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Implementation and Adoption Delays in Market Share Models Under Feedback Advertising Policies

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ABSTRACT This paper proposes definitions of implementation and adoption delays, arising from firm and client behaviors, in the context of market share dynamics. Information delay refers to the existence of lags in the information used for the advertising policy, while adoption delay refers to the existence of a lag in the effect of the advertising policy. These are natural lags in the flow of information and have not been considered in several models proposed in the literature. In this paper, these delays are introduced into recently proposed extensions of the Vidale-Wolfe-Deal and Lanchester models of market share dynamics subjected to affine advertising control policies. Conditions for stability of the equilibrium market share are derived. In addition, it is shown that Hopf bifurcations leading to oscillatory behavior exist for certain parameter values, and corresponding conditions for these are given. The main results are: the equilibrium market shares of the extended Vidale-Wolfe-Deal and Lanchester models are both robust to implementation delays, but, in the case of adoption delays, for both models, numerical results show that there is a critical value such that if the sum of the adoption delays exceeds this value, there is an onset of oscillations of market shares, through a Hopf bifurcation.

INDEX TERMS Advertising, affine feedback control, bifurcations, delays, duopolies, market share dynamics.

I. INTRODUCTION

Market share dynamics under advertising in duopolies have a long history, starting with the classical Vidale-Wolfe [1] and Arrow-Nerlove [2] models for monopolies, which have been generalized and studied intensively over the last sixty years, as can be seen in the survey [3]. A similar statement is true of Lanchester type models, first proposed in [4], and extended in [5], [6], for example. The focus has been on the derivation of optimal advertising policies (controls) for these classes of models (see, for example, [7]). We will refer to this class of as *market share dynamics models with controls*.

Few papers have considered the effect of delays on market share dynamics models with controls, even though there has been a considerable amount of research on Cournot and Bertrand duopoly models with delays [8]–[12]. Cournot type models generally consider the control variables to be the quantities of a product produced and sold and make assumptions on the inverse demand functions which determine price and, as a result, profit. In this sense, market share dynamics models, with advertising effort as a control variable which directly affects market share, are different from the Cournot type models. There is also a considerable amount of work on so-called ecologically inspired models of competitive dynamics, with delays, but *without control variables* (see, for example, [13]–[15]).

A salient feature of all existing market share dynamics models with controls is that they assume instant access to market information as well as an immediate effect of advertising. These are not realistic assumptions and consideration of

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delays in market behavior is necessary for a better description and understanding of market dynamics [16]–[20]. However, since delayed systems are infinite dimensional dynamical systems, important changes in the dynamics of systems can be produced, including oscillations, bifurcations, unstable and chaotic behaviors [21]–[24].

The contribution of this paper is to define two types of delays, named implementation and adoption delays, and introduce them into the Vidale-Wolfe-Deal and extended Lanchester models. Once this is done, as with all delay models, one of the central questions is to examine the effect of the delays on the asymptotic behavior of the system. To this end, the stability and bifurcation properties of the proposed delay models are then analyzed with regard to the control parameters and delay values. The main results can be summarized as follows:

- for implementation delays, the Vidale-Wolfe-Deal and extended Lanchester models have stable dynamics regardless of delay values;
- for adoption delays, the Vidale-Wolfe-Deal model can present Hopf bifurcations when the condition $\tau_1 + \tau_2 > 2\tau_c$ is satisfied. For the extended Lanchester model, the existence of Hopf bifurcations seems to be possible only when the condition $\tau_1 = \tau_2 = \tau_c$ is satisfied. τ_c denotes a critical value of the delay, which is defined in the sequel and τ_1, τ_2 are the adoption delays of firms 1 and 2.

The generality of the model proposed in this paper allows the important conclusion that it is more important to reduce the adoption delay than to get up-to-date information on market share and this point is explained below.

The results in this paper were obtained in the PhD thesis of the first author and are unpublished but, under university policy, available on the university thesis archive server [25].

II. DELAYS IN MARKET SHARE DYNAMICS MODELS

Delays can be found in several systems in diverse areas such as biology, economics, physics and the social sciences [26], [27]. Delays in the economy may arise in many ways: for example, a delay between the time an economic decision is made and the time when it produces results. Another example in which a delay crops up is in the estimation of an expected value, because the calculation depends on the current and *past* values of the variable in question [26].

For market share dynamics models, delays can arise in two ways: from the clients and the firms. In the particular case of a duopoly, it is often the case that clients and firms require past and present information in their decision-making processes [28]. In the consumer decision process, a time lag between the recognition of the necessity of a product and its purchase generally occurs. This lag or delay is generated as a result of many factors, internal or external to the consumer, such as age, social level, availability of time, information search, product prices or product quality [29]. On the other hand, the acquisition and processing of data, required in the decision-making process of firms, is costly and

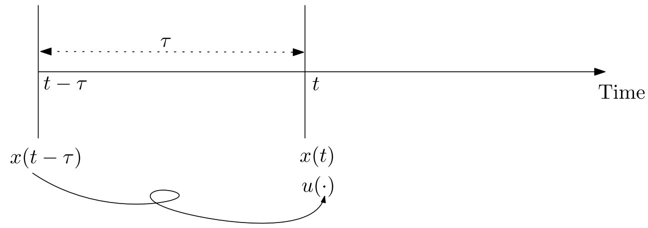


FIGURE 1. A timeline illustrating implementation delay: current time is t, at which (feedback) advertising effort $u(\cdot)$ will be applied. If the only market share information available is that of past instant $(t-\tau)$, this means that the advertising effort applied at time t can be expressed as $u(x(t-\tau))$, abbreviated to u_τ.

time-consuming, so it often happens that only delayed system data is available at the instant that the strategic decision has to be implemented [28]. Thus two types of delays are proposed in the following sections. A general market share dynamics model can be written as follows:

$$\dot{x} = f(x, u) - g(x) \tag{1}$$

where $f(x, u)$ is the term representing the growth of the market share as a function of the market share x and advertising effort u, and $g(x)$ is the decay term, representing decrease or loss of the market share over time. For example, the Vidale-Wolfe-Deal model [30] is obtained by choosing $x = (x_1, x_2)$, $f = ((1 - x_1 - x_2)u_1, (1 - x_1 - x_2)u_2)$, $g = (\lambda_1 x_1, \lambda_2 x_2)$.

This general model (1) will be used to introduce the two types of delays defined below.

A. IMPLEMENTATION DELAY IN ADVERTISING POLICY

In this section, inspired by the discussion in [9] (in the context of a Cournot type model with gradient dynamics), we make the following definition, assuming that the advertising effort is defined as a function of the market share (i.e., in feedback form):

Definition 1 (Implementation Delay): is said to occur when the market share information utilized to define advertising policy is lagged or delayed with respect to the instant when the latter is applied.

The timeline diagram (Figure 1) clarifies the definition:

From the definition, denoting $u(x(t-\tau))$ as u_τ, it follows that the following modification to (1):

$$\dot{x} = f(x, u_\tau) - g(x) \tag{2}$$

represents a market share dynamics model with uniform implementation delay, meaning that information on market share of each firm becomes available with the same delay to both firms. Nonuniform delays are possible (see sec. VI).

B. CONSUMER ADOPTION DELAY

Assuming that the effect of the advertising effort of a firm on clients is not immediate, the following definition is natural:

Definition 2 (Adoption Delay): is said to occur when advertising policy, put into effect at time t, only affects the dynamics at time $t + \tau$.

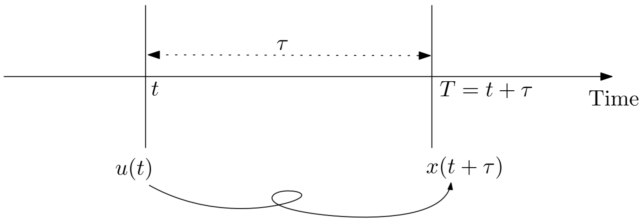


FIGURE 2. A timeline illustrating adoption delay: market share $x(T)$ at current time $T = t + \tau$ is affected by (feedback) advertising effort $u(x(t))$ is applied at time $t = T - \tau$, based on the known market share $x(t)$ at time t.

The timeline diagram (Figure 2) clarifies the definition:

Defining time $T = t + \tau$ to be the current time instant, the market share dynamics under adoption delay can be written as:

$$\dot{x}(T) = f\,(x(t), u(t)) - g(x(T)) \tag{3}$$

Since $x(t) = x(T - \tau) = x_\tau$ and $u(t) = u(T - \tau) = u_\tau$, equation (3) can be rewritten as:

$$\dot{x} = f\,(x_\tau, u_\tau) - g(x) \tag{4}$$

III. INSERTING IMPLEMENTATION AND ADOPTION DELAYS INTO MARKET SHARE DYNAMICS MODELS

The Vidale-Wolfe-Deal and extended Lanchester models recently studied in [31] are chosen to illustrate the insertion of implementation and adoption delays. For both models, the following notation is adopted: x_1, x_2 are the state variables representing the market shares of firm 1 and firm 2 respectively, u_1 and u_2 are the actions of firm 1 and firm 2 representing advertising effort and are assumed to have nonnegative values, λ_1 and λ_2 are the decay terms of firm 1 and firm 2. An *affine feedback advertising policy* is defined by the expression $u_i = k_i x_i + c_i$, $i = 1, 2$, where $k_i x_i$ denotes advertising effort proportional to the current market share, with k_i being the proportionality constant, and c_i denotes a constant advertising effort.

In regard to the choice of controls in this paper, the affine feedback advertising policies used in [31] are also used here. There are two reasons for this: the first is that the affine feedback advertising policy is intuitive, simple to implement and can be used to approximate any nonlinear control to first order and the second is that calculating optimal controls for differential games is quite complex, even without delays in the dynamics (see [7]). The Vidale-Wolfe-Deal and extended Lanchester models under affine feedback advertising from [31] are now recalled briefly, and followed by the models with implementation and adoption delays inserted.

Analysis and algebraic manipulation were performed in Maple software and the code is available in: http://ieee-dataport.org/4099

A. VIDALE-WOLFE-DEAL MODEL UNDER AFFINE ADVERTISING CONTROL POLICY

Assuming that the total population size is constant and normalized to 1 [32], Deal's extension [30] of the classical Vidale-Wolfe model to the case of a duopoly can be expressed as follows:

$$\begin{aligned}\dot{x}_1 &= (1 - x_1 - x_2)u_1 - \lambda_1 x_1\\ \dot{x}_2 &= (1 - x_1 - x_2)u_2 - \lambda_2 x_2\end{aligned} \tag{5}$$

Assuming that Vidale-Wolfe-Deal model is subject to affine feedback advertising policies, as proposed in [31], the model (5) becomes:

$$\begin{aligned}\dot{x}_1 &= -k_1 x_1^2 - k_1 x_1 x_2 + k_1 x_1 - \lambda x_1 + c_1 - c_1 x_1 - c_1 x_2\\ \dot{x}_2 &= -k_2 x_2^2 - k_2 x_1 x_2 + k_2 x_2 - \lambda x_2 + c_2 - c_2 x_1 - c_2 x_2\end{aligned} \tag{6}$$

B. EXTENDED LANCHESTER MODEL UNDER AFFINE ADVERTISING CONTROL POLICY

The classical Lanchester model [4], [33] can be understood as a special case of the Vidale-Wolfe model with competitive advertising in a saturated market [5]. This model was extended, by considerations involving a third population of undecided users and normalization of the total population, and subjected to an affine advertising policy in [31] to yield the following dynamics:

$$\begin{aligned}\dot{x}_1 &= -k_1 x_1^2 - k_2 x_1 x_2 - c_1 x_1 - c_2 x_1 + k_1 x_1 - \lambda x_1 + c_1\\ \dot{x}_2 &= -k_1 x_1 x_2 - k_2 x_2^2 - c_1 x_2 - c_2 x_2 + k_2 x_2 - \lambda x_2 + c_2\end{aligned} \tag{7}$$

IV. VIDALE-WOLFE-DEAL MODEL WITH DELAYS UNDER AFFINE ADVERTISING CONTROL POLICY

This section introduces the two types of delay into the first model recapitulated in the previous section. Stability and bifurcation results, as well as numerical examples illustrating the results, follow each model. Note that in order to facilitate analysis, equal delay values for both firms ($\tau_1 = \tau_2 = \tau$) were considered. However, the numerical examples presented relax this condition. All proofs of results are in appendices A and B.

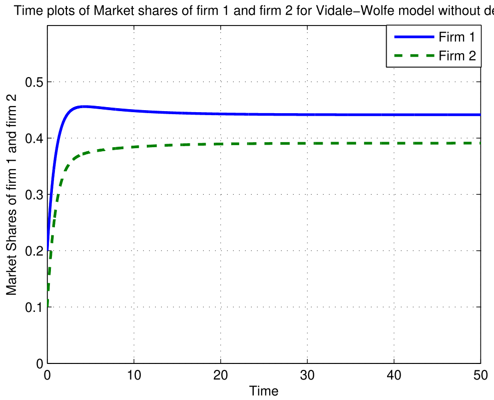
A. VIDALE-WOLFE-DEAL MODEL WITH IMPLEMENTATION DELAY

Following Definition 1, the Vidale-Wolfe-Deal model with uniform implementation delay τ, is expressed as follows:

$$\begin{aligned}\dot{x}_1 &= u_{1\tau}\,(1 - x_1 - x_2) - \lambda_1 x_1\\ \dot{x}_2 &= u_{2\tau}\,(1 - x_1 - x_2) - \lambda_2 x_2\end{aligned} \tag{8}$$

where the implementation delays affect the advertising policies (now denoted $u_{i\tau}$, $i = 1, 2$) as follows:

$$\begin{aligned}u_{1\tau} &= k_1 x_{1\tau} + c_1 = k_1 x_1\,(t - \tau) + c_1\\ u_{1\tau} &= k_2 x_{2\tau} + c_2 = k_2 x_2\,(t - \tau) + c_2\end{aligned}$$



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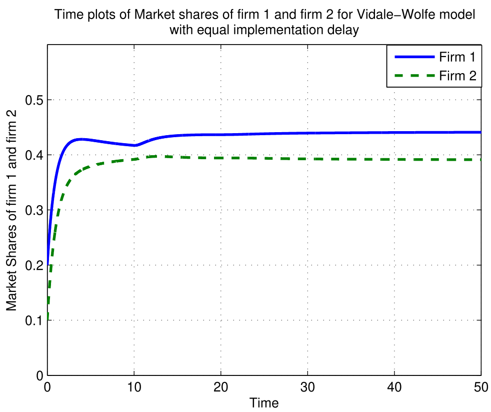
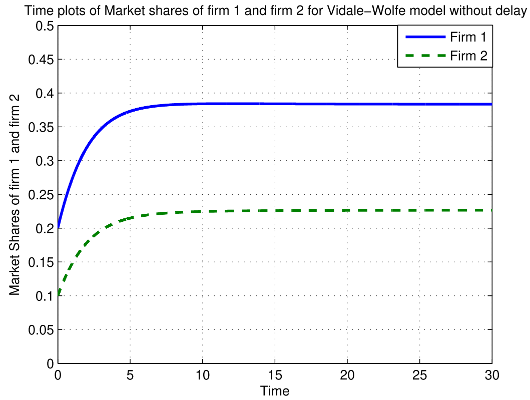


(b)

FIGURE 3. Evolution of market shares of firms x_1 and x_2 for Vidale-Wolfe-Deal model: (a) without delay ($\tau = 0$) and (b) with uniform implementation delay $\tau_1 = \tau_2 = \tau = 10$.



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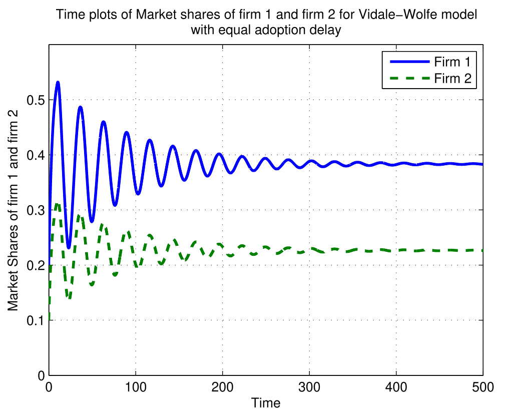


(b)

FIGURE 4. Evolution of market shares of firms x_1 and x_2 for: (a) Vidale-Wolfe-Deal model without delay ($\tau = 0$) and (b) Vidale-Wolfe-Deal model with adoption delay $\tau = 10$. Note that the delay free ($\tau = 0$) equilibrium point maintains its stability, when the delay is increased to $\tau = 10$.

Substituting the above expressions, model (8) can be expressed by:

$$\begin{aligned} \dot{x}_1 &= -k_1 x_1 x_{1\tau} - k_1 x_2 x_{1\tau} - k_1 x_{1\tau} - c_1 x_1 \\ &\quad - c_1 x_2 - \lambda x_1 + c_1 \\ \dot{x}_2 &= -k_2 x_1 x_{2\tau} - k_2 x_2 x_{2\tau} - k_2 x_{2\tau} - c_2 x_1 \\ &\quad - c_2 x_2 - \lambda x_2 + c_2 \end{aligned} \tag{9}$$

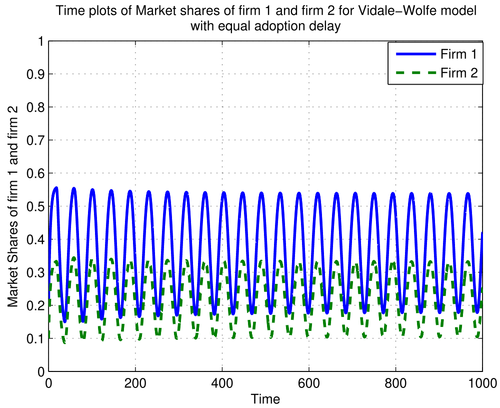
Based on stability analysis [34] and the Hopf bifurcation theorem [35] (detailed analysis and procedure are presented in Appendix A), Proposition 1 and Corollaries 1, 2 are obtained. Since the actual conditions referred to in the proposition and corollaries involve complicated algebraic conditions, they have been put in the appendix and are stated here in words. Proposition 1 and Corollary 1 assert that if a certain inequality, involving algebraic functions of the model parameters, holds then the delay free model has a stable equilibrium. Corollary 2 says that if the classical characteristic equation [36] has a positive solution and, in addition, a certain function involving the model parameters and trigonometric functions of the delay is nonzero, then the model with positive delay τ has a Hopf bifurcation. The exact mathematical statements of the propositions follow.

Proposition 1: The equilibrium point (x_1^*, x_2^*) for model (9) with delay value $\tau = 0$ is a stable equilibrium point when the conditions expressed in (31) hold.

Corollary 1: The equilibrium point (x_1^*, x_2^*) with equal parameter values for the affine advertising policies (i.e., $k_1 = k_2, c_1 = c_2$) for model (9) with delay value $\tau = 0$ is a stable equilibrium point when the conditions expressed in (35) hold.

Corollary 2: The model (9) with equal parameter values for the affine advertising policies (i.e., $k_1 = k_2, c_1 = c_2$) has a Hopf bifurcation for delay value $\tau > 0$ when the equation (39) has a positive solution and condition (43) holds.

Remark 1: Similar formulations, in words, can be made for all the subsequent propositions in the sequel, but will be omitted for brevity. The numerical examples serve to show that the conditions, derived in detail in the appendices, are not empty, in the sense that there are reasonable sets of model parameter values for which they are satisfied. In addition, the conditions $k_1 = k_2$, $c_1 = c_2$ are also relaxed in the numerical examples.



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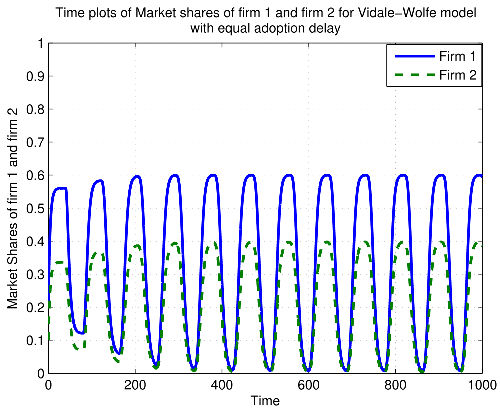
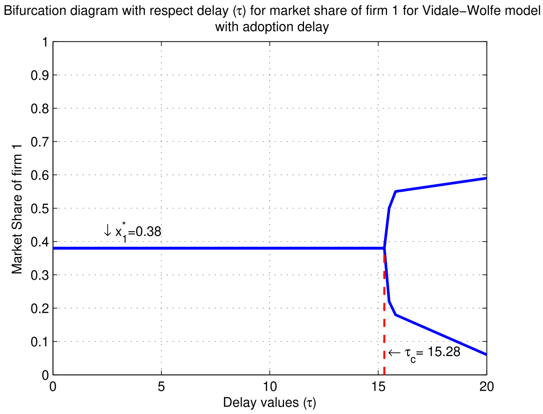


(b)

FIGURE 5. Evolution of market shares of firms x_1 and x_2 for: (a) Vidale-Wolfe model with adoption delay $\tau_c = 15.28$. Note that this value is the critical delay above which the bifurcation occurs. (b) Vidale-Wolfe-Deal model with adoption delay $\tau = 40$.



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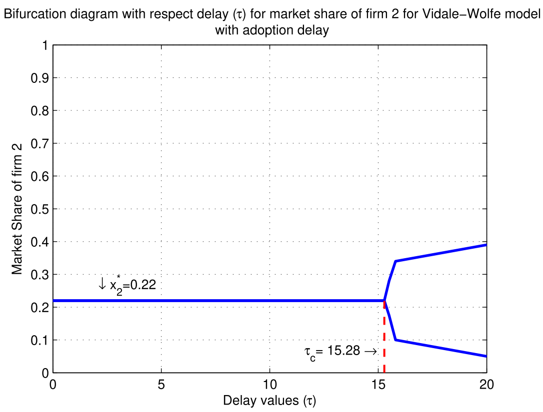


(b)

FIGURE 6. Bifurcation diagram with respect delay for Vidale-Wolfe model with adoption delay for: (a) market share of firm 1, (b) market share of firm 2.

B. NUMERICAL EXAMPLES FOR VIDALE-WOLFE-DEAL MODEL WITH IMPLEMENTATION DELAY

In this section, numerical examples that illustrate the Vidale-Wolfe-Deal model with implementation delay are given. The following parameter values are chosen: $x_1(0) = 0.2$, $x_2(0) = 0.1$, $\lambda = 0.2$, $k_1 = 0.4$, $c_1 = 0.35$, $k_2 = 0.17$, $c_2 = 0.4$. For these parameters, the equilibrium point is given by: $x_1^* = 0.44$, $x_2^* = 0.39$. First, analyzing for $\tau = 0$, substituting the chosen parameter values into equation (30) yields:

$$P_{vw}(\psi, \tau) = \Psi^2 + 1.2954\Psi + 0.174 \tag{10}$$

Hence, by Proposition 1, the equilibrium point is stable. Next, for the case when $\tau > 0$, substituting the chosen parameter values in equation (30), in which τ is permitted to be positive, leads to the characteristic equation:

$$P_{vw}(\psi, \tau) = \Psi^2 + 1.39\Psi - 0.09\Psi e^{-\psi\tau} + 0.001e^{-2\psi\tau} - 0.06e^{-\psi\tau} + 0.238 \tag{11}$$

Setting $\lambda = i\omega$ and solving, it turns out that the characteristic equation has no positive root. Thus, the model has no Hopf bifurcation. In Figure 3 numerical simulations for the Vidale-Wolfe-Deal model (9) are presented. Figure 3(a) shows the dynamics of the model without implementation delay ($\tau = 0$). Figure 3(b) illustrates the dynamics of the model for delay $\tau = 10$. In this figure we can see that the effect of implementation delay is *harmless*. Note that the delay-free equilibrium point maintains its stability for this value of delay.

C. VIDALE-WOLFE-DEAL MODEL WITH ADOPTION DELAY

From Definition 2, the Vidale-Wolfe-Deal model with adoption delay can be formulated as follows:

$$\begin{aligned} \dot{x}_1 &= u_{1\tau}\left(1 - x_{1\tau} - x_{2\tau}\right) - \lambda_1 x_1 \\ \dot{x}_2 &= u_{2\tau}\left(1 - x_{1\tau} - x_{2\tau}\right) - \lambda_2 x_2 \end{aligned} \tag{12}$$

where:

$$\begin{aligned} u_{1\tau} &= k_1 x_{1\tau} + c_1 = k_1 x_1\left(t - \tau_1\right) + c_1 \\ u_{2\tau} &= k_2 x_{2\tau} + c_2 = k_2 x_2\left(t - \tau_2\right) + c_2 \end{aligned}$$

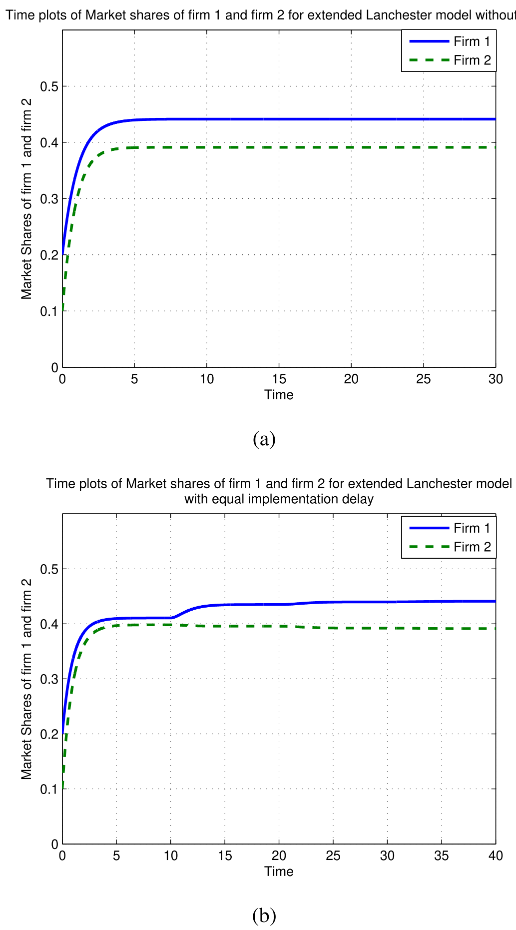


FIGURE 7. **Evolution of market shares of firms x_1 and x_2 for: (a) extended Lanchester model without delay ($\tau = 0$) and (b) extended Lanchester model with implementation delay $\tau = 10$. In both figures, it can be seen that the delay-free equilibrium points maintain their stability.**

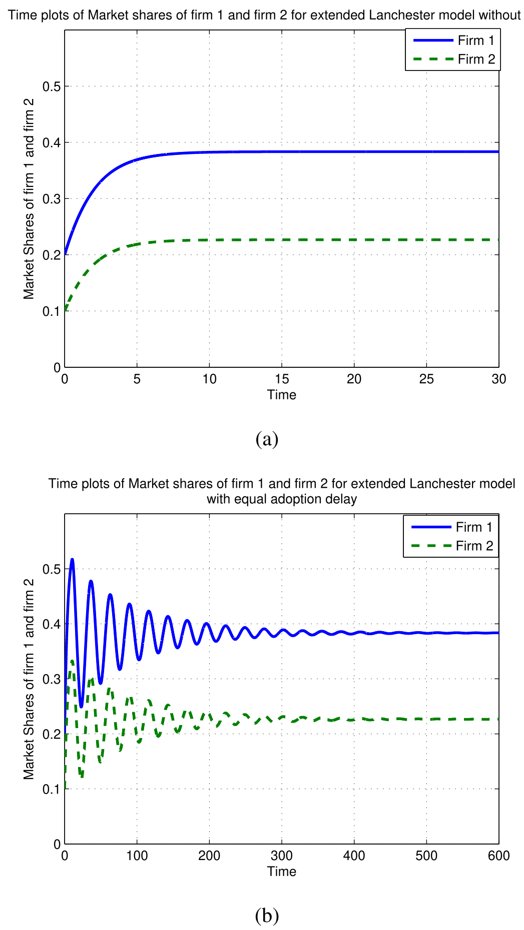


FIGURE 8. **Evolution of market shares of firms x_1 and x_2 for: (a) Extended Lanchester model without delay $\tau = 0$ and (b) Extended Lanchester model with adoption delay $\tau = 10$. For this case, it can be observed that the model presents oscillations but manages to maintain the stability of its equilibrium points.**

Substituting the above expressions into (12), the Vidale-Wolfe-Deal model with adoption delay is as follows:

$$\begin{aligned}\dot{x}_1 &= -x_{1\tau}^2 k_1 - x_{1\tau}x_{2\tau}k_1 - x_{1\tau}c_1 + x_{1\tau}k_1 \\ &\quad - x_{2\tau}c_1 - \lambda x_1 + c_1 \\ \dot{x}_2 &= -x_{1\tau}x_{2\tau}k_2 - x_{2\tau}^2 k_2 - x_{1\tau}c_2 - x_{2\tau}c_2 \\ &\quad + x_{2\tau}k_2 - \lambda x_2 + c_2 \end{aligned} \tag{13}$$

Then, based on stability analysis [34] and the Hopf bifurcation theorem [35] (detailed analysis and procedure are presented in Appendix B), the following propositions are obtained:

Proposition 2: The equilibrium point (x_1^*, x_2^*) for model (13) with delay value $\tau = 0$ is a stable equilibrium point when the conditions expressed in (47) hold.

Corollary 3: The equilibrium point (x_1^*, x_2^*) with equal parameter values for the affine advertising policies (i.e., $k_1 = k_2, c_1 = c_2$) of model (13) is a stable equilibrium point for $\tau = 0$ when the conditions expressed in (52) hold.

Corollary 4: The model (13) with equal parameter values for the affine advertising policies (i.e., $k_1 = k_2, c_1 = c_2$) has a Hopf bifurcation for delay value $\tau > 0$ when the equation (56) has a positive solution and condition (60) holds.

D. NUMERICAL RESULTS FOR VIDALE-WOLFE-DEAL MODEL WITH ADOPTION DELAY

Some numerical results for the Vidale-Wolfe-Deal model with adoption delay are now presented. The following parameter values are considered: $x_1(0) = 0.2$, $x_2(0) = 0.1$, $\lambda = 0.25$, $k_1 = 0.25$, $c_1 = 0.15$, $k_2 = 0.2$ and $c_2 = 0.1$. For these parameters, the equilibrium point is given by: $x_1^* = 0.383$, $x_2^* = 0.226$. Now, analyzing for $\tau = 0$, we substitute the parameter values in equation (46) and we get:

$$P_{vw}(\psi, \tau) = \Psi^2 + 0.71\Psi + 0.09 \tag{14}$$

Hence, from Proposition 2 it can be affirmed that equilibrium point is stable. Then, for the case when $\tau > 0$, substituting the parameter values in equation (46), in which τ is permitted to be positive, we obtain:

$$\begin{aligned} P_{vw}(\psi, \tau) &= \Psi^2 + 0.5\Psi + 0.21\Psi e^{-\psi\tau} - 0.02e^{-2\psi\tau} \\ &\quad + 0.05e^{-\psi\tau} + 0.06 \end{aligned} \tag{15}$$

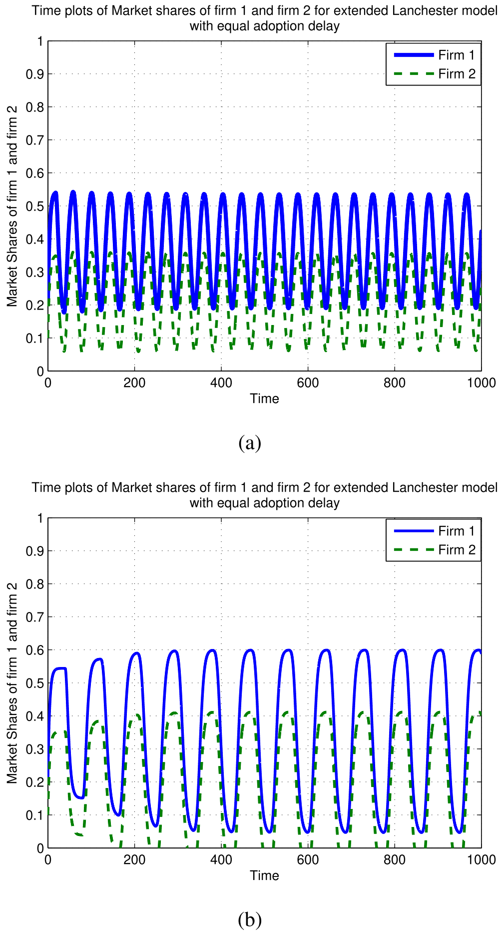


FIGURE 9. **Evolution of market shares of firms x_1 and x_2 for: (a) Extended Lanchester model with critical adoption delay $\tau_c = 14.98$ and (b) Extended Lanchester model with adoption delay $\tau = 40$.**

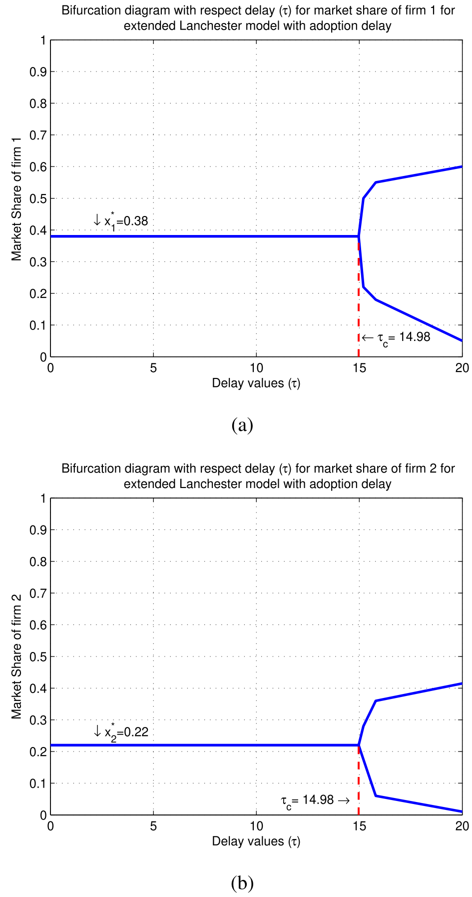


FIGURE 10. **Bifurcation diagram with respect to delay for extended Lanchester model with adoption delay for: (a) market share of firm 1, (b) market share of firm 2.**

Solving the equation for $\lambda = i\omega$ we have that one solution is given by $\omega = 0.17$ and $\tau = 15.28$ and $\left(\frac{d\lambda}{d\tau}\right) \neq 0$. Therefore, according to Hopf bifurcation Theorem it can be stated that the model has Hopf bifurcation. Figures 4 and 5 illustrate the numerical results for the Vidale-Wolfe-Deal model with adoption delay. Thus, in Figure 4(a) the dynamics of the model without delay ($\tau = 0$) is shown. Then, Figure 4(b) shows the dynamics of the model for $\tau = 10$, that is, an value delay less than the critical value. Note that in this case, the dynamics of the model has oscillations but the equilibrium point remains stable. Figure 5(a) illustrates the dynamics of the model for $\tau = 15.28$. This delay value is the critical delay value (τ_c) that produces existence of Hopf bifurcation. Finally, Figure 5(b) shows the dynamics of the model for a delay value ($\tau = 40$) greater than critical delay value. Notice that the model for this delay value has oscillations of increasing amplitude and the equilibrium point loses stability.

Figure 6 shows the bifurcation diagram with respect to the variation of the delay value. Maximum and minimum values of the market shares in the last 100 simulation steps in order to analyze the change in their behavior. The figures 6(a) and 6(b) show the behavior of the equilibrium point of the market shares for firm 1 and firm 2 respectively. In both figures it can be observed how the equilibrium point remains constant up to the critical delay value where the bifurcation begins.

V. EXTENDED LANCHESTER MODEL WITH DELAYS UNDER AFFINE ADVERTISING CONTROL POLICY

This section introduces the two types of delay into the model of Section 3.2. Similar to the previous section, stability and bifurcation results, as well as numerical examples illustrating the results, follow each model. Once again, in order to facilitate analysis, equal delay values for both firms ($\tau_1 = \tau_2 = \tau$) were considered. All proofs of results are in appendices C and D.

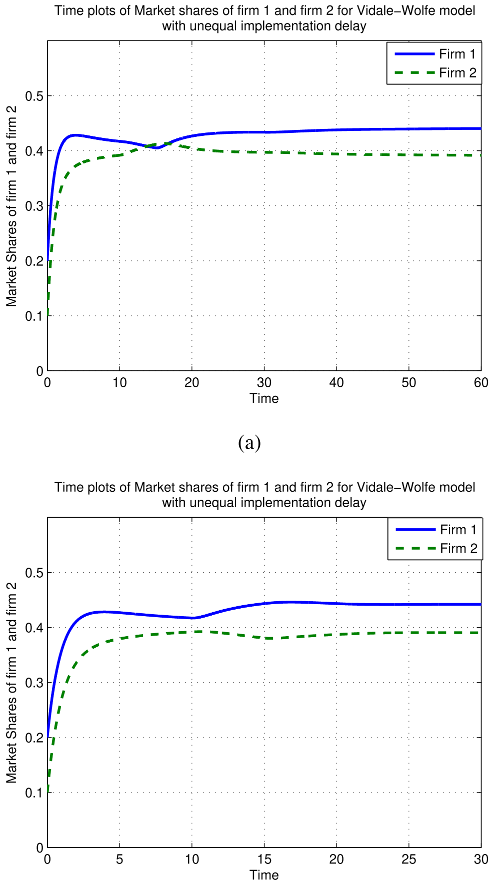
A. EXTENDED LANCHESTER MODEL WITH IMPLEMENTATION DELAY

From Definition 1, the extended Lanchester model with implementation delay can written as:

$$\begin{aligned} \dot{x}_1 &= u_{1\tau}(1 - x_1) - u_{2\tau}x_1 - \lambda_1 x_1 \\ \dot{x}_2 &= u_{2\tau}(1 - x_2) - u_{1\tau}x_2 - \lambda_2 x_2 \end{aligned} \tag{16}$$

where:

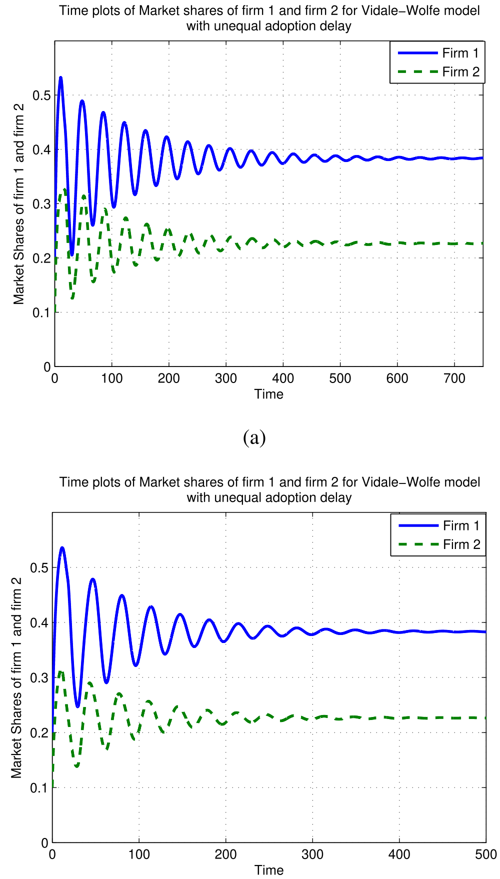
$$\begin{aligned} u_{1\tau} &= k_1 x_{1\tau} + c_1 = k_1 x_1(t - \tau_1) + c_1 \\ u_{2\tau} &= k_2 x_{2\tau} + c_2 = k_2 x_2(t - \tau_2) + c_2 \end{aligned}$$



(a)

(b)

FIGURE 11. Evolution of market shares of firms x_1 and x_2 for: (a) Vidale-Wolfe-Deal model with implementation delay $\tau_1 = 15$, $\tau_2 = 10$ (b) Vidale-Wolfe-Deal model with implementation delay $\tau_1 = 10$ and $\tau_2 = 15$. For this case, we can note that is no crossing in the time plots of market shares.



(a)

(b)

FIGURE 12. Evolution of market shares of firms x_1 and x_2 for: (a) Vidale-Wolfe-Deal model with adoption delay for $\tau_1 = 18$ and $\tau_2 = 10$ (b) Vidale-Wolfe-Deal model with adoption delay for $\tau_1 = 10$ and $\tau_2 = 18$.

Substituting the above expressions into model (16) we obtain:

$$\begin{aligned}\dot{x}_1 &= -k_1x_1x_{1\tau} - k_2x_1x_{2\tau} + k_1x_{1\tau} - c_1x_1 \\ &\quad - c_2x_1 - \lambda x_1 + c_1 \\ \dot{x}_2 &= -k_1x_2x_{1\tau} - k_2x_2x_{2\tau} + k_2x_{2\tau} - c_1x_2 \\ &\quad - c_2x_2 - \lambda x_2 + c_2 \end{aligned} \tag{17}$$

Then, based on stability analysis [34] and the Hopf bifurcation theorem [35] (detailed analysis and procedure are presented in Appendix C) we obtain the following propositions:

Proposition 3: The equilibrium point (x_1^*, x_2^*) for model (17) with delay value $\tau = 0$ is a stable equilibrium point when the conditions expressed in (64) hold.

Corollary 5: The equilibrium point (x_1^*, x_2^*) with equal parameter values for the affine advertising policies (i.e., $k_1 = k_2, c_1 = c_2$) is a stable equilibrium point for delay value $\tau = 0$ when the conditions expressed in (69) hold.

Corollary 6: Model (17) with equal parameter values for the affine advertising policies (i.e., $k_1 = k_2, c_1 = c_2$) has Hopf bifurcation for delay value $\tau > 0$ when equation (73) has a positive solution and condition (77) holds.

B. NUMERICAL RESULTS FOR EXTENDED LANCHESTER MODEL WITH IMPLEMENTATION DELAY

In this section, some numerical results for the extended Lanchester model with implementation delay are presented. The parameter values considered are similar to the Vidale-Wolfe-Deal model, that is: $x_1(0) = 0.2, x_2(0) = 0.1, \lambda = 0.2$, $k_1 = 0.4$, $c_1 = 0.35$, $k_2 = 0.17$, $c_2 = 0.4$. The equilibrium point for these parameters results: $x_1^* = 0.44$, $x_2^* = 0.39$. Now, analyzing for $\tau = 0$, the parameter values in equation (63) are substituted and it is obtained:

$$P_{el}(\psi, \tau) = \Psi^2 + 2.0569\Psi + 1.042 \tag{18}$$

Taking into account Proposition 3 it can be said that equilibrium point is stable. Then, when $\tau > 0$, substituting the parameter values above into equation (63), in which τ is permitted to be positive, we have:

$$\begin{aligned}P_{el}(\psi, \tau) &= \Psi^2 + 2.38\Psi - 0.32\Psi e^{-\psi\tau} + 0.01e^{-2\psi\tau} \\ &\quad - 0.39e^{-\psi\tau} + 1.42 \end{aligned} \tag{19}$$

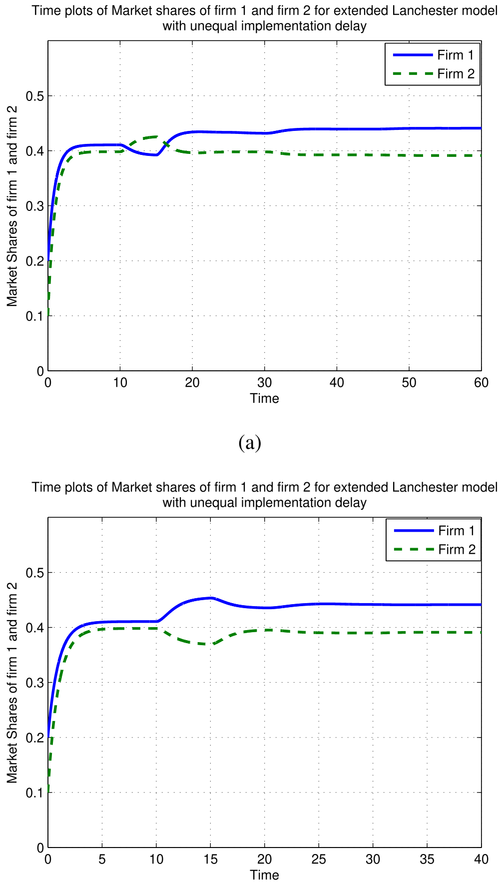


FIGURE 13. **Evolution of market shares of firms x_1 and x_2 for: (a) Extended Lanchester model with implementation delay for $\tau_1 = 15$ and $\tau_2 = 10$ (b) Extended Lanchester model with implementation delay for $\tau_1 = 10$ and $\tau_2 = 15$ For this case, we can note that market share values do not cross each other.**

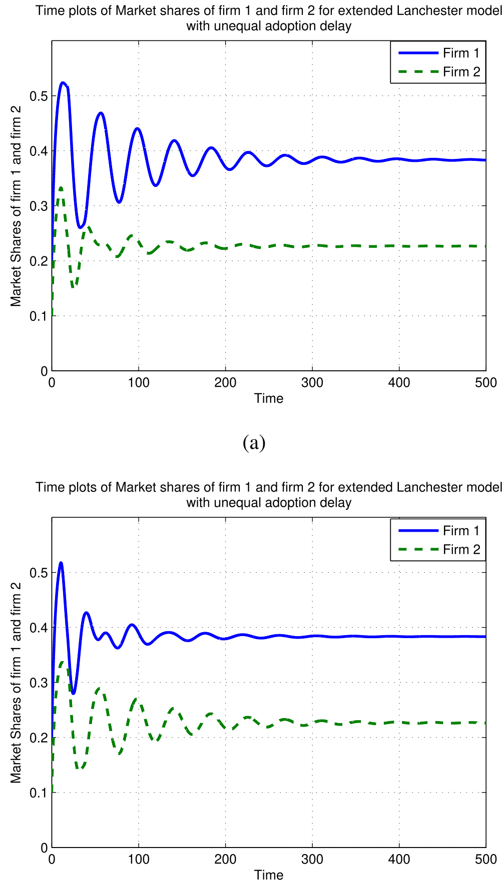


FIGURE 14. **Evolution of market shares of firms x_1 and x_2 for: (a) Extended Lanchester model with adoption delay for $\tau_1 = 18$ and $\tau_2 = 10$ (b) Extended Lanchester model with adoption delay for $\tau_1 = 10$ and $\tau_2 = 18$. In both models the equilibrium points remain stable after transients.**

Next, solving the equation for $\lambda = i\omega$ we find that the characteristic equation has no positive root. Therefore, it is concluded that the model has no Hopf bifurcation. In figure 7 numerical simulations for the extended Lanchester model with implementation delay are shown. Figure 7(a) illustrates the dynamics of the model without delay ($\tau = 0$). Figure 7(b) shows the dynamics of the model for delay value $\tau = 10$. Notice that the equilibrium point maintains its stability.

C. EXTENDED LANCHESTER MODEL WITH ADOPTION DELAY

From Definition 2, the extended Lanchester model with adoption delay can be written as follows:

$$\begin{aligned} \dot{x}_1 &= u_{1\tau}\left(1 - x_{1\tau}\right) - u_{2\tau}x_{1\tau} - \lambda_1 x_1 \\ \dot{x}_2 &= u_{2\tau}\left(1 - x_{2\tau}\right) - u_{1\tau}x_{2\tau} - \lambda_2 x_2 \end{aligned} \tag{20}$$

where:

$$\begin{aligned} u_{1\tau} &= k_1 x_{1\tau} + c_1 = k_1 x_1\left(t - \tau_1\right) + c_1 \\ u_{2\tau} &= k_2 x_{2\tau} + c_2 = k_2 x_2\left(t - \tau_2\right) + c_2 \end{aligned}$$

Substituting the above expressions, model (20) can be expressed by:

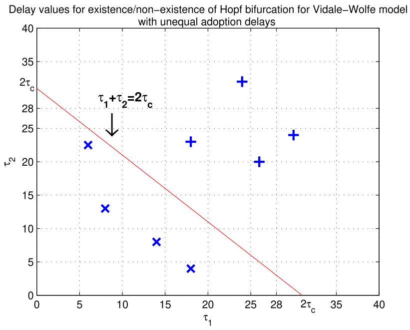
$$\begin{aligned} \dot{x}_1 &= -x_{1\tau}^2 k_1 - x_{1\tau}x_{2\tau}k_2 - x_{1\tau}c_1 \\ &\quad - x_{1\tau}c_2 + x_{1\tau}k_1 - \lambda x_1 + c_1 \\ \dot{x}_2 &= -x_{1\tau}x_{2\tau}k_1 - x_{2\tau}^2 k_2 - x_{2\tau}c_1 \\ &\quad - x_{2\tau}c_2 + x_{2\tau}k_2 - \lambda x_2 + c_2 \end{aligned} \tag{21}$$

Then, based on stability analysis [34] and the Hopf bifurcation theorem [35] (detailed analysis and procedure are presented in Appendix D) we obtain the following propositions:

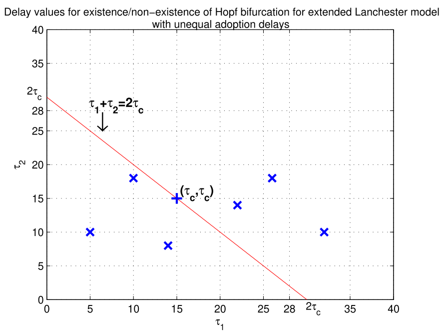
Proposition 4: The equilibrium point $\left(x_1^*, x_2^*\right)$ for model (21) with delay value $\tau = 0$ is a stable equilibrium point when the conditions expressed in (81) hold.

Corollary 7: The equilibrium point $\left(x_1^*, x_2^*\right)$ with equal parameter values for the affine advertising policies (i.e., $k_1 = k_2, c_1 = c_2$) of model (21) is a stable equilibrium point for delay value $\tau = 0$ when the conditions expressed in (86) hold.

Corollary 8: Model (21) with equal parameter values for the affine advertising policies (i.e., $k_1 = k_2, c_1 = c_2$) has



(a)



(b)

FIGURE 15. Existence of Hopf bifurcation for unequal adoption delay values for x_1 (τ_1) and x_2 (τ_2) for: (a) Vidale-Wolfe-Deal model and (a) Extended Lanchester model. The symbol $\times$ denotes non-existence while the symbol $+$ denotes existence of Hopf bifurcation.

Hopf bifurcation for delay value $\tau > 0$ when equation (90) has a positive solution and condition (94) holds.

D. NUMERICAL RESULTS FOR EXTENDED LANCHESTER MODEL WITH ADOPTION DELAY

Once again, the parameter values chosen are similar to the ones used for the Vidale-Wolfe-Deal model: $x_1(0) = 0.2$, $x_2(0) = 0.1$, $\lambda = 0.25$, $k_1 = 0.25$, $c_1 = 0.15$, $k_2 = 0.2$, $c_2 = 0.1$. For these parameters the equilibrium point is given by: $x_1^* = 0.383$, $x_2^* = 0.226$. First, analyzing for $\tau = 0$, the chosen parameter values are substituted in equation (80) to get:

$$P_{vw}(\psi, \tau) = \Psi^2 + 0.97\Psi + 0.23. \tag{22}$$

From Proposition 4 it follows that equilibrium point is stable. Next, analyzing when $\tau > 0$, substituting the parameter values in equation (80), in which τ is permitted to be positive, we have:

$$P_{vw}(\psi, \tau) = \Psi^2 + 0.5\Psi - 0.47\Psi e^{-\psi\tau} + 0.005e^{-2\psi\tau} - 0.11e^{-\psi\tau} + 0.06 \tag{23}$$

Then, solving equation (23) for $\lambda = i\omega$ we have that one solution is given by $\omega = 0.17$ and $\tau = 14.98$ and $\left(\frac{d\lambda}{d\tau}\right) \neq 0$. Therefore, it follows that the model has Hopf bifurcation. Figures 8 and 9 illustrate the numerical results for the extended Lanchester model with adoption delay. Figure 8(a) shows the dynamics of the model without delay ($\tau = 0$). Figure 8(b) illustrates the dynamics of the model for $\tau = 10$. Notice that, despite the appearance of oscillations,the equilibrium point remains stable. Figure 9(a) shows the dynamics of the model for $\tau = 14.98$. This delay value is the critical delay value (τ_c) which implies the existence of a Hopf bifurcation. Figure 9(b) illustrates the dynamics of the model for $\tau = 40$, and it is confirmed that unbounded oscillations occur and the equilibrium point becomes unstable.

Finally, Figure 10 shows the bifurcation diagram with respect to delay variation, which is similar analysis to the previous case shown in 6.

VI. VIDALE-WOLFE-DEAL MODEL AND EXTENDED LANCHESTER MODEL WITH UNEQUAL DELAYS UNDER AFFINE ADVERTISING CONTROL POLICY

In the previous sections, the Vidale-Wolfe-Deal and extended Lanchester models were analyzed considering the existence of two types of delays (implementation and adoption). In order to facilitate analysis, equal delay values for both firms ($\tau_1 = \tau_2$) were considered, however, this assumption often does not hold in practice (and analysis is difficult). Thus, in this section, some numerical simulations are shown, with unequal delay values, i.e., ($\tau_1 \neq \tau_2$).

A. VIDALE-WOLFE-DEAL MODEL WITH UNEQUAL DELAY VALUES UNDER AFFINE ADVERTISING CONTROL POLICY

Rewriting the equation (9) for the Vidale-Wolfe-Deal model with unequal implementation delays and defining $x_{1\tau}$ and $x_{2\tau}$ as follows:

$$\begin{aligned} x_{1\tau} &= x_1(t - \tau_1) \\ x_{2\tau} &= x_2(t - \tau_2) \end{aligned}$$

we have:

$$\begin{aligned} \dot{x}_1 &= -k_1 x_1 x_{1\tau} - k_1 x_2 x_{1\tau} - k_1 x_{1\tau} - c_1 x_1 \\ &\quad - c_1 x_2 - \lambda x_1 + c_1 \\ \dot{x}_2 &= -k_2 x_1 x_{2\tau} - k_2 x_2 x_{2\tau} - k_2 x_{2\tau} - c_2 x_1 \\ &\quad - c_2 x_2 - \lambda x_2 + c_2 \end{aligned} \tag{24}$$

Likewise, rewriting the equation (13) for the Vidale-Wolfe-Deal model with unequal adoption delays we obtain:

$$\begin{aligned} \dot{x}_1 &= -x_{1\tau}^2 k_1 - x_{1\tau} x_{2\tau} k_1 - x_{1\tau} c_1 + x_{1\tau} k_1 \\ &\quad - x_{2\tau} c_1 - \lambda x_1 + c_1 \\ \dot{x}_2 &= -x_{1\tau} x_{2\tau} k_2 - x_{2\tau}^2 k_2 - x_{1\tau} c_2 - x_{2\tau} c_2 \\ &\quad + x_{2\tau} k_2 - \lambda x_2 + c_2 \end{aligned} \tag{25}$$

Assume the same parameters as in section IV and unequal delay values ($\tau_1 \neq \tau_2$). Figures 11 and 12 show the numerical results for the Vidale-Wolfe-Deal model with implementation and adoption delay respectively but with different delay values for each firm. Figure 11(a) illustrates the dynamics of the model with implementation delay considering $\tau_1 > \tau_2$. In figure 11(b) the dynamics of the model when $\tau_1 < \tau_2$ is presented. Note that in both cases the equilibrium point remains stable and for $\tau_1 > \tau_2$ the dynamics has a crossing in time plots of market share trajectories. Figure 12(a) shows the dynamics of the model with adoption delay when $\tau_1 > \tau_2$. Finally, in figure 12(b) the dynamic of the model for $\tau_1 < \tau_2$ is ilustrated. Note that both models exhibit oscillations, but in both cases, the equilibrium points remain stable.

B. EXTENDED LANCHESTER MODEL WITH UNEQUAL DELAY VALUES UNDER AFFINE ADVERTISING CONTROL POLICY

Similar to the previous subsection, equation (17) which represents the extended Lanchester model with implementation delay is initially rewritten:

$$\begin{aligned}\dot{x}_1 &= -k_1x_1x_{1\tau} - k_2x_1x_{2\tau} + k_1x_{1\tau} - c_1x_1 - c_2 \\ &\quad x_1 - \lambda x_1 + c_1 \\ \dot{x}_2 &= -k_1x_2x_{1\tau} - k_2x_2x_{2\tau} + k_2x_{2\tau} - c_1x_2 - c_2 \\ &\quad x_2 - \lambda x_2 + c_2 \end{aligned} \tag{26}$$

Likewise, equation (21) for extended Lanchester model with adoption delay is rewritten:

$$\begin{aligned}\dot{x}_1 &= -x_{1\tau}^2k_1 - x_{1\tau}x_{2\tau}k_2 - x_{1\tau}c_1 - x_{1\tau}c_2 \\ &\quad + x_{1\tau}k_1 - \lambda x_1 + c_1 \\ \dot{x}_2 &= -x_{1\tau}x_{2\tau}k_1 - x_{2\tau}^2k_2 - x_{2\tau}c_1 - x_{2\tau}c_2 \\ &\quad + x_{2\tau}k_2 - \lambda x_2 + c_2 \end{aligned} \tag{27}$$

Suppose that, for the numerical examples, the same parameters as in section V and unequal delay values ($\tau_1 \neq \tau_2$) are chosen. Figures 13 and 14 illustrate the numerical results for the extended Lanchester model with implementation and adoption delay respectively for different delay values for each firm. Figure 13(a) shows the dynamics of the model with implementation delays such that $\tau_1 > \tau_2$. Then, in figure 13(b) we present the dynamics of the model assuming $\tau_1 < \tau_2$. Notice that in both cases the equilibrium point remains stable, but for $\tau_1 > \tau_2$, the dynamics present two crossings between the trajectories. Figure 14(a) presents the dynamics of the model with adoption delay for $\tau_1 > \tau_2$. Finally, figure 14(b) shows the dynamics of the model when $\tau_1 < \tau_2$. Note that both models present oscillations but, in both cases, the equilibrium points remain stable. Finally, figure 15 summarizes the results of numerical simulations performed for different values of τ_1 and τ_2 and leads to the formulation of conjectures about conditions for the existence of Hopf bifurcation in Vidale-Wolfe-Deal and extended Lanchester models with unequal adoption delays. The conjectures are as follows:

- for implementation delays, the Vidale-Wolfe-Deal and extended Lanchester models have stable dynamics regardless of delay values τ_1 and τ_2;
- for adoption delays, the Vidale-Wolfe-Deal model can present Hopf bifurcations when the condition $\tau_1 + \tau_2 > 2\tau_c$ is satisfied. For extended Lanchester model, the existence of Hopf bifurcations seems to be possible only when the condition $\tau_1 = \tau_2 = \tau_c$ is satisfied.

VII. CONCLUSION AND DISCUSSIONS

This paper defines the concept of implementation and adoption delays of firms and clients in market share dynamics. In particular, these two types of delays are introduced into the Vidale-Wolfe-Deal and extended Lanchester models recently studied by the authors [31], subject to affine feedback advertising policies. In the presence of implementation delays, the Vidale-Wolfe-Deal and extended Lanchester models present stable dynamics regardless of the delay values. On the other hand, for the case of identical adoption delays for both firms, the behavior of the Vidale-Wolfe-Deal and extended Lanchester models was analysed mathematically and can be summarized as follows: for both models, there is a critical value of delay τ_c for Hopf bifurcation; for $\tau < \tau_c$, in both models, market shares may have an oscillatory transient, but eventually settle at an equilibrium. As τ becomes progressively larger and eventually exceeds τ_c, bounded and eventually unbounded oscillations of market share occur and the equilibrium point becomes unstable. Finally, the case of unequal delay values for each firm is very complex and was therefore studied only through numerical simulations, leading to the formulation of some conjectures which may be of interest for future research.

From the viewpoint of a decision maker who might use these models to guide advertising policies, the results of this paper can be phrased as follows. Utilization of an old or delayed value of market share to compute advertising effort does not affect the equilibrium value of market share. However, it is critical to reduce the interval between advertising and its effect, i.e., the adoption delay, since this could lead to oscillations and even instability in market share. The important implication of these results is that it is more important to reduce the adoption delay than to obtain the most recent information on market share, since the former can cause instability and severe oscillation in market shares, but this does not happen with implementation delays.

Finally, as possible future work, it could be of interest to extend the analysis of stability and bifurcations with respect to the parameters of the affine feedback advertising policy (i.e., $k_i, c_i, i = 1, 2$).

APPENDIX A
STABILITY ANALYSIS FOR VIDALE-WOLFE-DEAL MODEL WITH IMPLEMENTATION DELAY

Assuming that $\tau_1 = \tau_2 = \tau$, the Jacobian matrix [34] with respect to equilibrium point for model (9) is given by:

$$J_{vw} = \begin{vmatrix} A_{vw} & B_{vw} \\ C_{vw} & D_{vw} \end{vmatrix} \tag{28}$$

where:

$$\begin{aligned} A_{vw} &= -k_1x_{1\tau} - c_1 - \lambda + e^{-\psi\tau}(-k_1x_1 - k_1x_2 + k_1) \\ B_{vw} &= -k_1x_{1\tau} - c_1 \\ C_{vw} &= -k_2x_{2\tau} - c_2 \\ D_{vw} &= -k_2x_{2\tau} - c_2 - \lambda + e^{-\psi\tau}(-k_2x_1 - k_2x_2 + k_2) \end{aligned}$$

At equilibrium:

$$\left(x_1^*(t), x_2^*(t)\right) = \left(x_1^*(t-\tau), x_2^*(t-\tau)\right) \tag{29}$$

Therefore, the stability of equilibrium points will be determined by the characteristic equation expressed by:

$$P_{vw}(\psi,\tau) = \Psi^2 + P_{1vw}\Psi + P_{2vw} + P_{3vw}e^{-2\psi\tau} + P_{4vw}\Psi e^{-\psi\tau} + P_{5vw}e^{-\psi\tau} \tag{30}$$

where:

$$\begin{aligned}
P_{1vw} &= k_1x_1 + k_2x_2 + c_1 + c_2 + 2\lambda \\
P_{2vw} &= k_1\lambda x_1 + k_2\lambda x_2 + c_1\lambda + c_2\lambda + \lambda^2 \\
P_{3vw} &= k_1k_2x_1^2 + 2k_1k_2x_1x_2 + k_1k_2x_2^2 - 2k_1k_2x_1 - 2k_1k_2x_2 \\
&\quad + k_1k_2 \\
P_{4vw} &= k_1x_1 + k_1x_2 + k2\times 1 + k2\times 2 - k1 - k2 \\
P_{5vw} &= k_1k_2x_1^2 + 2k_1k_2x_1x_2 + k_1k_2x_2^2 + c_1k_2x_1 + c_1k_2x_2 \\
&\quad + c_2k_1x_1 + c_2k_1x_2 - k_1k_2x_1 - k_1k_2x_2 + k_1\lambda x_1 \\
&\quad + k_1\lambda x_2 + k_2\lambda x_1 + k_2\lambda x_2 - c_1k_2 - c_2k_1 - k_1\lambda - k_2\lambda
\end{aligned}$$

STABILITY ANALYSIS OF THE CHARACTERISTIC EQUATION FOR $\tau = 0$

Based on the previous analysis we can establish that the stability criteria [37] can be established when:

$$\begin{aligned} P_{1vw} &> 0 \\ P_{2vw} &> 0 \end{aligned} \tag{31}$$

This completes the proof of Proposition 1.

STABILITY ANALYSIS OF THE CHARACTERISTIC EQUATION FOR $\tau = 0$ (SPECIAL CASE: $k_1 = k_2 = k$ AND $c_1 = c_2 = c$)

Considering the special case when: $k_1 = k_2 = k$ and $c_1 = c_2 = c$ we have:

$$x_1^* = -\frac{1}{4}\frac{-2c - \lambda + k + \sqrt{A_{apx}}}{k} \tag{32}$$

$$x_2^* = -\frac{1}{4}\frac{-2c - \lambda + k + \sqrt{A_{apx}}}{k} \tag{33}$$

where: $A_{apx} = c^2 + 2c\,(c + textlambda - k) + 8ck + (c + textlambda - k)^2$

In this case, the characteristic equation is given by:

$$P_{vw}(\psi,\tau) = \Psi^2 + P_{1vw}\Psi + P_{2vw} + P_{3vw}e^{-2\psi\tau} + P_{4vw}\Psi e^{-\psi\tau} + P_{5vw}e^{-\psi\tau} \tag{34}$$

where:

$$\begin{aligned}
P_{1vw} &= c + \frac{3}{2}\lambda + \frac{1}{2}\sqrt{E_{apx}} + \frac{k}{2} \\
P_{2vw} &= \frac{1}{2}k\lambda + c\lambda + \frac{1}{2}\lambda^2 + \frac{1}{2}\sqrt{E_{apx}} \\
P_{3vw} &= 2c^2 + 2c\lambda - \left(c - \frac{1}{2}\lambda - \frac{1}{2}k\right)\left(\sqrt{E_{apx}}\right) + \frac{1}{2}\lambda^2 \\
&\quad + \frac{1}{2}k^2 + 2ck \\
P_{4vw} &= \sqrt{E_{apx}} - 2c - k - \lambda \\
P_{5vw} &= -c\lambda - \frac{3}{2}k\lambda - \frac{1}{2}\lambda^2 + \frac{1}{2}\lambda\sqrt{E_{apx}}
\end{aligned}$$

where: $E_{apx} = 4c^2 + 4ck + 4c\lambda + k^2 - 2k\lambda + \lambda^2$

In this case, the characteristic equation is given by:

$$P_{vw}(\psi,\tau) = \Psi^2 + P_{6vw}\Psi + P_{7vw}$$

where:

$$\begin{aligned} P_{6vw} &= P_{1vw} + P_{4vw} \\ P_{7vw} &= P_{2vw} + P_{3vw} + P_{5vw} \end{aligned}$$

Then, the stability criteria [37] can be established when

$$\begin{aligned} P_{6vw} &> 0 \\ P_{7vw} &> 0 \end{aligned} \tag{35}$$

This completes the proof of Corollary 1.

STABILITY ANALYSIS OF THE CHARACTERISTIC EQUATION FOR $\tau > 0$ (SPECIAL CASE: $k_1 = k_2 = k$ AND $c_1 = c_2 = c$)

Returning to the analysis when $\tau > 0$, in this case the characteristic equation [36] is given by equation (34). Substituting $\Psi = iw$ in (34) yields:

$$P_{vw}(i\omega,\tau) = (i\omega)^2 + P_{1vw}(i\omega) + P_{2vw} + P_{3vw}e^{-2(i\omega)\tau} + P_{4vw}(i\omega)e^{-(i\omega)\tau} + P_{5vw}e^{-(i\omega)\tau} \tag{36}$$

Then, separating the real and imaginary parts, we have:

$$P_{3vw}\cos(2\omega\tau) + \omega P_{4vw}\sin(\omega\tau) = \omega^2 - P_{2vw} - P_{5vw}\cos(\omega\tau) \tag{37}$$

$$-P_{3vw}\sin(2\omega\tau) + \omega P_{4vw}\cos(\omega\tau) = -P_{1vw}\omega + P_{5vw}\sin(\omega\tau) \tag{38}$$

Solving equations (37) and (38), we obtain:

$$\begin{aligned}
0 &= -P_{4vw}\omega^3 + P_{4vw}^2\sin(\omega\tau)\,\omega^2 + P_{4vw}P_{2vw}w \\
&\quad + P_{4vw}P_{3vw}\cos(2\omega\tau)\,\omega - P_{1vw}P_{5vw}\omega \\
&\quad + P_{5vw}^2\sin(\omega\tau) + P_{3vw}P_{5vw}\sin(2\omega\tau)
\end{aligned} \tag{39}$$

Now, after rearrangement, the characteristic equation becomes:

$$P_{vw}(\psi,\tau) = \Psi^2 + P_{1vw}\Psi + P_{2vw} + e^{-\psi\tau}\left(P_{4vw}\Psi + P_{3vw}e^{-\psi\tau} + P_{5vw}\right) \tag{40}$$

Then, the second necessary condition for the existence of a Hopf Bifurcation [36] is formulated as:

$$\Re\left(\frac{d\lambda}{d\tau}\right) \neq 0 \tag{41}$$

Now, calculating $\left(\frac{d\lambda}{d\tau}\right)$ from (40), we get:

$$\left(\frac{d\lambda}{d\tau}\right) = \frac{E_{vw} + F_{vw}i}{G_{vw} + H_{vw}i} \tag{42}$$

where:

$$E_{vw} = -w\left(4P_{3vw}\cos^2(\omega\tau) + P_{4vw}\omega\sin(\omega\tau) + P_{5vw}\cos(\omega\tau)\right) + 2\omega P_{3vw}$$
$$F_{vw} = -w\left(4P_{3vw}\cos(\omega\tau)\sin(\omega\tau) - \cos(\omega\tau)P_{4vw}\omega\right) - \omega P_{5vw}\sin(\omega\tau)$$
$$G_{vw} = -4P_{3vw}\tau\cos(\omega\tau)\sin(\omega\tau) + \tau P_{4vw}\omega\cos(\omega\tau) - \tau P_{5vw}\sin(\omega\tau) + P_{4vw}\sin(\omega\tau) - 2w$$
$$H_{vw} = 4P_{3vw}\tau\cos^2(\omega\tau) + P_{4vw}\tau\omega\sin(\omega\tau) + P_{5vw}\tau\cos(\omega\tau) - P_{4vw}\cos(\omega\tau) - 2P_{3vw}\tau - P_{1vw}$$

Therefore:

$$\Re\left(\frac{d\lambda}{d\tau}\right) = \frac{E_{vw}G_{vw} + F_{vw}H_{vw}}{G_{vw}^2 + H_{vw}^2} \neq 0 \tag{43}$$

Thus, from the previous analysis, Corollary 2 is proved.

APPENDIX B
STABILITY ANALYSIS FOR VIDALE-WOLFE-DEAL MODEL WITH ADOPTION DELAY

Assuming that $\tau_1 = \tau_2 = \tau$, the Jacobian matrix with respect to equilibrium point for model (13) is given by:

$$J_{vw} = \begin{bmatrix} A_{vw} & B_{vw} \\ C_{vw} & D_{vw} \end{bmatrix} \tag{44}$$

where:

$$A_{vw} = -\lambda + e^{-\psi\tau}\left(-2x_{1\tau}k_1 - x_{2\tau}k_1 - c_1 + k_1\right)$$
$$B_{vw} = e^{-\psi\tau}\left(-x_{1\tau}k_1 - c_1\right)$$
$$C_{vw} = e^{-\psi\tau}\left(-x_{2\tau}k_2 - c_2\right)$$
$$D_{vw} = -\lambda + e^{-\psi\tau}\left(-x_{1\tau}k_2 - 2x_{2\tau}k_2 - c_2 + k_2\right)$$

At equilibrium it must hold that:

$$\left(x_1^*(t), x_2^*(t)\right) = \left(x_1^*(t-\tau), x_2^*(t-\tau)\right) \tag{45}$$

Stability of equilibrium points is determined by the characteristic equation [36]:

$$P_{vw}(\psi,\tau) = \Psi^2 + P_{1vw}\Psi + P_{2vw} + P_{3vw}e^{-2\psi\tau} + P_{4vw}\Psi e^{-\psi\tau} + P_{5vw}e^{-\psi\tau} \tag{46}$$

where:

$$P_{1vw} = 2\lambda^2$$
$$P_{2vw} = \lambda^2$$
$$P_{3vw} = 2k_1k_2x_1^2 + 4k_1k_2x_1x_2 + 2k_1k_2x_2^2 + c_1k_2x_1 + c_1k_2x_2 + c_2k_1x_1 + c_2k_1x_2 - 3k_1x_2 - 3k_1k_2x_2 - c_1k_2 - c_2k_1 + k_1k_2$$
$$P_{4vw} = 2k_1x_1 + k_1x_2 + k_2x_1 + 2k_2x_2 + c_1 + c_2 - k_1 - k_2$$
$$P_{5vw} = 2k_1\lambda x_1 + k_1\lambda x_2 + k_2\lambda x_1 + 2k_2\lambda x_2 + c_1\lambda + c_2\lambda - k_1\lambda - k_2\lambda$$

STABILITY ANALYSIS OF THE CHARACTERISTIC EQUATION FOR $\tau = 0$

Based on the previous analysis we can establish that the stability criteria [37] can be established when:

$$P_{1vw} > 0$$
$$P_{2vw} > 0 \tag{47}$$

This completes the proof of Proposition 2.

STABILITY ANALYSIS OF THE CHARACTERISTIC EQUATION FOR $\tau = 0$ (SPECIAL CASE: $k_1 = k_2 = k$ AND $c_1 = c_2 = c$)

Considering the special case when: $k_1 = k_2 = k$ and $c_1 = c_2 = c$, we obtain:

$$x_1^* = -\frac{1}{4}\frac{-2c - \lambda + k + \sqrt{B_{apx}}}{k} \tag{48}$$
$$x_2^* = -\frac{1}{4}\frac{-2c - \lambda + k + \sqrt{B_{apx}}}{k} \tag{49}$$

where: $B_{apx} = c^2 + 2c(c + \lambda - k) + 8ck + (c + \lambda - k)^2$

Therefore, the characteristic equation is given by:

$$P_{vw}(\psi,\tau) = \Psi^2 + P_{1vw}\Psi + P_{2vw} + P_{3vw}e^{-2\psi\tau} + P_{4vw}\Psi e^{-\psi\tau} + P_{5vw}e^{-\psi\tau} \tag{50}$$

where:

$$P_{1vw} = 2\lambda^2$$
$$P_{2vw} = \lambda^2$$
$$P_{3vw} = -\left(c - \lambda - \frac{1}{2}k\right)\left(\sqrt{E_{apx}}\right) + 2c^2 + 3c\lambda + \lambda^2 - \frac{1}{2}k\lambda + \frac{1}{2}k^2 + 2ck$$
$$P_{4vw} = -c - \frac{3}{2} - \frac{1}{2}k + \frac{3}{2}\sqrt{E_{apx}}$$
$$P_{5vw} = -c\lambda - \frac{3}{2}\lambda^2 - \frac{1}{2}k\lambda + \frac{3}{2}\lambda\sqrt{E_{apx}}$$

where: $E_{apx} = 4c^2 + 4ck + 4c\lambda + k^2 - 2k\lambda + \lambda^2$

In this case, the characteristic equation is given by:

$$P_{vw}(\psi,\tau) = \Psi^2 + P_{6vw}\Psi + P_{7vw} \tag{51}$$

where:

$$P_{6vw} = P_{1vw} + P_{4vw}$$
$$P_{7vw} = P_{2vw} + P_{3vw} + P_{5vw}$$

Stability conditions [36] are as follows:

$$P_{6vw} > 0$$
$$P_{7vw} > 0 \tag{52}$$

Completing the proof of Corollary 3

STABILITY ANALYSIS OF THE CHARACTERISTIC EQUATION FOR $\tau > 0$ (SPECIAL CASE: $k_1 = k_2 = k$ AND $c_1 = c_2 = c$)

In this case, the characteristic equation is given by equation (50). Substituting $\Psi = iw$ and in (50), we obtain:

$$P_{vw}(i\omega,\tau) = (i\omega)^2 + P_{1vw}(i\omega) + P_{2vw} + P_{3vw}e^{-2(i\omega)\tau} + P_{4vw}(i\omega)e^{-(i\omega)\tau} + P_{5vw}e^{-(i\omega)\tau} \quad (53)$$

Then, separating the real and imaginary parts, we have:

$$P_{3vw}\cos(2\omega\tau) + \omega P_{4vw}\sin(\omega\tau) = \omega^2 - P_{2vw} - P_{5vw}\cos(\omega\tau) \quad (54)$$

$$-P_{3vw}\sin(2\omega\tau) + \omega P_{4vw}\cos(\omega\tau) = -P_{1vw}\omega + P_{5vw}\sin(\omega\tau) \quad (55)$$

Solving equations (54) and (55), we get:

$$0 = -P_{4vw}\omega^3 + P_{4vw}^2\sin(\omega\tau)\,\omega^2 + P_{4vw}P_{2vw}w + P_{4vw}P_{3vw}\cos(2\omega\tau)\,\omega - P_{1vw}P_{5vw}\omega + P_{5vw}^2\sin(\omega\tau) + P_{3vw}P_{5vw}\sin(2\omega\tau) \quad (56)$$

After rearrangement, the characteristic equation becomes:

$$P_{vw}(\psi,\tau) = \Psi^2 + P_{1vw}\Psi + P_{2vw} + e^{-\psi\tau}\left(P_{4vw}\Psi + P_{3vw}e^{-\psi\tau} + P_{5vw}\right) \quad (57)$$

The second necessary condition for Hopf Bifurcation existence [36] is:

$$\Re\left(\frac{d\lambda}{d\tau}\right) \neq 0 \quad (58)$$

Now, calculating $\left(\frac{d\lambda}{d\tau}\right)$ from (57) we obtain:

$$\left(\frac{d\lambda}{d\tau}\right) = \frac{A_{vw} + B_{vw}i}{C_{vw} + D_{vw}i} \quad (59)$$

where:

$$\begin{aligned}
A_{vw} &= -w\Big(4P_{3vw}\cos^2(\omega\tau) + P_{4vw}\omega\sin(\omega\tau) \\
&\quad + P_{5vw}\cos(\omega\tau)\Big) + 2\omega P_{3vw} \\
B_{vw} &= -w\left(4P_{3vw}\cos(\omega\tau)\sin(\omega\tau) - \cos(\omega\tau)P_{4vw}\omega\right) \\
&\quad + \omega P_{5vw}\sin(\omega\tau) \\
C_{vw} &= -4P_{3vw}\tau\cos(\omega\tau)\sin(\omega\tau) + \tau P_{4vw}\omega\cos(\omega\tau) \\
&\quad - \tau P_{5vw}\sin(\omega\tau) + P_{4vw}\sin(\omega\tau) - 2w \\
D_{vw} &= 4P_{3vw}\tau\cos^2(\omega\tau) + P_{4vw}\tau\omega\sin(\omega\tau) \\
&\quad + P_{5vw}\tau\cos(\omega\tau) - P_{4vw}\cos(\omega\tau) - 2P_{3vw}\tau - P_{1vw}
\end{aligned}$$

Therefore:

$$\Re\left(\frac{d\lambda}{d\tau}\right) = \frac{A_{vw}C_{vw} + B_{vw}D_{vw}}{C_{vw}^2 + D_{vw}^2} \neq 0 \quad (60)$$

This establishes Corollary 4.

APPENDIX C
STABILITY ANALYSIS FOR EXTENDED LANCHESTER MODEL WITH IMPLEMENTATION DELAY

Assuming that $\tau_1 = \tau_2 = \tau$, the Jacobian matrix for model (17) is given by:

$$J_{el} = \begin{bmatrix} A_{el} & B_{el} \\ C_{el} & D_{el} \end{bmatrix} \quad (61)$$

where:

$$\begin{aligned}
A_{el} &= -k_1x_{1\tau} - k_2x_{2\tau} - c_1 - c_2 - \lambda + e^{-\psi\tau}(-k_1x_1 + k_1) \\
B_{el} &= -e^{-\psi\tau}(k_2x_1) \\
C_{el} &= -e^{-\psi\tau}(k_1x_2) \\
D_{el} &= -k_1x_{1\tau} - k_2x_{2\tau} - c_1 - c_2 - \lambda + e^{-\psi\tau}(-k_2x_2 + k_2)
\end{aligned}$$

At equilibrium:

$$\left(x_1^*(t), x_2^*(t)\right) = \left(x_1^*(t-\tau), x_2^*(t-\tau)\right) \quad (62)$$

Stability of equilibrium points is determined by the characteristic equation:

$$\Psi^2 + P_{1el}\Psi + P_{2el} + P_{3el}e^{-2\psi\tau} + P_{4el}\Psi e^{-\psi\tau} + P_{5el}e^{-\psi\tau} \quad (63)$$

where:

$$\begin{aligned}
P_{1el} &= 2k_1x_1 + 2k_2x_2 + 2c_1 + 2c_2 + 2\lambda \\
P_{2el} &= k_1^2x_1^2 + 2k_1k_2x_1x_2 + k_2^2x_2^2 + 2c_1k_1x_1 + 2c_1k_2x_2 \\
&\quad + 2c_2k_1x_1 + 2c_2k_2x_2 + 2k_1\lambda x_1 + 2k_2\lambda x_2 + c_1^2 \\
&\quad + 2c_1c_2 + 2c_1\lambda + c_2^2 + 2c_2\lambda + \lambda^2 \\
P_{3el} &= -k_1k_2x_1 - k_1k_2x_2 + k_1k_2 \\
P_{4el} &= k_1x_1 + k_2x_2 - k_1 - k_2 \\
P_{5el} &= k_1x_1^2 + 2k_1k_2x_1x_2 + k_2x_2^2 + c_1k_1x_1 + c_1k_2x_2 \\
&\quad + c_2k_1x_1 + c_2k_2x_2 - k_1^2x_1 - k_1k_2x_1 - k_1k_2x_2 \\
&\quad + k_1\lambda x_1 - k_2^2x_2 + k_2\lambda x_2 - c_1k_1 - c_1k_2 \\
&\quad - c_2k_1 - c_2k_2 - k_1\lambda - k_2\lambda
\end{aligned}$$

STABILITY ANALYSIS OF THE CHARACTERISTIC EQUATION FOR $\tau = 0$

Based on the previous analysis we can establish that the stability criteria [37] can be established when:

$$\begin{aligned}
P_{1el} &> 0 \\
P_{2el} &> 0
\end{aligned} \quad (64)$$

This completes the proof of Proposition 3.

STABILITY ANALYSIS OF THE CHARACTERISTIC EQUATION FOR $\tau = 0$ (SPECIAL CASE: $k_1 = k_2 = k$ AND $c_1 = c_2 = c$)

Considering the special case when: $k_1 = k_2 = k$ and $c_1 = c_2 = c$ we have:

$$x_1^* = -\frac{1}{4}\frac{-2c - \lambda + k + \sqrt{C_{apx}}}{k} \quad (65)$$

$$x_2^* = -\frac{1}{4}\frac{-2c - \lambda + k + \sqrt{C_{apx}}}{k} \quad (66)$$

where: $C_{apx} = c^2 + 2c(c + textlambda - k) + 8ck + (c + textlambda - k)^2$

In this case, the characteristic equation is given by:

$$P_{el}(\psi, \tau) = \Psi^2 + P_{1el}\Psi + P_{2el} + P_{3el}e^{-2\psi\tau} + P_{4el}\Psi e^{-\psi\tau} + P_{5el}e^{-\psi\tau} \quad (67)$$

where:

$$\begin{aligned} P_{1el} &= 2c + k + \lambda + \sqrt{E_{apx}} \\ P_{2el} &= 2c\lambda + \frac{1}{2}\lambda^2 + \frac{1}{2}k^2 + 2c^2 + 2ck + \left(c + \frac{1}{2}\lambda + \frac{1}{2}k\right)\left(\sqrt{E_{apx}}\right) \\ P_{3el} &= ck + \frac{1}{2}k\lambda + \frac{1}{2}k^2 - \frac{1}{2}k\sqrt{E_{apx}} \\ P_{4el} &= -c - \frac{1}{2}\lambda - \frac{3}{2}k + \frac{1}{2}\sqrt{E_{apx}} \\ P_{5el} &= -\frac{1}{2}k^2 - ck - \frac{3}{2}k\lambda - \frac{1}{2}k\sqrt{E_{apx}} \end{aligned}$$

where: $E_{apx} = 4c^2 + 4ck + 4c\lambda + k^2 - 2k\lambda + \lambda^2$

In this case, the characteristic equation is given by:

$$P_{el}(\psi, \tau) = \Psi^2 + P_{6el}\Psi + P_{7el} \quad (68)$$

where:

$$\begin{aligned} P_{6el} &= P_{1el} + P_{4el} \\ P_{7el} &= P_{2el} + P_{3el} + P_{5el} \end{aligned}$$

Thus, stability conditions are given by:

$$\begin{aligned} P_{6el} &> 0 \\ P_{7el} &> 0 \end{aligned} \quad (69)$$

This completes the proof of Corollary 5

STABILITY ANALYSIS OF THE CHARACTERISTIC EQUATION FOR $\tau > 0$ (SPECIAL CASE: $k_1 = k_2 = k$ AND $c_1 = c_2 = c$)

In this case, the characteristic equation is given by equation (67). Substituting $\Psi = iw$ in (67), we obtain:

$$P_{el}(i\omega, \tau) = (i\omega)^2 + P_{el}(i\omega) + P_{2el} + P_{3el}e^{-2(i\omega)\tau} + P_{4el}(i\omega)e^{-(i\omega)\tau} + P_{5el}e^{-(i\omega)\tau} \quad (70)$$

Solving the equation and next separating the real and imaginary parts, we get:

$$P_{3el}\cos(2\omega\tau) + \omega P_{4el}\sin(\omega\tau) = \omega^2 - P_{2el} - P_{5el}\cos(\omega\tau) \quad (71)$$

$$-P_{3el}\sin(2\omega\tau) + \omega P_{4el}\cos(\omega\tau) = -P_{1el}\omega + P_{5el}\sin(\omega\tau) \quad (72)$$

Solving equations (71) and (72), we have:

$$\begin{aligned} 0 = &-P_{4el}\omega^3 + P_{4el}^2\sin(\omega\tau)\,\omega^2 + P_{4el}P_{2el}\omega \\ &+ P_{4el}P_{3el}\cos(2\omega\tau)\,\omega - P_{1el}P_{5el}\omega \\ &+ P_{5el}^2\sin(\omega\tau) + P_{3el}P_{5el}\sin(2\omega\tau) \end{aligned} \quad (73)$$

Rearranging the characteristic equation yields:

$$P_{el}(\psi, \tau) = \Psi^2 + P_{1el}\Psi + P_{2el} + e^{-\psi\tau}\left(P_{4el}\Psi + P_{3el}e^{-\psi\tau} + P_{5el}\right) \quad (74)$$

The second necessary condition for Hopf Bifurcation existence [36] is:

$$\Re\left(\frac{d\lambda}{d\tau}\right) \neq 0 \quad (75)$$

Now, calculating $\left(\frac{d\lambda}{d\tau}\right)$ from (74) we get:

$$\left(\frac{d\lambda}{d\tau}\right) = \frac{A_{el} + B_{el}i}{C_{el} + D_{el}i} \quad (76)$$

where:

$$\begin{aligned} A_{el} &= -w\Big(4P_{3el}\cos^2(\omega\tau) + P_{4el}\omega\sin(\omega\tau) + P_{5el}\cos(\omega\tau)\Big) + 2\omega P_{3el} \\ B_{el} &= -w\left(4P_{3el}\cos(\omega\tau)\sin(\omega\tau) - \cos(\omega\tau)P_{4el}\omega\right) - \omega P_{5el}\sin(\omega\tau) \\ C_{el} &= -4P_{3el}\tau\cos(\omega\tau)\sin(\omega\tau) + \tau P_{4el}\omega\cos(\omega\tau) - \tau P_{5el}\sin(\omega\tau) + P_{4el}\sin(\omega\tau) - 2w \\ D_{el} &= 4P_{3el}\tau\cos^2(\omega\tau) + P_{4el}\tau\omega\sin(\omega\tau) + P_{5el}\tau\cos(\omega\tau) - P_{4el}\cos(\omega\tau) - 2P_{3el}\tau - P_{1el} \end{aligned}$$

Therefore:

$$\Re\left(\frac{d\lambda}{d\tau}\right) = \frac{A_{el}C_{el} + B_{el}D_{el}}{C_{el}^2 + D_{el}^2} \neq 0 \quad (77)$$

This establishes Corollary 6.

APPENDIX D
STABILITY ANALYSIS FOR EXTENDED LANCHASTER MODEL WITH ADOPTION DELAY

Once again, assuming that $\tau_1 = \tau_2 = \tau$, the Jacobian matrix for model (21) is given by:

$$J_{el} = \begin{bmatrix} A_{el} & B_{el} \\ C_{el} & D_{el} \end{bmatrix} \quad (78)$$

where:

$$\begin{aligned} A_{el} &= -\lambda + e^{-\psi\tau}\left(-2x_{1\tau}k_1 - x_{2\tau}k_2 - c_1 - c_2 + k_1\right) \\ B_{el} &= -e^{-\psi\tau}x_{1\tau}k_2 \\ C_{el} &= -e^{-\psi\tau}x_{2\tau}k_1 \\ D_{el} &= -\lambda + e^{-\psi\tau}\left(2x_{1\tau}k_1 - 2x_{2\tau}k_2 - c_1 - c_2 + k_2\right) \end{aligned}$$

At equilibrium it must hold that:

$$\left(x_1^*(t), x_2^*(t)\right) = \left(x_1^*(t-\tau), x_2^*(t-\tau)\right) \quad (79)$$

Stability of equilibrium points is determined by the characteristic equation:

$$P_{el}(\psi, \tau) = \Psi^2 + P_{1el}\Psi + P_{2el} + P_{3vw}e^{-2\psi\tau} + P_{4el}\Psi e^{-\psi\tau} + P_{5el}e^{-\psi\tau} \quad (80)$$

where:

$$\begin{aligned} P_{1el} &= 2\lambda \\ P_{2el} &= \lambda^2 \\ P_{3el} &= 2k_1^2x_1^2 + 4k_1k_2x_1x_2 + 2k_2x_2^2 + 3c_1k_1x_1 + 3c_1k_2x_2 \\ &\quad + 3c_2k_1x_1 + 3c_2k_2x_2 - k_1^2x_1 - 2k_1k_2x_1 - 2k_1k_2x_2 \\ &\quad - k_2^2x_2 + c_1^2 + 2c_1c_2 - c_1k_1 - c_1k_2 + c_2^2 - c_2k_1 \\ &\quad - c_2k_2 + k_1k_2 \\ P_{4el} &= 3k_1x_1 + 3k_2x_2 + 2c_1 + 2c_2 - k_1 - k_2 \\ P_{5el} &= 3k_1\lambda x_1 + 3k_2\lambda x_2 + 2c_1\lambda + 2c_2\lambda - k_1\lambda - k_2\lambda \end{aligned}$$

STABILITY ANALYSIS OF THE CHARACTERISTIC EQUATION FOR $\tau = 0$

Based on the previous analysis we can establish that the stability criteria [37] can be established when:

$$\begin{aligned} P_{1el} &> 0 \\ P_{2el} &> 0 \end{aligned} \tag{81}$$

This completes the proof of Proposition 4.

STABILITY ANALYSIS OF THE CHARACTERISTIC EQUATION FOR $\tau = 0$ (SPECIAL CASE: $k_1 = k_2 = k$ AND $c_1 = c_2 = c$)

Considering the special case when: $k_1 = k_2 = k$ and $c_1 = c_2 = c$ we obtain:

$$x_1^* = -\frac{1}{4}\frac{-2c - \lambda + k + \sqrt{D_{apx}}}{k} \tag{82}$$

$$x_2^* = -\frac{1}{4}\frac{-2c - \lambda + k + \sqrt{D_{apx}}}{k} \tag{83}$$

where: $D_{apx} = c^2 + 2c\,(c + textlambda - k) + 8ck + (c + textlambda - k)^2$

In this case, the characteristic equation is given by:

$$P_{el}(\psi, \tau) = \Psi^2 + P_{1el}\Psi + P_{2el} + P_{3el}e^{-2\psi\tau} + P_{4el}\Psi e^{-\psi\tau} + P_{5el}e^{-\psi\tau} \tag{84}$$

where:

$$\begin{aligned} P_{1el} &= 2\lambda \\ P_{2el} &= \lambda^2 \\ P_{3el} &= -\left(c - \lambda - \frac{1}{2}k\right)\left(\sqrt{E_{apx}}\right) + 2c^2 + c\lambda + \lambda^2 \\ &\quad - \frac{1}{2}k\lambda + \frac{1}{2}k^2 + 2ck \\ P_{4el} &= c - \frac{3}{2}\lambda - \frac{1}{2}k + \frac{3}{2}\sqrt{E_{apx}} \\ P_{5el} &= c\lambda - \frac{3}{2}\lambda^2 - \frac{1}{2}k\lambda + \frac{3}{2}\lambda\sqrt{E_{apx}} \end{aligned}$$

where: $E_{apx} = 4c^2 + 4ck + 4c\lambda + k^2 - 2k\lambda + \lambda^2$

In this case, the characteristic equation is given by:

$$P_{el}(\psi, \tau) = \Psi^2 + P_{6el}\Psi + P_{7el} \tag{85}$$

where:

$$\begin{aligned} P_{6el} &= P_{1el} + P_{4el} \\ P_{7el} &= P_{2el} + P_{3el} + P_{5el} \end{aligned}$$

Thus the stability is established when:

$$\begin{aligned} P_{6el} &> 0 \\ P_{7el} &> 0 \end{aligned} \tag{86}$$

This completes the proof of Corollary 7

STABILITY ANALYSIS OF THE CHARACTERISTIC EQUATION FOR $\tau > 0$ (SPECIAL CASE: $k_1 = k_2 = k$ AND $c_1 = c_2 = c$)

In this case, the characteristic equation is given by equation (84). Substituting $\Psi = iw$ in (84), we get:

$$P_{el}(i\omega, \tau) = (i\omega)^2 + P_{1el}\,(i\omega) + P_{2el} + P_{3el}e^{-2(i\omega)\tau} + P_{4el}(i\omega)e^{-(i\omega)\tau} + P_{5el}e^{-(i\omega)\tau} \tag{87}$$

Then, separating the real and imaginary parts, we have:

$$P_{3el}\cos(2\omega\tau) + \omega P_{4el}\sin(\omega\tau) = \omega^2 - P_{2el} - P_{5el}\cos(\omega\tau) \tag{88}$$

$$-P_{3el}\sin(2\omega\tau) + \omega P_{4el}\cos(\omega\tau) = -P_{1el}\omega + P_{5el}\sin(\omega\tau) \tag{89}$$

Solving equations (88) and (89), we obtain:

$$\begin{aligned} 0 &= -P_{4el}\omega^3 + P_{4el}^2\sin(\omega\tau)\,\omega^2 + P_{4el}P_{2el}\omega \\ &\quad + P_{4el}P_{3el}\cos(2\omega\tau)\,\omega - P_{1el}P_{5el}\omega + P_{5el}^2\sin(\omega\tau) \\ &\quad + P_{3el}P_{5el}\sin(2\omega\tau) \end{aligned} \tag{90}$$

Now, rearranging the characteristic equation we get:

$$P_{el}(\psi, \tau) = \Psi^2 + P_{1el}\Psi + P_{2el} + e^{-\psi\tau}\left(P_{4el}\Psi + P_{3el}e^{-\psi\tau} + P_{5el}\right) \tag{91}$$

The second necessary condition for the existence of a Hopf bifurcation [36] is:

$$\Re\left(\frac{d\lambda}{d\tau}\right) \neq 0 \tag{92}$$

Now, calculating $\left(\frac{d\lambda}{d\tau}\right)$ from (91) we obtain:

$$\left(\frac{d\lambda}{d\tau}\right) = \frac{A_{el} + B_{el}i}{C_{el} + D_{el}i} \tag{93}$$

where:

$$\begin{aligned} A_{el} &= -w\Big(4P_{3el}\cos^2(\omega\tau) + P_{4el}\omega\sin(\omega\tau) \\ &\quad + P_{5el}\cos(\omega\tau)\Big) + 2\omega P_{3el} \\ B_{el} &= -w\,(4P_{3el}\cos(\omega\tau)\sin(\omega\tau) - \cos(\omega\tau)\,P_{4el}\omega \\ &\quad + P_{5el}\sin(\omega\tau)\,) \\ C_{el} &= -4P_{3el}\tau\cos(\omega\tau)\sin(\omega\tau) + \tau P_{4el}\omega\cos(\omega\tau) \\ &\quad - \tau P_{5el}\sin(\omega\tau) + P_{4el}\sin(\omega\tau) - 2w \\ D_{el} &= 4P_{3el}\tau\cos^2(\omega\tau) + P_{4el}\tau\omega\sin(\omega\tau) + P_{5el}\tau\cos(\omega\tau) \\ &\quad - P_{4el}\cos(\omega\tau) - 2P_{3el}\tau - P_{1el} \end{aligned}$$

Therefore:

$$\Re\left(\frac{d\lambda}{d\tau}\right) = \frac{A_{el}C_{el} + B_{el}D_{el}}{C_{el}^2 + D_{el}^2} \neq 0 \tag{94}$$

This concludes the proof of Corollary 8

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