability problem w.r.t. the delay parameter τ under different values of d_1 and d_2.

We first address the case $d_1 = d_2 = 0$, which exactly corresponds to the pointwise-delay model. We may adopt some existing approaches and have that the stability set of τ is $[0, 0.7854)$.



(a) Image (b) Argument

FIGURE 6. Image and argument of $f(\lambda, d_1)$ **when** $d_1 = d_2 = 0.1$ **for example 2.**



(a) For $\omega \in [0, 5]$ (b) For $\omega \in [0, 25]$

FIGURE 7. FSCs when $d_1 = d_2 = 0.1$ **for example 2.**

As mentioned in Subsection III-B, there is a fixed simple characteristic root $\lambda = 0$ and it does not affect the FSCs.

By using the method mentioned in Subsection III-C, we have that $NU(0.1) = NU(0.1 + \varepsilon) = 0$. As λ varies along a proper Jordan curve (see [13] for details), the image of $f(\lambda, d_1)$ is given in Fig. 6(a) and the argument of $f(\lambda, d_1)$ (denoted by $\arg(f(\lambda, d_1)) \in (-\pi, \pi]$) is given in Fig. 6(b), where the abscissa axis denotes the proportion of the completion of λ's track along the Jordan curve.

The remaining analysis is as follows.

The FSCs for $d_1 = d_2 = 0.1$ are shown in Fig. 7(a) and Fig. 7(b). According to the discussions in Subsection IV-A, there are totally 19 FSCs. Specifically, they are all coinciding. One may easily prove it in light of the factorization technique to be discussed in the next subsection.

From the FSCs, we can detect the CIRs: $\lambda_\alpha = 1.9869j$ with the CDs $\tau_{\alpha,k} = 0.7906 + 3.1624k$ for all $\alpha = 0, \ldots, 18$, where $k \in \mathbb{N}$.

According to Theorem 1, it follows that $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) = +1$ for all $\alpha = 0, \ldots, 18$ and $k \in \mathbb{N}$. Hence, $U_{\lambda_\alpha} = +1$ for all $\alpha = 0, \ldots, 18$.

In light of Theorem 2, for any $\tau > d_1$ which is not a critical delay, $NU(\tau)$ for the car-following system can be explicitly expressed as

$$NU(\tau) = 0 + \sum_{\alpha=0}^{18} NU_\alpha(\tau),$$

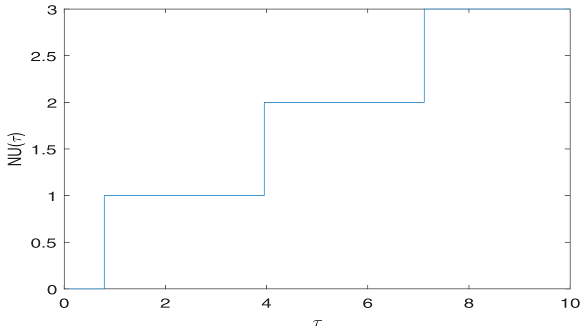


FIGURE 8. $NU(\tau)$ when $d_1 = d_2 = 0.1$ for example 2.

TABLE 3. Stability results for example 2.

$d_1 = d_2 =$	Stability τ-set	Range of θ with $\kappa(\theta) > 0$
0	$[0, 0.7854)$	
0.01	$[0.01, 0.7855)$	$[0, 0.02]$ to $(0.7755, 0.7955)$
0.05	$[0.05, 0.7867)$	$[0, 0.1]$ to $(0.7367, 0.8367)$
0.1	$[0.1, 0.7906)$	$[0, 0.2]$ to $(0.6906, 0.8906)$
0.15	$[0.15, 0.7970)$	$[0, 0.3]$ to $(0.6470, 0.9470)$
0.2	$[0.2, 0.8057)$	$[0, 0.4]$ to $(0.6057, 1.0057)$
0.25	$[0.25, 0.8165)$	$[0, 0.5]$ to $(0.5665, 1.0665)$
0.3	$[0.3, 0.8293)$	$[0, 0.6]$ to $(0.5293, 1.1293)$

where

$$NU_\alpha(\tau) = \begin{cases} 0, & \tau < \tau_{\alpha,0}, \\ 2\left\lceil \dfrac{\tau - \tau_{\alpha,0}}{2\pi/\omega_\alpha} \right\rceil, & \tau > \tau_{\alpha,0}, \end{cases} \quad \alpha = 0, \ldots, 18.$$

Therefore, there is one and only one stability τ-interval: $[0.1, 0.7906)$. The $NU(\tau)$ distribution is plotted in Fig. 8.

For other values of d_1 and d_2, we directly list the results for the complete stability problem w.r.t. τ in Table 3. ■

It is worth to mention that the results in the above two examples are analytical without any conservatism.

C. SOME DISCUSSIONS

In the above examples, all the CIRs are simple and non-degenerate. Otherwise, the analysis of the CIRs' asymptotic behavior will be much more complicated. Such examples for UDDSs can be found in [13]. It is worth mentioning that the frequency-sweeping approach covers the general case: Multiple and/or degenerate CIRs are allowed.

It is seen from Table 2 and Table 3 that the UDDS model may give rise to a larger stability τ-set as well as a larger allowable range of delay effect than the pointwise-delay system counterpart. The positive effect of introducing the UDDS model is illustrated.

The examples demonstrate that it is simple to implement the frequency-sweeping approach, although the car-following systems under consideration are with high dimensions.

The problem considered in this paper is a specific consensus one. For instance, there is not a non-delayed term in the expression (3). The results of this paper inspire us to study more general consensus problems for multi-agent systems by using the frequency-sweeping approach. For the consensus studies of multi-agent systems with delays, one may refer to e.g., [19], [22], [26], [31], and [33].

In the literature, the factorization of the characteristic function is often used. For instance, the characteristic function $f(\lambda, \tau)$ (4) can be factorized as

$$f(\lambda, \tau) = \prod_{i=1}^{n} (\lambda - \lambda_i(J)\mu(\lambda)e^{-\tau\lambda}),$$

where $\lambda_i(J)$ stands for the i-th eigenvalue of J with $\lambda_1(J) = 0$.

With the above factorization, we may study the complete stability problem for each factorized characteristic equation $\lambda - \lambda_i(J)\mu(\lambda)e^{-\tau\lambda} = 0$. The common stability τ-set of all the factorized characteristic equations expect the one corresponding to $\lambda_1(J)$ is the stability τ-set of the car-following system.

It is worth to note that some $\lambda_i(J)$ may be complex numbers. That is, some factorized characteristic functions are *complex-coefficient quasipolynomials*, which leads to some technical points to be specifically investigated. In the future, the frequency-sweeping approach could be used with the factorization, and some deeper properties may be explored in this way.

V. CONCLUDING REMARKS

In this paper, we study the stability of car-following systems with delays. The delays are modelled as uniformly distributed ones. Such models are more realistic than the commonly-used pointwise-delay counterparts and meanwhile the analysis is more complicated.

We adopt in this paper the frequency-sweeping approach recently proposed for uniformly distributed delay systems (UDDSs). With some technical novelties, a modified version of frequency-sweeping approach is obtained. It is shown that the complete stability problem w.r.t. the delay parameter τ now can be systematically solved and that the implementation of the frequency-sweeping approach is quite simple.

The problem under consideration in this paper belongs to the consensus problem for multi-agent systems. Inspired by the results of this paper, in the future, we may study some more general consensus problems by using the frequency-sweeping approach.

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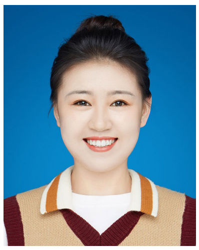

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