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Fast Underdetermined DOA Estimation Based on Generalized MRA via Original Covariance Vector Sparse Reconstruction

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ABSTRACT Minimum redundancy array (MRA) has the maximum aperture with continuous difference co-array among various sparse arrays with same number of physical sensors, but it is hard to calculate the sensor position of MRA and realize array design by using MRA. To solve those problem, generalized MRA is proposed with mutual coupling limitation and easy calculation method of sensor position. Based on proposed array configuration, a high-precision underdetermined direction of arrival (DOA) estimation method is proposed with reduced computational complexity. In this method, fast covariance matrix reconstruction is achieved by trace norm minimization with more accurate covariance estimation. Based on the Toeplitz property of covariance matrix in uniform array, a new sparse representation model is established with reduced dimension of covariance vector and faster DOA estimation is achieved via convex optimization. In addition, the proposed method can also be used for underdetermined DOA estimation of other sparse arrays. Using simulation experiments, we demonstrate that the proposed sparse array configuration has superiority over other sparse arrays and the proposed method can outperform most existing methods in terms of underdetermined DOA estimation accuracy and efficiency.

INDEX TERMS Underdetermined DOA estimation, sparse array, minimum redundancy array, matrix reconstruction, sparse reconstruction.

I. INTRODUCTION

Recently, direction of arrival (DOA) estimation based on sparse array has been widely studied in multiple input multiple output (MIMO) radar [1], [2] and underwater acoustic scenarios [3], [4] for the reason that sparse array can achieve DOA estimation with more signals than sensors. Although minimum redundancy array (MRA) [5] is able to deal with more sources than sensors, there is usually no corresponding MRA structure under some aperture, so it is difficult to realize the array design by using MRA. Lately, several new sparse array structures with closed form expression were proposed, such as the nested array (NA) [6], [7], coprime array (CPA) [8]–[10], generalized coprime array (GCPA) [11], super nested array (SNA) [12], generalized nested array (GNA) [13] and their combinations [14]. However, these sparse arrays all have less consecutive virtual sensors in the difference co-array [15]–[18] than the MRA under the same number of physical sensors while the number of consecutive virtual sensors is the main factor of underdetermined DOA estimation performance. Hence, we extend the definition of the MRA and propose a new array geometry of generalized minimum redundancy array (GMRA) with high degrees of freedom and low coupling effect. The proposed array is more suitable for array designing than the MRA.

Underdetermined DOA estimation of sparse array can be achieved by using sparsity-based methods. Based on the basis pursuit de-noising and sparse recovery, Malioutov combined singular value decomposition (SVD) with L1-norm function to provide the new estimator called L1-SVD [19] and it can reduce the computational burden via SVD. Based on [19], Yin introduced the idea of sparse representation of array

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covariance vector (SRACV) and proposed a new method called L1-SRACV [20]. Liu proposed a DOA estimation method named covariance matrix sparse representation (CMSR) with L1-norm [21] by sparsely representing the half elements of the vectorization of covariance matrix with error constraint. Zhou proposed a sparsity-based DOA estimator for sparse array by minimizing the difference between the spatially smoothed covariance matrix and the sparsely reconstructed covariance matrix [22]. Wu then proposed another two DOA estimators for off-grid signals [23], [24]. Although plenty of sparsity-based estimators have also been proposed above to improve the estimation accuracy, many of them suffer from the inaccurate covariance matrix because of limited number of snapshots. So we introduce the covariance matrix reconstruction into the sparsity-based DOA estimator to achieve more accurate DOA estimation in this paper.

Based on the difference co-array, spatial smoothing MUSIC (SS-MUSIC) [25] has been carried out which is in fact the reconstruction of elements in covariance matrix of sparse array based on the Toeplitz characteristic of covariance matrix in uniform array as explained in [26], [27]. Covariance matrix interpolation approach (CMIA) has also been carried out with nuclear norm minimization by interpolating additional sensors to the discontinuous different co-array [28]–[33] and make full use of all difference co-array output while SS-MUSIC can only use the continuous part. Covariance matrix reconstruction approach (CMRA) [34]–[37] was proposed via low rank matrix de-noising framework [38] with error constraint. All methods aforementioned can achieve accurate covariance matrix estimation by nuclear norm minimization. However, their computational burden is much heavy, so we reduce the constraints and solve the norm minimization problem by closed form solution. Theoretical analysis and simulation experiments are carried out to demonstrate the advantages of the proposed sparse array configuration and algorithm.

The main contributions of this paper as follows.

1) We put forward a new sparse array configuration based on minimum redundancy array, called generalized minimum redundancy array (GMRA) with high degrees of freedom and low coupling effect.
2) We propose a fast and accurate covariance matrix estimation method with closed-form expression based on matrix reconstruction and linear equation system.
3) We introduce covariance matrix reconstruction into sparsity-based DOA estimator for underdetermined DOA estimation with high precision.

Notations: Through this paper, scalars, vectors, matrices, and sets are denoted by lowercase letters, lowercase letters in boldface, uppercase letters in boldface, and letters in blackboard, respectively. $E[\bullet]$ and $vec(\bullet)$ mean the expectation and vectorization operator, respectively. The superscripts †, *, T, and H denote the Pseudo inverse, conjugate, transpose, and conjugate transpose, respectively. The symbol ⊗ represents Kronecker product. The rank norm, L1-norm, L2-norm, nuclear norm, trace norm, and Frobenius norm are respectively denoted by $\|\|_0$, $\|\|_1$, $\|\|_2$, $\|\|_*$, $\|\|_T$, and $\|\|_F$.

The remainder of this paper is organized as follows. In Section II, the generalized minimum redundancy array is proposed. In Section III, original covariance matrix reconstruction is solved with fast calculation method and high-precision underdetermined DOA estimation is achieved with the proposed original covariance vector sparse representation (OCVSR). In Section IV, performance analysis is carried out. In Section V, simulation experiments are carried out to test the advantages of the proposed sparse array configuration and algorithm. We make conclusions in Section VI.

II. SPARSE ARRAY DESIGN

Sparse arrays can be regarded as the sparse selection of uniform arrays, so there is a corresponding uniform array with same aperture for each sparse array and the corresponding uniform array is called as original uniform array.

A. SPARSE ARRAY MODEL

The locations of the original uniform array with M physical sensors can be given as

$$\begin{aligned}\mathbb{P}_U &= \{ud, 0 \le u \le M-1\}\\ &= \{p_{u1}, p_{u2}, \cdots, p_{uM}\}\end{aligned} \tag{1}$$

where $d = \lambda_0/2$ stands for the minimum spacing between sensors and λ_0 is the wavelength of carrier. P_{ui} means the position of the ith sensor in original uniform linear array.

With coprime integers M_1 and M_2, the sensors of CPA and GCPA are positioned at

$$\begin{aligned}\mathbb{P}_{CPA} &= \{M_1 n_2 d, 0 \le n_2 \le M_2-1\}\\ &\quad \cup \{M_2 n_1 d, 1 \le n_1 \le M_1-1\}\end{aligned} \tag{2}$$

$$\begin{aligned}\mathbb{P}_{GCPA} &= \{M_1 n_2 d, 0 \le n_2 \le M_2-1\}\\ &\quad \cup \{M_2 n_1 d, 1 \le n_1 \le 2M_1-1\}\end{aligned} \tag{3}$$

Just like the descriptions in [12], NA and SNA with integers N_1 and N_2 are located at

$$\begin{aligned}\mathbb{P}_{NA} &= \{n_1 d, 0 \le n_1 \le N_1-1\}\\ &\quad \cup \{((N_1+1)\,n_2-1)\,d, 1 \le n_2 \le N_2\}\end{aligned} \tag{4}$$

$$\begin{aligned}\mathbb{P}_{SNA} &= \mathbb{P}_1 \cup \mathbb{P}_2 \cup \mathbb{P}_3 \cup \mathbb{P}_4 \cup \{(N_2(N_1+1)-2)\,d\}\\ &\quad \cup \{(l\,(N_1+1)-1)\,d, 2 \le l \le N_2\}\end{aligned} \tag{5}$$

where

$$\begin{cases}\mathbb{P}_1 = \{2ld, 0 \le l \le F_1\}\\ \mathbb{P}_2 = \{(N_1-2l-1)\,d, 0 \le l \le F_2\}\\ \mathbb{P}_3 = \{(N_1+2+2l)\,d, 0 \le l \le F_3\}\\ \mathbb{P}_4 = \{(2N_1-2l-1)\,d, 0 \le l \le F_4\}\end{cases}$$

$$\begin{bmatrix}F_1\\F_2\\F_3\\F_4\end{bmatrix} = \begin{cases}[r, r-1, r-1, r-2], & N_1 = 4r\\ [r, r-1, r-1, r-1], & N_1 = 4r+1\\ [r+1, r-1, r, r-2], & N_1 = 4r+2\\ [r, r, r, r-1], & N_1 = 4r+3\end{cases} \tag{6}$$

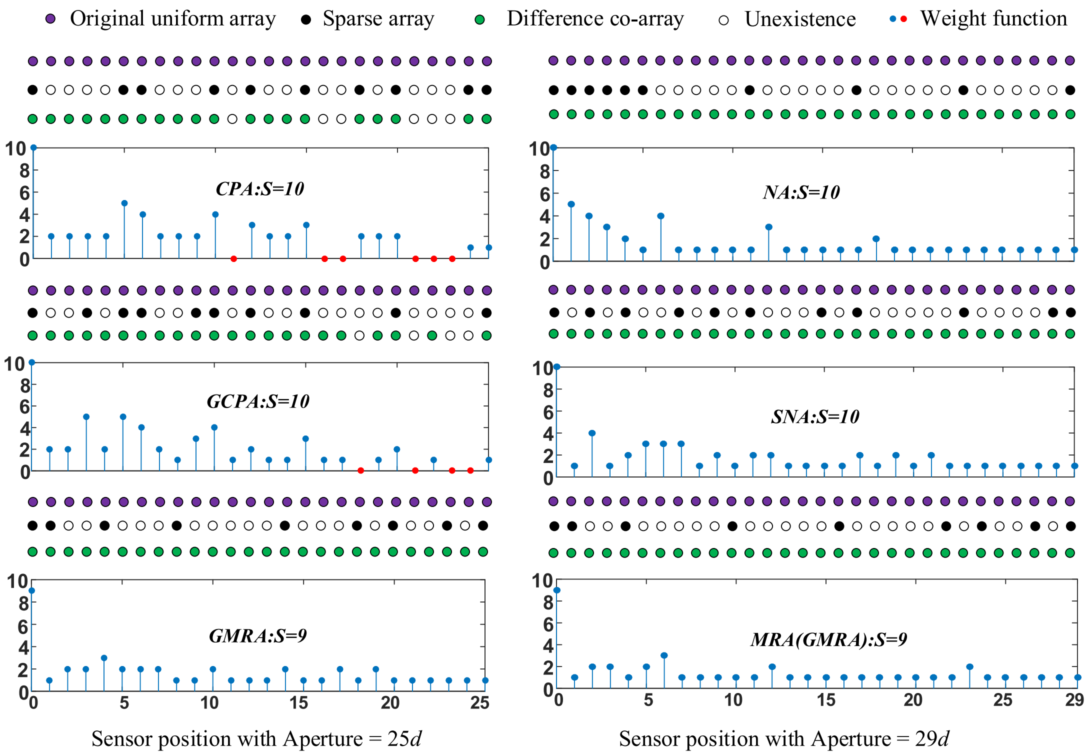

FIGURE 1. An Example for the sensors position of sparse array and corresponding original uniform array, difference co-array, weight function, Including CPA, GCPA, NA, SNA, MRA and GMRA.

There is one example for the four sparse array above with $N_1 = N_2 = 5$ and $M_1 = 5$, $M_2 = 6$ or $M_2 = 3$ as illustrated in Fig.1.

The sensor positions of sparse linear array with S physical sensors are

$$\mathbb{P}_S = \{p_{s1} = 0, p_{s2}, \cdots, p_{sS}\} \tag{7}$$

Definition 1 (Difference Co-Array): For a sparse array, its difference co-array $\mathbb{P}_D$ is defined as

$$\mathbb{P}_D = \left\{p_{si} - p_{sj} = md\right\}, \quad \forall i, j = 1, 2, \cdots S \tag{8}$$

Closed-form expressions for mutual coupling matrix $\boldsymbol{C}_S$ in [39], [40]

$$C_{ij} = \begin{cases} c_{|i-j|} & |i-j| \le m_{\max} \\ 0 & |i-j| > m_{\max} \end{cases} \tag{9}$$

where $i, j \in \{1, 2, \cdots, S\}$ and mutual coupling coefficients $c_0, c_1, \cdots, c_{m_{\max}}$ satisfy $1 = |c_0| > |c_1| > \cdots > \left|c_{m_{\max}}\right|$. $m_{\max}$ is the max sensor spacing with coupling effect.

The difference co-array set which contains all pairs $\left(p_{si}, p_{sj}\right)$ contributing to the difference co-array is

$$\mathbb{M}(m) = \left\{\left(p_{si}, p_{sj}\right) \in \mathbb{P}_S^2 | p_{si} - p_{sj} = md\right\} \tag{10}$$

Definition 2 (Weight Function): The weight function is defined as the number of sensor pairs that lead to difference co-array index m, which is exactly the cardinality of $\mathbb{M}(m)$

$$w(m) = |\mathbb{M}(m)|, \quad md \in \mathbb{P}_D \tag{11}$$

The illustration of corresponding difference co-array and weight function of sparse array are also illustrated in Fig.1.

The aperture of original uniform array is equal to the aperture of sparse array, that means $p_{sS} = p_{uM}$.

Based on the sensors position of sparse array and corresponding original uniform array, we can get the selective matrix $\boldsymbol{\Gamma} \in \mathbb{R}^{S \times M}$ whose row can be calculated as follows,

$$\boldsymbol{\Gamma}_{\{:,i\}} = \begin{cases} \boldsymbol{0}_{S\times 1} & p_{ui} \notin \mathbb{P}_S \\ \boldsymbol{\rho}_j & p_{ui} \in \mathbb{P}_S \end{cases} \tag{12}$$

where $\boldsymbol{\rho}_j$ is unit vector in which the jth element is 1 if $p_{ui} = p_{sj}$.

B. GENERALIZED MINIMUM REDUNDANCY ARRAY

As illustrated in Fig.1, NA, SNA and MRA are fully augmentable array (FAA) [41] which are hole-free in difference co-array, while there are some holes in difference co-array of partially augmentable arrays (PAA) [42], like CPA and GCPA.

For a given number of elements, the sensor locations of MRA are chosen such that the covariance matrix will contain entries with minimum redundancy which means minimum possible repeatable entries. The redundancy of sparse array is

$$R_e = \frac{S\left(S-1\right)/2}{\rho_{\max}} \tag{13}$$

where $\rho_{\max}$ is the ratio of aperture to unit spacing d. There are no holes in difference co-array of MRA and the redundancy

TABLE 1. Aperture of MRAs under different number of sensors.

Sensor Number	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Aperture/d	3	6	9	13	17	23	29	36	43	50	58	68	80	91	101

of MRA is minimum under the same number of physical sensors.

The DOA estimation accuracy and degrees of freedom (DOFs) mainly depend on the array aperture, so we need to design the array configuration under a given aperture. However the aperture of MRA is discontinuous as illustrated in Table 1 and there are no MRA structure under many aperture, so it is hard to apply the structure of MRA in array design. To find the sparse linear arrays with minimum redundancy of arbitrary aperture, we extend the definition of MRA and propose the concept of generalized minimum redundancy criterion.

Definition 3 (Generalized Minimum Redundancy Array): Generalized Minimum Redundancy Array (GMRA) is the FAA with minimum redundancy and the least mutual coupling effect under the same aperture.

Based on the formula (9), the mutual coupling of sensors mainly depends on the physical element spacing and decreases sharply with the increase of the element spacing. So we can compare the mutual coupling effect of different sparse arrays by comparing their weight function.

The calculation steps of sensor position in GMRA under arbitrary aperture M_d are described as Table 2.

TABLE 2. The calculation steps of sensor position in GMRA under arbitrary aperture.

Input	Given arbitrary aperture M_d
Output	sensor position of corresponding GMRA $\mathbb{P}_S$
Step 1	look up the maximum number no more than M_d in the second row of Table 1, the corresponding value S is the number of physical sensors in GMRA;
Step 2	Based on permutation and combination, all sparse arrays with aperture M_d and S physical sensors are found, then we can calculate the corresponding difference co-array and choose the sparse array with hole-free in difference co-array. After that, all FAAs can be obtained with aperture M_d and S physical sensors.
Step 3	Calculate the weight function of FAAs and compare the weights of difference co-array from small to large until there are only one FAA left which is actually the GMRA.

Based on Table 2, we can get the sensors position of GMRA with aperture less than $40d$. As illustrated in Fig.2, it is easy to find an GMRA under arbitrary aperture less than $40d$. The DOFs and corresponding sensor number are also shown in Fig.2. GMRA owns least physical sensors under the same aperture compared with other sparse arrays. So the GMRA is most economic and can achieve the same DOA estimation performance as other sparse linear arrays with less sensors.

As illustrated in Fig.2 and Fig.3, some MRAs are also GMRAs.

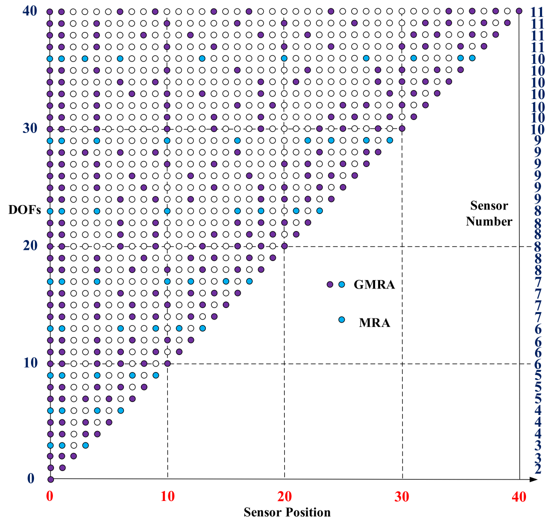

FIGURE 2. DOFs and corresponding sensor number of GMRA when aperture is no more than $40d$.

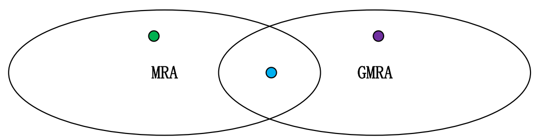

FIGURE 3. The relationship of GMRA and MRA.

C. SIGNAL MODEL

Assume K uncorrelated narrowband sources from directions of $-\pi/2 < \theta_k < \pi/2$ for $k = 1, \ldots, K$ impinge on the original uniform array. Then, the output of original uniform array is given by,

$$\boldsymbol{x}(t) = \sum_{k=1}^{K} \boldsymbol{a}(\theta_k) s_k(t) + \boldsymbol{n}(t) = \boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{n}(t) \tag{14}$$

where $\boldsymbol{A} = \left[\boldsymbol{a}(\theta_1) \cdots \boldsymbol{a}(\theta_K)\right]$ denotes the array manifold matrix and θ_k is azimuth angle of the kth source. $\boldsymbol{a}(\theta_k)$ denotes the spatial steering vector and can be expressed as

$$\boldsymbol{a}(\theta_k) = [e^{\frac{-j(2\pi p_{u1} \sin\theta_k)}{\lambda_0}}, \cdots, e^{\frac{-j(2\pi p_{uM} \sin\theta_k)}{\lambda_0}}]^T \tag{15}$$

$$\boldsymbol{s}(t) = [s_1(t), \cdots, s_k(t)]^T \tag{16}$$

where signal vector $\boldsymbol{s}(t)$ and additive noise vector $\boldsymbol{n}(t)$ are zero-mean uncorrelated random vectors satisfying $E\left[\boldsymbol{s}(t)\boldsymbol{s}(t)^H\right] = \boldsymbol{R}_s$ and $E\left[\boldsymbol{n}(t)\boldsymbol{n}(t)^H\right] = \boldsymbol{\delta}_n$, respectively. $\boldsymbol{R}_s$ and $\boldsymbol{\delta}_n$ denotes the signal power matrix and the noise power matrix, respectively. j is an imaginary number.

Under the assumption that the source and the noise are uncorrelated spatially and temporally, the original covariance matrix $\boldsymbol{R}_{xx}$ and de-noising original covariance matrix $\boldsymbol{T}_{xx}$ of the original uniform array can be obtained as

$$\begin{aligned}\boldsymbol{R}_{xx} &= E[\boldsymbol{x}(t)\boldsymbol{x}^H(t)] = \sum_{k=1}^{K}\delta_k\boldsymbol{a}(\theta_k)\boldsymbol{a}^H(\theta_k) + \delta_n\boldsymbol{I}_M \\ &= \boldsymbol{A}\boldsymbol{R}_s\boldsymbol{A}^H + \delta_n\boldsymbol{I}_M = \boldsymbol{T}_{xx} + \delta_n\boldsymbol{I}_M \end{aligned} \tag{17}$$

where $\boldsymbol{R}_s = diag\left(\delta_1 \cdots \delta_k \cdots \delta_K\right)$ and δ_k is the power of kth source. $\boldsymbol{I}_M$ is M dimensional identity matrix. $\boldsymbol{R}_{xx}$ and $\boldsymbol{T}_{xx}$ are both Toeplitz and Hermitian matrices as described in [12].

The sparse array output vector can be expressed as

$$\begin{aligned}\boldsymbol{y}(t) &= \sum_{k=1}^{K}\boldsymbol{a}_S(\theta_k)s_k(t) + \boldsymbol{n}_s(t) = \boldsymbol{A}_s\boldsymbol{s}(t) + \boldsymbol{n}_s(t) \\ &= \boldsymbol{\Gamma}\boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{\Gamma}\boldsymbol{n}(t) = \boldsymbol{\Gamma}(\boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{n}(t)) \\ &= \boldsymbol{\Gamma}\boldsymbol{x}(t) \end{aligned} \tag{18}$$

$$\boldsymbol{a}_S(\theta_k) = [e^{\frac{-j(2\pi p_{s1}\sin\theta_k)}{\lambda_0}}, \cdots, e^{\frac{-j(2\pi p_{sS}\sin\theta_k)}{\lambda_0}}]^T \tag{19}$$

where $\boldsymbol{a}_S(\theta_k)$is the spatial steering vector of sparse array.

The covariance matrix of sparse array output is derived as

$$\boldsymbol{R}_{yy} = \boldsymbol{\Gamma}\boldsymbol{A}\boldsymbol{R}_s\boldsymbol{A}^H\boldsymbol{\Gamma}^H + \delta_n\boldsymbol{I}_S = \boldsymbol{\Gamma}\boldsymbol{R}_{xx}\boldsymbol{\Gamma}^H + \delta_n\boldsymbol{I}_S \tag{20}$$

where $\boldsymbol{I}_S$ is S dimensional identity matrix and covariance matrix of sparse array can be approximated with L snapshots as

$$\hat{\boldsymbol{R}}_{yy} = \frac{1}{L}\sum_{i=1}^{L}\boldsymbol{y}(t_i)\boldsymbol{y}^H(t_i) \tag{21}$$

where δ_n is equal to the minimum eigenvalue of $\hat{\boldsymbol{R}}_{yy}$.

III. THE PROPOSED ALGORITHM

In this section, we will carry out the fast calculation of original covariance matrix and then provide a sparse representation model based on estimated covariance vector. Finally, underdetermined DOA estimation is achieved by proposed original covariance vector sparse representation (OCVSR).

A. ORIGINAL COVARIANCE MATRIX RECONSTRUCTION

The estimation error between $\boldsymbol{R}_{yy}$ and $\hat{\boldsymbol{R}}_{yy}$ is defined as $\boldsymbol{E} = \hat{\boldsymbol{R}}_{yy} - \boldsymbol{R}_{yy}$. Based on the law of large numbers, we can get that the vectorization of estimation error obeys normal distribution as follows.

$$vec(\boldsymbol{E}) \sim AsN\left(0, \boldsymbol{W}\right) \tag{22}$$

where $\hat{\boldsymbol{W}} = \frac{1}{L}\hat{\boldsymbol{R}}_{yy}^T \otimes \hat{\boldsymbol{R}}_{yy}$.

Then we can derive the following properties,

$$\left\|\hat{\boldsymbol{W}}^{-\frac{1}{2}}vec(\boldsymbol{E})\right\|_2^2 \sim As\chi^2\left(|S|^2\right) \tag{23}$$

where $As\chi^2\left(|S|^2\right)$ denotes the asymptotic chi-square distribution with $|S|^2$ degrees of freedom. So we can get a limitation for the estimation error as follows

$$\left\|\hat{\boldsymbol{W}}^{-\frac{1}{2}}vec(\boldsymbol{E})\right\|_2^2 \leq \eta^2 \tag{24}$$

where regularization parameter η can be calculated by using MATLAB routine "chi2inv(1-w,$|S|^2$)" and we set w = 0.001 as described in [20].

Then we can estimate $\boldsymbol{T}_{xx}$ by solving the following norm minimization problem.

$$\begin{aligned} &\min_{\boldsymbol{T}_{xx}\in\mathbb{C}^{M\times M}} \|\boldsymbol{T}_{xx}\|_0 \\ &s.t. \left\|\hat{\boldsymbol{W}}^{-\frac{1}{2}}vec(\boldsymbol{E})\right\|_2^2 \leq \eta^2, \quad \boldsymbol{T}_{xx} \geq 0 \end{aligned} \tag{25}$$

which is then relaxed into the following convex optimization model by replacing the rank norm with the trace norm. Because the estimated covariance matrix is positive semidefinite in large probability [43] when the number of snapshots is more than three times of the number of sensors, the constraint of positive semidefinite is eliminated.

$$\begin{aligned} &\min_{\boldsymbol{T}_{xx}\in\mathbb{C}^{M\times M}} \|\boldsymbol{T}_{xx}\|_T \\ &s.t. \left\|\hat{\boldsymbol{W}}^{-\frac{1}{2}}vec(\boldsymbol{E})\right\|_2^2 \leq \eta^2 \end{aligned} \tag{26}$$

Then we can solve the trace norm minimization problem by approximate solution of linear equation instead of CMRA and CMIA with heavy computational burden.

B. FAST IMPLEMENTATION

We can transform the trace norm minimization problem in (26) into the LASSO problem with Lagrange form just like [36]

$$\begin{aligned} &\min_{\boldsymbol{T}_{xx}\in\mathbb{C}^{M\times M}} \lambda\|\boldsymbol{T}_{xx}\|_T + \frac{1}{2}\left\|\hat{\boldsymbol{W}}^{-\frac{1}{2}}vec(\boldsymbol{E})\right\|_2^2 \\ &= \min_{\boldsymbol{T}_{xx}\in\mathbb{C}^{M\times M}} \|(\lambda\boldsymbol{I} - \boldsymbol{C})\boldsymbol{T}_{xx}\|_T + \frac{1}{2}\|\boldsymbol{T}_{xx}\boldsymbol{C}\boldsymbol{T}_{xx}\boldsymbol{C}\|_T \end{aligned} \tag{27}$$

where $\boldsymbol{C} = \boldsymbol{\Gamma}^T\hat{\boldsymbol{R}}_{yy}^{-1}\boldsymbol{\Gamma}$.

According to the Karush-Kuhn-Tucker(KKT) condition in convex optimization theory in [44], the optimal solution of (27) satisfying

$$T\left[\lambda\boldsymbol{I} - \boldsymbol{C}\right] = T\left[\boldsymbol{C}\boldsymbol{T}_{xx}\boldsymbol{C}\right] \tag{28}$$

where $T[\bullet]$ is Toeplitz operation and the diagonal elements at all levels are averaged to form a new vector.

The formula (28) is actually a linear equation system with $2M - 1$ variables. To solve it, we transform the left side as

$$T\left[\boldsymbol{C}\boldsymbol{T}_{xx}\boldsymbol{C}\right] = \underbrace{\begin{bmatrix}\Phi_{M,:}^* \\ \vdots \\ \Phi_{2,:}^* \\ \Phi\end{bmatrix}}_{\boldsymbol{H}}\begin{bmatrix}\boldsymbol{t} \\ t_1^* \\ \vdots \\ t_{M-1}^*\end{bmatrix} \tag{29}$$

$$\Phi = \begin{bmatrix} T^T\left[\boldsymbol{J}_{:,1:M}\boldsymbol{J}_{1:M,:}\right] \\ T^T\left[\boldsymbol{J}_{:,1:M-1}\boldsymbol{J}_{2:M,:}\right] \\ \vdots \\ T^T\left[\boldsymbol{J}_{:,1}\boldsymbol{J}_{M,:}\right] \end{bmatrix} \tag{30}$$

where $\boldsymbol{t} = \left[t_{M-1} \cdots t_1 \; t_0\right]^T$.

To transform the whole matrix $\boldsymbol{H}$ into block matrix, we insert a zero vector into the Mth column of the matrix and we can get

$$T\left[\boldsymbol{C}\boldsymbol{T}_{xx}\boldsymbol{C}\right]=\left[\boldsymbol{H}_1\ \boldsymbol{H}_2\right]\begin{bmatrix}\boldsymbol{t}\\ \boldsymbol{t}^*\end{bmatrix} \tag{31}$$

It is obvious that (30) is complex linear equation system, so we transform it to real linear equation system. Assume $\boldsymbol{q}=T\left[\boldsymbol{C}\boldsymbol{T}_{xx}\boldsymbol{C}\right]$, then we can get

$$\underbrace{\begin{bmatrix}\mathcal{R}(\boldsymbol{q})\\ \mathcal{I}(\boldsymbol{q})\end{bmatrix}}_{\boldsymbol{q}_r}=\underbrace{\begin{bmatrix}\mathcal{R}(\boldsymbol{H}_1+\boldsymbol{H}_2) & \mathcal{I}(\boldsymbol{H}_2-\boldsymbol{H}_1)\\ \mathcal{I}(\boldsymbol{H}_1+\boldsymbol{H}_2) & \mathcal{R}(\boldsymbol{H}_1-\boldsymbol{H}_2)\end{bmatrix}}_{\boldsymbol{H}_r}\underbrace{\begin{bmatrix}\mathcal{R}(\boldsymbol{t})\\ \mathcal{I}(\boldsymbol{t})\end{bmatrix}}_{\boldsymbol{t}_r} \tag{32}$$

where $\mathcal{R}(\bullet)$ means real part and $\mathcal{I}(\bullet)$ means imaginary part.

Then we can get $\boldsymbol{t}_r=\boldsymbol{H}_r^{\dagger}\boldsymbol{q}_r$ and the de-nosing original covariance matrix as follows

$$\boldsymbol{T}_{xx}=\begin{bmatrix}t_0 & t_1^* & \cdots & t_{M-1}^*\\ t_1 & t_0 & \cdots & t_{M-2}^*\\ \vdots & \vdots & \ddots & \vdots\\ t_{M-1} & t_{M-2} & \cdots & t_0\end{bmatrix} \tag{33}$$

Based on the Toeplitz property of de-nosing original covariance matrix above, we can find there are only M unique elements in $\boldsymbol{T}_{xx}$ with M^2 elements and the covariance vector $\boldsymbol{t}$ with M elements contain all the information of the de-noising original covariance matrix. So we can just apply the sparse reconstruction of M elements instead of $S^2/2$ elements in CMSR.

In next subsection, we will sparsely represent the original covariance vector estimated above and achieve underdetermined DOA estimation by proposed OCVSR.

C. ORIGINAL COVARIANCE VECTOR SPARSE REPRESENTATION

According to the formula (17) and (32), the element of $\boldsymbol{t}$ can be derived as

$$t_i=\sum_{k=1}^{K}\delta_k e^{-j\pi i\sin\theta_k},\quad 0\le i\le M-1 \tag{34}$$

We can flip the covariance vector $\boldsymbol{t}$ upside down and obtain a new covariance vector $\boldsymbol{t}_f=\left[t_0\ \cdots\ t_{M-2}\ t_{M-1}\right]^T$, which is named as original covariance vector and can be represented as

$$\boldsymbol{t}_f=\begin{bmatrix}1 & \cdots & 1\\ e^{-j\pi\sin\theta_1} & \cdots & e^{-j\pi\sin\theta_K}\\ \vdots & \ddots & \vdots\\ e^{-j\pi(M-1)\sin\theta_1} & \cdots & e^{-j\pi(M-1)\sin\theta_K}\end{bmatrix}\begin{bmatrix}\delta_1\\ \vdots\\ \delta_K\end{bmatrix}=\boldsymbol{A}\boldsymbol{\delta} \tag{35}$$

where $\boldsymbol{\delta}$ is the set of the signal power estimates and $\boldsymbol{t}_f$ can be deviated from the combination of array responding matrix $\boldsymbol{A}$ and $\boldsymbol{\delta}$ with the perturbation $\boldsymbol{\varepsilon}=\boldsymbol{t}_f-\boldsymbol{A}\boldsymbol{\delta}$.

Original covariance vector $\boldsymbol{t}_f$ can be represented on an over-complete spatial dictionary just like follows.

$$\boldsymbol{t}_f=\boldsymbol{A}_G\boldsymbol{\delta}_G+\boldsymbol{\varepsilon}=\left[\boldsymbol{a}(\theta_{G1})\cdots\boldsymbol{a}(\theta_{Gg})\right]\begin{bmatrix}\delta_{S1}\\ \vdots\\ \delta_{Sg}\end{bmatrix}+\begin{bmatrix}\varepsilon_1\\ \vdots\\ \varepsilon_M\end{bmatrix} \tag{36}$$

where $\boldsymbol{\varepsilon}$ is the perturbation derived from finite-length snapshots and $\boldsymbol{A}_G$ is an over-complete dictionary formed by $\boldsymbol{a}(\theta_{Gi})$ with $\{\boldsymbol{a}(\theta_1)\cdots\boldsymbol{a}(\theta_K)\}\subset\{\boldsymbol{a}(\theta_{G1})\cdots\boldsymbol{a}(\theta_{Gg})\}$.

The estimation of original covariance vector is perturbation-contaminated due to finite sampling, so we can derive the following convex optimization problem based on L1-norm. Then we can get the DOA θ_k by solving convex optimization problem.

$$\min_{\boldsymbol{\delta}_G\in\mathbb{C}^{g\times 1}}\|\boldsymbol{\delta}_G\|_1\quad s.t.\ \left\|\hat{\boldsymbol{t}}_f-\boldsymbol{A}_G\boldsymbol{\delta}_G\right\|_2^2\le\beta^2 \tag{37}$$

while threshold β is calculated based on the perturbation of the original covariance vector and aims to restrict the fitting error between the practical and hypothetical models. And it can be estimated as following.

$$\beta=\mu\times(MVar(\varepsilon_1))^{\frac{1}{2}} \tag{38}$$

PROOF See Appendix A.

DOAs can be retrieved by solving above optimization and this estimator is named as original ovariance vector sparse representation (OCVSR). The procedure of OCVSR is described in Table 3.

TABLE 3. Procedure of DOA estimation for GMRA based on OCVSR.

Input	Given sparse array output $\boldsymbol{y}(t)$, sensors position $\mathbb{P}_S$
Output	Estimated θ_k, δ_k
step 1	Approximating the covariance matrix $\hat{\boldsymbol{R}}_{yy}$ and $\hat{\delta}_n$ with formula (21)
step 2	Estimated original covariance vector $\boldsymbol{t}_f$ with formula (32)
step 3	Calculating the perturbation β with formula (38)
step 4	Recovering the sparse signal components $\boldsymbol{\delta}_G$ with formula (37) via cvx toolbox
step 5	Obtaining the DOA estimates θ_k by searching the peaks of $\boldsymbol{\delta}_G$;

IV. PERFORMANCE ANALYSIS

In this section, a brief performance analysis is given to have more quantitative analysis of the proposed OCVSR compared with existing algorithms.

A. COMPLEXITY ANALYSIS

The proposed OCVSR has low computational complexity compared to the existing approaches CMIA [30], CMRA [36] and CMSR [21], respectively. Here, a more detailed comparison on the number of multipliers will be made to demonstrate the computational saving.

CMRA with nuclear norm needs $O\left(M^6\right)$ operations, CMIA with nuclear norm need $O\left(3M^6\right)$ operations, but the covariance matrix reconstruction in OCVSR needs $O(M^3)$ operations by contrast.

The Eigen-decomposition in MUSIC requires $O(M^3)$ operations. The sparse reconstruction for the array covariance vector in CMSR needs $O\left(M^3(S^2/2)^3\right)$ operations, while the sparse reconstruction of original covariance vector in OCVSR only requires $O(M^6)$ operations.

TABLE 4. Computational complexity of different DOA estimators.

CMRA+MUSIC	CMIA+MUSIC	CMSR	OCVSR
$O(M^6+M^3)$	$O(3M^6+M^3)$	$O(M^3S^6/8)$	$O(M^6+M^3)$

B. ALGORITHMS COMPARISON

Compared with the CMSR, the number of the variables which needs to sparsely represent in OCVSR is reduced which means reduced computational complexity. CMRA with MUSIC and CMIA with MUSIC have higher estimation accuracy than CMSR and can handle more sources for more precise original covariance matrix estimation.

However, CMIA and CMRA which combined with MUSIC requires the extra process of signal number detection, while there is no need to know the prior knowledge of signal number in CMSR and OCVSR. The inaccurate estimation of the signals number has a great impact on the DOA estimation performance and could lead to a significant decrease in the estimation performance.

V. SIMULATION RESULTS

In this section, the performance of the proposed GMRA is compared with the advanced CPA, GCPA, NA under same aperture and the performance of the proposed method OCVSR for underdetermined DOA estimation with comparison to CMIA, CMRA and CMSR is evaluated. We choose the four sparse arrays as illustrated as Fig.1. The interval of the over-complete sparse representation grids and space search interval of MUSIC are all set to be 0.1°.

There are K equal-power uncorrelated far-field narrow-band signals which are assumed to be uniformly distributed in $[-53.02°, 53.02°]$ unless otherwise stated, and DOAs are not restricted to lie on the pre-defined grids so signals are in off-grid model just like [45]. It should be noted that all experiments in this paper are in underdetermined cases which means $K > S$. When $K = 11 > S$, we achieve underdetermined DOA estimation. When $K = 21 > 2S$, it means that the number of signals which the estimators can handle exceeds twice number of sensors.

A. UNDERDETERMINED ESTIMATION PERFORMANCE

We first carry out a simple simulation with normalized spatial spectrum by SS-MUSIC to compare the performance of different sparse arrays under mutual coupling effect with $c_1 = 0.5 - 0.4i$ and $m_{max} = 1$. CPA, GCPA and NA with $S = 10$ are compared to GMRA with $S = 9$. The spatial

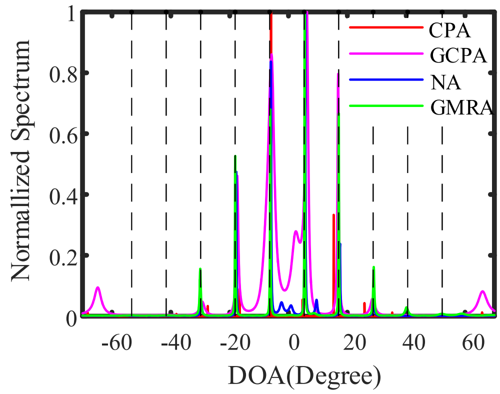

FIGURE 4. Spatial spectrum of different sparse array by SS-MUSIC with *SNR*=10dB and *L*=500 under mutual coupling effect.

spectrums of the four sparse arrays with $K = 10$ estimable signals are described as below.

As depicted in Fig.4, GMRA has the best DOA estimation performance under mutual coupling.

We then carry out a simple simulation with normalized spatial spectrum by OCVSR to illustrate the underdetermined DOA estimation performance of different sparse arrays. The spatial spectrums of the four sparse arrays with maximum number of estimable signals are described as below.

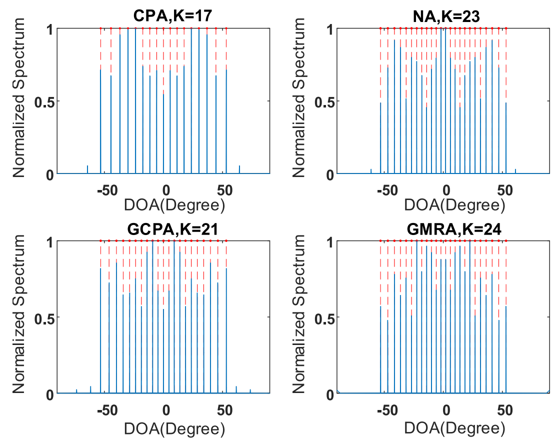

FIGURE 5. Spatial spectrum of different sparse array by OCVSR with *SNR*=10dB and *L*=500.

As depicted in Fig.5, the number of signals are all exceeds the number of sensors, so OCVSR has strong adaptability to underdetermined DOA estimation with different sparse arrays. As we can see, although signals are in off-grid situation, OCVSR can alleviate the effect of basis mismatch to a certain extent.

It is clear from Fig.5 that NA and GMRA can achieve underdetermined DOA estimation with greater number of signals than CPA and GCPA, so it can be concluded that the larger aperture of the sparse arrays means higher precision as well as more DOFs. GMRA can address even greater number of signals with less sensors than NA. The estimation performance of OCVSR based on GMRA is compared with existing methods and spatial spectrums of experiment are presented in Fig.6.

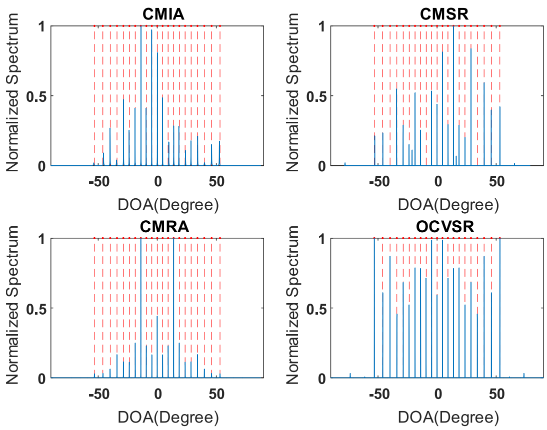

FIGURE 6. Spatial spectrum of GMRA by different methods with $K=21$, $SNR=10$dB and $L=200$.

For DOA estimation of FAA, CMIA with MUSIC is equal to SS-MUSIC. It is observed from Fig.6 that CMIA and CMRA can achieve more accurate DOA estimation than CMSR but the signal power estimation is poor compared with the true value of signal power. There are some fakepeaks in CMSR and the number of signals is not estimated correctly. Compared with CMIA and CMRA, OCVSR achieve more accurate signal power estimation with nearly the same accurate DOA estimation.

B. ESTIMATION ACCURACY ANALYSIS

In order to test the performance of DOA estimators better, the Monte-Carlo trials are carried out. The number of Monte-Carlo trials is set to be $I = 500$ for each experiment. The definition of root mean square error (RMSE) is

$$RMSE = \sqrt{\frac{1}{IK}\sum_{i=1}^{I}\sum_{k=1}^{K}\left(\hat{\theta}_k(i) - \theta_k\right)^2} \tag{39}$$

In the second subsection, the accuracy estimation capability of the proposed OCVSR is examined by calculating RMSE versus different number of snapshots or SNR by using proposed GMRA. We set SNR=10dB while L varying from 50 to 550 and $L = 500$ when SNR varying from -10dB to 10dB.

From Fig.7, we can see that OCVSR can achieve much more accurate DOA estimation with lower RMSE than CMSR under different conditions. So it can be concluded than the covariance matrix is helpful for the improvement of estimation accuracy of sparsity-based DOA estimator. It can also be seen that OCVSR can achieve little lower RMSE than CMRA in most cases.

As depicted in Fig.7, CMIA and CMSR suffer obvious performance degradation when $L < 300$, while CMRA and proposed OCVSR achieve nearly the same RMSE when $L > 100$. It means OCVSR is more robust to few snapshot.

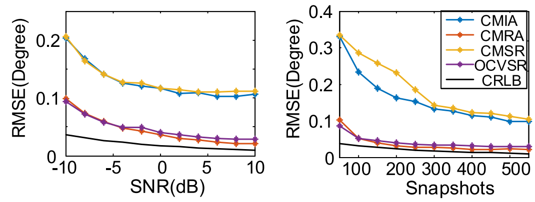

FIGURE 7. RMSE for GMRA via different methods with different SNR or snapshots.

The accuracy estimation capability of OCVSR is also illustrated with different number of signals varying from 3 to 23 and angle interval varying form 4° to 8° when there are more sources than sensors. The simulation results of Monte-Carlo trials are illustrated in Fig.8.

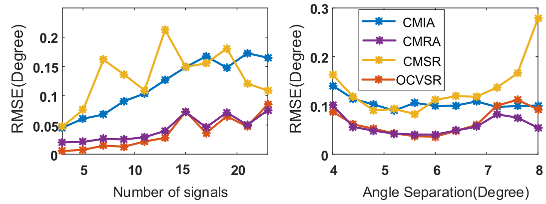

FIGURE 8. RMSE for GMRA via different methods with different signal number or angle separation.

As depicted in Fig.8, The estimation error increases with the growth of the number of signals. For CMIA and CMRA which are combined with MUSIC, the more the signals number, the less the noise vector, and the lower the estimation accuracy of the noise subspace. For CMSR and OCVSR which are based on sparse reconstruction, the more the signals number, the smaller the sparsity of signal power vector, the lower the reconstruction accuracy of signal power vector.

We can also find the estimation accuracy is not sensitive to the change of angle separation when it exceeds 4°.

C. COMPUTATIONAL EFFICIENCY COMPARISION

In the last experiment, the average running times of each method for one trial underdetermined DOA estimation under different SNR, number of snapshots and signals are compared. CPU time of different methods for one trial is counted in Table 5 with $K = 11$, $SNR = 10$dB and $L = 500$ by using GMRA. For each trial, we can find OCVSR spends less time than CMSR and spends nearly the same time of CMRA with MUSIC.

TABLE 5. CPU time of different methods for one trial underdetermined DOA estimation.

Methods	CMRA	CMIA	CMSR	cvx OCVSR	OCVSR
Time(s)	1.1968	3.1139	5.6379	2.2418	1.2062

The simulation result of Monte-Carlo trials under different conditions is also illustrated in Fig.8 and we set with $K = 11$, $SNR = 10$dB when the number of snapshots$L$ varying from

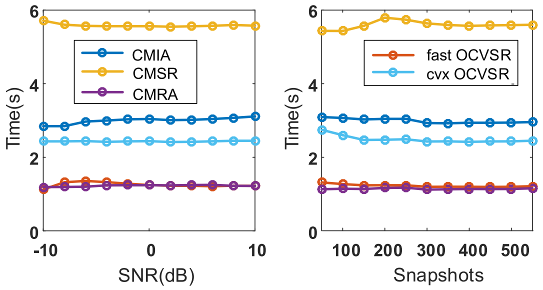

FIGURE 9. **Computational time via different SNR, Snapshots and Number of signals.**

50 to 550 while $K = 11$, $L = 100$ when SNR varying from 10dB to 10 dB by using GMRA.

It is clear from Fig.9 that CPU time of DOA estimation by OCVSR is robust to the change of SNR, snapshots and the number of signals and fast covariance matrix estimation is helpful to reduce the computational burden. OCVSR nearly spend the same time as CMRA and achieve underdetermined DOA estimation without the knowledge of signal number.

Based on the simulation results analysis above, we can conclude that the OCVSR has better estimation efficiency compared to the most existing approaches and has good computational efficiency.

VI. CONCLUSION

In this paper, the underdetermined DOA estimation of sparse array has been transformed into the conventional DOA estimation of uniform array with the same aperture. And we find one reason why sparse array can achieve DOA estimation with more sources than sensors is the redundancy of Toeplitz covariance matrix of uniform array. With the appropriate array structure, sparse array can address same number of sources as uniform array. Based on the MRA, a generalized MRA, named as GMRA, was proposed with high degrees of freedom and low mutual coupling. Different form the MRA, the GMRA always has corresponding array structure under various aperture. So it can be used to design the sparse array configuration easily. Experiment result for underdetermined DOA performance of different sparse arrays demonstrated the superiority of GMRA in degrees of freedom and mutual coupling.

To achieve more accurate underdetermined DOA estimation of GMRA without the knowledge of signal number, a sparsity-based DOA estimation method, named as OCVSR, was proposed. With fast covariance matrix reconstruction, OCVSR can achieve more accurate DOA estimation without increasing computational burden than other sparsity-based methods. It is due to the reduced dimension of covariance vector based on the redundancy of Toeplitz covariance matrix of uniform array. In addition to achieve high-precision DOA estimation of GMRA, OCVSR can also achieve underdetermined DOA estimation of other sparse arrays. However, OCVSR has little high requirements of sufficient snapshots and can only address uncorrelated sources now. A future work is to expand the adaptability to coherent signals and achieve 2D DOAs with sparse parallel array or sparse planar array.

APPENDIX A
THE PROOF OF THE EQUATION (38)

When limited snapshots are collected, the covariance of original uniform array output can be estimated from (32), and the pth element of the original covariance vector is

$$\begin{aligned}\hat{t}_p &= \frac{1}{L}\sum_{t=1}^{L}\left[\sum_{k_1}^{K} s_{k_1}(t)e^{-j\pi p\sin\theta_{k_1}} + n_p(t)\right] \\ &\quad\times\left[\sum_{k_1}^{K} s_{k_2}(t)e^{-j\pi\sin\theta_{k_2}} + n_1(t)\right] \\ &= \sum_{k=1}^{K}\left(\frac{1}{L}\sum_{t=1}^{L}|s_k(t)|^2\right)e^{-j\pi(p-1)\sin\theta_k} \\ &\quad+\frac{1}{L}\sum_{t=1}^{L}\sum_{k_1=1}^{K}\sum_{\substack{k_2=1\\k_1\neq k_2}}^{K} s_{k_1}(t)\,s_{k_2}^{*}(t)\,e^{-j\pi\left(p\sin\theta_{k_1}-\sin\theta_{k_2}\right)} \\ &\quad+\frac{1}{L}\sum_{t=1}^{L}\sum_{k=1}^{K} s_k(t)\,n_1^{*}(t)\,e^{j\pi\sin\theta_k} \\ &\quad+\frac{1}{L}\sum_{t=1}^{L}\sum_{k=1}^{K} s_k^{*}(t)\,n_p(t)\,e^{-j\pi p\sin\theta_k} \\ &\quad+\frac{1}{L}\sum_{t=1}^{L} n_p(t)\,n_1^{*}(t) \\ &= \sum_{k=1}^{K}\delta_k e^{-j\pi(p-1)\sin\theta_k} + t_p^{(1)} + t_p^{(2)} + t_p^{(3)} \end{aligned} \tag{40}$$

The perturbation of $\hat{t}_p$ in $\hat{\boldsymbol{t}}_f$ is $\varepsilon_p = t_p^{(1)} + t_p^{(2)} + t_p^{(3)}$. When the number of snapshots L is adequately large, ε_p is approximately circular complex Gaussian distributed according to the law of large numbers.

As the incident signals and the additive noise are mutually independent, one can easily conclude

$$E\left(\varepsilon_p\right) = E\left(t_p^{(1)}\right) + E\left(t_p^{(2)}\right) + E\left(t_p^{(3)}\right) = 0 \tag{41}$$

The estimate of the original covariance vector is perturbation-contaminated on account of limited snapshots and the variance of ε_p can be estimated as

$$\begin{aligned} Var\left(\varepsilon_p\right) &= E\left(t_p^{(1)}t_p^{(1)*}\right) + E\left(t_p^{(2)}t_p^{(1)*}\right) + E\left(t_p^{(3)}t_p^{(1)*}\right) \\ &\quad+E\left(t_p^{(1)}t_p^{(3)*}\right) + E\left(t_p^{(2)}t_p^{(2)*}\right) + E\left(t_p^{(3)}t_p^{(2)*}\right) \\ &\quad+E\left(t_p^{(1)}t_p^{(3)*}\right) + E\left(t_p^{(2)}t_p^{(3)*}\right) + E\left(t_p^{(3)}t_p^{(3)*}\right) \\ &= E\left(t_p^{(1)}t_p^{(1)*}\right) + E\left(t_p^{(2)}t_p^{(2)*}\right) + E\left(t_p^{(3)}t_p^{(3)*}\right) \\ &= \frac{1}{L}\sum_{k=1}^{K}\delta_k\left(\sum_{\substack{k'=1\\k'\neq k}}^{K}\delta_{k'}\right) + \frac{2}{L}\delta_n\sum_{k=1}^{K}\eta_k + \frac{1}{L}\delta_n^2 \end{aligned} \tag{42}$$

Then we can get

$$E\left(\left\|\hat{t}_f - A_G\delta_G\right\|_2^2\right) = \sum_{p=1}^{M} E\left(\left|\varepsilon_p\right|^2\right) = M\,Var\,(\varepsilon_1) \quad (43)$$

Based on the fitting error's distribution, threshold can be obtained as

$$\beta = \mu \times M\,Var\,(\varepsilon_1) \quad (44)$$

where μ is a weighting factor and is set to be 1 based on experience in the simulations.

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