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Relaxed Stabilization Conditions for TS Fuzzy Systems With Optimal Upper Bounds for the Time Derivative of Fuzzy Lyapunov Functions

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ABSTRACT This paper initially proposes an optimization problem and after presents its optimal solution. Then, this result is applied to obtain relaxed conditions to design controllers for nonlinear plants described by Takagi-Sugeno (TS) models, based on fuzzy Lyapunov function (FLF) and Linear Matrix Inequalities (LMI). The FLF is given by $V(x(t)) = x(t)^T P(\alpha(x(t)))x(t)$, where $x(t)$ is the plant state vector, $P(\alpha(x(t))) = \alpha_1(x(t))P_1 + \alpha_2(x(t))P_2 + \cdots + \alpha_r(x(t))P_r$, $P_i = P_i^T > 0$ and $\alpha_i(x(t))$ is the weight related to the local model i in the representation of the plant by TS fuzzy models, for $i = 1, 2, \cdots, r$. When one calculates the time derivative of this $V(x(t))$, it appears the term $x(t)^T \dot{P}(\alpha(x(t)))x(t)$, that is usually handled using conservative upper bounds, supposing that the bounds of the time derivative of $\alpha_i(x(t))$, $i = 1, 2, \cdots, r$, are available. The main result of this paper is a procedure to obtain optimal upper bounds for the term $x(t)^T \dot{P}(\alpha(x(t)))x(t)$, such that they contemplate the maximum value and are always smaller than or equal to the maximum value. It is a relevant result on this subject, because these optimal upper bounds do not add any constraint. With these optimal upper bounds, a relaxed design method for stabilization of TS fuzzy models is proposed. Two numerical examples illustrate the effectiveness of this procedure.

INDEX TERMS Fuzzy Lyapunov function (FLF), Takagi-Sugeno (TS) fuzzy systems, linear matrix inequalities (LMIs), fuzzy control, stability, stabilization.

I. INTRODUCTION

In the real and non-idealized world, most of dynamic systems found have a nonlinear nature. Such characteristic complicates these systems analysis, but now there are a lot of tools for control design and analysis of nonlinear systems [1], [2].

Besides the system analysis theory, another important area in the last decades was the description of systems, using fuzzy logic theory [3]. In the specific case of nonlinear control theory, the authors believe that the most important application of fuzzy logic was the Takagi-Sugeno (TS) fuzzy systems [4].

With the evolution of processors, fact that allowed greater data processing in less time, especially in the 90's, new control techniques were adopted [5], offering newer issues on the system's analysis and controller design. Therefore, the creation of a new and powerful tool was necessary: the Linear Matrix Inequalities, or LMIs [6], [7]. The idea of this strategy in the design of controllers was the description of project's condition through LMIs, in which when there is a solution, it can be effectively obtained through convex programming. Therefore, the design of the controllers became a convex optimization problem, subject that has been already widely studied and it is not hard to be solved [8], [9]. The LMIs-based designs of controllers for TS fuzzy models expanded the possibility of finding feasible solutions [10], [11], through relaxed procedures initially presented in [7].

The design of controllers based on LMIs also allows to simultaneously consider that the plant has, for instance,

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uncertain parameters, time delays, input subject to saturation and to specify performance indexes such as $\mathscr{H}_\infty$ and $\mathscr{H}_2$. However, sometimes it is not possible to obtain feasible solutions and then the controllers gains, because the LMI specifications of the plant constraints and of the aforementioned performance indexes are only sufficient and conservative conditions. Therefore, the study of relaxed conditions for designing LMI-based controllers is an important subject in this field, because it offers more general design procedures.

In [12], a fuzzy Lyapunov function approach was proposed, using multiple positive definite matrices P_j, which will be explained later, offering a relaxation in the stability analysis of TS fuzzy systems, when compared with the results based on quadratic Lyapunov functions that use only one positive definite matrix P. This new approach allowed advances on the theory of fuzzy TS systems [13]. However, it also introduced a new issue: the treatment of the time derivatives of the weights that combine the local models in the TS representation of the plant.

Using theories like the operating region [14], [15], it is possible to calculate upper and lower bounds of the time derivatives of these weights. Nowadays, usually the stability analysis and stabilization techniques for TS fuzzy systems still stand on the use of these maximums and minimums, without any bounds adaptation [16]–[18]. Such characteristic of these techniques makes the search for feasibility conservative, once they can consider cases that would never happen, given the main characteristic of the fuzzy Lyapunov function: the sum of the time derivatives of the weights that combine the local models in the TS representation of the plant must be equal to zero.

In this paper, a simple procedure to find optimal upper bounds of a term that appears in the time derivative of fuzzy Lyapunov functions (FLF), in the design of controllers for nonlinear systems described by TS fuzzy models is proposed. These upper bounds are optimal because they contemplate the maximum value of this term, and are always smaller than or equal to this maximum value. It is a relevant result on this subject, because these optimal upper bounds do not add any constraint. With these optimal upper bounds, a relaxed design method for stabilization of TS fuzzy models is proposed. It means that, for this problem, it is not possible to obtain less conservative upper bounds than the proposed in this paper.

In Section II were developed theoretical results that are the basis of the proposed relaxed design conditions for TS fuzzy plants. It was formulated an optimization problem and after are presented details about its optimal solution. Furthermore, in Section III this optimization result is applied to obtain relaxed conditions to design controllers for nonlinear plants described by TS fuzzy models, based on FLF and LMI. Two numerical examples illustrate the effectiveness of this procedure in Section IV and finally, the conclusions are given in Section V.

Throughout this manuscript, define the set $K_r = \{1, 2, \cdots, r\}$ and $M > 0$ ($M < 0, M \geqslant 0, M \leqslant 0$) means the matrix M is positive definite (negative, positive semi-definite, negative semi-definite, respectively). In order to simplify the notation, sometimes the weights $\alpha_i(x(t))$ related to the local model i in the representation of the plant by TS fuzzy models will be represented by α_i, for $i = 1, 2, \cdots, r$.

II. PROBLEM STATEMENT

In this section, an optimization problem regarding a general cost function will be presented along with its solution, providing a better view on the generality of problems with this format. Later, this optimization method will be used to provide relaxed conditions in the design of controllers for nonlinear plants described by TS fuzzy models, using fuzzy Lyapunov functions.

First, consider a set o parameters $h_i \in \mathbb{R}$, $i \in \mathbb{K}_r$, the known real constants $\phi_{i,1}$, $\phi_{i,2}$, C and that the following conditions hold:

$$\phi_{i,1} > 0,\ \phi_{i,2} > 0,\ i \in \mathbb{K}_r,$$
$$-\phi_{i,1} \leqslant h_i \leqslant \phi_{i,2},\ i \in \mathbb{K}_r, \tag{1}$$
$$\sum_{i=1}^{r} h_i = C, \tag{2}$$

with $K_r = \{1, 2, \cdots, r\}$.

In this study, the set of weights h_i, $i \in \mathbb{K}_r$, is such that each h_i, $i \in \mathbb{K}_r$, can assume any value satisfying (1), in a way that the other elements $h_j, j \in \{1, 2, \cdots, i-1, i+1, \cdots, r-1, r\}$ exist satisfying (1), and (2) is respected.

To illustrate better this fact, suppose that for an arbitrary element $i \in \mathbb{K}_r$, $h_i = \phi_{i,2}$. Therefore, there exist h_j, $j \in \{1, 2, \cdots, i-1, i+1, \cdots, r-1, r\}$ satisfying (1), such that the condition (2) is respected:

$$\begin{aligned}\sum_{j=1}^{r} h_j &= h_1 + \cdots + h_{i-1} + h_i + h_{i+1} + \cdots + h_{r-1} + h_r \\ &= h_1 + \cdots + h_{i-1} + \phi_{i,2} + h_{i+1} + \cdots + h_{r-1} + h_r \\ &= C,\ i \in \mathbb{K}_r. \end{aligned} \tag{3}$$

Thus, from (1) $-\phi_{i,1} \leqslant h_i$ for all $i \in \mathbb{K}_r$, as of (3),

$$\begin{aligned} &-\phi_{1,1} - \phi_{2,1} - \cdots - \phi_{i-1,1} + \phi_{i,2} - \phi_{i+1,1} \\ &\quad - \cdots - \phi_{r-1,1} - \phi_{r,1} \leqslant h_1 + h_2 + \cdots + h_{i-1} + \phi_{i,2} \\ &\quad + h_{i+1} + \cdots + h_{r-1} + h_r = C,\ i \in \mathbb{K}_r. \end{aligned} \tag{4}$$

Similarly, for an arbitrary element $i \in \mathbb{K}_r$, suppose that $h_i = -\phi_{i,1}$. Therefore, there exist $h_j, j \in \{1, 2, \cdots, i-1, i+1, \cdots, r-1, r\}$ satisfying (1), such that the condition (2) is respected:

$$\sum_{j=1}^{r} h_j = h_1 + \cdots + h_{i-1} - \phi_{i,1} + h_{i+1} + \cdots + h_r = C. \tag{5}$$

Thus, from (1) $\phi_{i,2} \geqslant h_i$ for all $i \in \mathbb{K}_r$, as of (5),

$$\begin{aligned} &\phi_{1,2} + \phi_{2,2} + \cdots + \phi_{i-1,2} - \phi_{i,1} + \phi_{i+1,2} \\ &\quad + \cdots + \phi_{r-1,2} + \phi_{r,2} \geqslant h_1 + h_2 + \cdots + h_{i-1} - \phi_{i,1} \end{aligned}$$

$$+ h_{i+1} + \cdots + h_{r-1} + h_r = C. \quad (6)$$

Consider now a set of vectors $T \in \mathbb{R}^r$, given by:

$$\upsilon = \{T \in \mathbb{R}^r / T = [T_1 \ T_2 \ \cdots \ T_r]^T, \\ T_i > 0 \text{ for all } i \in \mathbb{K}_r\}. \quad (7)$$

With the definitions and analyzes described before, it is possible to establish the statement of the problem and present its solution.

Remark 1: Let α_i, $i \in \mathbb{K}_r$, be the normalized weights regarding the membership function of each rule of a plant described by a TS fuzzy model. Then, in the application of the results of this section for obtaining relaxed conditions in the design of controllers for this plant, using FLF, it will be considered that $h_i = \dot{\alpha}_i(t)$, $i \in \mathbb{K}_r$, and also that conditions (1), (2), (4) and (6) hold, with $C = 0$. This result will be presented in Theorem 3 of Section III.

A. DEFINITION OF THE PROBLEM AND SOLUTION ANALYSIS

Consider the cost function

$$J(h) = \sum_{i=1}^{r} h_i T_i, \quad (8)$$

defined for all the weights $h_i \in \mathbb{R}$ satisfying (1), (2), (4) and (6), with the variables T_i described in (7).

Define the following Ω set:

$$\Omega = \{h = [h_1 \ h_2 \cdots h_r]^T \in \mathbb{R}^r / h_i, i \in \mathbb{K}_r, \\ \text{satisfy (1), (2), (4) and (6)}\}. \quad (9)$$

Define the optimal weights $h_i^*(T) \in \Omega$, $i \in \mathbb{K}_r$, with T_i and T given in (7), such as

$$\max_{h\in\Omega} J(h) = \sum_{i=1}^{r} h_i^*(T) T_i. \quad (10)$$

As will be seen in Remark 2, for all h_i satisfying (1), (2), (4) and (6), and the variables T_i defined in (7), the optimal weights $h_i^*(T) \in \Omega$ described in (10) are not constant and depend on T.

Thus, assuming that T defined in (7) is not available, the problem is to find optimal upper bounds for $J(h)$ defined in (8), composed by $m \geqslant 1$ sets of r constant weights h_{ji}^*, $j \in \mathbb{K}_m$ and $i \in \mathbb{K}_r$, such that (1), (2),(4) and (6) hold, and using the notation $\bar{h}_j^* = [h_{j_1}^* h_{j_2}^* \cdots h_{j_r}^*]^T \in \Omega$ for $j \in \mathbb{K}_m$, the following conditions are satisfied:

$$\max_{h\in\Omega} J(h) \geqslant J(\bar{h}_j^*) = \sum_{i=1}^{r} h_{ji}^* T_i, \forall j \in \mathbb{K}_m \quad (11)$$

and

$$\max_{h\in\Omega} J(h) = \max_{j\in\mathbb{K}_m} J(\bar{h}_j^*) = \max_{j\in\mathbb{K}_m} \sum_{i=1}^{r} h_{ji}^* T_i. \quad (12)$$

Remark 2: It is important to observe that the optimal weights $h_i^*(T)$ in (10), $i \in \mathbb{K}_r$, depend on the values of the vector T defined in (7). To exemplify this fact, suppose that $T_i = \epsilon_1 > 0$, $T_j = \epsilon_2 > 0$, where i is arbitrary such that $i \in \mathbb{K}_r$ and $j \in \{1, 2, \cdots, i-1, i+1, \cdots, r-1, r\}$. Thus, note that when $\epsilon_1 >> \epsilon_2$, from (8), $J(h)$ will be approximately equal to $h_i^*(T)T_i$ and the maximum of $J(h)$ for all $h \in \Omega$, defined in (9), will be close to $\phi_{i,2}T_i$, because from (1) the maximum value of $h_i^*(T)$ is $\phi_{i,2}$. The analysis made in equations (3) and (4) guarantees the existence of h_j, $j \in \{1, 2, \cdots, i-1, i+1, \cdots, r-1, r\}$ satisfying (1), such that the condition (2) is respected. Now, this analysis works for all $i \in \mathbb{K}_r$. However, if one adopts $h_i^*(T) = \phi_{i,2}$, for all $i \in \mathbb{K}_r$, note that from the condition (2) one should have $\phi_{1,2} + \phi_{2,2} + \cdots + \phi_{r,2} = C$, which is impossible when observing the analysis made in (5) and (6), because $\phi_{i,2} > 0 > -\phi_{i,1}$, and consequently this sum would be greater than C. Thus, in this case, there are not constant values of $h_i^*(T)$, $i \in \mathbb{K}_r$, that satisfy (10), if one considers all possible unknown vectors T defined in (7). This fact shows that the optimal weights $h_i^*(T)$, $i \in \mathbb{K}_r$, depend on the value of the vector T.

As the vector T is considered not to be available, it is not possible to obtain the optimal weights $h_i^*(T)$, $i \in \mathbb{K}_r$, satisfying the equation (10).

In order to find optimal upper bounds for $J(h)$, given by (10) with constant weights $h_{ij}^*(T)$, $i \in \mathbb{K}_r$, $j \in \mathbb{K}_m$, which satisfy (11) and (12), one idea is to split the set υ of vectors T described in (7) in m subsets, defined by υ_j, $j \in \mathbb{K}_m$, in a way that for $T \in \upsilon_j$, the following conditions hold:

$$\max_{\substack{h\in\Omega \\ T\in\upsilon_j}} J(h) = \sum_{i=1}^{r} h_{ij}^* T_i, j \in \mathbb{K}_m, \quad (13)$$

where the weights h_{ij}^*, for all $i \in \mathbb{K}_r$, $j \in \mathbb{K}_m$, are constants, and $\bar{h}_j^* = [h_{j_1}^* \cdots h_{j_r}^*]^T \in \Omega$ for all $j \in \mathbb{K}_m$.

Additionally, if one chooses these sets in a way that $\upsilon = \upsilon_1 \cup \upsilon_2 \cup \cdots \cup \upsilon_m$, then every element of υ belongs to at least one set υ_j, $j \in \mathbb{K}_m$. Hence, one can use Lemma 1 presented below, to obtain an optimal upper bound of $J(h)$, such that (11) and (12) hold.

Lemma 1: Consider the condition described in (14), with $J(h)$ defined in (8), for all $T \in \upsilon$ and $h \in \Omega$. It is always true but will be useful in this lemma:

$$J(h) = \sum_{i=1}^{r} h_i T_i \leqslant \max_{\substack{h\in\Omega \\ T\in\upsilon}} J(h). \quad (14)$$

Now, suppose that

$$\max_{\substack{h\in\Omega \\ T\in\upsilon_j}} J(h) = \sum_{i=1}^{r} h_{ji}^* T_i, \text{ for all } j \in \mathbb{K}_m, \quad (15)$$

where $\bar{h}_j^* = [h_{j_1}^* h_{j_2}^* \cdots h_{j_r}^*]^T \in \Omega$, for all $j \in \mathbb{K}_m$, are constant vectors and $\upsilon = \upsilon_1 \cup \upsilon_2 \cup \cdots \cup \upsilon_m$.

Thus, the condition (14) is equivalent to

$$J(h) \leqslant \sum_{i=1}^{r} h_{ji}^* T_i, \text{ for all } j \in \mathbb{K}_m. \quad (16)$$

Additionally,

$$\underset{\substack{h\in\Omega\\T\in\upsilon}}{max} J(h) = \underset{j\in\mathbb{K}_m}{max} J(\bar{h}_j^*) = \underset{j\in\mathbb{K}_m}{max}\sum_{i=1}^{r} h_{ij}^* T_i. \quad (17)$$

Proof: From (8), (14) and (15) and the fact that $\upsilon = \upsilon_1 \cup \upsilon_2 \cup \cdots \cup \upsilon_m$, thus for all $T \in \upsilon$ and $h \in \Omega$, one has:

$$J(h) \leqslant \underset{\substack{h\in\Omega\\T\in\upsilon}}{max} J(h)$$

$$\underset{\upsilon=\upsilon_1\cup\upsilon_2\cup\cdots\cup\upsilon_m}{\longleftrightarrow} J(h) \leqslant \underset{j\in\mathbb{K}_m}{max}\ \underset{\substack{h\in\Omega\\T\in\upsilon_j}}{max} J(h)$$

$$\underset{(15)}{\longleftrightarrow} J(h) \leqslant \sum_{i=1}^{r} h_{ji}^* T_i, \text{ for all } j \in \mathbb{K}_m. \quad (18)$$

In (18) it was considered the following fact: the condition $z(\omega) \leqslant \max\{z_1(\omega), z_2(\omega), \cdots, z_m(\omega)\}$, where $z(\omega), z_1(\omega), z_2(\omega), \cdots, z_m(\omega) \in \mathbb{R}$ and $\omega = [\omega_1\ \omega_2 \cdots \omega_r]^T \in \mathbb{R}^r$, is equivalent to $z(\omega) \leqslant z_i(\omega)$, for all $i \in \mathbb{K}_m$.

The verification of the condition given in (17) comes from the fact that $\upsilon = \upsilon_1 \cup \upsilon_2 \cup \cdots \cup \upsilon_m$. Therefore, the maximum of $J(h)$ will be the maximum of the maximums for T in $\upsilon_1, \upsilon_2, \cdots, \upsilon_m$, described in (15).

The proof is concluded. ■

Thus, note that the conditions (11) and (12) are also satisfied.

Remark 3: For now on, the problem consists in finding the optimal weights $\bar{h}_j^* = [h_{j1}^* h_{j2}^* \cdots h_{jr}^*]^T \in \Omega$, $j \in \mathbb{K}_m$ and $i \in \mathbb{K}_r$, in a way that for all the possible values of the variables T_i defined in (7) there are $m \geqslant 1$ upper bounds for $J(h)$, represented by $J(\bar{h}_j^*)$, $j \in \mathbb{K}_m$, presented in (11) and (12). These upper bounds are optimal because they never exceed the maximum value of $J(h)$, for all $h \in \Omega$, and contemplate the maximum value of $J(h)$ for all $h \in \Omega$. Lemma 1 shows that these upper bounds offer equivalent conditions for the optimal upper bound stated in (14).

The solution of this problem is important in the design of controllers for nonlinear systems described by Takagi-Sugeno fuzzy models, since the available methods consider conservative upper bounds for $J(h)$. Therefore, the proposed solution will allow the design of controllers for this class of systems to be less conservative. Thus, it is possible to obtain better performance indexes or equal to the currently available ones.

B. SOLUTION OF THE PROBLEM

Analyzing the problem described in Subsection II-A, as well as the analyzes made indicating a possible procedure for the obtaining of a solution using Lemma 1, a first result would be towards the following question: assuming that the variables $T_i > 0$, $i \in \mathbb{K}_r$, are known constants, what would be the optimal weights h_i^*, $i \in \mathbb{K}_r$, that satisfy (10)?

The answer to this question is presented in Algorithm 1, that offers a methodical procedure for the obtaining of these optimal weights h_i^*, $i \in \mathbb{K}_r$, when $T_i > 0$, $i \in \mathbb{K}_r$, are known.

An interesting remark is that Algorithm 1 does not need the explicit knowledge of T_i, $i \in \mathbb{K}_r$, but only the knowledge of the position of these elements when put in decreasing order. For instance, for $r = 3$, suppose that T_1, T_2 e T_3 are such as $T_2 \geqslant T_1 \geqslant T_3$. Thus, this information, considering the knowledge of the real constants $\phi_{i,1}$, $\phi_{i,2}$ and C for $i \in \mathbb{K}_r$ is sufficient for Algorithm 1 to obtain the optimal weights h_1^*, h_2^* and h_3^*, that satisfy the condition (10).

Now, one should remember that T_i, $i \in \mathbb{K}_r$ in the problem described in Section II-A, are unknown variables, which may assume any positive values. One idea to obtain an optimal upper bound for $J(h)$ in (8) is to consider all possible combinations of distinct decreasing orders of T_i, $i \in \mathbb{K}_r$, and apply Algorithm 1 for each one of them. The number of distinct decreasing orders combination of T_i, $i \in \mathbb{K}_r$, is obtained by the permutation of r elements, which is equal to $m = r! = r.(r-1).(r-2).\cdots 2.1$. Define $\bar{h}_j^* = [h_{j1}^* h_{j2}^* \cdots h_{jr}^*]^T$, the vector of optimal weights, which maximize $J(h)$, for the decreasing order j, $j \in \mathbb{K}_m$. That means that for each one of these decreasing orders, defined as j, $j \in \mathbb{K}_m$, then $\underset{h\in\Omega}{max}\ J(h) = J(\bar{h}_j^*)$.

Thus, note that, as for all $T_i > 0$, $i \in \mathbb{K}_r$, one of these combinations will match with the correct decreasing order, because all possible combinations are considered. Therefore, one has:

$$\underset{h\in\Omega}{max}\ J(h) = \underset{j\in\mathbb{K}_m}{max}\ J(\bar{h}_j^*). \quad (19)$$

With this procedure, $m = r!$ upper bounds are obtained from $J(h)$ defined in (8), such that at least one of them will match with $\underset{h\in\Omega}{max}\ J(h)$. Hence, this set of upper bounds can be considered optimal, as it contemplates the $\underset{h\in\Omega}{max}\ J(h)$ and it is never greater than the $\underset{h\in\Omega}{max}\ J(h)$, because from (19), $J(\bar{h}_j^*) \leqslant \underset{h\in\Omega}{max}\ J(h)$, $\forall j \in \mathbb{K}_m$. Thus, this is a solution for the problem described in Subsection II-A.

This procedure can be formalized, using Lemma 1, considering that the set υ_j corresponds to the decreasing order j of T_i, $i \in \mathbb{K}_r$, for all $j \in \mathbb{K}_m$. In that way, as the $m = r!$ combinations of T_i, $i \in \mathbb{K}_r$, contemplate all the possible cases, one has from (7) that $\upsilon = \upsilon_1 \cup \upsilon_2 \cup \cdots \cup \upsilon_m$, and Lemma 1 can be applied to obtain an optimal solution for the problem.

Next an algorithm that offers a methodical procedure for obtaining the optimal weights h_{ij}^*, $i \in \mathbb{K}_r$, for the decreasing order j of T_i, defined by the set υ_j, $j \in \mathbb{K}_m$, is presented, such that condition (15) holds.

C. ALGORITHM 1

Define the set:

$$\Psi_r = \{\omega = [\omega_1\omega_2\cdots\omega_r]^T \in \mathbb{R}^r / \omega_i \in \mathbb{K}_r \text{ for all } i \in \mathbb{K}_r \text{ and } \omega_1 \neq \omega_2 \neq \cdots \neq \omega_r\}. \quad (20)$$

Note that, for any value of the variables $T_i > 0$, $i \in \mathbb{K}_r$, defined in (7), it is always possible to put them in decreasing order, and using the definition (20), to obtain $\omega \in \Psi_r$ such as:

$$T_{\omega_r} \geqslant T_{\omega_{r-1}} \geqslant \cdots \geqslant T_{\omega_2} \geqslant T_{\omega_1}, \omega = [\omega_1 \omega_2 \cdots \omega_r]^T \in \Psi_r. \quad (21)$$

For instance, suppose that $r = 3$ with $T_1 = 4$, $T_2 = 6$ and $T_3 = 5$. Thus, from (21), $\omega_3 = 2$, $\omega_2 = 3$ and $\omega_1 = 1$ in a way that $T_{\omega_3} = T_2 = 6 \geqslant T_{\omega_2} = T_3 = 5 \geqslant T_{\omega_1} = T_1 = 4$ and therefore $\omega = [\omega_1\ \omega_2\ \omega_3]^T = [1\ 3\ 2]^T \in \Psi_3$. Observe that in this case the condition (21) was satisfied.

Define:

$$J(h_\omega) = \sum_{i=1}^{r} h_{\omega_i} T_{\omega_i}. \quad (22)$$

The problem consists of finding a set of optimal weights for $h_\omega = [h_{\omega_1}\ h_{\omega_2}\ \cdots\ h_{\omega_r}]^T \in \Omega$ defined in (9), represented by $h^*_\omega = [h^*_{\omega_1}\ h^*_{\omega_2}\ \cdots\ h^*_{\omega_r}]^T$, in way that for the variables $T_{\omega_i} > 0$, $\omega_i \in \mathbb{K}_r$, defined in (7) and satisfying (21), the condition below holds:

$$\underset{h_\omega \in \Omega}{max} J(h_\omega) = J(h^*_\omega) = \sum_{i=1}^{r} h^*_{\omega_i} T_{\omega_i}. \quad (23)$$

The central idea of the algorithm consists in the assignment of the higher possible values for h_{ω_i}, $i \in \mathbb{K}_r$, starting with $\omega_i = \omega_r$, then after with $\omega_i = \omega_{r-1}$ and so on. This strategy takes into account the expression of $J(h_\omega)$ defined in (22) and the decreasing order presented in (21), in which all the higher values of T_{ω_i} are $T_{\omega_r}, T_{\omega_{r-1}}, \cdots, T_{\omega_1}$, recalling that the objective is to find optimal weights h^*_ω to maximize $J(h_\omega)$, in a way that (23) holds.

The procedure must yet consider that $h^*_\omega = [h^*_{\omega_1}\ h^*_{\omega_2}\ \cdots\ h^*_{\omega_r}]^T \in \Omega$ given by (9) and that $\omega = [\omega_1\ \omega_2 \cdots \omega_r]^T \in \Psi_r$ defined in (20). Therefore, from (1) and (2), $0 > -\phi_{\omega_i,1} \leqslant h^*_{\omega_i} \leqslant \phi_{\omega_i,2} > 0$, for all $i \in \mathbb{K}_r$, and $h^*_{\omega_1} + h^*_{\omega_2} + \cdots + h^*_{\omega_r} = C$. In that way, the lower values of h_{ω_i} will be reserved for h_{ω_1}, $h_{\omega_2}, \cdots$, associated with the lower values of T_{ω_i}, which are T_{ω_1}, then T_{ω_2} and so on, as can be seen in (21).

Now the two steps of the algorithm will be presented:

- Step 1: Consider that the vector $\omega = [\omega_1\ \omega_2 \cdots \omega_r]^T \in \Psi_r$ is known and define the sum S_p below, observing that it is composed by the terms associated with the maximums and minimums of h_{ω_i}, for all $\omega_i \in \mathbb{K}_r$, because from (1), $-\phi_{\omega_i,1} \leqslant h_{\omega_i} \leqslant \phi_{\omega_i,2}$, for all $i \in \mathbb{K}_r$:

$$\begin{aligned} S_p &= -\phi_{\omega_1,1} - \phi_{\omega_2,1} - \cdots - \phi_{\omega_{p-2},1} - \phi_{\omega_{p-1},1} \\ &\quad + \phi_{\omega_p,2} + \phi_{\omega_{p+1},2} + \cdots + \phi_{\omega_{r-1},2} + \phi_{\omega_r,2}. \end{aligned} \quad (24)$$

Initially, it will be determined $k \in \mathbb{K}_r$ such that:

$$\begin{cases} S_p \leqslant C, \text{ for } k \leqslant p \leqslant r, \text{ and} \\ S_{k-1} > C. \end{cases} \quad (25)$$

Note that, from the aforementioned analysis, which resulted in condition (4), thus for $i = \omega_r$ in (4) and $\omega = [\omega_1\ \omega_2 \cdots \omega_r]^T \in \Psi_r$, one has from (24) that $S_r = -\phi_{\omega_1,1} - \cdots - \phi_{\omega_{r-1},1} + \phi_{\omega_r,2} \leqslant C$.

Now, observe that, from the analysis previously made which resulted in condition (6), thus for $i = \omega_1$ in (6) and $\omega = [\omega_1\ \omega_2 \cdots \omega_r]^T \in \Psi_r$, one has from (24) that $-\phi_{\omega_1,1} + \phi_{\omega_2,2} + \cdots + \phi_{\omega_{r-1},1} + \phi_{\omega_r,2} \geqslant C$. Therefore, since from (1), $0 > -\phi_{\omega_i,1} \leqslant h_{\omega_i} \leqslant \phi_{\omega_i,2} > 0$, one obtains from (24) that $S_1 = \phi_{\omega_1,2} + \phi_{\omega_2,2} + \cdots + \phi_{\omega_r,2} > -\phi_{\omega_1,1} + \phi_{\omega_1,2} + \cdots + \phi_{\omega_r,2} \geqslant C$. Thus, $S_1 > C$.

Therefore, it was verified that $S_r \leqslant C$ and $S_1 > C$. Additionally, observe that, from (1) and (24),

$$S_{i+1} - S_i = -\phi_{\omega_i,1} - \phi_{\omega_i,2} < 0, i = 1, 2, \cdots r-1. \quad (26)$$

Ergo, $S_r < S_{r-1} < S_{r-2} < \cdots < S_2 < S_1$. In that way, if $S_k \leqslant C$, then $S_r < S_{r-1} < S_{r-2} < \cdots < S_{k+1} < S_k \leqslant C$. Similarly, if $S_{k-1} > C$, then $S_1 > S_2 > \cdots > S_{k-2} > S_{k-1} > C$. These facts assure the existence of k satisfying the condition $1 < k \leqslant r$ such that the conditions established in (25) hold.

- Step 2: Obtained k, $1 \leqslant k \leqslant r$, in a way that the conditions (25) are satisfied, then, for a given $\omega = [\omega_1\ \omega_2 \cdots \omega_r]^T \in \Psi_r$, the optimal weights, when (21) holds, are the ones described below:

$$\begin{cases} h^*_{\omega_i} = -\phi_{\omega_i,1}, \ i = 1, 2, \cdots, k-3, k-2, \\ h^*_{\omega_j} = \phi_{\omega_j,2}, \ j = k, k+1, \cdots, r-1, r, \\ h^*_{\omega_{k-1}} = C - \sum_{i=1}^{k-2}(-\phi_{\omega_i,1}) - \sum_{j=k}^{r} \phi_{\omega_j,2}. \end{cases} \quad (27)$$

Observe from (27) that $h^*_{\omega_1} + h^*_{\omega_2} + \cdots + h^*_{\omega_{r-1}} + h^*_{\omega_r} = C$ and therefore, these weights satisfy the condition (2).

Note also that the set of weights defined in (25) and (27) are called optimum weights, because, as will be shown in Theorem 1, they satisfy the condition (23).

As seen in Step 1, considering (24) and (25), $S_k \leqslant C$, so $S_r < S_{r-1} < S_{r-2} < \cdots < S_{k+1} < S_k \leqslant C$ and also $S_{k-1} > C$, therefore $S_1 > S_2 > \cdots > S_{k-2} > S_{k-1} > C$.

Analyzing the sums S_k and S_{k-1}, note that the only difference between them is that in S_k there is the term $-\phi_{\omega_{k-1},1}$, that was substituted by the term $\phi_{\omega_{k-1},2}$. Thus, the relation between S_k and S_{k-1} was obtained, and presented in (26). To formalize the existence of $h^*_{\omega_{k-1}}$ described in (25) in a way that (1) is satisfied, the following analysis will be performed.

As from (25), $S_k \leqslant C$ and $S_{k-1} > C$, then $S_k - C \leqslant 0$ and $S_{k-1} - C > 0$, and it is possible to obtain $\alpha \in [0, 1]$, such as the convex combination above is null:

$$\alpha(S_{k-1} - C) + (1-\alpha)(S_k - C) = 0. \quad (28)$$

Really, from (28) it is possible to obtain the value of α:

$$\alpha = \frac{(-S_k + C)}{(-S_k + C) + (S_{k-1} - C)}, \quad (29)$$

and observe that $\alpha \in [0, 1]$, because, as seen, $(-S_k + C) \geqslant 0$ and $(S_{k-1} - C) > 0$. In that way, (28) corresponds to a convex combination equal to zero.

Now, substituting S_{k-1} and S_k, defined in (24), in (28), one has from (26) that:

$$\begin{aligned} &\alpha(S_{k-1} - C) + (1-\alpha)(S_k - C) \\ &\quad = S_k + \alpha(S_{k-1} - S_k) - C \end{aligned}$$

$$
\begin{aligned}
&= S_k + \alpha(\phi_{\omega_{k-1},1} + \phi_{\omega_{k-1},2}) - C \\
&= -\phi_{\omega_1,1} - \cdots - \phi_{\omega_{k-2},1} + [\alpha\phi_{\omega_{k-1},2} \\
&\quad +(1-\alpha)(-\phi_{\omega_{k-1},1})] + \phi_{\omega_k,2} + \cdots + \phi_{\omega_r,2} - C = 0.
\end{aligned} \tag{30}
$$

Note that from (29) and from (26), that $\alpha = \frac{-S_k+C}{\phi_{\omega_{k-1},1}+\phi_{\omega_{k-1},2}}$. Therefore, from (30) and S_k defined in (24), one has that the corresponding term $h^*_{\omega_{k-1}}$ is given by:

$$
\begin{aligned}
&[\alpha\phi_{\omega_{k-1},2} + (1-\alpha)(-\phi_{\omega_{k-1},1})] \\
&= \left[\frac{-S_k + C}{\phi_{\omega_{k-1},1} + \phi_{\omega_{k-1},2}}\right]\phi_{\omega_{k-1},2} \\
&\quad + \left[1 - \frac{-S_k + C}{\phi_{\omega_{k-1},1} + \phi_{\omega_{k-1},2}}\right](-\phi_{\omega_{k-1},1}) \\
&= -S_k + C - \phi_{\omega_{k-1},1} \\
&= \phi_{\omega_1,1} + \cdots + \phi_{\omega_{k-2},1} - \phi_{\omega_k,2} - \cdots - \phi_{\omega_r,2} + C \\
&= h^*_{\omega_{k-1}}.
\end{aligned} \tag{31}
$$

Analyzing (31), note that $h^*_{\omega_{k-1}}$ defined in (27) satisfies that condition $h^*_{\omega_{k-1}} = [\alpha\phi_{\omega_{k-1},2}+(1-\alpha)(-\phi_{\omega_{k-1},1})]$. Therefore, as $h^*_{\omega_{k-1}}$ is convex combination of $\phi_{\omega_{k-1},2}$ and $-\phi_{\omega_{k-1},1}$, then one can affirm that $-\phi_{\omega_{k-1},1} \leqslant h^*_{\omega_{k-1}} \leqslant \phi_{\omega_{k-1},2}$.

Thus, the existence of $h^*_{\omega_{k-1}}$ such that (1) holds and the optimal weights in (27) are such that $h^*_{\omega} = [h^*_{\omega_1}\ h^*_{\omega_2}\ \cdots\ h^*_{\omega_r}]^T \in \Omega$, with $\omega = [\omega_1\ \omega_2\ \cdots\ \omega_r]^T \in \Psi_r$, and also considering that (21) holds. Hence, h^*_ω satisfies the conditions (1) and (2).

Briefly, the steps of Algorithm 1 are given by:

- **Step 1 -** Obtain k satisfying the conditions described in (25) through the definition of the S_p sums in (24). That can be done calculating all the sums S_p starting with S_r, the after S_{r-1}, and so on, until one finds S_k, which is the last sum that is smaller than or equal to C. This procedure represents the trying of making the function $J(h)$ the more positive as it can be, through the association of the maximum weights to the highest possible number of terms that compose the function. This fact follows from the specified optimal values for $h^*_{\omega_j} = \phi_{\omega_j,2}$ in (27), for $j = k, k+1, \cdots, r-1, r$, that are equal to the respective maximums of h_{ω_j} as described in (1). When it is not possible anymore to associate a maximum weight to its respective term, since the existing condition is not respected anymore, this weight must be adapted so the function $J(h)$ can be maximized, taking into account (1) and (2).
- **Step 2 -** After obtaining k that satisfied the conditions (25), then calculate the optimal weights $h_{\omega_i}, i \in \mathbb{K}_r$ for a given $\omega = [\omega_1\ \omega_2\ \cdots\ \omega_r]^T \in \Psi_r$ from (27).

Next, it will be demonstrated that the weights defined in (25) and (27) are optimal, because they satisfy the conditions established in (23).

Theorem 1: Consider that the variables $T_{\omega_i} > 0, i \in \mathbb{K}_r$, satisfy the condition (21). Therefore, the weights defined in (25) and (27) are optimal, as they satisfy the condition (23).

Proof: From $J(h_\omega)$ defined in (22), for $h_\omega = h^*_\omega = [h^*_{\omega_1}\ h^*_{\omega_2}\ \cdots\ h^*_{\omega_r}]^T$ and the weights defined in (25) and (27), then

$$
\begin{aligned}
J(h^*_\omega) &= \sum_{i=1}^{r} h^*_{\omega_i} T_{\omega_i} \\
&= \sum_{i=1}^{k-2}(-\phi_{\omega_i,1})T_{\omega_i} + h^*_{\omega_{k-1}}T_{\omega_{k-1}} + \sum_{j=k}^{r}\phi_{\omega_j,2}T_{\omega_j},
\end{aligned} \tag{32}
$$

where $h^*_{\omega_{k-1}}$ was defined in (27).

Now consider an arbitrary vector $h_\omega = [h_{\omega_1}\ h_{\omega_2}\ \cdots\ h_{\omega_r}]^T \in \Omega$, that, without loss of generality, can be defined in the following way: $h_{\omega_1} = -\phi_{\omega_1,1} + \Delta\phi_{\omega_1}$, $h_{\omega_2} = -\phi_{\omega_2,1} + \Delta\phi_{\omega_2}, \cdots, h_{\omega_{k-2}} = -\phi_{\omega_{k-2},1} + \Delta\phi_{\omega_{k-2}}$, $h_{\omega_k} = \phi_{\omega_k,2} - \Delta\phi_{\omega_k}, \cdots, h_{\omega_{r-1}} = \phi_{\omega_{r-1},2} - \Delta\phi_{\omega_{r-1}}$, $h_{\omega_r} = \phi_{\omega_r,2} - \Delta\phi_{\omega_r}$. As from (1), $-\phi_{\omega_i,1} \leqslant h_{\omega_i} \leqslant \phi_{\omega_i,2}$, for all $i \in \mathbb{K}_r$ and as $h_\omega \in \Omega$, then necessarily $\Delta\phi_{\omega_1}, \Delta\phi_{\omega_2}, \cdots, \Delta\phi_{\omega_{k-3}}, \Delta\phi_{\omega_{k-2}}$ and $\Delta\phi_{\omega_k}, \Delta\phi_{\omega_{k+1}}, \cdots, \Delta\phi_{\omega_{r-1}}, \Delta\phi_{\omega_r}$ are all positive or null.

Observe yet that $h_{\omega_{k-1}}$ was not defined, because from (2), is given by:

$$
\begin{aligned}
h_{\omega_{k-1}} &= C - (h_{\omega_1} + h_{\omega_2} + \cdots + h_{\omega_{k-2}} + h_{\omega_k} \\
&\quad + \cdots + h_{\omega_{r-1}} + h_{\omega_r}) \\
&= C - (-\phi_{\omega_1,1} + \Delta\phi_{\omega_1} - \phi_{\omega_2,1} + \Delta\phi_{\omega_2} \\
&\quad - \cdots - \phi_{\omega_{k-2},1} + \Delta\phi_{\omega_{k-2}} + \phi_{\omega_k,2} - \Delta\phi_{\omega_k} \\
&\quad + \cdots + \phi_{\omega_{r-1},2} - \Delta\phi_{\omega_{r-1}} + \phi_{\omega_r,2} - \Delta\phi_{\omega_r}) \\
&= [C + \phi_{\omega_1,1} + \phi_{\omega_2,1} + \cdots + \phi_{\omega_{k-2},1} - \phi_{\omega_k,2} \\
&\quad - \cdots - \phi_{\omega_{r-1},2} - \phi_{\omega_r,2}] \\
&\quad -[\Delta\phi_{\omega_1} + \Delta\phi_{\omega_2} + \cdots + \Delta\phi_{\omega_{k-2}} - \Delta\phi_{\omega_k} \\
&\quad - \cdots - \Delta\phi_{\omega_{r-1}} - \Delta\phi_{\omega_r}] \\
&= h^*_{\omega k-1} - [\Delta\phi_{\omega_1} + \Delta\phi_{\omega_2} + \cdots + \Delta\phi_{\omega_{k-2}} \\
&\quad -\Delta\phi_{\omega_k} - \cdots - \Delta\phi_{\omega_r}],
\end{aligned} \tag{33}
$$

considering the definition of $h^*_{\omega_{k-1}}$ in (27).

Therefore, for this $h_\omega = [h_{\omega_1}\ h_{\omega_2}\ \cdots\ h_{\omega_r}]^T$, note that from (22), (32) and (33),

$$
\begin{aligned}
&J(h^*_\omega) - J(h_\omega) \\
&= \sum_{i=1}^{r} h^*_{\omega_i}T_{\omega_i} - \sum_{i=1}^{r} h_{\omega_i}T_{\omega_i} \\
&= \left\{\sum_{i=1}^{k-2} -\phi_{\omega_i,1}T_{\omega_i} + h^*_{\omega_{k-1}}T_{\omega_{k-1}} + \sum_{j=k}^{r}\phi_{\omega_j,2}T_{\omega_j}\right\} \\
&\quad - \left\{\sum_{i=1}^{k-2}(-\phi_{\omega_i,1} + \Delta\phi_{\omega_i})T_{\omega_i} + h_{\omega_{k-1}}T_{\omega_{k-1}}\right. \\
&\qquad \left. + \sum_{j=k}^{r}(\phi_{\omega_j,2} - \Delta\phi_{\omega_j})T_{\omega_j}\right\} \\
&= -\left\{\sum_{i=1}^{k-2}\Delta\phi_{\omega_i,1}T_{\omega_i} + [-(\Delta\phi_{\omega_1} + \cdots + \Delta\phi_{\omega_{k-2}}\right.
\end{aligned}
$$

$$- \Delta\phi_{\omega_k} - \cdots - \Delta\phi_{\omega_{r-1}} - \Delta\phi_{\omega_r})]T_{\omega_{k-1}} \\ \left. - \sum_{j=k}^{r} \Delta\phi_{\omega_j,2} T_{\omega_j} \right\}. \quad (34)$$

Now, from $\Delta\phi_{\omega_i} > 0$ for all $i \in \mathbb{K}_r$, $i \neq k-1$, and from (21) $T_{\omega_l} \geqslant T_{\omega_q}$ for all q e l with $l \geqslant q$. Therefore, $\Delta\phi_{\omega_j} T_{\omega_j} \geqslant \Delta\phi_{\omega_j} T_{\omega_{k-1}}$, for $j = k, k+1, \cdots, r$ and $-\Delta\phi_{\omega_i} T_{\omega_i} \geqslant -\Delta\phi_{\omega_i} T_{\omega_{k-1}}$, for $i = 1, 2, \cdots, k-2$. Thus, one has from (34) that

$$\begin{aligned} J(h_\omega^*) - J(h_\omega) &= -\sum_{i=1}^{k-2} \Delta\phi_{\omega_i} T_{\omega_i} + (\Delta\phi_{\omega_1} + \cdots + \Delta\phi_{\omega_{k-2}} \\ &\quad -\Delta\phi_{\omega_k} - \cdots - \Delta\phi_{\omega_{r-1}} - \Delta\phi_{\omega_r}) T_{\omega_{k-1}} \\ &\quad + \sum_{j=k}^{r} \Delta\phi_{\omega_j} T_{\omega_j} \\ &\geqslant -\sum_{i=1}^{k-2} \Delta\phi_{\omega_i} T_{\omega_{k-1}} + (\Delta\phi_{\omega_1} + \cdots + \Delta\phi_{\omega_{k-2}} \\ &\quad -\Delta\phi_{\omega_k} - \cdots - \Delta\phi_{\omega_{r-1}} \\ &\quad -\Delta\phi_{\omega_r}) T_{\omega_{k-1}} + \sum_{j=k}^{r} \Delta\phi_{\omega_j} T_{\omega_{k-1}} \\ &= 0. \end{aligned} \quad (35)$$

Therefore, $J(h_\omega^*) - J(h_\omega) \geqslant 0$, which implies that $J(h_\omega^*) \geqslant J(h_\omega)$. As h_ω e $h_\omega^* \in \Omega$ was defined in an arbitrary way, one can conclude that the condition (23) is satisfied. The proof is concluded. ■

Example 1: Consider the cost function below:

$$J(h) = \sum_{i=1}^{3} h_i T_i, \quad (36)$$

defined for all weights $h_i \in \mathbb{R}$ satisfying (1), (2), (4) and (6), where the variables T_1, T_2 and T_3 were defined in (7). Also, consider that the weights h_1, h_2 and h_3 have the following bounds:

$$\begin{aligned} -\phi_{1,1} &= -4 \leqslant h_1 \leqslant 2 = \phi_{1,2} \\ -\phi_{2,1} &= -2 \leqslant h_2 \leqslant 3 = \phi_{2,2} \\ -\phi_{3,1} &= -3 \leqslant h_3 \leqslant 5 = \phi_{3,2} \\ h_1 + h_2 &+ h_3 = C = 0. \end{aligned} \quad (37)$$

To determine an optimal upper bound for the cost function presented in (36), the steps of Algorithm 1 will be used.

Initially, it is necessary to define the set Ψ presented in (20):

$$\Psi_3 = \{\omega = [\omega_1\ \omega_2\ \omega_3]^T \in \mathbb{R}^3 / \omega_i \in \mathbb{K}_3 \\ \text{for all } i \in \mathbb{K}_3 \text{ e } \omega_1 \neq \omega_2 \neq \omega_3\}. \quad (38)$$

In this example, the decreasing order of the variables T_1, T_2 and T_3 is not known. Therefore, as cited before, one must consider all possible combinations of decreasing orders of the terms T_1, T_2 e T_3. In this case one obtains $m = r! = 3! = 6$ possible combinations:

$$\begin{cases} T_3 \geqslant T_2 \geqslant T_1 \\ T_3 \geqslant T_1 \geqslant T_2 \\ T_2 \geqslant T_3 \geqslant T_1 \\ T_2 \geqslant T_1 \geqslant T_3 \\ T_1 \geqslant T_3 \geqslant T_2 \\ T_1 \geqslant T_2 \geqslant T_3. \end{cases} \quad (39)$$

First, it will be considered that $T_3 \geqslant T_2 \geqslant T_1$. Therefore, from (21) $\omega_1 = 1$, $\omega_2 = 2$, $\omega_3 = 3$ and $\omega = [\omega_1\ \omega_2\ \omega_3]^T = [1\ 2\ 3]^T$

Step 1: Considering that, in this case, $\omega = [1\ 2\ 3]^T$, the parameters defined in (37) and the definition of S_p in (24), one can calculate the sums S_1, S_2 e S_3:

$$\begin{aligned} S_3 &= -\phi_{1,1} - \phi_{2,1} + \phi_{3,2} = -4 - 2 + 5 = -1 \\ S_2 &= -\phi_{1,1} + \phi_{2,2} + \phi_{3,2} = -4 + 3 + 5 = 4 \\ S_1 &= \phi_{1,2} + \phi_{2,2} + \phi_{3,2} = 2 + 3 + 5 = 10. \end{aligned} \quad (40)$$

From (40), one can observe that S_3 is the first sum minor or equal to $C = 0$. Thus, from (25), one can define $S_{k-1} = S_2$ and $k = 3$.

Step 2: Obtained k, satisfying (25), one can specify the optimal weights, as in (27):

$$\begin{cases} h_{\omega_1}^* = h_1^* = -\phi_{1,1} = -4, \\ h_{\omega_3}^* = h_3^* = \phi_{3,2} = 5, \\ h_{\omega_2}^* = h_2^* = C - \sum_{i=1}^{k-2}(-\phi_{\omega_i,1}) - \sum_{j=k}^{r} \phi_{\omega_j,2} \\ \qquad\qquad = C + \phi_{1,1} - \phi_{3,2} = -1. \end{cases} \quad (41)$$

Note that all optimal weights obtained in (41) respect the limits presented in (37). Therefore, for the case of $T_3 \geqslant T_2 \geqslant T_1$, the cost function (36) will have its optimal upper bound in the following way:

$$\begin{aligned} \max J(h) = J(h^*) &= \sum_{i=1}^{3} h_i^* T_i = h_1^* T_1 + h_2^* T_2 + h_3^* T_3 \\ &= -4T_1 - 1T_2 + 5T_3. \end{aligned} \quad (42)$$

Next, one must consider a different decreasing order for T_1, T_2 e T_3, such as $T_3 \geqslant T_1 \geqslant T_2$ for instance, and take the two Algorithm 1 steps again, resulting in an optimal upper bound for $J(h)$ for this specific case where $T_3 \geqslant T_1 \geqslant T_2$.

To simplify this demonstration, the table below is presented, in which all optimal upper bounds are described for each possible decreasing order combination of the terms T_1, T_2 e T_3, following the permutation order presented in (39).

Remark 4: The sum S_1 is equal to the sum of all maximum limits presented in (37). Hence, it is independent from the considered permutation. It this case, $S_1 = 10$, and because of that this term was omitted from Table 1.

Once considered all possible decreasing order combinations for T_1, T_2 and T_3, it was proved before that one of

TABLE 1. Calculation of all possible sums S_k and optimal weights for Example 1.

Permutation j	S_2	S_3	h^*_{j1}	h^*_{j2}	h^*_{j3}	$max\ J(h)$
$T_3 \geqslant T_2 \geqslant T_1$ $(j=1)$	4	-1	-4	-1	5	$-4T_1 - 1T_2 + 5T_3$
$T_3 \geqslant T_1 \geqslant T_2$ $(j=2)$	3	-1	-3	-2	5	$-3T_1 - 2T_2 + 5T_3$
$T_2 \geqslant T_3 \geqslant T_1$ $(j=3)$	4	-4	-4	3	1	$-4T_1 + 3T_2 + T_3$
$T_2 \geqslant T_1 \geqslant T_3$ $(j=4)$	2	-4	0	3	-3	$0T_1 + 3T_2 - 3T_3$
$T_1 \geqslant T_3 \geqslant T_2$ $(j=5)$	5	-3	2	-2	0	$2T_1 - 2T_2 + 0T_3$
$T_1 \geqslant T_2 \geqslant T_3$ $(j=6)$	2	-3	2	1	-3	$2T_1 + 1T_2 - 3T_3$

these cases will match the real decreasing order of these variables and, therefore, the maximum of $J(h)$ will already have been contemplated in the problem solution. Moreover, as stated before, from Lemma 1 for $m = 6$, the values of the 6 upper bounds presented in Table 1 are always smaller than or equal to the maximum of $J(h)$ in (36), considering the constraints (37) and T defined in (7).

III. MAIN CONTROL THEORY RESULTS

Consider the Takagi-Sugeno continuous-time fuzzy system described by the fuzzy rules

$$Rule\ i: \begin{cases} \text{If} \quad z_1(t) \text{ is } M_1^i \textit{ and} \cdots \textit{and } z_p(t) \text{ is } M_p^i, \\ \text{Then} \quad \dot{x}(t) = A_i x(t) + B_i u(t), \end{cases} \tag{43}$$

with $i \in \mathbb{K}_r$ denoting the rule's number. For the ith rule, $M_i^j, j \in \mathbb{K}_p$ denotes the fuzzy sets, $z(t)$ denotes the premise variables, $x(t)$ is the state vector, $u(t)$ is the control signal, A_i and B_i are local systems matrices. For now on, it will be considered that $\alpha_i = \alpha_i(x(t))$, $i \in \mathbb{K}_r$. The set of rules (43) can also be represented as

$$\dot{x}(t) = A(\alpha)x(t) + B(\alpha)u(t) = \sum_{i=1}^{r} \alpha_i [A_i x(t) + B_i u(t)], \tag{44}$$

where α_i, $i \in \mathbb{K}_r$, are the weights regarding the local models, respecting the following characteristics:

$$\sum_{i=1}^{r} \alpha_i = 1, \sum_{i=1}^{r} \dot{\alpha}_i = 0, \alpha_i \geqslant 0, \text{ for all } i \in \mathbb{K}_r, \tag{45}$$

with $\dot{\alpha}_i$, $i \in \mathbb{K}_r$ denoting the time derivative of α_i, that is $\frac{d\alpha(t)}{dt}$.

Consider also the fuzzy Lyapunov function (FLF) described as

$$V(x(t)) = \sum_{i=1}^{r} \alpha_i x^T(t) P_i x(t), \tag{46}$$

where that P_i, $i \in \mathbb{K}_r$, are positive defined matrices.

The fuzzy controller is given by:

$$u(t) = -K(\alpha)x(t) = -\sum_{j=1}^{r} \alpha_j K_j x(t). \tag{47}$$

Therefore, the system (44) with the control law (47) can be rewritten as

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} \alpha_i (A_i x(t) + B_i u(t)) \\ &= \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j (A_i x(t) - B_i K_j x(t)). \end{aligned} \tag{48}$$

In [19], the authors presented results towards the stabilization of TS fuzzy systems using the FLF, as the following theorem (Theorem 6 in [19]).

Theorem 2: [19] Let $\mu > 0$ be a given scalar. Consider that α_ρ respects (45), and $|\dot{\alpha}_\rho| \leqslant \phi_\rho$, for all $\rho \in \mathbb{K}_r$. The system (44) is stabilizable by the fuzzy controller (47), with gains given by $K_i = S_i^T R^{-T}$, if there are symmetric matrices Γ_i, Y, and any matrices R, S_i satisfying the following LMIs

$$\Gamma_i > 0, \tag{49}$$

$$\Gamma_i + Y > 0 \ (i = 1, 2, \cdots, r), \tag{50}$$

$$\Xi_{ii} < 0 \ (i = 1, 2, \cdots, r), \tag{51}$$

$$\bar{\Xi}_{ij} < 0 \ (i < j = 1, 2, \cdots, r), \tag{52}$$

where:

$$\Xi_{ij} = \begin{bmatrix} \Gamma_\phi - A_i R^T - R A^T + B_i S_j^T + S_j B_i^T & * \\ \Gamma_i - \mu(A_i R^T - B_i S_j^T) + R & \mu(R + R^T) \end{bmatrix}, \tag{53}$$

$$\bar{\Xi}_{ij} = \Xi_{ij} + \Xi_{ji}, \tag{54}$$

$$\Gamma_\phi = \sum_{\rho=1}^{r} \phi_\rho (\Gamma_\rho + Y). \tag{55}$$

Proof: See [19]. ■

Although it presents important results over the stabilization of TS fuzzy systems using the FLF, it is important to notice that Theorem 2 has its proof based on the following condition:

$$\begin{aligned} &\dot{V}(x(t)) \\ &= \sum_{i=1}^{r} \alpha_i \left[2x^T(t) P_i \dot{x}(t) + \sum_{\rho=1}^{r} \dot{\alpha}_\rho x^T(t)(P_\rho + Y)x(t) \right] \\ &\leqslant \sum_{i=1}^{r} \alpha_i \left[2x^T(t) P_i \dot{x}(t) + x^T(t) \underbrace{\sum_{\rho=1}^{r} \phi_\rho (P_\rho + Y)}_{P_\phi} x(t) \right], \end{aligned} \tag{56}$$

which is true, since $(P_\rho + Y) > 0$ for all $\rho \in \mathbb{K}_r$, and the time derivatives of the normalized weights related to the local models cannot be greater than their maximums values. From (45), observe that $\dot{\alpha}_1 + \dot{\alpha}_2 + \cdots + \dot{\alpha}_r = 0$. Then, for any slack symmetric matrix Y, one has that the term from the equality (56) given by $(\dot{\alpha}_1 + \dot{\alpha}_2 + \cdots + \dot{\alpha}_r)x(t)^T Yx(t)$ is equal to zero. However, it is a conservative thought, since all the derivatives of the normalized weights related to the local models cannot achieve its maximums at the same time, as it would not respect (45). Therefore, it does not represent the reality of what is happening with the system.

Because of that, it is proposed in Theorem 3 a different way of dealing with these time derivatives of the normalized weights related to the local models, using Theorem 1, assuring now that the conditions (45) are always respected and providing an optimal upper bound for the term related to $\dot{P}(\alpha) = \sum_{i=1}^{r} \dot{\alpha}_i P_i$.

Theorem 3: Let $\mu > 0$ be a given scalar. Consider that α_ρ respects (45), with $-\phi_{\rho,1} \leqslant \dot{\alpha}_\rho \leqslant \phi_{\rho,2}$, where $\phi_{\rho,1}$ and $\phi_{\rho,2}$ are known real constants, and defining $h_\rho = \dot{\alpha}_\rho$, for all $\rho \in \mathbb{K}_r$, then the conditions (1), (2), (4) and (6) hold, for $C = 0$. Furthermore, suppose that there exist matrices $P_\rho = P_\rho^T > 0$, define now $T_\rho = x(t)^T P_\rho x(t)$, for all $\rho \in \mathbb{K}_r$. Note that T_ρ is positive for $x(t) \neq 0$, and then apply Algorithm 1 for each one of the k decreasing order of $T_1, T_2, \cdots, T_r$, where $k \in \mathbb{K}_m$ and $m = r!$, to obtain the weights of the optimal upper bound $h_{k_1}^* = \dot{\alpha}_{k_1}^*, h_{k_2}^* = \dot{\alpha}_{k_2}^*, \cdots, h_{k_r}^* = \dot{\alpha}_{k_r}^*$, for $J(h)$ given in (22), represented by $J(h) \leqslant \sum_{\rho=1}^{r} h_{k\rho}^* T_\rho$, for all $k \in \mathbb{K}_m$. The system (44) is stabilizable by the fuzzy controller (47), with gains given by $K_i = S_i^T R^{-T}$, and also $P_i = R^{-1}\Gamma_i R^{-T}$, if there are symmetric matrices Γ_i, Y, and any matrices R, S_i satisfying the following LMIs

$$\Gamma_i > 0, \ (i = 1, 2, \cdots, r), \tag{57}$$

$$\Xi_{iik} < 0 \ (i = 1, 2, \cdots, r; k = 1, 2, \cdots, m), \tag{58}$$

$$\bar{\Xi}_{ijk} < 0 \ (i < j = 1, 2, \cdots, r; k = 1, 2, \cdots, m), \tag{59}$$

where:

$$\Xi_{ijk} = \begin{bmatrix} \Gamma_k^* - A_i R^T - RA^T + B_i S_j^T + S_j B_i^T & * \\ \Gamma_i - \mu(A_i R^T - B_i S_j^T) + R & \mu(R + R^T) \end{bmatrix}, \tag{60}$$

$$\bar{\Xi}_{ijk} = \Xi_{ijk} + \Xi_{jik}, \tag{61}$$

$$\Gamma_k^* = \sum_{\rho=1}^{r} \dot{\alpha}_{k\rho}^* (\Gamma_\rho). \tag{62}$$

Proof: Let the fuzzy Lyapunov function (FLF) candidate be described as:

$$V(x(t)) = \sum_{i=1}^{r} \alpha_i x^T(t) P_i x(t), \tag{63}$$

where matrices $P_i = P_i^T > 0$, for all $i \in \mathbb{K}_r$.

Consider also, from (48), the null product below

$$2[x^T(t)M + \dot{x}^T(t)\mu M] \times \left[\dot{x} - \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i \alpha_j (A_i - B_i K_j) x(t)\right] = 0. \tag{64}$$

Taking the time derivative of (63), one obtains

$$\dot{V}(x) = 2\sum_{i=1}^{r} \alpha_i x^T(t) P_i \dot{x}(t) + x^T(t) \underbrace{\sum_{\rho=1}^{r} \dot{\alpha}_\rho P_\rho}_{\dot{P}(\alpha)} x(t). \tag{65}$$

Using Theorem 1 for processing the term in (65) related to $\dot{P}(\alpha)$, which can be seen as a cost function of the bounded weights $\dot{\alpha}_i$, for $i \in \mathbb{K}_r$, one has that

$$\begin{aligned} x^T(t)\dot{P}(\alpha)x(t) &= \sum_{\rho=1}^{r} \dot{\alpha}_\rho x^T(t) P_\rho x(t) \\ &\leqslant \sum_{\rho=1}^{r} \dot{\alpha}_{k\rho}^* x^T(t) P_\rho x(t) \end{aligned} \tag{66}$$

where $\dot{\alpha}_{k\rho}^*$, $k \in \mathbb{K}_m$, $m = r!$, are the optimal weights calculated by the algorithm presented in Theorem 1 for every case of decreasing order k of $x(t)^T P_i x(t)$, $i \in \mathbb{K}_r$. For this application of Theorem 1 and the aforementioned Algorithm 1, define $h_\rho = \dot{\alpha}_\rho$ and $T_\rho = x(t)^T P_\rho x(t)$, for all $\rho \in \mathbb{K}_r$. Then, observe that, from (65), the term $x(t)^T \dot{P}(\alpha)x(t)$ is equal to the cost function $J(h)$ defined in (8). Moreover, from the conditions stated in Theorem 3, the conditions (1), (2), (4) and (6) hold, for $C = 0$. Thus, one can use the results from Algorithm 1 to obtain m optimal upper bounds of $J(h)$ for each k decreasing order of $T_1, T_2, \cdots, T_r$, where $k \in \mathbb{K}_m$, and $m = r!$. Now, for a decreasing order $k \in \mathbb{K}_m$, defined as in (13), and for the optimal weights $h_{k\rho}^* = \dot{\alpha}_{k\rho}^*$, this optimal upper bound for $J(h)$ is equal to $\sum_{\rho=1}^{r} \dot{\alpha}_{k\rho}^* T_\rho$, as presented in the end of (66). Theorem 1 shows that these weights are optimal and Lemma 1 assures that if one considers the m mentioned upper bounds, then it is equivalent to say that $J(h)$ is smaller than or equal to its maximum value.

Now, adding the null term (64), from (66) one has that for $k \in \mathbb{K}_m$,

$$\begin{aligned} \dot{V}(x) &\leqslant 2\sum_{i=1}^{r} \alpha_i x^T(t) P_i \dot{x}(t) + x^T(t) \underbrace{\sum_{\rho=1}^{r} \dot{\alpha}_{k\rho}^* P_\rho}_{P_k^*} x(t) \\ &\quad + 2[x^T(t)M + \dot{x}^T(t)\mu M] \\ &\quad \times \left[\dot{x} - \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i\alpha_j (A_i - B_i K_j) x(t)\right] \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i\alpha_j [2x^T(t) P_i \dot{x}(t) + x^T(t) P_k^* \dot{x}(t) \\ &\quad + 2x^T(t) M \dot{x}(t) + 2\dot{x}^T(t)\mu M \dot{x}(t) \\ &\quad - 2x^T(t) M \{A_i - B_i K_j\} x(t) \\ &\quad - 2\dot{x}^T(t)\mu M \{A_i - B_i K_j\} x(t)]. \end{aligned} \tag{67}$$

Using the following vector

$$\xi \triangleq [x^T(t) \ \ \dot{x}^T(t)]^T, \tag{68}$$

(67) can be rewritten as

$$\dot{V}(x) \leqslant \sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_j\xi^T\Lambda_{ijk}\xi = \sum_{i=j}^{r}\alpha_i^2\xi^T\Lambda_{iik}\xi + \sum_{i<j}^{r}\alpha_i\alpha_j\xi^T\underbrace{[\Lambda_{ijk}+\Lambda_{jik}]}_{\bar{\Lambda}_{ijk}}\xi, \tag{69}$$

where

$$\Lambda_{ijk} \triangleq \begin{bmatrix}\lambda_1 & * \\ \lambda_2 & \lambda_3\end{bmatrix},$$
$$\lambda_1 = P_k^* - M(A_i - B_iK_j) - (A_i - B_iK_j)M^T,$$
$$\lambda_2 = P_i - \mu M(A_i - B_iK_j^T) + M^T,$$
$$\lambda_3 = \mu(M + M^T). \tag{70}$$

If $\Lambda_{iik} < 0$ and $\bar{\Lambda}_{ijk} < 0$, for all $i, j \in \mathbb{K}_r$ and $k \in \mathbb{K}_m$, the equilibrium point $x = 0$ of the controlled system (44) and (47), described in (48), is asymptotically stable. Pre and post-multiplying Λ_{ijk} and $\bar{\Lambda}_{ijk}$ by the nonsingular matrices $\text{diag}(M^{-1}, M^{-1})$ and $\text{diag}(M^{-T}, M^{-T})$, respectively, and pre and post-multiplying (57) by M^{-1} and M^{-T}, respectively, the following inequalities are obtained:

$$M^{-1}P_iM^{-T} > 0,$$
$$\Theta_{iik} < 0\ (i = 1, 2, \cdots, r;\ k = 1, 2, \cdots, m),$$
$$\bar{\Theta}_{ijk} < 0\ (i < j = 1, 2, \cdots, r;\ k = 1, 2, \cdots, m) \tag{71}$$

with $\bar{\Theta}_{ijk} = \Theta_{ijk} + \Theta_{jik}$ and

$$\Theta_{ijk} \triangleq \begin{bmatrix}\gamma_1 & * \\ \gamma_2 & \gamma_3\end{bmatrix},$$
$$\gamma_1 = M^{-1}P_k^*M^{-T} - (A_i - B_iK_j)M^{-T} - M^{-1}(A_i - B_iK_j),$$
$$\gamma_2 = M^{-1}P_iM^{-T} - \mu(A_i - B_iK_j^T)M^{-T} + M^{-1},$$
$$\gamma_3 = \mu(M^{-1} + M^{-T}). \tag{72}$$

Considering the following definitions:

$$R \triangleq M^{-1},\ \Gamma_i \triangleq RP_iR^T,\ \Gamma_k^* \triangleq RP_k^*R^T,\ S_j \triangleq RK_j^T, \tag{73}$$

the LMIs (57) - (59) are obtained, imposing that $\dot{V}(x) < 0$ for $x \neq 0$. Therefore, if they hold, the equilibrium point $x = 0$ of the controlled system (44) and (47) is asymptotically stable, with closed loop gains given by $K_j = S_j^T R^{-T}$, for $j \in \mathbb{K}_r$. ■

Remark 5: It is important to notice that Theorem 3 is able to process non-symmetric bounds of the time derivatives of the weights α_i, $i \in \mathbb{K}_r$ (which is not possible with Theorem 2). Moreover, it also provides necessary and sufficient conditions for the stabilization of nonlinear TS fuzzy systems, using a FLF and the stability analysis presented in the proof of Theorem 3. It follows because the proposed optimal upper bounds (66) for the term $x(t)^T\dot{P}(\alpha)x(t)$ that appears in $\dot{V}(x(t))$, described in (65), considering the analyzes presented in Theorem 1 and Lemma 1, do not add any constraint and, therefore, is the least conservative that can be found. Also, Theorem 3 is based on cases that can happen, compared to Theorem 2, that only considers a single case that can never happen, where all maximums of the time derivatives of the normalized weights of the local models are achieved by system at the same time, representing a moment that $\sum_{i=1}^{r}\dot{\alpha}_i \neq 0$, which would not respect the conditions (45).

Remark 6: It is also important to observe that both studied methods relay on the knowledge of the bounds of the time derivatives of the normalized weights related to the local models of the representation of the plant by a TS fuzzy model. Thus, Theorems 2 and 3 cannot be used if these bounds are not available. This is an assumption usually adopted in the use of Fuzzy Lyapunov Functions for obtaining conditions to design controllers. In the literature one can find two procedures to specify these bounds. In the first, the designer chooses these bounds with sufficiently high values, design the controller gains for instance using the conditions given in Theorems 2 and 3, and after the design the controlled system is simulated for initial conditions in the desired operation region, to check if the adopted bounds hold [20]. In the second, additional analyses offer systematic procedures for the specification of these limits [12], [21].

Remark 7: The proposed procedure has one limitation related to computational effort, when compared to other available methods. For instance, in Theorem 2 the upper bound was described with only one condition represented by Γ_ϕ. Now, in the optimum conditions proposed in Theorem 3, there exist $m = r!$ upper bounds, represented by Γ_k^*, $k = 1, 2, \ldots, m$, where r is the number of local models of the plant described by a Takagi-Sugeno fuzzy model. It is not possible to obtain less conservative upper bounds than the proposed in this paper and used in Theorem 3, but it pays the price in computational effort compared to other methods. Additionally, if the bounds of the time derivatives of the normalized weights related to the local model system are not symmetric, then Theorem 2 can not be used. It is another advantage of Theorem 3. In Section IV are presented simulation results and additional analyses regarding this subject.

IV. NUMERICAL EXAMPLES

Consider the following TS nonlinear system presented in [19], given by (44), with $r = 2$ and

$$A_1 = \begin{bmatrix}3,6 & -1,6 \\ 6,2 & -4,3\end{bmatrix},\quad A_2 = \begin{bmatrix}-a & -1,6 \\ 6,2 & -4,3\end{bmatrix},$$
$$B_1 = \begin{bmatrix}-0,45 \\ -3\end{bmatrix},\quad B_2 = \begin{bmatrix}-b \\ -3\end{bmatrix}.$$

Theorems 2 and 3 were applied to verify the stabilizable region for $a \in [0, 5]$ and $b \in [1.2, 1.5]$, considering $\mu = 0.04$ and $\phi_{1,1} = \phi_{1,2} = \phi_{2,1} = \phi_{2,2} = \phi_1 = \phi_2 = 1$, as it was used in [19]. Using the Matlab$^{\text{TM}}$ tool YALMIP and the SeDuMi solver, the results for this trial are shown in Figure 1. In this case $r = 2$ and $m = r! = 2$. In Table 2 are presented the sums and the optimum weights obtained for the design of the controller gains based on the conditions from Theorem 3.

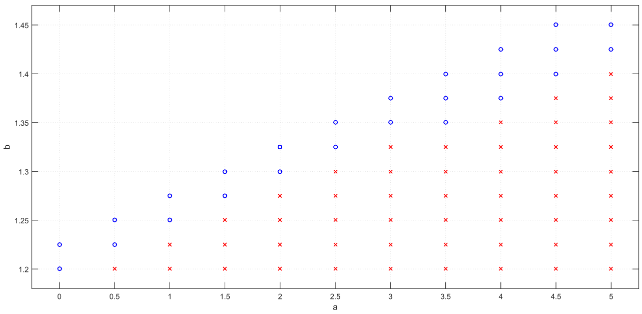

FIGURE 1. Stabilizable regions provided by Theorem 2 (×) and Theorem 3 (∘,×).

TABLE 2. Calculation of all possible sums S_k and optimal weights for the numeric example with $r = 2$.

Permutation k	S_2	$\dot{\alpha}^*_{k1}$	$\dot{\alpha}^*_{k2}$	P^*_k
$x(t)^T P_2 x(t) \geqslant x(t)^T P_1 x(t)$ $(k = 1)$	0	-1	1	$-P_1 + P_2$
$x(t)^T P_1 x(t) \geqslant x(t)^T P_2 x(t)$ $(k = 2)$	0	1	-1	$P_1 - P_2$

Remark 8: Note that, despite considering a simple system, with equal symmetric bounds of the time derivatives of the weights α_1 and α_2, the conditions provided by Theorem 3 already shows a wider stabilizable region than the one obtained from Theorem 2. If the bounds of the time derivatives of the normalized weights of the local models were not symmetric, Theorem 2 could not be used, showing another advantage of Theorem 3. Theorem 3 is based on the upper bounds proposed in Section II, Algorithm 1, that are optimum upper bounds as proved in Theorem 1. It means that it is not possible to obtain less conservative upper bounds than the proposed in Section II and used in Theorem 3, but it pays the price in computational effort compared to other methods. Therefore, as the difference between the results presented in Theorems 2 and 3 is only on the specification of the aforementioned upper bounds, then when the conditions of Theorem 2 hold, the conditions of Theorem 3 also hold. It follows because the upper bound described for Theorem 2 produces a single term $\Gamma_\phi = \sum_{\rho=1}^{r} \phi_\rho(\Gamma_\rho + Y)$, using all the maximums of the time derivatives of the normalized weights of the local models ($\phi_\rho = 1$) at the same time to produce this upper bound, as shown in (55), considering a situation that will never happen, since the conditions (45) must always be satisfied. Therefore, adopting this upper bound it brings constraints to the search of a solution to the problem. Theorem 3 produces in this case $m = r! = 2! = 2$ upper bounds and in this example they are represented by the terms $-P_1 + P_2$ for $k = 1$ and $P_1 - P_2$ for $k = 2$, as described in Table 2. Therefore, it translates in more computational effort to search for a solution, but only consider cases that can happen, taking the conditions (45) into account before forming those upper bounds. One can observe that the upper bounds shown in Table 2 for P^*_k, $k = 1, 2$, related to Theorem 3, are always smaller than the upper bound produced by Theorem 2.

TABLE 3. Calculation of all possible sums S_k and optimal weights for the numeric example with $r = 3$.

Permutation k	S_2	S_3	$\dot{\alpha}^*_{k1}$	$\dot{\alpha}^*_{k2}$	$\dot{\alpha}^*_{k3}$	P^*_k
$x(t)^T P_3 x(t) \geqslant x(t)^T P_2 x(t) \geqslant x(t)^T P_1 x(t)$ $(k = 1)$	1	-1	-1	0	1	$-P_1 + 0P_2 + P_3$
$x(t)^T P_3 x(t) \geqslant x(t)^T P_1 x(t) \geqslant x(t)^T P_2 x(t)$ $(k = 2)$	1	-1	0	-1	1	$0P_1 - P_2 + P_3$
$x(t)^T P_2 x(t) \geqslant x(t)^T P_3 x(t) \geqslant x(t)^T P_1 x(t)$ $(k = 3)$	1	-1	-1	1	0	$-P_1 + P_2 + 0P_3$
$x(t)^T P_2 x(t) \geqslant x(t)^T P_1 x(t) \geqslant x(t)^T P_3 x(t)$ $(k = 4)$	1	-1	0	1	-1	$0P_1 + P_2 - P_3$
$x(t)^T P_1 x(t) \geqslant x(t)^T P_3 x(t) \geqslant x(t)^T P_2 x(t)$ $(k = 5)$	1	-1	1	-1	0	$P_1 - P_2 + 0P_3$
$x(t)^T P_1 x(t) \geqslant x(t)^T P_2 x(t) \geqslant x(t)^T P_3 x(t)$ $(k = 6)$	1	-1	1	0	-1	$P_1 + 0P_2 - P_3$

Consider now the following TS nonlinear system with $r = 3$ described by (44) and with

$$A_1 = \begin{bmatrix} 3,6 & -1,6 \\ 6,2 & -4,3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -a & -1,6 \\ 6,2 & -4,3 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -3,6 & -1,6 \\ a & -4,3 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0,45 \\ -3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -b \\ -3 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -0,45 \\ -b \end{bmatrix}.$$

Theorems 2 and 3 were applied to verify the stabilizable regions for $a \in [-5, 5]$ and $b \in [-1.1, -0.6]$, considering $\mu = 0.04$ and $\phi_{1,1} = \phi_{1,2} = \phi_{2,1} = \phi_{2,2} = \phi_{3,1} = \phi_{3,2} = \phi_1 = \phi_2 = \phi_3 = 1$, as it was used in [19]. Using the Matlab^{TM} tool YALMIP and the SeDuMi solver, the results of this trial are shown in Figure 2. In this case $r = 3$ and $m = r! = 6$. In Table 3 are presented the sums and the optimum weights obtained for the design of the controller gains based on the conditions from Theorem 3.

Remark 9: The sum S_1 was omitted from Tables 2 and 3 because it is independent from the considered permutation in both cases. For the case where $r = 2$, $S_1 = 2$. For the case where $r = 3$, $S_1 = 3$.

Note now that, increasing the order of the system, even though still only using equal bounds of the time derivatives of the weights α_1, α_2 and α_3, the feasible region obtained with conditions from Theorem 2 was smaller than the region obtained using Theorem 3, that is based on an optimal procedure to find the optimal weights of the upper bound P^*_k, for all $k \in \mathbb{K}_m$, with $m = r!$. This procedure increased

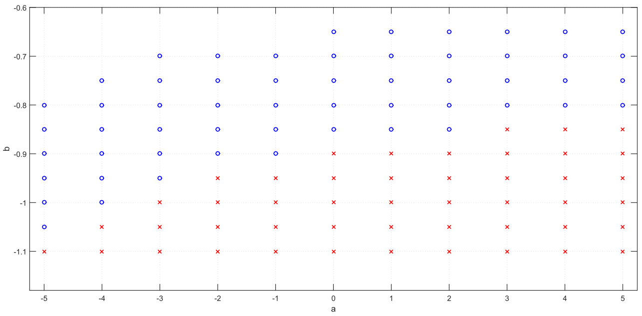

FIGURE 2. Stabilizable regions provided by Theorem 2 (×) and Theorem 3 (∘,×).

the computational effort for the solving of the problem, but provided a positive impact on the result, with a much bigger feasible region than that obtained with the conditions from Theorem 2. It is important to remember that all possible upper bounds produced by Theorem 3 are smaller than the one produced by Theorem 2, that considers only a condition that does not satisfy the existence conditions (45) of the TS fuzzy system (44).

V. CONCLUSION

This paper presented a simple procedure to find optimal upper bounds of a term that appears in the time derivative of fuzzy Lyapunov functions, in the design of controllers for nonlinear systems described by Takagi-Sugeno fuzzy models. These upper bounds are optimal because they contemplate the maximum value of this term, and are always smaller than or equal to this maximum value. It is a relevant result on this subject, because these optimal upper bounds do not add any constraint. With these optimal upper bounds, a relaxed design method for stabilization of TS fuzzy models is proposed. Two numeric examples illustrate the effectiveness of this procedure. Although some new methods related to this control problem have appeared in recent literature, such as [22] and [21], it is important to highlight that they are based only on conditions to make these upper bounds more relaxed, but never optimal. Furthermore, in [21], were considered only symmetric intervals for the time derivatives of the normalized weights related to the local models. In [22], it was adopted an increased area for the search of a solution, including cases that could not realistic happen, in order to get a balance between relaxed design conditions and computational performance. It is not possible to obtain less conservative upper bounds than the proposed in Theorem 3, but it pays the price in computational effort compared to other methods.

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