IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received April 19, 2021, accepted April 29, 2021, date of publication May 4, 2021, date of current version May 26, 2021.

Digital Object Identifier 10.1109/ACCESS.2021.3077387

Stabilization of Periodic Switched k-Valued Logical Networks

YAN GAO [1], CHENCHEN LIU [2], AND JIAQI WANG [1]
[1]School of P.E., Institute of Control and Systems, Shandong University, Jinan 250061, China
[2]School of Medical Information Engineering, Jining Medical University, Jining 272067, China

Corresponding author: Chenchen Liu (lchenchen1979@hotmail.com)

This work was supported in part by the National Natural Science Foundation of China under Grant 2017JJ20986, in part by the China Postdoctoral Science Foundation under Grant 2017M622169, and in part by the Future Project for Youth Scholar of Shandong University under Grant 2017WLJH17.

ABSTRACT The stabilization of periodic switched k-valued logical networks is investigated in this paper, and some new results are presented. The system considered consists of several k valued logical networks and these networks run in a periodic switching law. First, by using the Cheng product of matrices, a periodic switched k-valued logical (control) network is transformed into a discrete dynamic system which is written as an algebraic form. Second, the switching-state space and the switching-input-state space are defined. Then combining with the algebraic form, some necessary and sufficient conditions for the stability and the stabilization are obtained. An algorithm to find the input sequence that stabilizes the system is also provided. Finally, illustrative examples are given to support the proposed new results.

INDEX TERMS k-valued logical networks, periodic switched, stability, stabilization, control design.

I. INTRODUCTION

Gene regulatory network, a network of genes and their interactions within a cell or a particular genome, is the mechanism that controls the gene expression in an organism. Boolean networks (BN), originally proposed by Kauffman in 1969 [1], becomes a powerful tool for describing, analyzing and simulating cellular networks. It has attracted a lot of attention and interest from many biologists, physicists and system scientists [2]. By expressing the complex connections and interactions between genes through simple logical relationships, BN has a high degree of abstraction, and can reflect the system's rich kinetic behaviors. With the emergence of many high-quality research results [3]–[5], it is further confirmed that BN is very effective in simulating practical processes. Therefore, the study of control problem has important theoretical and practical significance.

Recently, a new matrix product, namely, the semi-tensor product of matrices, is proposed. Here we call it Cheng product due to the fact that it was proposed by Prof. Cheng [6]. Cheng product extends the ordinary matrix multiplication to any two matrices, which not only preserves all the basic properties of the original matrix multiplication, but also has a certain degree of commutativity. These properties make the matrix method easy to be applied to logical functions, high-dimensional arrays and nonlinear problems. So the Boolean logic equation can be transformed into the discrete dynamic equation, which is more convenient to describe and study. Using Cheng product, BNs can be successfully analyzed and many essential results have been obtained [7]–[11]. Similar to linear systems, some properties have been discussed, including their controllability, observability and synchronization [12]–[16].

In actual gene networks, the multi-mode switching phenomenon usually appears. For switched systems, the switching mechanism can improve the control capability. Refs. [17], [18] investigated the controllability and observability of switched Boolean control networks under the switching signal. Ref. [19] studied the set stability and stabilization of switched BNs with state-based switching. However, the switching signals of the above results are the general signals. Sometimes the switching signals actually are generated by external intervention or constraints. For example, the cycle of day and night makes biological system evolutive and alive in a cyclic environment. Ref. [20] studied the reachability and controllability of periodic switched BNs where the switching signals are periodic signals. Under the periodic

The associate editor coordinating the review of this manuscript and approving it for publication was Yang Tang.

special switching signals, the property of switched BNs is meaningful.

k-valued logical network consists of a family of k-valued logical variables and a set of k-valued logical functions. It is an important networked system since it can be regarded as the extension of BNs. The k-valued logical network has become an effective tools in researching evolutionary games [21]–[23], discrete event systems [24], [25] and finite automata [26], [27]. 2-valued BNs could describe the switching characteristics of biological networks by 0 and 1. While 3-valued logical networks can describe catalytic, inactive and inhibitory behaviors of cell activities. So far, some fundamental problems of them have been deeply studied, such as the stability at point attractors and stability at dynamic attractors, see [28], [29] for details. To the best of our knowledge, the stability and stabilization of k-valued logical networks is still relatively little from the perspective of periodic switching signals. So inspired by that, we investigate the stability and stabilization of periodic switched k-valued logical networks. Some contributions are as follows.

(1) Under a periodic switching law, a system composed by several k-valued logical networks is considered. By Cheng product and its properties, the system is converted as an algebraic form.

(2) Two spaces, the switching-state space and the switching-input-state space, are defined. Combining with the algebraic form, some necessary and sufficient conditions for the stability and the stabilization of k-valued logical networks are obtained.

(3) An algorithm to find the input sequence that stabilizes the k-valued logical network is also provided.

The contents of this paper consist of two parts: one is stabilization problem, which is the key content of this paper, and the other is some theoretical analysis derived from the study of stabilization problem.

(4a) The methods, provided for solving the stabilization problem, are based on k-valued logical networks, which is a generalization of BNs. Besides, the corresponding results can be also applied to periodic switched BNs.

(4b) Many remarks, proposed in the process of studying stabilization problem, indicates the possibility that relevant results can be generalized to other networks and situations. In addition, pure mathematical issues derived from some problems, which are the unique properties of the periodic switched k-valued logical networks, are worthy of further study by interested readers.

The paper is organized as following. In Section II, we recall some preliminaries on Cheng product and problem formulation which will be used in later sections. For the periodic switched k-valued logical networks, the main results of this paper, stability, stabilization and optimal control design for stabilization are discussed in Section III. Moreover, the application of the design in solving the stabilization problem is shown. At last, a brief conclusion is given in section IV.

A. NOTATIONS

The following symbols will be used throughout the paper: $\mathbb{R}$ and $\mathbb{Z}$ denote the sets of real and positive integer numbers, respectively.

1) $\mathrm{Col}_j(M)$ and $\mathrm{Row}_i(M)$ stand for the j^{th} column and the i^{th} row of the matrix M, respectively, and $\mathrm{Col}(M)$ is the set of columns of matrix M.
2) δ_k^i denotes the i^{th} column of the k-dimensional identity matrix I_k.
3) A matrix $L \in \mathcal{L}_{m\times n}$ is called a logical matrix if $Col(L) \subseteq \mathcal{L}_{m\times n}$, $\mathcal{L}_{m\times n}$ represents the set of all $m \times n$ logical matrices.
4) $\mathcal{D}_k := \{0, 1, 2, \dots, k-1\}$.
5) $\Delta_k := \{\delta_k^i \mid i = 1, 2, \dots, k\}$, where δ_k^i represents the i-th column of identity matrix I_k.
6) $\otimes$ represents the Kronecker product of matrices. If $A = (a_{ij}) \in R^{m\times n}$ and $B = (b_{ij}) \in R^{p\times q}$, then

$$A \otimes B = \begin{bmatrix} a_{11} \times B & a_{12} \times B & \cdots & a_{1n} \times B \\ a_{21} \times B & a_{22} \times B & \cdots & a_{2n} \times B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} \times B & a_{m2} \times B & \cdots & a_{mn} \times B \end{bmatrix},$$

where

$$a_{ij} \times B = \begin{bmatrix} a_{ij} \times b_{11} & a_{ij} \times b_{12} & \cdots & a_{ij} \times b_{1t} \\ a_{ij} \times b_{21} & a_{ij} \times b_{22} & \cdots & a_{ij} \times b_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ a_{ij} \times b_{s1} & a_{ij} \times b_{s2} & \cdots & a_{ij} \times b_{st} \end{bmatrix}.$$

II. PRELIMINARIES

This section gives some necessary preliminaries on Cheng product and periodic switched k-valued logical networks, which will be used in the following.

A. CHENG PRODUCT

Definition 1 [30]: The Cheng product of matrices $A \in \mathbb{R}_{m\times n}$ and $B \in \mathbb{R}_{s\times t}$ is defined as

$$A \ltimes B = (A \otimes I_{\frac{p}{n}})(B \otimes I_{\frac{p}{s}}), \quad (1)$$

where p is the least common multiple of n and s.

Proposition 1 [30]: Cheng product satisfies combination law and pseudo exchange law:

(1) Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, $C \in \mathbb{R}^{r\times s}$, then

$$(A \ltimes B) \ltimes C = A \ltimes (B \ltimes C). \quad (2)$$

(2) Let $X \in \mathbb{R}^{t\times 1}$, $A \in \mathbb{R}^{m\times n}$, then

$$X \ltimes A = (I_t \otimes A) \ltimes X. \quad (3)$$

Proposition 2 [30]: If $X \in \Delta_k$, then

$$XX = \Phi_k \ltimes X, \quad (4)$$

where $\Phi_k = [\delta_k^1\delta_k^1 \ \ \delta_k^2\delta_k^2 \ \ \cdots \ \ \delta_k^k\delta_k^k]$ is the base-k power-reducing matrix.

TABLE 1. Three structure matrices in Example 1.

Operator	Structure matrix
$\neg$	$M_n = \delta_3[3\ 2\ 1]$
$\wedge$	$M_c = \delta_3[1\ 2\ 3\ 2\ 2\ 3\ 3\ 3\ 3]$
$\vee$	$M_d = \delta_3[1\ 1\ 1\ 1\ 2\ 2\ 1\ 2\ 3]$

If $X, Y \in \Delta_k$, then

$$E_k XY = X, \tag{5}$$

where $E_k = [I_k\ I_k\ \cdots\ I_k]$ is the base-k dummy operator.

Lemma 1 [30]: (1) Let $f(x_1, x_2, \ldots, x_n)$ be a logic function. Then there exists a unique logic matrix $M_f \in \mathcal{L}_{2\times 2^n}$, called structure matrix of f, such that

$$f(x_1, x_2, \ldots, x_n) = M_f \ltimes x, \tag{6}$$

where $x = \ltimes_{i=1}^{n} x_i$.

(2) Assume

$$\begin{cases} y = M_y \ltimes_{i=1}^{n} x_i, \\ z = M_z \ltimes_{i=1}^{n} x_i, \end{cases} \tag{7}$$

where $x_i \in \Delta$, $i = 1, 2, \ldots, n$, $M_y, M_z \in \mathcal{L}_{2\times 2^n}$. Then

$$yz = (M_y * M_z) \ltimes_{i=1}^{n} x_i, \tag{8}$$

where $M_y * M_z = [Col_1(M_y) \otimes Col_1(M_z), Col_2(M_y) \otimes Col_2(M_z), \ldots, Col_{2^n}(M_y) \otimes Col_{2^n}(M_z)]$.

Proposition 3:

$$\neg X = 1 - X,$$
$$X \wedge Y = \min\{X, Y\},$$
$$X \vee Y = \max\{X, Y\},$$

and

$$\neg \sim M_n = \delta_k[k\ k-1\ \cdots\ 1],$$
$$\wedge \sim M_c = \delta_k[1\ 2\ \cdots\ k\ 2\ 2\ \cdots\ k\ \cdots\ k\ k\ \cdots\ k],$$
$$\vee \sim M_d = \delta_k[1\ 1\ \cdots\ 1\ 1\ 1\ \cdots\ 2\ \cdots\ 1\ 2\ \cdots\ k].$$

Example 1: The structure matrices of three common logical operators ($k = 3$) is shown in Table 1.

Proposition 4: Let $P_k = A_1 A_2 \ldots A_k$, $A_i \in \Delta$, then

$$P_k^2 = \Phi_k P_k, \tag{9}$$

where

$$\Phi_k = \prod_{i=1}^{k} \mathrm{I}_{2^{i-1}} \otimes \left[(\mathrm{I}_2 \otimes W_{[2,2^{k-i}]}) M_r\right]. \tag{10}$$

B. PERIODIC SWITCHED k-VALUED LOGICAL NETWORKS

We consider a periodic switched k-valued logical networks with n state nodes $\{x_1, x_2, \ldots, x_n\}$. The state nodes take values from $\mathcal{D}_k$. Its dynamics can be described as

$$\begin{cases} x_1(t+1) = f_1^{\sigma(t)}(x_1(t), \ldots, x_n(t)), \\ x_2(t+1) = f_2^{\sigma(t)}(x_1(t), \ldots, x_n(t)), \\ \quad\vdots \\ x_n(t+1) = f_n^{\sigma(t)}(x_1(t), \ldots, x_n(t)), \\ \sigma(t) = t \mod l+1, \end{cases} \tag{11}$$

where x_i is state variable, $f_i^j : \mathcal{D}^n \mapsto \mathcal{D}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, l$ logical functions, $\sigma(t) : \mathbb{N} \mapsto \{1, 2, \ldots, l\}$ is the switching signal.

We make a preliminary analysis about switching signals $\sigma(t)$. According to the formula $\sigma(t) = t \mod l + 1$, we can know that l systems evolve in a fixed order. After the system state is updated by l steps, it enters the repeated evolution process again. For example, when $l = 3$, the system runs according to

$$t = 0,\ \Sigma_{\sigma(0)} = \Sigma_1 = \{f_1^1, f_2^1, \ldots, f_n^1\};$$
$$t = 1,\ \Sigma_{\sigma(1)} = \Sigma_2 = \{f_1^2, f_2^2, \ldots, f_n^2\};$$
$$t = 2,\ \Sigma_{\sigma(2)} = \Sigma_3 = \{f_1^3, f_2^3, \ldots, f_n^3\};$$
$$t = 3,\ \Sigma_{\sigma(3)} = \Sigma_1 = \{f_1^1, f_2^1, \ldots, f_n^1\};$$
$$t = 4,\ \Sigma_{\sigma(4)} = \Sigma_2 = \{f_1^2, f_2^2, \ldots, f_n^2\};$$
$$t = 5,\ \Sigma_{\sigma(5)} = \Sigma_3 = \{f_1^3, f_2^3, \ldots, f_n^3\};$$
$$\cdots$$

III. MAIN RESULTS

A. MATRIX EXPRESSION

Next, we use the Cheng product of matrices to transform the system form Boolean function form to algebraic form. Using Lemma 1, we have

$$\begin{cases} x_1(t+1) = M_1^{\sigma(t)} x_1(t) x_2(t) \cdots x_n(t), \\ x_2(t+1) = M_2^{\sigma(t)} x_1(t) x_2(t) \cdots x_n(t), \\ \quad\vdots \\ x_n(t+1) = M_n^{\sigma(t)} x_1(t) x_2(t) \cdots x_n(t). \end{cases}$$

where $M_i^{\sigma(t)}$ is the structure matrix of $f_i^{\sigma(t)}$, $i = 1, 2, \ldots, n$. Let $x(t) = x_1(t)x_2(t)\cdots x_n(t)$, multiply the left and right formulas above and get the following algebraic form:

$$\Sigma_{\sigma(t)} :\ x(t+1) = M_{\sigma(t)} x(t), \tag{12}$$

where $M_{\sigma(t)} = M_1^{\sigma(t)} * M_2^{\sigma(t)} * \cdots * M_n^{\sigma(t)} \in \mathcal{L}_{k^n \times k^n}$.

Identify $\sigma(t) \sim \delta_l^{t \mod l+1}$, then a new logical matrix is formed by all matrices $M_1, M_2, \ldots, M_l$:

$$M = [M_1\ M_2\ \cdots\ M_l] \in \mathcal{L}_{k^n \times lk^n}. \tag{13}$$

Hence, (12) can be rewritten as:

$$\Sigma_{\sigma(t)} :\ x(t+1) = M\sigma(t)x(t), \tag{14}$$

where $\sigma(t) \in \Delta_l$. In this paper, we use σ, x, u to represent both their quantitative forms and vector forms with a slight abuse.

B. STABILITY ANALYSIS

This section considers global stability. Stability is an important property of the system, which describes the future direction of the system. A system is said to be globally stable if it globally converges to a fixed point [31]. In other words, it has a fixed point as the only attractor.

The global stability of periodic switched k-valued logical networks is defined by the following mathematics symbol.

Definition 2: System (14) is said to be globally stable to $\delta_{k^n}^d \in \Delta_{k^n}$, if for any initial state $x(0) \in \Delta_{2^n}$, $x(t) = \delta_{k^n}^d$ holds for some $s \in \mathbb{N}$, $t \geq s$.

In order to study this property, we introduce the concept of reachability

Definition 3: Consider system (14), its switching-state space is defined as

$$S = \{(\sigma, x)|\sigma \in \{1, 2, \ldots, l\},\quad x = (x_1, x_2, \ldots, x_n) \in \mathcal{D}_k^n\}. \tag{15}$$

1) Let $p_i = (\sigma_i, x^i) \in S$, under vector form, if (p_1, p_2) satisfies

$$\begin{cases} x^2 = M\sigma_1 x^1, \\ E\sigma_2 = E\sigma_1 \mod l+1, \end{cases} \tag{16}$$

where $E := [1, 2, \cdots, l]$, then call (p_1, p_2) a directed edge. The set of directed edges depending on switching is remarked as $V \subseteq S \times S$.

2) If $(p_i, p_{i+1}) \in V, i = 1, 2, \ldots, \theta - 1$, then $(p_1, p_2, \ldots, p_\theta)$ is a path, denoted by $p_1 \to p_\theta$. p_θ is reachable from p_1 after $\theta - 1$ steps.

Note that a path with two edges $(p_1 = (\sigma_1, x^1), p_2 = (\sigma_2, x^2), p_3 = (\sigma_3, x^3))$ means that the σ_1-th subsystem transforms x^1 to x^2, and then the σ_2-th subsystem transforms x^2 to x^3. Though x^3 is reachable from x^1, the transformation of state depends on the running subsystem.

For example, the σ_2-th subsystem transforms x^2 to some state but x^3. From Definition 3, if $p_1 \to p_\theta$, one has

$$\begin{aligned} x^2 &= M\sigma_1 x^1,\ E\sigma_2 = E\sigma_1 \mod l+1, \\ x^3 &= M\sigma_2 x^2,\ E\sigma_3 = E\sigma_2 \mod l+1, \\ &\vdots \\ x^\theta &= M\sigma_{\theta-1} x^{\theta-1},\ E\sigma_\theta = E\sigma_{\theta-1} \mod l+1, \end{aligned}$$

then

$$\begin{aligned} x^\theta &= M\sigma_{\theta-1} x^{\theta-1} \\ &= M\sigma_{\theta-1} M\sigma_{\theta-2} \cdots M\sigma_1 x^1. \end{aligned}$$

By resetting the corresponding subscript, the following properties can be obtained.

Proposition 5: Let $x^{i_j} = \delta_{k^n}^{i_j}, j = 1, 2, \ldots, \theta$, then $(p_{i_1}, p_{i_2}, \ldots, p_{i_\theta})$ is a path if and only if

$$[M_{i_1+\theta_2-1} \cdots M_{i_1+1} M_{i_1} (\overline{M}_{i_1})^{\theta_1}]_{i_\theta, i_1} = 1, \tag{17}$$

where $\overline{M}_j = M_{j-1} M_{j-2} \cdots M_1 M_l \cdots M_{j+1} M_j$, $\theta = \theta_1 l + \theta_2$ and $\theta_2 = \theta \mod l$.

The next theorem can be easily obtained by Proposition 5 and the arbitrariness of x_0.

Theorem 1: Consider system (14), state $\delta_{k^n}^d$ is reachable from any initial state if and only if there exists an integer $s \in \mathbb{N}$, such that

$$M_{s_2} M_{s_2-1} \cdots M_1 \overline{M}_1 = \delta_{k^n}[d\ d\ \cdots\ d], \tag{18}$$

where $s = s_1 l + s_2$ and $s_2 = s \mod l$.

Proof: Under the periodic switching signals, the initial state of the system circulates along the order of $1, 2, \ldots, l$. Given an initial state $x(0) = x_0$, the system state $x(s)$ is

$$\begin{aligned} x(s) &= M\sigma(s-1)x(s-1) \\ &= M\sigma(s-1)M\sigma(s-2)x(s-2) \\ &\vdots \\ &= M\sigma(s-1)M\sigma(s-2)\cdots M\sigma(0)x_0. \end{aligned}$$

From Proposition 5, state $x(s) = \delta_{k^n}^d$ is reachable from state $x(0) = x_0$ if and only if

$$\begin{aligned} \delta_{k^n}^d &= x(s) = M\sigma(s-1)M\sigma(s-2)\cdots M\sigma(0)x_0 \\ &= M_{s_2} M_{s_2-1} \cdots M_1 (M_l M_l \cdots M_1)^{s_1} x_0 \\ &= M_{s_2} M_{s_2-1} \cdots M_1 \overline{M}_1 x_0. \end{aligned}$$

By the arbitrariness of initial state x_0, the conclusion can be obtained.

Proposition 6: If system (14) is globally stable, the upper bound of s is $k^n l - 1$.

Proof: Consider the system states at integral multiple time of l,

$$x(0),\ x(l),\ x(2l),\ x(3l),\ \cdots,\ x(k^n l), \tag{19}$$

then definitely exists $0 \leq k_1 < k_2 \leq k^n$, such that $x(k_1 l) = x(k_2 l)$. So begin from time $k_2 l$, the state trajectory repeats this trajectory $x(k_1 l), x(k_1 l + 1), \ldots, x(k_2 l - 1)$. Hence, the upper bound of s is $k^n l - 1$

Proposition 7: If system (14) is globally stable to $\delta_{k^n}^d$, then $\delta_{k^n}^d$ is a fixed point of each subsystem, i.e.,

$$[M_i]_{d,d} = 1,\ i = 1, 2, \ldots, l. \tag{20}$$

Proof: Apply the reduction method. Assume there exists $i \in \{1, 2, \ldots, l\}$ satisfying $[M_i]_{d,d} \neq 1$. Then $[M_i]_{d',d} = (\delta_{k^n}^{d'})^{\mathrm{T}} M_i \delta_{k^n}^d = 1$ for some $d' \neq d$. This means that the i-th subsystem changes state $\delta_{k^n}^d$ to $\delta_{k^n}^{d'}$, then the system is stable, this is a contradiction.

Theorem 2: System (14) is globally stable to $\delta_{k^n}^d$ if and only if (18) and (20) hold.

According to system (14), we can get

$$\begin{cases} x(1) = M_1 x(0), \\ x(2) = M_2 x(1) = M_2 M_1 x(0), \\ \vdots \\ x(l) = M_l x(1-1) = M_l M_{l-1} \cdots M_2 M_1 x(0). \end{cases} \tag{21}$$

Denote $\overline{M}_1 = M_l M_{l-1} \cdots M_2 M_1$, then we have

$$x((t+1)l) = \overline{M}_1 x(tl),\ t = 0, 1, 2, 3, \ldots. \tag{22}$$

Let $\bar{x}(t) := x(tl)$, $t = 0, 1, 2, 3, \ldots$. Then the following theorem which gives a necessary condition of globally stable is derived.

Proposition 8: If system (14) is globally stable to $\delta_{k^n}^d$, then $\delta_{k^n}^d$ is the unique fixed point of system

$$\bar{x}(t+1) = \overline{M}_1 \bar{x}(t). \tag{23}$$

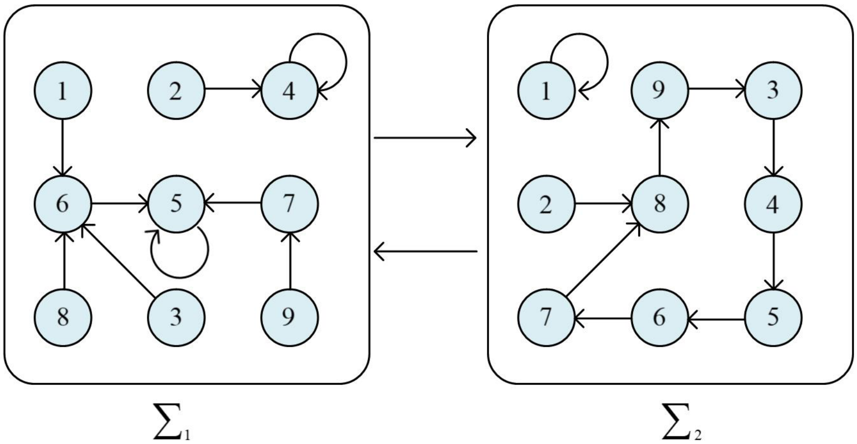

FIGURE 1. **The state transition diagram of Example 2.**

Proof: Since the number of elements in Δ_{k^n} is k^n and

$$\bar{x}(k^n) = \overline{M}_1 \bar{x}(k^n - 1) = \overline{M}_1^{k^n} \bar{x}(0), \tag{24}$$

If (14) is globally stable to $\delta_{k^n}^d$, then

$$\bar{x}(k^n) = \bar{x}(k^n + 1) = \cdots = \delta_{k^n}^d. \tag{25}$$

Assume $\delta_{k^n}^{d'} \neq \delta_{k^n}^d$ is also a fixed point of (23), then the system state will remain

$$\bar{x}(k^n) = \bar{x}(k^n + 1) = \cdots = \delta_{k^n}^d \tag{26}$$

if $x(0) = \delta_{k^n}^{d'}$, which is a contradiction.

It is noted that this proposition is just a necessary condition and the sufficiency is not established. The following gives a counterexample.

Example 2: Consider periodic switched 3-valued logic network system (14) with 2 state nodes and 2 subsystems, where

$$M_1 = \begin{bmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&0&0\\ 0&0&0&0&1&1&1&0&0\\ 1&0&1&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix}, \tag{27}$$

$$M_2 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0 \end{bmatrix}. \tag{28}$$

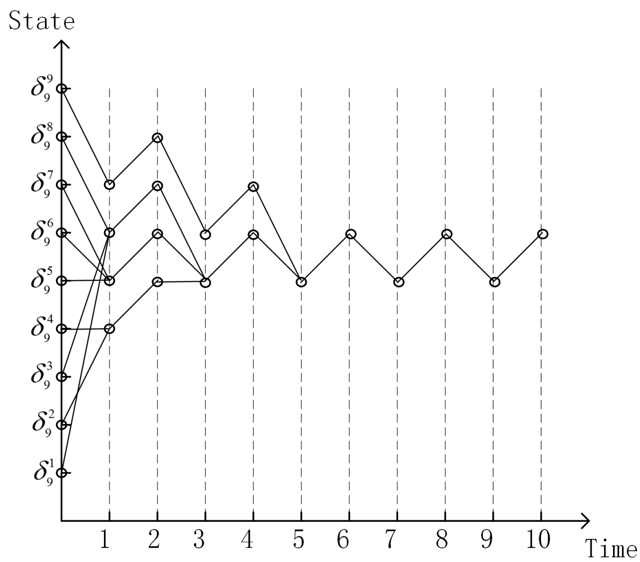

FIGURE 2. **The corresponding system state trajectory of Example 2.**

The state transition process of this system is shown in Fig. 1. We omit the subscript "δ_9" for simplicity. It is easy to get that

$$\overline{M}_1 = \begin{bmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&0&0\\ 0&0&0&0&1&1&1&0&0\\ 1&0&1&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix}, \tag{29}$$

and $[\overline{M}_1]_{6,6} = 1$. Hence, δ_9^6 is the unique fixed point of system (23). But system (14) is not globally stable to δ_9^6 since $M_1\delta_9^6 = \delta_9^5$ and $M_2\delta_9^5 = \delta_9^6$. The corresponding system state trajectory is shown in Fig. 2.

C. STABILIZATION ANALYSIS

For the stabilization of the system, it is important to study whether there is an input sequence, so that after a period of time, the state of the system is stable at a certain state. Consider the following periodic switched k-valued logic control network:

$$\begin{cases} x_1(t+1) = g_1^{\sigma(t)}(u_1(t), \dots, u_m(t), x_1(t), \dots, x_n(t)), \\ x_2(t+1) = g_2^{\sigma(t)}(u_1(t), \dots, u_m(t), x_1(t), \dots, x_n(t)), \\ \quad \vdots \\ x_n(t+1) = g_n^{\sigma(t)}(u_1(t), \dots, u_m(t), x_1(t), \dots, x_n(t)), \\ \sigma(t) = t \mod l+1, \end{cases} \tag{30}$$

where $x_i (i = 1, 2, \dots, n)$, $u_j (j = 1, 2, \dots, m)$ are state variables and input variables, respectively. $g_i^{\sigma(t)} : \mathcal{D}^{m+n} \mapsto \mathcal{D}, i = 1, \dots, n$ are logical functions and $\sigma(t) : \mathbb{N} \mapsto \{1, 2, \dots, l\}$ is the switching signal. According to Lemma 1, the algebraic form of system (30) is:

$$\begin{cases} x_1(t+1) = M_1^{\sigma(t)} u(t)x(t), \\ x_2(t+1) = M_2^{\sigma(t)} u(t)x(t), \\ \quad \vdots \\ x_n(t+1) = M_n^{\sigma(t)} u(t)x(t), \end{cases} \tag{31}$$

where $u(t) = u_1(t) \cdots u_m(t)$, $x(t) = x_1(t) \cdots x_n(t)$ and $M_i^{\sigma(t)}$ are the structure matrices of $g_i^{\sigma(t)}$, $(i = 1, 2, \dots, n)$.

Furthermore, system (30) is equivalent to

$$\Sigma_{\sigma(t)} : x(t+1) = M_{\sigma(t)} u(t)x(t), \tag{32}$$

where $M_{\sigma(t)} = M_1^{\sigma(t)} * M_2^{\sigma(t)} * \cdots * M_n^{\sigma(t)} \in \mathcal{L}_{k^n \times k^n}$.

The difference from before is that the matrix $M_{\sigma(t)}$, here is an order logic matrix. Thus, we have

$$\begin{cases} M = [M_1 \; M_2 \; \cdots \; M_l] \in \mathcal{L}_{k^n \times lk^{m+n}}, \\ M_j = [M_1^j \; M_2^j \; \cdots \; M_{k^n}^j], \end{cases} \tag{33}$$

Hence, (32) can be rewritten as:

$$\Sigma_{\sigma(t)} : x(t+1) = M\sigma(t)u(t)x(t), \tag{34}$$

where $\sigma(t) \in \Delta_l$, $u(t) \in \Delta_{k^m}$.

We introduce the previously defined switch-state space into input.

Definition 4: Consider system (32), its switching-input-state space is defined as

$$\begin{aligned} S &= \{(\sigma, u, x) | \sigma \in \{1, 2, \dots, l\}, \\ u &= (u_1, u_2, \dots, u_m) \in \mathcal{D}_k^m, \\ x &= (x_1, x_2, \dots, x_n) \in \mathcal{D}_k^n\}. \end{aligned}$$

Based on this space, if $(p_1, p_2) \in V \subseteq S \times S$ is a directed path, under vectors form, $p_1 = (\sigma_1, u^1, x^1)$ and $p_2 = (\sigma_2, u^2, x^2)$ should satisfy

$$\begin{cases} x^2 = M\sigma_1 u^1 x^1, \\ E\sigma_2 = E\sigma_1 \mod l+1, \end{cases} \tag{35}$$

where $E = [1, 2, 3, \dots, l]$.

Accordingly, if $(p_1, p_2, \dots, p_\theta)$ is a path, then

$$\begin{aligned} x^2 &= M\sigma_1 u^1 x^1, \; E\sigma_2 = E\sigma_1 \mod l+1, \\ x^3 &= M\sigma_2 u^2 x^2, \; E\sigma_3 = E\sigma_2 \mod l+1, \\ &\cdots \\ x^\theta &= M\sigma_{\theta-1} u^{\theta-1} x^{\theta-1}, \; E\sigma_\theta = E\sigma_{\theta-1} \mod l+1. \end{aligned}$$

then

$$\begin{aligned} x^\theta &= M\sigma_{\theta-1} u^{\theta-1} x^{\theta-1} \\ &= M\sigma_{\theta-1} u^{\theta-1} M\sigma_{\theta-2} u^{\theta-2} \cdots M\sigma_1 u^1 x^1. \end{aligned}$$

According to the method of paper [17] to define the switching-input-state incidence matrix. The range of switching signal $\sigma(t)$ has k values. Hence, there are $l \times k^m \times k^n$ points in switching-input-state space.

Definition 5: Consider system (32), its switching-input-state incidence matrix T is an $lk^{m+n} \times lk^{m+n}$ Boolean matrix, defined as

$$T = \begin{cases} 1, & (p_j, p_i) \in V \text{ is a directed path}, \\ 0, & \text{otherwise}. \end{cases} \tag{36}$$

Proposition 9: The switching-input-state incidence matrix of system (32) is

$$T = [\delta_l^2 T_1 \; \delta_l^3 T_2 \; \cdots \; \delta_l^l T_{l_1} \; \delta_l^1 T_l], \tag{37}$$

where $T_i = \mathbf{1}_{k^m} \ltimes M_i$.

Proof: For any two points, $p_i, p_j \in S$, $p_i = \delta_l^{i_1} \delta_{k^m}^{i_2} \delta_{k^n}^{i_3}$, $p_j = \delta_k^{j_1} \delta_{k^m}^{j_2} \delta_{k^n}^{j_3}$, from Definition 4, (p_j, p_i) is a directed edge, if and only if M_{j_1} and $u(t) = \delta_{k^m}^{j_2}$, $\delta_{k^n}^{j_3}$ can be updated by $\delta_{k^n}^{i_3}$ in one step, thus

$$\begin{aligned} \delta_{k^n}^{i_3} &= M_{j_1} \delta_{k^m}^{j_2} \delta_{k^n}^{j_3}, \\ \delta_l[2, 3, \dots, l, 1] \delta_l^{j_1} &= \delta_l^{i_1}. \end{aligned}$$

It follows that

$$T = \begin{bmatrix} 0 & 0 & 0 & \cdots & \mathbf{1}_{k^m} M_l \\ \mathbf{1}_{k^m} M_1 & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{1}_{k^m} M_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \mathbf{1}_{k^m} M_{l-1} & 0 \end{bmatrix}.$$

Setting $T_i = \mathbf{1}_{k^m} M_i$, $i = 1, 2, \dots, l$, the proof is completed.

This proposition well explains the structure of matrix T of periodic switched k-valued logic network (32) and the origin of its definition (Definition 5).

Definition 6: Suppose $A \in \mathcal{M}_{m \times n}$, if there is a positive integer r which can divide m, satisfying $\mathrm{Row}_{i+r}(A) = \mathrm{Row}_i(A)$, $1 \le i \le m - r$, then the matrix A is called as row periodic matrix, and the period is r.

For example, $T_{\sigma(t)}$ is a row periodic matrix with periodic k^m. Noting that

$$\begin{aligned} T_{\sigma(t)} &= \mathbf{1}_{k^m} M_{\sigma(t)}, \\ T_{\sigma(t)} \ltimes B &= \mathbf{1}_{k^m} \ltimes M_{\sigma(t)} \ltimes B \end{aligned}$$

$$= \mathbf{1}_{k^m} \ltimes (M_{\sigma(t)} \ltimes B),$$

Therefore if a matrix is a row periodic matrix with r rows, right multiply this matrix by any matrix, and the result is still a row periodic matrix with r rows.

Remark 1: For the period of the row period matrix which is defined here, there is no strict restrictions. However, from a mathematical point of view, we can restrict the period value to the minimum value of all periods, and we want to know whether the right multiplication matrix change this minimum period? The reader who is interested in this can discuss it further.

Remark 2: The constraint problem is related to the switching problem to a certain extent. For example, the periodic switching of traffic lights at intersections is associated with the direction restriction of vehicles. Therefore, it is an interesting topic to study the relationship between switched BCNs and restricted BCNs. Reference [32] recorded some works on BNs and Boolean control networks in which states or inputs are admissible or constrained. The results in [32] can be used as the point of penetration for interested readers.

Theorem 3: Consider system (32), if the initial state is $x(0) = \delta_{k^n}^j$, then $x(0)$ has been updated θ times, and get $x(\theta) = \delta_{k^n}^i$ which path number is c

$$[(\delta_{k^m}^1)^{\mathrm{T}} T_{\theta_2} T_{\theta_2-1} \dots T_1 \overline{T}_1^{\theta_1} \mathbf{1}_{k^m}]_{i,j} = c, \tag{38}$$

where $\overline{T}_1 = T_l T_{l-1} \dots T_1$, $\theta = \theta_1 l + \theta_2$, $\theta_2 = \theta \mod l$.

Proof: This theorem is proved by induction. When $\theta = 1, \sigma(\theta) = 1$, one has

$$\begin{aligned}[(\delta_{k^m}^1)^{\mathrm{T}} T_1 \mathbf{1}_{k^m}]_{i,j} &= [(\delta_{k^m}^1)^{\mathrm{T}} \mathbf{1}_{k^m} M_1 \mathbf{1}_{k^m}]_{i,j} \\ &= [M_1 \mathbf{1}_{k^m}]_{i,j},\end{aligned}$$

where $M_1 \mathbf{1}_{k^m}$ records all of the possible inputs. The conclusion obviously holds.

Suppose that the conclusion is established after θ steps of update. Now consider the case where it updates by $\theta+1$ steps, the path from $x(0)$ to $x(\theta+1)$ by $\theta+1$ steps, these steps can be decomposed into the path from $x(0)$ to each $x(\theta)$ by θ times updating and from $x(\theta)$ to $x(\theta+1)$ by one step.

At the given time θ, the evolved system is $M_{\sigma(\theta)}$. At this time, there are k^{m+n} choices, so from time $x(0)$ to $x(\theta+1)$ the number of paths is

$$\begin{aligned}&\sum_{s=1}^{k^{m+n}} ((\delta_{k^m}^1)^{\mathrm{T}} T_{\theta_2+1})_{i,s} [T_{\theta_2} T_{\theta_2-1} \dots T_1 \overline{T}_1^{j} \mathbf{1}_{k^m}]_{s,j} \\ &\qquad = [(\delta_{k^m}^1)^{\mathrm{T}} T_{\theta_2} T_{\theta_2-1} \dots T_1 \overline{T}_1^{\theta_1} \mathbf{1}_{k^m}]_{i,j}.\end{aligned}$$

By mathematical induction, the theorem is proved. Obviously, we can get the following results:

Theorem 4: Consider system (32), the state $x(\theta) = \delta_{k^n}^i$ is reachable from the initial state $x(0) = \delta_{k^n}^j$ if and only if

$$[(\delta_{k^m}^1)^{\mathrm{T}} T_{\theta_2} T_{\theta_2-1} \dots T_1 \overline{T}_1^{\theta_1} \mathbf{1}_{k^m}]_{i,j} > 0. \tag{39}$$

Next we consider the relationship between the above several matrices. $T_{\sigma(t)} = \mathbf{1}_{k^m} M_{\sigma(t)}$, $L_{\sigma(t)} = M_{\sigma(t)} \mathbf{1}_{k^m}$.

Proposition 10: $T_{\sigma(t)}$ and $L_{\sigma(t)}$ satisfy:

$$T_{\theta_2} T_{\theta_2-1} \dots T_1 \overline{T}_1^{\theta_1} = \mathbf{1}_{k^m} \overline{L}_{\theta_2+1}^{\theta_1} L_{\theta_2} L_{\theta_2-1} \cdots L_2 M_1, \tag{40}$$

$$L_{\theta_2} L_{\theta_2-1} \dots L_1 \overline{L}_1^{\theta_1} = M_{\theta_2} T_{\theta_2-1} T_{\theta_2-2} \cdots T_1 \overline{T}_1^{\theta_1} \mathbf{1}_{k^m}, \tag{41}$$

where

$$\begin{aligned}\overline{T}_k &= T_{k-1} \cdots T_1 T_l \cdots T_{k+1} T_k, \\ \overline{L}_k &= L_{k-1} \cdots L_1 L_l \cdots L_{k+1} L_k.\end{aligned}$$

Proof: According to Cheng product, one has

$$\begin{aligned}&T_{\theta_2} T_{\theta_2-1} \dots T_1 \overline{T}_1^{\theta_1} \\ &\quad = \underbrace{\mathbf{1}_{k^m} M_{\theta_2} \mathbf{1}_{k^m} M_{\theta_2-1} \cdots \mathbf{1}_{k^m} M_1}_{\theta} \\ &\quad = \mathbf{1}_{k^m} \underbrace{M_{\theta_2} \mathbf{1}_{k^m} M_{\theta_2-1} \mathbf{1}_{k^m} \cdots \mathbf{1}_{k^m}}_{\theta-1} M_1 \\ &\quad = \mathbf{1}_{k^m} \underbrace{L_{\theta_2} L_{\theta_2-1} \cdots L_2}_{\theta-1} M_1 \\ &\quad = \mathbf{1}_{k^m} \overline{L}_{\theta_2+1}^{\theta_1} L_{\theta_2} L_{\theta_2-1} \cdots L_2 M_1.\end{aligned}$$

The proof of other formula is similar to that.

The global stabilization problem of system (32) is to find an input sequence such that the system globally converges to a fixed point.

Definition 7: System (32) is said to be global stabilizable to $\delta_{k^n}^d \in \Delta_{k^n}$, if there exists an input sequence $u(0), u(1), \dots$, such that for any initial $x(0) \in \Delta_{2^n}$, $x(t) = \delta_{k^n}^d$ holds for some $s \in \mathbb{N}$, $t \geq s$.

From Definition 7, we can know that the problem of stabilization is divided into two processes. One is that the system can stabilize at a fixed point after a finite number of steps, and the other is that the system has such a fixed point.

Theorem 5: $\delta_{k^n}^d$ is a fixed point of system (32) if and only if there exists an input sequence $u(\sigma(t))$ such that

$$[M\sigma(t) u(\sigma(t))]_{d,d} = 1. \tag{42}$$

A fixed point of system (32) means it is a common fixed point of all subsystems. Hence, we can proof this theorem simply by contradiction. When the system runs to a certain subsystem Σ_i, and the state is stabilized to $\delta_{k^n}^d$, then input $\delta_{k^m}^{j_i}$, can be taken to make the state unchanged. Continue this process, run to the subsystem Σ_{i+1}, and take input $\delta_{k^m}^{j_{i+1}}, \dots,$ Therefore, as long as the initial state of the system can be reached $\delta_{k^n}^d$, then the input sequence of the system repeats the input sequence of the above structure.

Theorem 6: System (32) is global reachable to $\delta_{k^n}^d$ if and only if there exists an input sequence $u_0, u_1, \dots, u_\theta$, such that

$$M_{\theta_2} u_\theta M_{\theta_2-1} u_{\theta-1} \cdots M_1 u_0 = \delta_{k^n}[d\ d\ \cdots\ d], \tag{43}$$

where $\theta = \theta_1 l + \theta_2$ and $\theta_2 = \theta \mod l$.

Proof: From system (32), one has

$$\begin{aligned}x(t) &= M\sigma(t-1) u(t-1) x(t-1) \\ &= M\sigma(t-1) u(t-1) M\sigma(t-2) u(t-2) \cdots \\ &\quad M\sigma(0) u(0) x(0)\end{aligned}$$

$$
\begin{aligned}
&= M\sigma(t-1)u(t-1)M\sigma(t-2)u(t-2)\cdots \\
&\quad M_1 u(0)x(0), \\
x(\theta) &= M_{\theta_2}u(\theta)M_{\theta_2-1}u(\theta-1)\cdots M_1 u(0)x(0). \qquad (44)
\end{aligned}
$$

The conclusion can be obtained by the arbitrariness of initial state $x(0)$.

By Theorem 6 and Theorem 5, two processes of the global stabilization problem can be done. (42) and (43) could construct an input sequence $u_0, u_1, \ldots, u_\theta,\ u(1), u(2), \ldots, u(1), u(2), \ldots, u(l), \ldots$ such that system (32) is global stabilizable to $\delta_{k^n}^d$. Hence, combining Theorem 5 and Theorem 6, we can obtain the necessary and sufficient conditions for the system to be stabilized.

Theorem 7: System (32) is global stabilizable if and only if formulas (42) and (43) hold.

Remark 3: When the system is stable at some observable states, the set stability and stabilization problems become the detectability. Ref. [33], [34] proposed the data form to investigate the detectability and obtained several criteria via novel form. Ref. [33] preliminarily analyzed the relationship between detectability and stabilization. Interested readers can further investigate this problem, and also continue to study the detectability of k-valued logical networks, which are all challenging topics.

Next we design an algorithm for finding the control sequence based on (42) and (43).

Algorithm 1 Consider system (32), Assuming that the system can be stabilized to a fixed point $\delta_{k^n}^d \in \Delta_{k^n}$ through s steps,

Step 1: Compute $s = s_1 l + s_2$, where $s_2 = s \mod l$;

Step 2: Replace θ in (43) with s, find $u_0, u_1, \ldots, u_s$ satisfying (43);

Step 3: Find an input sequence $\delta_{k^m}^{j_1}, \delta_{k^m}^{j_2}, \ldots, \delta_{k^m}^{j_l}$ satisfying

$$
[M_i \delta_{k^m}^{j_i}]_{d,d} = 1,\ i = 1, 2, \ldots, l; \qquad (45)
$$

Step 4: The control sequence $u(0), u(1), \ldots, u(s), \ldots$ can be designed as:

$$
u_0, u_1, \ldots, u_s, \underline{\delta_{k^m}^{j_{s+1}}, \delta_{k^m}^{j_{s+2}}, \ldots, \delta_{k^m}^{l}, \delta_{k^m}^{1}, \ldots, \delta_{k^m}^{j_s}}, \qquad (46)
$$

where the underlined part of the input sequence is repeated in subsequent input sequences.

Remark 4: Reference [35] proposed the input network, in which controls are logical variables satisfying certain logical rule. Notice that an input network will eventually become some limit cycles after finite steps, which is equivalent to periodic switching. The methods, proposed in this paper, are based on k-valued logical networks, which is a generalization of traditional BNs. And the corresponding results can be also applicable to periodic switched BNs. Compared with reference [20], in which the reachability and controllability of periodic switched BNs with periodic signals were investigated, but the problem of stabilization was not analyzed in detail. Our methods can not only effectively deal with the reachability and controllability problems in [20], but also settle the stabilization problem.

As we have analyzed in Remarks 2-4, the periodic switched k-valued logical network has many special properties, including the minimal periodic problem of row periodic matrix, the connection of switched BCNs and restricted BCNs, the relationship between detectability and stabilization, which are worth making efforts for further investigation.

BNs can be seen as a special case of k-valued logical networks. Then one example on BN is given to verify the results of this paper.

Example 3: Consider the periodic switched 2-valued logic network, which consists of three subsystems:

$$
\Sigma_1 : \begin{cases} x_1(t+1) = x_2(t)\bar{\vee}u(t), \\ x_2(t+1) = x_1(t) \wedge x_2(t), \end{cases} \qquad (47)
$$

$$
\Sigma_2 : \begin{cases} x_1(t+1) = x_2(t) \vee u(t), \\ x_2(t+1) = x_1(t) \to x_2(t), \end{cases} \qquad (48)
$$

$$
\Sigma_3 : \begin{cases} x_1(t+1) = x_2(t) \wedge u(t), \\ x_2(t+1) = x_1(t) \wedge x_2(t), \end{cases} \qquad (49)
$$

The algebraic form of the three subsystems is as follows:

$$
\begin{aligned}
\Sigma_1 &: x(t+1) = M_1 u(t)x(t), \qquad (50)\\
\Sigma_2 &: x(t+1) = M_2 u(t)x(t), \qquad (51)\\
\Sigma_3 &: x(t+1) = M_3 u(t)x(t), \qquad (52)
\end{aligned}
$$

where

$$
M_1 = \begin{bmatrix} 0&0&0&0&1&0&0&0 \\ 0&1&0&1&0&0&1&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&1&0&1 \end{bmatrix},
$$

$$
M_2 = \begin{bmatrix} 1&0&1&1&1&0&1&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&0&0 \end{bmatrix}
$$

and

$$
M_3 = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&1&0&1&0&1&1&1 \end{bmatrix}.
$$

Similar to Example 2, we draw a state transition diagram in Fig. 3. Then we can get

$$
\Sigma_{\sigma(t)} :\ x(t+1) = M\sigma(t)u(t)x(t), \qquad (53)
$$

where

$$
\begin{aligned}
M &= [M_1\ M_2\ M_3] \\
&= \left[\begin{matrix} 0&0&0&0&1&0&0&0&1&0&1&1 \\ 0&1&0&1&0&0&1&0&0&1&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&1&0&1&0&0&0&0 \end{matrix}\right. \\
&\qquad \left.\begin{matrix} 1&0&1&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&1&0 \\ 0&0&0&1&0&0&0&0&1&0&0&0 \\ 0&1&0&0&0&1&0&1&0&1&1&1 \end{matrix}\right].
\end{aligned}
$$

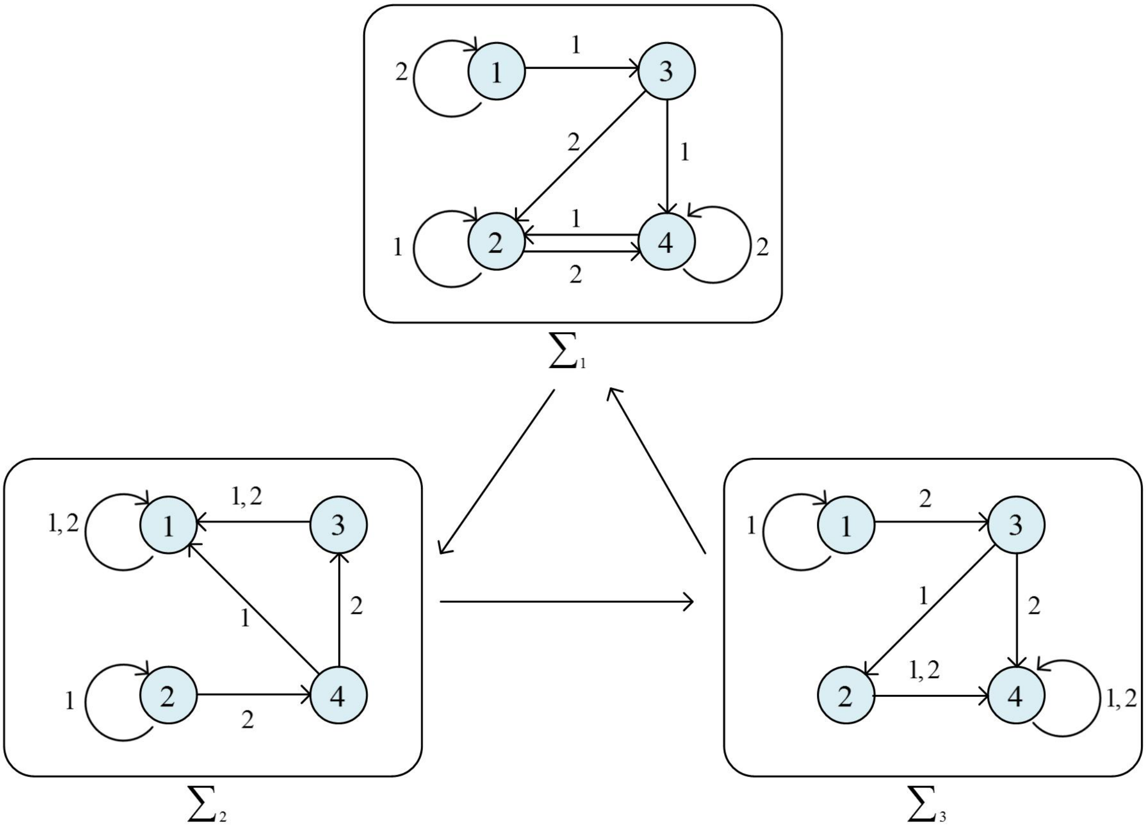

FIGURE 3. The state transition diagram of Example 3.

It is easy to know that

$$M\delta_3^1\delta_2^2 = M_1\delta_2^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$M\delta_3^2\delta_2^2 = M_2\delta_2^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$M\delta_3^3\delta_2^1 = M_3\delta_2^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

We can see

$$[M\delta_3^1\delta_2^2]_{1,1} = 1,$$
$$[M\delta_3^2\delta_2^2]_{1,1} = 1,$$
$$[M\delta_3^3\delta_2^1]_{1,1} = 1.$$

Therefore, by **Theorem 5**, δ_4^1 is the fixed point of system (60).

As long as the initial state of the system reaches δ_4^1, as long as the corresponding control input (Σ_1, δ_2^2), (Σ_2, δ_2^1), (Σ_3, δ_2^1), the system state can be stabilized at δ_4^1.

After simple calculation, we can get

$$M_2\delta_2^1M_1\delta_2^1M_3\delta_2^1M_2\delta_2^1M_1\delta_2^2$$
$$= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

According to **Theorem 6**, for any initial state of the system, we can design the input sequence

$$\delta_2^2, \delta_2^1, \delta_2^1, \delta_2^2, \delta_2^1,$$

such that this system is global reachable to δ_4^1.

In summary, in order to stabilize the system state to δ_4^1, we can design the input sequence through Algorithm 1 as follows:

$$u(0), u(1), u(2), u(3), u(4), \underline{u(5), u(6), u(7)}$$
$$= \delta_2^2, \delta_2^1, \delta_2^1, \delta_2^2, \delta_2^1, \underline{\delta_2^1, \delta_2^2, \delta_2^2}.$$

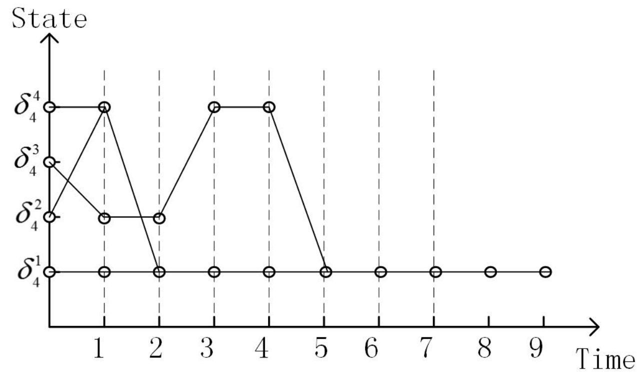

FIGURE 4. The corresponding system state trajectory of Example 3.

The corresponding system state trajectory is shown in Fig. 4.

IV. CONCLUSION

The stability and stabilization of periodic switched k-valued logic network were preliminarily explored in this paper. As an effective tool, the Cheng product and its properties plays an important role for this work. Based on Cheng product, the system was transformed as an algebraic form. Under the framework of the switching-state space and the switching-input-state space, some new matrices were constructed. Then some necessary and sufficient conditions for the stability and stabilization of periodic switched k-valued logic networks were obtained. Moreover, the stabilizer design algorithm was also provided. Finally, two examples were given to verify the validity of the proposed method.

Future works like observability and controllability of periodic switched k-valued logic networks are challenging. From Proposition 7 and Theorem 5 in the paper, we can see that this state is a fixed point of each subsystem. Stabilization is a stronger concept than controllability, which requires not only that the system state can reach a certain state, but also that the state remains unchanged after reaching it. For the observability problem, the output data before the system stabilizes to some state is more important.

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YAN GAO received the B.S. and M.S. degrees in mathematics and applied mathematics from Qufu Normal University, Qufu, China, and the Ph.D. degree in management science and engineering from the Nanjing University of Aeronautics and Astronautics, Nanjing, China. She is currently an Associate Professor with Shandong University, Jinan, China, where she is working for Mathematics Postdoctoral Research Station. She has authored with the Future Scholar of Shandong University. Her research interests include Boolean networks, fuzzy logic, fuzzy decision making, and their applications.

CHENCHEN LIU received the B.S. and M.S. degrees in mathematics and applied mathematics from Qufu Normal University, Qufu, China. He is currently pursuing the Ph.D. degree with Jining Medical University, Jining, China. He is also an Associate Professor in mathematics Jining Medical University. His research interests include medical information engineering, Boolean networks, fuzzy logic, fuzzy decision making, and their applications.

JIAQI WANG received the B.S. degree in optoelectronic information science and engineering and Chinese language and literature from the University of Jinan, Jinan, China.