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Interdependency of Complex Fuzzy Neighborhood Operators and Derived Complex Fuzzy Coverings

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ABSTRACT Complex fuzzy coverings (CFCs) are the natural mixture of the complex fuzzy sets (CFSs) and coverings, which are the modified versions of the coverings by replacing crisp sets with CFSs. This manuscript aims to explore the complex fuzzy neighborhood operators (CFNOs) by introducing the notions such as β-neighborhood system (β-NO), complex fuzzy β-minimal description (CFβ-MND), and complex fuzzy β-maximal description (CFβ-MXD). First, we explore the complex fuzzy β-covering approximation space (CFβ-CAS) and then we propose the above notions and investigate their properties. Additionally, we construct the CFNOs based on the complex fuzzy β-coverings (CFβ-Cs). Finally, the CFβ-Cs were derived by using CFNOs, and their properties are considered. These all notions are also verified with the help of suitable examples to show that the presented approaches are extensive, reliable, and proficient techniques.

INDEX TERMS Complex fuzzy β-coverings, complex fuzzy neighborhood operators, complex fuzzy sets, rough sets.

I. INTRODUCTION

Decision-making is a proficient technique to manage awkward and unreliable information in numerous realistic issues. The decision-making procedure is received extensive attention from different scholars and various scholars have utilized it in the environment of separated areas. In a genuine decision-making technique, a significant issue is how to communicate the characteristic worth all the more effectively and precisely. In reality, in light of the multifaceted nature of decision-making issues and the fuzziness of decision-making rules, it isn't sufficient to communicate trait estimations of options by precise qualities. For this, the theory of fuzzy set (FS) was discovered by Zadeh [1] contains the truth degree in the form of the element of the unit interval. Numerous scholars have utilized the FS theory in separated fields [2]–[5].

As the FS thinks about just the truth degree yet doesn't weigh on the phase term bit of the information substances, which similarly expect an equivalent part in evaluating the article in the decision-making technique. Nonetheless, in reality, it is consistently difficult to communicate the assessment of the truth degree by a careful incentive in an FS. In such cases, it very well may be simpler to portray dubiousness and vulnerability utilizing two-dimensional data rather than a single one. Subsequently, an expansion of the current speculations may be incredibly significant to portray the vulnerabilities due to his/her hesitant judgment in complex decision-making issues. For this, the theory of complex fuzzy set (CFS) was discovered by Ramot *et al.* [6] contains the truth degree in the form of complex numbers whose real and imaginary parts are belonging to the unit interval. By presenting this subsequent measurement, the total data can be extended in one set, and thus, loss of data can be maintained a strategic distance from. To outline the criticalness of the stage term, consider an illustration of a specific organization that chooses to put in new information preparing and

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examination programming. For this, the organization counsels a specialist who gives the data concerning (a) alternate choices of programming (b) relating programming adaptation. The organization needs to choose the most ideal alternative(s) of programming with its most recent form at the same time. Here, the issue is two-dimensional, to be specific, to choose the ideal option of programming and its most adaptation. This issue can't be demonstrated precisely utilizing the customary FS hypothesis. So the most ideal approach to speak to the entirety of the data given by the master is by utilizing the CFS hypothesis. The adequacy terms in CFS might be utilized to give an organization's choice concerning the elective programming and the stage terms might be utilized to speak to an organization's choice concerning programming form. Numerous scholars have utilized the CFS theory in separated fields [7]–[11].

The theory of rough set (RS) was initially discovered by Pawlak [12] as a genuine method for information revelation and data handling, in which the central technique comprises of relations that show to data frameworks or decision-making. In RS, two principal factors influence the description capacity of data frameworks or decision-making tables: set approximation and information reduction. From one perspective, given a subset of the universe, two perceptible sets called lower and upper approximations are explored to approximate the subset. Then again, under the rule of keeping the set approximations unaltered, information reduction is led to eliminating the excess credits from the data framework or decision-making table to get some less complex standards than the first data framework or decision-making table. Numerous scholars have utilized the RS theory in separated fields [13]–[19].

Let $\widetilde{\mho} = \{\mho_1, \mho_2, \ldots, \mho_m\}$ with $\mho_i \in \mathcal{F}\left(\acute{U}\right)(i = 1, 2, \ldots, m)$ expresses the family of the fuzzy power set of $\acute{U}$ is called fuzzy$\beta-$ C of $\acute{U}$ if for each $\beta \in (0, 1]$ such that $\left(\cup_{i=1}^{m}\mho_i\right)(\mathfrak{s}) = 1$ for each $\mathfrak{s} \in \acute{U}$. The pair $\left(\acute{U}, \widetilde{\mho}\right)$ expresses the fuzzy $\beta-$Covering approximation spaces. Additionally, If set $\widetilde{\mho} = \{\mho_1, \mho_2, \ldots, \mho_m\}$ with $\mho_i \in \mathcal{F}\left(\acute{U}\right)(i = 1, 2, \ldots, m)$ expresses the family of the fuzzy power set of $\acute{U}$ is called fuzzy $\beta-$ Covering of $\acute{U}$ if for each $\beta \in (0, 1]$ such that $\left(\cup_{i=1}^{m}\mho_i\right)(\mathfrak{s}) \geq \beta$ for each $\mathfrak{s} \in \acute{U}$. The pair $\left(\acute{U}, \widetilde{\mho}\right)$ expresses the CF$\beta-$CAS. Additionally, we will explore two covering based rough sets, which was elaborated by:

$$\widetilde{\mathcal{N}}^{\beta}_{\widetilde{\mho}}(\mathfrak{s}) = \left\{\mho \in \widetilde{\mho} : \mho(\mathfrak{s}) \geq \beta\right\}$$

Further, for any covering approximation space $\left(\acute{U}, \mho\right)$, the minimal description of $\mathfrak{s}$ is elaborated by:

$$\mathcal{M}d_{\mho}(\mathfrak{s}) = \left\{\begin{array}{l}\ddot{O} \in \mho : \mathfrak{s} \in \ddot{O} \wedge \\ \left(\forall \mathcal{S} \in \mho \wedge \mathfrak{s} \in \mathcal{S} \wedge \mathcal{S} \subseteq \ddot{O} \Longrightarrow \ddot{O} = \mathcal{S}\right)\end{array}\right\}$$

Similarly, the maximal description of $\mathfrak{s}$ is elaborated by:

$$\mathcal{MD}_{\mho}(\mathfrak{s}) = \left\{\begin{array}{l}\ddot{O} \in \mho : \mathfrak{s} \in \ddot{O} \wedge \\ \left(\forall \mathcal{S} \in \mho \wedge \mathfrak{s} \in \mathcal{S} \wedge \mathcal{S} \supseteq \ddot{O} \Longrightarrow \ddot{O} = \mathcal{S}\right)\end{array}\right\}$$

The geometrical expressions of the elaborated approaches are discussed in the form of Figure 1.

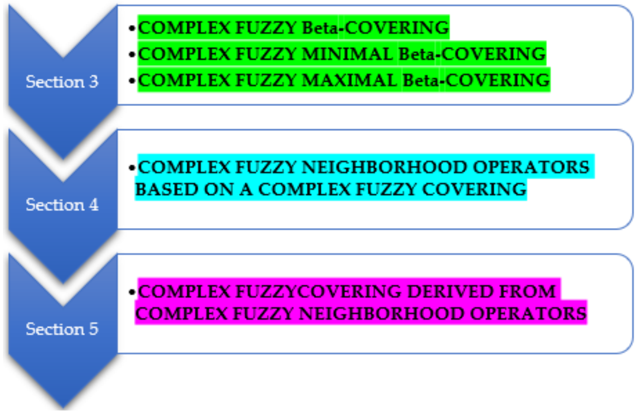

FIGURE 1. Graphical expressions of the explored approaches.

The CFS is a simplification of the FS in which the amplitude term offers the degree of belongings of an object while the phase term explains the periodicity. These phase terms differentiate the CFS from the traditional FS theories. In FS theory, the data are accomplished with the reimbursement of only the degree of the belongings while the part of periodicity is completely ignored. Hence, this may result in the loss of knowledge through the decision-making procedures based on certain cases. To additional demonstrate the theory of phase terms, we take an example. Suppose a person wants to purchase a car under crucial factors such as its model and its production dates. Since the model of each car moves with the evolution of the production dates and hence to select or decision regarding choosing the optimal car is a decision-making process under these two factors simultaneously.

Also, such types of problems cannot be modeled accurately with traditional theories. However, CFS theory is well suited for such classes of problems where the amplitude terms may be used to provide a decision about the model of a car while the phase terms concerning its production dates. Henceforth, a CFS is a more generalized continuation of the existing theories such as FSs. Additionally, D'eer *et al.* [8] explored some neighborhood operators and their derived coverings, we use these operators is to elaborate on some new six types of operators and also utilized their coverings such as maximal and minimal $\beta-$coverings are to show the reliability consistency of the investigated approaches. When we consider complex fuzzy types of information's then the existing types of operators in [8] are not able to manage with it. But when we consider the existing fuzzy types of information's then the explored types of operators can manage with it due to its structure. The main objectives of this article are summarized as follow:

1. To explore the CFNOs by introducing the notions such as β-NS, CFβ-MND, and CFβ-MXD. First, we explore

the CFβ-CAS and then we propose the above notions and investigate their properties.

2. Additionally, we construct the CFNOs based on the CFβ-Cs.
3. Finally, the CFβ-Cs were derived by using CFNOs, and their properties are considered.
4. These all notions are also verified with the help of suitable examples to show that the presented approaches are extensive reliable and proficient techniques.

The main structure of this manuscript is discussed in the following way: In section 2, we recall some theories like covering-based rough sets and complex fuzzy sets. In section 3, we present the idea of CF$\beta-$ C, CF$\beta-$MND, and CF$\beta-$MXD, and $\beta-$NS in CF$\beta-$CAS. In section 4, we present the idea of complex fuzzy neighborhood operators in the environment of rough sets theory based on the modifications such as $\beta-$neighborhood system, CF$\beta-$MND, CF$\beta-$MXD. Additionally, and the CFNOs are discussed below. In section 5, the complex fuzzy β-coverings were derived by using complex fuzzy neighborhood operators and their properties are considered. These all notions are also verified with the help of suitable examples to show that the presented approaches are extensive reliable and proficient techniques.

II. PRELIMINARIES

In this study, we recall some theories like covering-based rough sets and complex fuzzy sets. The NO [13] is elaborated by $\mathcal{N}:\text{Ű}\rightarrow\mathcal{P}\left(\text{Ű}\right)$, where $\mathcal{P}\left(\text{Ű}\right)$ shows the family of the subsets of fix set Ű. Additionally, some properties for $\mathcal{N}$ are followed as a NO $\mathcal{N}$ is reflexive i.e. $\S\in\mathcal{N}(\S)$ for each $\S\in\text{Ű}$; a NO $\mathcal{N}$ is symmetric if $\S\in\mathcal{N}\left(\S'\right)\Leftrightarrow\S'\in\mathcal{N}(\S),\forall\S,\S'\in\text{Ű}$; a NO $\mathcal{N}$ is transitive if $\S\in\mathcal{N}\left(\S'\right)\Leftrightarrow\mathcal{N}(\S)\subseteq\mathcal{N}\left(\S'\right),\forall\S,\S'\in\text{Ű}$.

Definition 1 ([16]): Let $\mho$ be the family of the subsets of fix set Ű. If $\theta\notin\mho$ and $\cup\mho=\text{Ű}$, then $\mho$ is called covering of Ű, and the pair $\left(\text{Ű},\mho\right)$ is called covering approximation space (CAS).

Definition 2 ([17]): For any CAS$\left(\text{Ű},\mho\right)$, the neighborhood of $\S$ is elaborated by:

$$\mathcal{N}_{\mho}(\S)=\cap\left\{\ddot{\bar{O}}\in\mho:\S\in\ddot{\bar{O}}\right\},\quad \S\in\text{Ű} \tag{1}$$

Definition 3 ([18]): For any CAS$\left(\text{Ű},\mho\right)$, the complementary neighborhood of $\S$ is elaborated by:

$$\mathcal{M}_{\mho}\left(\S'\right)=\left\{\S\in\text{Ű}:\S'\in\mathcal{N}_{\mho}(\S)\right\} \tag{2}$$

From the above hypothesis, we get if $\S\in\mathcal{N}_{\mho}(\S)$ and $\S\in\mathcal{M}_{\mho}(\S)$, then the relation between neighborhood and complimentary neighborhood is discussed is follow as: for any CAS $\left(\text{Ű},\mho\right)$ and for any $\S,\S'\in\text{Ű},\S\in\mathcal{M}_{\mho}\left(\S'\right)\Leftrightarrow\S'\in\mathcal{N}_{\mho}(\S)$.

Definition 4 ([17]): For any CAS$\left(\text{Ű},\mho\right)$, the minimal description of $\S$ is elaborated by:

$$\mathcal{M}d_{\mho}(\S)=\left\{\begin{array}{l}\ddot{\bar{O}}\in\mho:\S\in\ddot{\bar{O}}\wedge\\ \left(\forall\mathcal{S}\in\mho\wedge\S\in\mathcal{S}\wedge\mathcal{S}\subseteq\ddot{\bar{O}}\Longrightarrow\ddot{\bar{O}}=\mathcal{S}\right)\end{array}\right\} \tag{3}$$

Similarly, the maximal description of $\S$ is elaborated by:

$$\mathcal{MD}_{\mho}(\S)=\left\{\begin{array}{l}\ddot{\bar{O}}\in\mho:\S\in\ddot{\bar{O}}\wedge\\ \left(\forall\mathcal{S}\in\mho\wedge\S\in\mathcal{S}\wedge\mathcal{S}\supseteq\ddot{\bar{O}}\Longrightarrow\ddot{\bar{O}}=\mathcal{S}\right)\end{array}\right\} \tag{4}$$

Definition 5 ([19]): For any CAS $\left(\text{Ű},\mho\right)$, the indiscernible neighborhood of $\S$ is elaborated by:

$$Friends_{\mho}(\S)=\cup\left\{\ddot{\bar{O}}\in\mho:\S\in\ddot{\bar{O}}\in\mho\right\},\quad \S\in\text{Ű} \tag{5}$$

Similarly, the close friends of $\S$ is elaborated by:

$$CFriends_{\mho}(\S)=\cup\left\{\mathcal{M}_{\mho}(\S)\right\},\quad \S\in\text{Ű} \tag{6}$$

Additionally, based on the covering $\mho$ with $\S\in\text{Ű}$, we choose three neighborhoods systems for $\S$. First, we choose the $\mathcal{N}_{\mho}(\S)=\left\{\ddot{\bar{O}}\in\mho:\S\in\ddot{\bar{O}}\right\}$, and further, we choose the $Md_{\mho}(\S)$ and $\mathcal{MD}_{\mho}(\S)$. By using these neighborhoods, Yao and Yao [13] discovered four neighborhood operators based on the covering $\mho$ are discussed below:

$$\begin{aligned}\mathcal{N}^1_{\mho}(\S)&=\cap\left\{\ddot{\bar{O}}\in\mho:\ddot{\bar{O}}\in\mathcal{M}d_{\mho}(\S)\right\}\\&=\cap\left\{\ddot{\bar{O}}\in\mho:\ddot{\bar{O}}\in\mathcal{N}_{\mho}(\S)\right\}\end{aligned} \tag{7}$$

$$\begin{aligned}\mathcal{N}^2_{\mho}(\S)&=\cup\left\{\ddot{\bar{O}}\in\mho:\ddot{\bar{O}}\in\mathcal{M}d_{\mho}(\S)\right\}\\&=CFriends_{\mho}(\S)\end{aligned} \tag{8}$$

$$\mathcal{N}^3_{\mho}(\S)=\cap\left\{\ddot{\bar{O}}\in\mho:\ddot{\bar{O}}\in\mathcal{MD}_{\mho}(\S)\right\} \tag{9}$$

$$\begin{aligned}\mathcal{N}^4_{\mho}(\S)&=\cup\left\{\ddot{\bar{O}}\in\mho:\ddot{\bar{O}}\in\mathcal{MD}_{\mho}(\S)\right\}\\&=\cup\left\{\ddot{\bar{O}}\in\mho:\ddot{\bar{O}}\in\mathcal{N}_{\mho}(\S)\right\}=Friends_{\mho}(\S)\end{aligned} \tag{10}$$

The Eq. (7) to Eq. (10) is reflexive, the Eq. (10) is symmetric, and the Eq. (7) and Eq. (9) are transitive. By using the covering $\mho$, six types of coverage were also presented by Yao and Yao [13], which are discussed below:

$$\mho^1=\cup\left\{\mathcal{M}d_{\mho}(\S):\S\in\text{Ű}\right\} \tag{11}$$

$$\mho^2=\cup\left\{\mathcal{MD}_{\mho}(\S):\S\in\text{Ű}\right\} \tag{12}$$

$$\begin{aligned}\mho^3&=\left\{\cap\mathcal{M}d_{\mho}(\S):\S\in\text{Ű}\right\}=\left\{\cap\mathcal{N}_{\mho}(\S):\S\in\text{Ű}\right\}\\&=\left\{\mathcal{N}^1_{\mho}(\S):\S\in\text{Ű}\right\}\end{aligned} \tag{13}$$

$$\begin{aligned}\mho^4&=\left\{\cup\mathcal{MD}_{\mho}(\S):\S\in\text{Ű}\right\}=\left\{\cup\mathcal{N}_{\mho}(\S):\S\in\text{Ű}\right\}\\&=\left\{\mathcal{N}^4_{\mho}(\S):\S\in\text{Ű}\right\}=\left\{Friends_{\mho}(\S):\S\in\text{Ű}\right\}\end{aligned} \tag{14}$$

$$\mho^5 = \mho - \left\{\bar{\bar{o}} \in \mho : \left(\exists\, \mho' \subseteq \mho - \left\{\bar{\bar{o}}\right\}\right)\left(\bar{\bar{o}} = \cap \mho'\right)\right\} \quad (15)$$

$$\mho^6 = \mho - \left\{\bar{\bar{o}} \in \mho : \left(\exists\, \mho' \subseteq \mho - \left\{\bar{\bar{o}}\right\}\right)\left(\bar{\bar{o}} = \cup \mho'\right)\right\} \quad (16)$$

Definition 6 ([6]): A CFS $\mathcal{A}$ is a function defined form fix set $\acute{U}$ to the unit disc in a complex plane i.e. $\mathcal{A}(\mathfrak{S}) = \mathcal{A}_{RP}(\mathfrak{S}).e^{i2\pi(\mathcal{A}_{IP}(\mathfrak{S}))}$ with the condition that is $\mathcal{A}_{RP}(\mathfrak{S}), \mathcal{A}_{IP}(\mathfrak{S}) \in [0,1]$. The family of all complex fuzzy subsets of $\acute{U}$ is expressed by $\mathcal{F}(\acute{U})$ is called the power set of $\acute{U}$. Additionally, if we choose the $\mathcal{A}, \mathcal{B} \in \mathcal{F}(\acute{U})$, then $\mathcal{A} \subseteq \mathcal{B}$ if $\mathcal{A}(\mathfrak{S}) \subseteq \mathcal{B}(\mathfrak{S}) \Longrightarrow \mathcal{A}_{RP}(\mathfrak{S}) \subseteq \mathcal{B}_{RP}(\mathfrak{S})$ and $\mathcal{A}_{IP}(\mathfrak{S}) \subseteq \mathcal{B}_{IP}(\mathfrak{S}), \mathcal{A} = \mathcal{B}$ if $\mathcal{A}(\mathfrak{S}) = \mathcal{B}(\mathfrak{S}) \Longrightarrow \mathcal{A}_{RP}(\mathfrak{S}) = \mathcal{B}_{RP}(\mathfrak{S})$ and $\mathcal{A}_{IP}(\mathfrak{S}) = \mathcal{B}_{IP}(\mathfrak{S})$. Additionally, the union, intersection, and compliment of any two complex number is elaborated by:

$$\begin{aligned}\mathcal{A} \cup \mathcal{B} &= \mathcal{A}(\mathfrak{S}) \cup \mathcal{B}(\mathfrak{S}) \\ &= max\left(\mathcal{A}_{RP}(\mathfrak{S}), \mathcal{B}_{RP}(\mathfrak{S})\right).e^{i2\pi\left(max\left(\mathcal{A}_{IP}(\mathfrak{S}), \mathcal{B}_{IP}(\mathfrak{S})\right)\right)}\end{aligned} \quad (17)$$

$$\begin{aligned}\mathcal{A} \cap \mathcal{B} &= \mathcal{A}(\mathfrak{S}) \cap \mathcal{B}(\mathfrak{S}) \\ &= min\left(\mathcal{A}_{RP}(\mathfrak{S}), \mathcal{B}_{RP}(\mathfrak{S})\right).e^{i2\pi\left(min\left(\mathcal{A}_{IP}(\mathfrak{S}), \mathcal{B}_{IP}(\mathfrak{S})\right)\right)}\end{aligned} \quad (18)$$

$$\mathcal{A}^c = 1 - \mathcal{A}(\mathfrak{S}) = 1 - \mathcal{A}_{RP}(\mathfrak{S}).e^{i2\pi\left(1-\mathcal{A}_{IP}(\mathfrak{S})\right)} \quad (19)$$

III. COMPLEX FUZZY $\beta-$ COVERING, COMPLEX FUZZY $\beta-$ MINIMAL COVERING, AND COMPLEX FUZZY $\beta-$ MAXIMAL COVERING

This study aims to present the idea of complex fuzzy $\beta-$covering (CF$\beta-$C), complex fuzzy $\beta-$minimal description (CF$\beta-$MND), complex fuzzy $\beta-$maximal description (CF$\beta-$MXD), and $\beta-$neighborhood systems ($\beta-$NS) in complex fuzzy $\beta-$CAS (CF$\beta-$CAS).

A. RELATIONSHIP BETWEEN $\beta-$NEIGHBORHOOD SYSTEM, COMPLEX FUZZY $\beta-$ MINIMAL DESCRIPTION, AND COMPLEX FUZZY $\beta-$ MAXIMAL DESCRIPTION

Find the relationships between these notions and their properties are also discussed. Finally, we discovered which two CF$\beta-$C generates the same (CF$\beta-$MND), CF$\beta-$MXD, and $\beta-$NS for any $\mathfrak{S} \in \acute{U}$. The notion of CF$\beta-$C $\breve{\mho}$ is discussed below.

Definition 7: A set $\breve{\mho} = \{\mho_1, \mho_2, \ldots, \mho_m\}$ with $\mho_i \in \mathcal{F}(\acute{U})\,(i = 1, 2, \ldots, m)$ expresses the family of the complex fuzzy power set of $\acute{U}$ is called CF$\beta-$ C of $\acute{U}$ if for each $\beta = \beta_{RP}e^{i2\pi(\beta_{IP})}$ where $\beta_{RP}, \beta_{IP} \in (0,1]$ such that $\left(\cup_{i=1}^{m}\mho_i\right)(\mathfrak{S}) \geq \beta$ for each $\mathfrak{S} \in \acute{U}$. The pair $\left(\acute{U}, \breve{\mho}\right)$ expresses the CF$\beta-$CAS. Additionally, we will explore the $\beta-$NS, which is presented below.

Definition 8: For any CF$\beta-$CAS$\left(\acute{U}, \breve{\mho}\right)$, the $\beta-$ NS of $\mathfrak{S}$ is elaborated by:

$$\widetilde{\mathcal{N}}^{\beta}_{\breve{\mho}}(\mathfrak{S}) = \left\{\mho \in \breve{\mho} : \mho(\mathfrak{S}) \geq \beta\right\} \quad (20)$$

For any complex fuzzy covering $\breve{\mho}$ must be CF $\beta-$ C based on any $\beta = \beta_{RP}e^{i2\pi(\beta_{IP})}$ where $\beta_{RP}, \beta_{IP} \in (0,1]$. For any $\mathfrak{S} \in \acute{U}$, there is $\mathfrak{C}\left(\breve{\mho}, \mathfrak{S}\right) = \coprod_{\beta = \beta_{RP}e^{i2\pi(\beta_{IP})},\ \beta_{RP}, \beta_{IP} \in (0,1]} \widetilde{\mathcal{N}}^{\beta}_{\breve{\mho}}(\mathfrak{S})$, where $\mathfrak{C}\left(\breve{\mho}, \mathfrak{S}\right) = \left\{\mho \in \breve{\mho} : \mho(\mathfrak{S}) \geq \beta\right\}$. On other hand, based on any $\beta = \beta_{RP}e^{i2\pi(\beta_{IP})}$ where $\beta_{RP}, \beta_{IP} \in (0,1]$ such that $C\left(\breve{\mho}, \mathfrak{S}\right) = \overbrace{\mathcal{N}}^{\beta}_{\breve{\mho}}(\mathfrak{S})$, for any $\mathfrak{S} \in \acute{U}$. Then we recall $\widetilde{\mathcal{N}}^{\beta}_{\breve{\mho}}(\mathfrak{S})$ covers $\mathfrak{C}\left(\breve{\mho}, \mathfrak{S}\right)$ for suitable values of $\beta = \beta_{RP}e^{i2\pi(\beta_{IP})}$ where $\beta_{RP}, \beta_{IP} \in (0,1]$. If we choose the values of $\beta_{RP} = \beta_{IP} = 1$, then $\widetilde{\mathcal{N}}^{\beta}_{\breve{\mho}}(\mathfrak{S}) \subseteq \mathfrak{C}$. For example, we choose a fixed set $\acute{U} = \{\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3\}$ based on the complex fuzzy covering of $\acute{U}$ i.e., $\breve{\mho} = \{\mho_1, \mho_2, \mho_3\}$, where $\mho_1 = \frac{0e^{i2\pi(0)}}{\mathfrak{S}_1} + \frac{0.1e^{i2\pi(0.11)}}{\mathfrak{S}_2} + \frac{1e^{i2\pi(1)}}{\mathfrak{S}_3}$, $\mho_2 = \frac{1e^{i2\pi(1)}}{\mathfrak{S}_1} + \frac{0.5e^{i2\pi(0.51)}}{\mathfrak{S}_2} + \frac{0.7e^{i2\pi(0.71)}}{\mathfrak{S}_3}$, and $\mho_3 = \frac{0e^{i2\pi(0)}}{\mathfrak{S}_1} + \frac{1e^{i2\pi(1)}}{\mathfrak{S}_2} + \frac{0e^{i2\pi(0)}}{\mathfrak{S}_3}$. From the above analysis, it is clear that $\mathfrak{C}\left(\breve{\mho}, \mathfrak{S}_1\right) = \{\mho_2\}, \mathfrak{C}\left(\breve{\mho}, \mathfrak{S}_2\right) = \{\mho_1, \mho_2, \mho_3\}$, and $C\left(\breve{\mho}, \mathfrak{S}_3\right) = \{\mho_1, \mho_2\}$. For any $\beta = \beta_{RP}e^{i2\pi(\beta_{IP})}$ where $\beta_{RP}, \beta_{IP} \in (0,1]$, $\breve{\mho}$ is a CF$\beta-$ C on $\acute{U}$. If we choose the values of $\beta_{RP} = \beta_{IP} = 0.1$, then $\mathfrak{C}\left(\breve{\mho}, \mathfrak{S}_1\right) = \{\mho_2\}, \mathfrak{C}\left(\breve{\mho}, \mathfrak{S}_2\right) = \{\mho_1, \mho_2, \mho_3\}$, and $\mathfrak{C}\left(\breve{\mho}, \mathfrak{S}_3\right) = \{\mho_1, \mho_2\}$. If we choose the values of $\beta = \beta_{RP}e^{i2\pi(\beta_{IP})}$ where $\beta_{RP}, \beta_{IP} \in (0, 0.1]$, then $\widetilde{\mathcal{N}}^{\beta}_{\breve{\mho}}(\mathfrak{S}_i) = \mathfrak{C}\left(\breve{\mho}, \mathfrak{S}_i\right), i = 1, 2, 3$.

Additionally, by using the idea of CFSs and CF$\beta-$C is to explore the CF$\beta-$MND, CF$\beta-$MXD, and the new kind of complex fuzzy covering-based rough set model, which are discussed below.

Definition 9: For any CF$\beta-$CAS $\left(\acute{U}, \breve{\mho}\right)$ with $\breve{\mho} = \{\mho_1, \mho_2, \ldots, \mho_m\}$, the CF$\beta-$MND $\widetilde{\mathcal{M}}d^{\beta}_{\breve{\mho}}(\mathfrak{S})$ is elaborated by:

$$\begin{aligned}&\widetilde{\mathcal{M}}d^{\beta}_{\breve{\mho}}(\mathfrak{S}) \\ &\quad = \left\{\mho \in \breve{\mho} : (\mho(\mathfrak{S}) \geq \beta)\right. \\ &\qquad \left.\wedge \left(\forall \mathcal{D} \in \breve{\mho} \wedge \mathcal{D}(\mathfrak{S}) \geq \beta \wedge \mathcal{D} \subseteq \mho \Longrightarrow \mho = \mathcal{D}\right)\right\}\end{aligned} \quad (21)$$

Similarly, the CF$\beta-$MXD $\widetilde{\mathcal{M}\mathcal{D}}^{\beta}_{\breve{\mho}}(\mathfrak{S})$ is elaborated by:

$$\begin{aligned}&\widetilde{\mathcal{M}\mathcal{D}}^{\beta}_{\breve{\mho}}(\mathfrak{S}) \\ &\quad = \left\{\mho \in \breve{\mho} : (\mho(\mathfrak{S}) \geq \beta)\right. \\ &\qquad \left.\wedge \left(\forall \mathcal{D} \in \breve{\mho} \wedge \mathcal{D}(\mathfrak{S}) \geq \beta \wedge \mathcal{D} \supseteq \mho \Longrightarrow \mho = \mathcal{D}\right)\right\}\end{aligned} \quad (22)$$

By using the Def. (8), the CF$\beta-$MND $\widetilde{\mathcal{M}d}^{\beta}_{\widetilde{\mho}}(\mathfrak{S})$ and the CF$\beta-$MXD $\widetilde{\mathcal{MD}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S})$ are elaborated by:

$$\widetilde{\mathcal{M}d}^{\beta}_{\widetilde{\mho}}(\mathfrak{S}) = \Big\{\mho \in \widetilde{\mathcal{N}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S}) : \forall \mathcal{D} \in \widetilde{\mathcal{N}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S}) \wedge \mathcal{D} \subseteq \mho \Longrightarrow \mho = \mathcal{D}\Big\} \tag{23}$$

$$\widetilde{\mathcal{MD}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S}) = \Big\{\mho \in \widetilde{\mathcal{N}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S}) : \forall \mathcal{D} \in \widetilde{\mathcal{N}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S}) \wedge \mathcal{D} \supseteq \mho \Longrightarrow \mho = \mathcal{D}\Big\} \tag{24}$$

For example, we choose a fixed set $\acute{\mho} = \{\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \mathfrak{S}_4, \mathfrak{S}_5, \mathfrak{S}_6\}$ based on the complex fuzzy covering of $\acute{\mho}$ i.e., $\widetilde{\mho} = \{\mho_1, \mho_2, \mho_3, \mho_4, \mho_5\}$ then see Table 1.

TABLE 1. Represents the values of three neighborhood systems.

u	$\widetilde{\mathcal{N}}^{0.51}_{\widetilde{\mathfrak{C}}}(u_i)$	$\widetilde{\mathcal{M}d}^{0.51}_{\widetilde{\mathfrak{C}}}(u)$	$\widetilde{\mathcal{MD}}^{0.51}_{\widetilde{\mathfrak{C}}}(u)$
u_1	$\{\mho_1, \mho_2, \mho_5\}$	$\{\mho_1, \mho_2\}$	$\{\mho_2, \mho_5\}$
u_2	$\{\mho_3, \mho_4\}$	$\{\mho_3, \mho_4\}$	$\{\mho_3, \mho_4\}$
u_3	$\{\mho_2, \mho_5\}$	$\{\mho_2, \mho_5\}$	$\{\mho_2, \mho_5\}$
u_4	$\{\mho_1, \mho_2, \mho_4, \mho_5\}$	$\{\mho_1, \mho_2, \mho_4\}$	$\{\mho_2, \mho_4, \mho_5\}$
u_5	$\{\mho_4, \mho_5\}$	$\{\mho_4, \mho_5\}$	$\{\mho_4, \mho_5\}$
u_6	$\{\mho_1, \mho_2, \mho_5\}$	$\{\mho_1, \mho_2\}$	$\{\mho_2, \mho_5\}$

where,

$$\mho_1 = \frac{0.71e^{i2\pi(0.72)}}{\mathfrak{S}_1} + \frac{0.11e^{i2\pi(0.12)}}{\mathfrak{S}_2} + \frac{0.31e^{i2\pi(0.32)}}{\mathfrak{S}_3} + \frac{0.51e^{i2\pi(0.52)}}{\mathfrak{S}_4} + \frac{0.31e^{i2\pi(0.32)}}{\mathfrak{S}_5} + \frac{0.61e^{i2\pi(0.62)}}{\mathfrak{S}_6},$$

$$\mho_2 = \frac{0.51e^{i2\pi(0.52)}}{\mathfrak{S}_1} + \frac{0.11e^{i2\pi(0.12)}}{\mathfrak{S}_2} + \frac{0.81e^{i2\pi(0.82)}}{\mathfrak{S}_3} + \frac{0.61e^{i2\pi(0.62)}}{\mathfrak{S}_4} + \frac{0.41e^{i2\pi(0.42)}}{\mathfrak{S}_5} + \frac{0.71e^{i2\pi(0.72)}}{\mathfrak{S}_6},$$

$$\mho_3 = \frac{0.21e^{i2\pi(0.22)}}{\mathfrak{S}_1} + \frac{0.71e^{i2\pi(0.72)}}{\mathfrak{S}_2} + \frac{0.21e^{i2\pi(0.22)}}{\mathfrak{S}_3} + \frac{0.11e^{i2\pi(0.12)}}{\mathfrak{S}_4} + \frac{0.21e^{i2\pi(0.22)}}{\mathfrak{S}_5} + \frac{0.31e^{i2\pi(0.32)}}{\mathfrak{S}_6},$$

$$\mho_4 = \frac{0.41^{i2\pi(042)}}{\mathfrak{S}_1} + \frac{0.61e^{i2\pi(0.62)}}{\mathfrak{S}_2} + \frac{0.41e^{i2\pi(0.42)}}{\mathfrak{S}_3} + \frac{0.81^{i2\pi(0.82)}}{\mathfrak{S}_4} + \frac{0.71e^{i2\pi(0.72)}}{\mathfrak{S}_5} + \frac{0.41e^{i2\pi(0.42)}}{\mathfrak{S}_6},$$

$$\mho_5 = \frac{0.91e^{i2\pi(0.92)}}{\mathfrak{S}_1} + \frac{0.41e^{i2\pi(0.42)}}{\mathfrak{S}_2} + \frac{0.71e^{i2\pi(0.72)}}{\mathfrak{S}_3} + \frac{0.91e^{i2\pi(0.92)}}{\mathfrak{S}_4} + \frac{0.61e^{i2\pi(0.62)}}{\mathfrak{S}_5} + \frac{0.61e^{i2\pi(0.62)}}{\mathfrak{S}_6}.$$

where $\beta = 0.5e^{i2\pi(0.51)}$ where $0 < \beta_{RP}, \beta_{IP} \leq 0.73$, then $\overbrace{\mathcal{N}}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho}}(\mathfrak{S}_i)$, $\widetilde{\mathcal{M}d}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho}}(\mathfrak{S})$, and $\widetilde{\mathcal{MD}}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho}}(\mathfrak{S})$ $(i = 1, 2, 3, 4, 5, 6)$ are listed in Table 1. It is easy to examine that $\overbrace{\mathcal{N}}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho}}(\mathfrak{S}_i) = \widetilde{\mathcal{M}d}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho}}(\mathfrak{S}) \cup \widetilde{\mathcal{MD}}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho}}(\mathfrak{S})$, for any $i = 1, 2, 3, 4, 5, 6$. By using the Def. (9), we conclude the following theories. The complex fuzzy minimal and complex fuzzy maximal descriptions are elaborated below.

$$Md_{\mho}\left(\widetilde{\mho}, \mathfrak{S}\right) = \Big\{\ddot{\mho} \in \mathfrak{C}(\mho, \mathfrak{S}) : (\forall \mathcal{S} \in \mathfrak{C}(\mho, \mathfrak{S})) \left(\mathcal{S}(\mathfrak{S}) = \ddot{\mho}(\mathfrak{S})\right), \mathcal{S} \subseteq \ddot{\mho} \Longrightarrow \mathcal{S} = \ddot{\mho}\Big\} \tag{25}$$

$$\mathcal{MD}_{\mho}\left(\widetilde{\mho}, \mathfrak{S}\right) = \Big\{\ddot{\mho} \in \mathfrak{C}(\mho, \mathfrak{S}) : (\forall \mathcal{S} \in \mathfrak{C}(\mho, \mathfrak{S})) \left(\mathcal{S}(\mathfrak{S}) = \ddot{\mho}(\mathfrak{S})\right), \mathcal{S} \supseteq \ddot{\mho} \Longrightarrow \mathcal{S} = \ddot{\mho}\Big\} \tag{26}$$

By using the Def. (8), the CF$\beta-$MND $\widetilde{\mathcal{M}d}^{\beta}_{\widetilde{\mho}}(\mathfrak{S})$ and the CF$\beta-$MXD $\widetilde{\mathcal{MD}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S})$ are elaborated by:

$$\widetilde{\mathcal{M}d}^{\beta}_{\widetilde{\mho}}(\mathfrak{S}) = \Big\{\mho \in \widetilde{\mathcal{N}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S}) : \forall \mathcal{D} \in \widetilde{\mathcal{N}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S}) \wedge \mathcal{D} \subseteq \mho \Longrightarrow \mho = \mathcal{D}\Big\} \tag{27}$$

$$\widetilde{\mathcal{MD}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S}) = \Big\{\mho \in \widetilde{\mathcal{N}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S}) : \forall \mathcal{D} \in \widetilde{\mathcal{N}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S}) \wedge \mathcal{D} \supseteq \mho \Longrightarrow \mho = \mathcal{D}\Big\} \tag{28}$$

It is not difficult to verify that $\mho(\mathfrak{S}) = \mathcal{D}(\mathfrak{S})$ is not necessary to the idea of CF$\beta-$MND and the CF$\beta-$MXD. Form the above analysis, it is clear that the above four definitions are different from each other.

Example 1: We choose a fix set $\acute{\mho} = \{\mho_1, \mho_2\}$ based on the complex fuzzy covering of $\acute{\mho}$ i.e., $\widetilde{\mho} = \Big\{\mho_1, \mho_2, \mho_3, \mho_4, \mho_5\Big\}$, where $\mho_1 = \frac{1e^{i2\pi(1)}}{\mathfrak{S}_1} + \frac{0.91e^{i2\pi(0.92)}}{\mathfrak{S}_2}$, $\mho_2 = \frac{0.7e^{i2\pi(0.72)}}{\mathfrak{S}_1} + \frac{0.11e^{i2\pi(0.12)}}{\mathfrak{S}_2}$.$\mho_3 = \frac{0.71e^{i2\pi(0.72)}}{\mathfrak{S}_1} + \frac{0.5e^{i2\pi(0.51)}}{\mathfrak{S}_2}$, $\mho_4 = \frac{0.71e^{i2\pi(0.72)}}{\mathfrak{S}_1} + \frac{0.21e^{i2\pi(0.22)}}{\mathfrak{S}_2}$, and $\mho_5 = \frac{0e^{i2\pi(0)}}{\mathfrak{S}_1} + \frac{0.51e^{i2\pi(0.52)}}{\mathfrak{S}_2}$.

From the above analysis, it is clear that $\mathfrak{C}\left(\widetilde{\mho}, \mathfrak{S}_1\right) = \{\mho_1, \mho_2, \mho_3, \mho_4\}$, $\mathfrak{C}\left(\widetilde{\mho}, \mathfrak{S}_2\right) = \{\mho_1, \mho_2, \mho_3, \mho_4, \mho_5\}$,

$$Md_{\mho}\left(\widetilde{\mho}, \mathfrak{S}_1\right) = \{\mho_1, \mho_4\}, Md_{\mho}\left(\widetilde{\mho}, \mathfrak{S}_2\right) = \{\mho_2, \mho_4, \mho_5\},$$

$$\mathcal{MD}_{\mho}\left(\widetilde{\mho}, \mathfrak{S}_1\right) = \{\mho_1, \mho_2\}, \mathcal{MD}_{\mho}\left(\widetilde{\mho}, \mathfrak{S}_2\right) = \{\mho_1, \mho_2, \mho_4\}.$$

In fact, $\mathfrak{S}_4 \subseteq \mathfrak{S}_1$, $\mathfrak{S}_4 \subseteq \mathfrak{S}_2$, and $\mathfrak{S}_5 \subseteq \mathfrak{S}_2$. For choosing the value of $\beta = 0.21e^{i2\pi(0.22)}$, where $0.21, 0.22 \in (0, 1]$. Then $\overbrace{\mathcal{N}}^{0.21e^{i2\pi(0.22)}}_{\widetilde{\mho}}(\mathfrak{S}_1) = \{\mho_1, \mho_2, \mho_3, \mho_4\}$, $\overbrace{\mathcal{N}}^{0.21e^{i2\pi(0.22)}}_{\widetilde{\mho}}(\mathfrak{S}_2) = \{\mho_1, \mho_2, \mho_3, \mho_4, \mho_5\}$, $\widetilde{\mathcal{M}d}^{0.21e^{i2\pi(0.22)}}_{\widetilde{\mho}}(\mathfrak{S}_1) = \{\mho_4\}$, $\widetilde{\mathcal{M}d}^{0.21e^{i2\pi(0.22)}}_{\widetilde{\mho}}(\mathfrak{S}_2) = \{\mho_4, \mho_5\}$, $\widetilde{\mathcal{MD}}^{0.21e^{i2\pi(0.22)}}_{\widetilde{\mho}}(\mathfrak{S}_1) = \{\mho_1, \mho_2\}$, $\widetilde{\mathcal{MD}}^{0.21e^{i2\pi(0.22)}}_{\widetilde{\mho}}(\mathfrak{S}_2) = \{\mho_1, \mho_2\}$.

To examine the relationships between $\widetilde{\mathcal{N}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S})$, $\widetilde{\mathcal{M}d}^{\beta}_{\widetilde{\mho}}(\mathfrak{S})$, and $\widetilde{\mathcal{MD}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S})$ are discussed with the help of the following propositions.

Proposition 1: For any CF$\beta-$CAS $\left(\acute{U}, \overset{\frown}{\mho}\right)$ with $\mho \in \widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$, then there exists $\mho_1 \in \widetilde{\mathcal{M}d}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$ and $\mho_2 \in \widetilde{\mathcal{M}\mathcal{D}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$, such that $\mho_1 \subseteq \mho \subseteq \mho_2$.

Proof: When $\mho' \nsubseteq \mho$, for any $\mho' \in \widetilde{\mathcal{M}d}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) - \{\mho\}$, then from $\mho(\mathfrak{s}) \geq \beta$ that $\in \widetilde{\mathcal{M}d}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$, therefore there exists $\mho_1 \in \widetilde{\mathcal{M}d}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$ such that $\mho_1 \subseteq \mho$. Similarly, $\mho'' \nsubseteq \mho$, for any $\mho'' \in \widetilde{\mathcal{M}\mathcal{D}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) - \{\mho\}$, then from $\mho(\mathfrak{s}) \geq \beta$ that $\mho \in \widetilde{\mathcal{M}\mathcal{D}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$, therefore there exists $\mho_2 \in \widetilde{\mathcal{M}\mathcal{D}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$ such that $\mho \subseteq \mho_2$.

Proposition 2: For any CF$\beta-$CAS $\left(\acute{U}, \overset{\frown}{\mho}\right)$, then $\widetilde{\mathcal{M}d}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) \subseteq \widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$ and $\widetilde{\mathcal{M}\mathcal{D}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) \subseteq \widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$ for any $\mathfrak{s} \in \acute{U}$, are hold obviously by using the Def. (8) and Def. (9).

Proposition 3: For any CF$\beta-$CAS$\left(\acute{U}, \overset{\frown}{\mho}\right)$, then $\cap\widetilde{\mathcal{M}d}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) = \cap\widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$ and $\cup\widetilde{\mathcal{M}\mathcal{D}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) = \cup\widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$ for any $\mathfrak{s} \in \acute{U}$.

Proof: Based on Proposition 1, for any $\mho \in \widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$, then there exists $\mho_1 \in \widetilde{\mathcal{M}d}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$ and $\mho_2 \in \widetilde{\mathcal{M}\mathcal{D}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$, such that $\mho_1 \subseteq \mho \subseteq \mho_2$. Then $\cap\widetilde{\mathcal{M}d}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) \subseteq \cap\widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$ and $\cup\widetilde{\mathcal{M}\mathcal{D}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) \supseteq \cup\widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$ for any $\mathfrak{s} \in \acute{U}$. Similarly, based on Proposition 2, $\widetilde{\mathcal{M}d}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) \subseteq \widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$ and $\widetilde{\mathcal{M}\mathcal{D}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) \subseteq \widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$ for any $\mathfrak{s} \in \acute{U}$. Then $\cap\widetilde{\mathcal{M}d}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) \subseteq \cap\widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$ and $\cup\widetilde{\mathcal{M}\mathcal{D}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) \subseteq \cup\widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$ for any $\mathfrak{s} \in \acute{U}$. For any $\mathfrak{s} \in \acute{U}$, we have

$$\begin{aligned}\cap\widetilde{\mathcal{M}d}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) &= \cap\Big\{\mho \in \widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) : \forall\mathcal{D} \in \widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) \\ &\qquad \wedge \mathcal{D} \subseteq \mho \Longrightarrow \mho = \mathcal{D}\Big\} = \cap\overbrace{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s});\\ \cup\widetilde{\mathcal{M}\mathcal{D}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) &= \cup\Big\{\mho \in \widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) : \forall\mathcal{D} \in \widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) \\ &\qquad \wedge \mathcal{D} \supseteq \mho \Longrightarrow \mho = \mathcal{D}\Big\} = \cup\overbrace{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}).\end{aligned}$$

Hence, $\cap\widetilde{\mathcal{M}d}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) = \cap\widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$ and $\cup\widetilde{\mathcal{M}\mathcal{D}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) = \cup\widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$ for any $\mathfrak{s} \in \acute{U}$.

By the proof of proposition 2, the necessary and sufficient condition for $\widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) = \widetilde{\mathcal{M}d}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) = \widetilde{\mathcal{M}\mathcal{D}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$ for any $\mathfrak{s} \in \acute{U}$ is explored. Additionally, we can explore the complex fuzzy $\beta-$approximation space.

Definition 10: For any CF$\beta-$CAS$\left(\acute{U}, \overset{\frown}{\mho}\right)$, then we call $\overset{\frown}{\mho}$ a semi-reduced complex fuzzy $\beta-$covering if $\mho_1, \mho_2 \in \overset{\frown}{\mho}$ and $\mho_1 \subseteq \mho_2$, then $\mho_1 = \mho_2$.

Additionally, for any CF$\beta-$CAS$\left(\acute{U}, \overset{\frown}{\mho}\right)$, then for any $\mathfrak{s} \in \acute{U}$, $\widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) = \widetilde{\mathcal{M}d}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) = \widetilde{\mathcal{M}\mathcal{D}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$ iff $\overset{\frown}{\mho}$ a semi-reduced complex fuzzy $\beta-$covering. Or, For any CF$\beta-$CAS$\left(\acute{U}, \overset{\frown}{\mho}\right)$, if for any $\mathfrak{s} \in \acute{U}$, $\left|\overbrace{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})\right| = 1$ then $\widetilde{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) = \widetilde{\mathcal{M}d}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s}) = \widetilde{\mathcal{M}\mathcal{D}}^{\beta}_{\overset{\frown}{\mho}}(\mathfrak{s})$. For any CF$\beta-CAS\left(\acute{U}, \overset{\frown}{\mho}\right)$, if $0 < \beta_1 \leq \beta_2 \leq \beta$, then

$$\overbrace{\mathcal{N}}^{\beta_2}_{\overset{\frown}{\mho}}(\mathfrak{s}) \subseteq \overbrace{\mathcal{N}}^{\beta_1}_{\overset{\frown}{\mho}}(\mathfrak{s}), \quad \text{for any } \mathfrak{s} \in \acute{U} \tag{29}$$

$$\widetilde{\mathcal{M}d}^{\beta_2}_{\overset{\frown}{\mho}}(\mathfrak{s}) \subseteq \widetilde{\mathcal{M}d}^{\beta_1}_{\overset{\frown}{\mho}}(\mathfrak{s}), \quad \text{for any } \mathfrak{s} \in \acute{U} \tag{30}$$

$$\widetilde{\mathcal{M}\mathcal{D}}^{\beta_2}_{\overset{\frown}{\mho}}(\mathfrak{s}) \subseteq \widetilde{\mathcal{M}\mathcal{D}}^{\beta_1}_{\overset{\frown}{\mho}}(\mathfrak{s}), \quad \text{for any } \mathfrak{s} \in \acute{U} \tag{31}$$

B. INTERDEPENDENCY OF $\beta-$ NEIGHBORHOOD SYSTEM

For any two CF$\beta-$Cs $\overset{\frown}{\mho}_1, \overset{\frown}{\mho}_2$ on fix set $\acute{U}$ by using the value of parameters $\beta = \beta_{RP}e^{i2\pi(\beta_{IP})}$ where $\beta_{RP}, \beta_{IP} \in (0, 1]$. If $\overbrace{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}_1}(\mathfrak{s}) = \overbrace{\mathcal{N}}^{\beta}_{\overset{\frown}{\mho}_2}(\mathfrak{s})$, for any $\mathfrak{s} \in \acute{U}$, then $\overset{\frown}{\mho}_1$ is not necessarily equal to $\overset{\frown}{\mho}_2$. These laws are also verified with the help of Example 2, which is discussed below.

Example 2: For any CF$\beta-$CAS $\left(\acute{U}, \overset{\frown}{\mho}\right)$ in Example 1, and $\mho' = \overset{\frown}{\mho} \cup \{\mho_6\}$, where

$$\mho_6 = \frac{0.41e^{i2\pi(0.42)}}{\mathfrak{s}_1} + \frac{0.31e^{i2\pi(0.32)}}{\mathfrak{s}_2} + \frac{0.31e^{i2\pi(0.32)}}{\mathfrak{s}_3} + \frac{0.41e^{i2\pi(0.42)}}{\mathfrak{s}_4} + \frac{0.21e^{i2\pi(0.22)}}{\mathfrak{s}_5} + \frac{0.11e^{i2\pi(0.12)}}{\mathfrak{s}_6},$$

From the above analysis, it is clear that the $\overset{\frown}{\mho}'$ is a CF$\beta-$C on fix set $\acute{U}$, $(0 < \beta \leq 0.73)$, then $\overbrace{\mathcal{N}}^{0.5e^{i2\pi(0.51)}}_{\overset{\frown}{\mho}'}(\mathfrak{s}_i)$, $i = 1, 2, 3, 4, 5, 6$, are discussed in the form of Table 2.

We easily find that, $\overbrace{\mathcal{N}}^{0.5e^{i2\pi(0.51)}}_{\overset{\frown}{\mho}'}(\mathfrak{s}_i) = \overbrace{\mathcal{N}}^{0.5e^{i2\pi(0.51)}}_{\overset{\frown}{\mho}}(\mathfrak{s}_i)$, for any $i = 1, 2, 3, 4, 5, 6$, but $\overset{\frown}{\mho}' \neq \overset{\frown}{\mho}$. For two CF$\beta-$Cs for fix set $\acute{U}$ by using the conditions to generate some $\beta-$neighborhood system. For this, we present some ideas of CF$\beta-$CAS.

Definition 11: For any CF$\beta-$CAS $\left(\acute{U}, \overset{\frown}{\mho}\right)$ and $\mho \in \overset{\frown}{\mho}$. If $(\mho)(\mathfrak{s}) < \beta$ for each $\mathfrak{s} \in \acute{U}$, $\beta = \beta_{RP}e^{i2\pi(\beta_{IP})}$ where $\beta_{RP}, \beta_{IP} \in (0, 1]$, then $\mho$ expresses the $\beta-$independent element of $\overset{\frown}{\mho}$, otherwise, $\mho$ expresses the $\beta-$dependent element of $\overset{\frown}{\mho}$.

Example 3: For any CF$\beta-$CAS $\left(\acute{U}, \overset{\frown}{\mho}'\right)$ in Example 2, then $\mho_6$ expresses the $0.5e^{i2\pi(0.51)}-$independent element of $\overset{\frown}{\mho}$, otherwise, $\mho_i$ $(i = 1, 2, 3, 4, 5)$ expresses the $0.5e^{i2\pi(0.51)}-$ *the*dependent element of $\overset{\frown}{\mho}$.

Additionally, for any CF$\beta-$CAS$\left(\acute{U}, \overset{\frown}{\mho}\right)$, if $\mho$ expresses the $\beta-$independent element of $\overset{\frown}{\mho}$, then $\overset{\frown}{\mho} - \{\mho\}$ is also CF$\beta-$C of fix set $\acute{U}$. Similarly, for any CF$\beta-$CAS $\left(\acute{U}, \overset{\frown}{\mho}\right)$, if $\mho$ expresses the $\beta-$independent element of $\overset{\frown}{\mho}$, and $\mho_1 \in \overset{\frown}{\mho} - \{\mho\}$, then $\mho_1$ expresses the $\beta-$independent element iff it is $\beta-$independent element of $\overset{\frown}{\mho} - \{\mho\}$.

TABLE 2. Representation of the values of the $\mathcal{N}^{0.5E2\pi(0.51)}_{\widetilde{\mho}'}(\S_i)$, I = 1,2,3,4,5,6.

$\mathcal{U}$	u_1	u_2	u_3	u_4	u_5	u_6
$\widetilde{\mathcal{N}}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mathcal{C}}'}(u_i)$	$\{\mho_1,\mho_2,\mho_5\}$	$\{\mho_3,\mho_5\}$	$\{\mho_2,\mho_5\}$	$\{\mho_1,\mho_2,\mho_4,\mho_5\}$	$\{\mho_4,\mho_5\}$	$\{\mho_1,\mho_2,\mho_5\}$

TABLE 3. Representation of the values of the above $\beta-$ covering neighborhood systems.

$\mathcal{U}$	u_1	u_2	u_3	u_4	u_5	u_6
$\widetilde{\mathcal{N}}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mathcal{C}}''}(u_i)$	$\{\mho_1,\mho_2,\mho_5\}$	$\{\mho_3,\mho_4,\mho_7\}$	$\{\mho_2,\mho_5\}$	$\{\mho_1,\mho_2,\mho_4,\mho_5\}$	$\{\mho_4,\mho_5\}$	$\{\mho_1,\mho_2,\mho_5\}$
$\widetilde{\mathcal{M}d}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mathcal{C}}''}(u_i)$	$\{\mho_1,\mho_2\}$	$\{\mho_3,\mho_4\}$	$\{\mho_2,\mho_5\}$	$\{\mho_1,\mho_2,\mho_4\}$	$\{\mho_4,\mho_5\}$	$\{\mho_1,\mho_2\}$
$\widetilde{\mathcal{MD}}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mathcal{C}}''}(u_i)$	$\{\mho_2,\mho_5\}$	$\{\mho_4,\mho_7\}$	$\{\mho_2,\mho_5\}$	$\{\mho_2,\mho_4,\mho_5\}$	$\{\mho_4,\mho_5\}$	$\{\mho_2,\mho_5\}$
$\widetilde{\mathcal{N}}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mathcal{C}}'''}(u_i)$	$\{\mho_1,\mho_2,\mho_5\}$	$\{\mho_3,\mho_4,\mho_8\}$	$\{\mho_2,\mho_5\}$	$\{\mho_1,\mho_2,\mho_4,\mho_5\}$	$\{\mho_4,\mho_5\}$	$\{\mho_1,\mho_2,\mho_5\}$
$\widetilde{\mathcal{M}d}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mathcal{C}}'''}(u_i)$	$\{\mho_1,\mho_2\}$	$\{\mho_4,\mho_8\}$	$\{\mho_2,\mho_5\}$	$\{\mho_1,\mho_2,\mho_4\}$	$\{\mho_4,\mho_5\}$	$\{\mho_1,\mho_2\}$
$\widetilde{\mathcal{MD}}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mathcal{C}}'''}(u_i)$	$\{\mho_2,\mho_5\}$	$\{\mho_3,\mho_4\}$	$\{\mho_2,\mho_5\}$	$\{\mho_2,\mho_4,\mho_5\}$	$\{\mho_4,\mho_5\}$	$\{\mho_2,\mho_5\}$

Definition 12: For any CF$\beta-$CAS $\left(\acute{\mathrm{U}},\widetilde{\mho}\right)$ and $\widetilde{\mathcal{B}} \subseteq \widetilde{\mho}$, if $\widetilde{\mho} - \widetilde{\mathcal{B}}$ is the set of $\beta-$independent element of $\widetilde{\mho}$, then $\widetilde{\mathcal{B}}$ expresses the $\beta-$basis of $\widetilde{\mho}$, and expressed by $\mathcal{B}^{\beta}\left(\widetilde{\mho}\right)$.

Definition 13: For any CF$\beta-$CAS$\left(\acute{\mathrm{U}},\widetilde{\mho}\right)$, if each element of $\widetilde{\mho}$ is a $\beta-$dependent element i.e., $\mathcal{B}^{\beta}\left(\widetilde{\mho}\right) = \widetilde{\mho}$, then $\widetilde{\mho}$ is $\beta-$dependent; otherwise, $\widetilde{\mho}$ is $\beta-$independent.

Example 4: For any CF$\beta-$CAS $\left(\acute{\mathrm{U}},\widetilde{\mho}'\right)$ in Example 2, then $\mathcal{B}^{0.5e^{i2\pi(0.51)}}\left(\widetilde{\mho}'\right) = \{\mho_1,\mho_2,\mho_3,\mho_4,\mho_5\}$. Hence, $\widetilde{\mho}'$ is $0.5e^{i2\pi(0.51)}-$independent. Furthermore, For any CF$\beta-$CAS $\left(\acute{\mathrm{U}},\widetilde{\mho}'\right)$ in Example 1, then $\mathcal{B}^{0.5e^{i2\pi(0.51)}}\left(\widetilde{\mho}\right) = \{\mho_1,\mho_2,\mho_3,\mho_4,\mho_5\} = \widetilde{\mho}$. Hence, $\widetilde{\mho}$ is $0.5e^{i2\pi(0.51)}-$dependent. Additionally, based on the following theories, we show that there is no influence on the $\beta-$neighborhood system after ignoring the $\beta-$independent element from the CF$\beta-$C. For any CF$\beta-$CAS $\left(\acute{\mathrm{U}},\widetilde{\mho}\right)$ with for any $\S \in \acute{\mathrm{U}}$, then

$$\widetilde{\mathcal{N}}^{\beta}_{\widetilde{\mho}}(\S) = \widetilde{\mathcal{N}}^{\beta}_{\mathcal{B}^{\beta}\left(\widetilde{\mho}\right)}(\S) \tag{32}$$

$$\widetilde{\mathcal{M}d}^{\beta}_{\widetilde{\mho}}(\S) = \widetilde{\mathcal{M}d}^{\beta}_{\mathcal{B}^{\beta}\left(\widetilde{\mho}\right)}(\S) \tag{33}$$

$$\widetilde{\mathcal{MD}}^{\beta}_{\widetilde{\mho}}(\S) = \widetilde{\mathcal{MD}}^{\beta}_{\mathcal{B}^{\beta}\left(\widetilde{\mho}\right)}(\S) \tag{34}$$

For any two CF$\beta-$CAS $\widetilde{\mho}_1$ and $\widetilde{\mho}_2$ based on $\acute{\mathrm{U}}$, then for any $\S \in \acute{\mathrm{U}}$, $\widetilde{\mathcal{N}}^{\beta}_{\widetilde{\mho}_1}(\S) = \widetilde{\mathcal{N}}^{\beta}_{\widetilde{\mho}_2}(\S)$ iff $\mathcal{B}^{\beta}\left(\widetilde{\mho}_1\right) = \mathcal{B}^{\beta}\left(\widetilde{\mho}_2\right)$. Based on the above theory, we get the following three different theories.

1. For any two $\beta-$independent CF$\beta-$Cs $\widetilde{\mho}_1$ and $\widetilde{\mho}_2$ based on $\acute{\mathrm{U}}$, if for any $\S \in \acute{\mathrm{U}}$, $\widetilde{\mathcal{N}}^{\beta}_{\widetilde{\mho}_1}(\S) = \widetilde{\mathcal{N}}^{\beta}_{\widetilde{\mho}_2}(\S)$ iff $\widetilde{\mho}_1 = \widetilde{\mho}_2$.
2. For any two CF$\beta-$Cs $\widetilde{\mho}_1$ and $\widetilde{\mho}_2$ based on $\acute{\mathrm{U}}$, if for any $\S \in \acute{\mathrm{U}}$, $\mathcal{B}^{\beta}\left(\widetilde{\mho}_1\right) = \mathcal{B}^{\beta}\left(\widetilde{\mho}_2\right)$, then $\widetilde{\mathcal{M}d}^{\beta}_{\widetilde{\mho}_1}(\S) = \widetilde{\mathcal{M}d}^{\beta}_{\widetilde{\mho}_2}(\S)$.
3. For any two CF$\beta-$Cs $\widetilde{\mho}_1$ and $\widetilde{\mho}_2$ based on $\acute{\mathrm{U}}$, if for any $\S \in \acute{\mathrm{U}}$, $\mathcal{B}^{\beta}\left(\widetilde{\mho}_1\right) = \mathcal{B}^{\beta}\left(\widetilde{\mho}_2\right)$, then $\widetilde{\mathcal{MD}}^{\beta}_{\widetilde{\mho}_1}(\S) = \widetilde{\mathcal{MD}}^{\beta}_{\widetilde{\mho}_2}(\S)$.

The reverse processes of Eq. (2) and Eq. (3) are not held. For this, we illustrate Example 5, which is discussed below.

Example 5: For any CF$\beta-$CAS $\left(\acute{\mathrm{U}},\widetilde{\mho}\right)$ in Example 1, $\widetilde{\mho''} = \widetilde{\mho} \cup \{\mho_7\}$ and $\widetilde{\mho'''} = \widetilde{\mho} \cup \{\mho_8\}$, where

$$\mho_7 = \frac{0.41e^{i2\pi(0.42)}}{\S_1} + \frac{0.81e^{i2\pi(0.82)}}{\S_2} + \frac{0.31e^{i2\pi(0.32)}}{\S_3} + \frac{0.41e^{i2\pi(0.42)}}{\S_4} + \frac{0.41^{i2\pi(0.42)}}{\S_5} + \frac{0.11e^{i2\pi(0.12)}}{\S_6},$$

$$\mho_8 = \frac{0.11e^{i2\pi(0.12)}}{\S_1} + \frac{0.71e^{i2\pi(0.72)}}{\S_2} + \frac{0.21e^{i2\pi(0.22)}}{\S_3} + \frac{0.31^{i2\pi(0.32)}}{\S_4} + \frac{0.11^{i2\pi(0.12)}}{\S_5} + \frac{0.11e^{i2\pi(0.12)}}{\S_6},$$

From the above analysis, it is clear that the $\widetilde{\mho'''}$ and $\widetilde{\mho''}$ are the CF$\beta-$Cs for fix set $\acute{\mathrm{U}}$, $(0 < \beta \leq 0.73)$, then $\mathcal{B}^{\beta}\left(\widetilde{\mho'''}\right) = \widetilde{\mho''}$ and $B^{\beta}\left(\widetilde{\mho'''}\right) = \widetilde{\mho'''}$. Furthermore, $\widetilde{\mathcal{N}}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho''}}(\S_i)$, $\widetilde{\mathcal{M}d}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho''}}(\S_i)$, $\widetilde{\mathcal{MD}}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho''}}(\S_i)$, $\widetilde{\mathcal{N}}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho'''}}(\S_i)$, $\widetilde{\mathcal{M}d}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho'''}}(\S_i)$, and $\widetilde{\mathcal{MD}}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho'''}}(\S_i)$, $i = 1,2,3,4,5,6.$ are discussed in the form of Table 3.

We easily find that, $\widetilde{\mathcal{M}d}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho''}}(\S_i) = \widetilde{\mathcal{M}d}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho}}(\S_i)$ and $\widetilde{\mathcal{MD}}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho'''}}(\S_i) = \widetilde{\mathcal{MD}}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho}}(\S_i)$ for any $i = 1,2,3,4,5,6.$

TABLE 4. Representation of the values of the $\widetilde{\mathcal{M}}d^{0.5\,e^{i2\pi(0.51)}}_{\widetilde{\mho}'}(\S_i)$, $i = 1, 2, 3, 4, 5, 6$.

u	u_1	u_2	u_3	u_4	u_5	u_6
$\widetilde{\mathcal{M}}d^{0.5e^{i2\pi(0.51)}}_{\widetilde{c}'}(u_i)$	$\{\mho_1, \mho_2\}$	$\{\mho_3, \mho_4\}$	$\{\mho_2, \mho_5\}$	$\{\mho_1, \mho_2, \mho_4\}$	$\{\mho_4, \mho_5\}$	$\{\mho_1, \mho_2\}$

C. INTERDEPENDENCY OF COMPLEX FUZZY $\beta-$ MINIMAL DESCRIPTION

For any two CF$\beta-$Cs $\widetilde{\mho}_1, \widetilde{\mho}_2$ on fix set $\acute{U}$ by using the value of parameters $\beta = \beta_{RP}e^{i2\pi(\beta_{IP})}$ where $\beta_{RP}, \beta_{IP} \in (0, 1]$. If $\widetilde{\mathcal{M}}d^{\beta}_{\widetilde{\mho}_1}(\S) = \widetilde{\mathcal{M}}d^{\beta}_{\widetilde{\mho}_2}(\S)$, for any $\S \in \acute{U}$, then $\widetilde{\mho}_1$ is not necessarily equal to $\widetilde{\mho}_2$. These laws are also verified with the help of Example 6, which is discussed below.

Example 6: For any CF$\beta-$CAS $\left(\acute{U}, \widetilde{\mho}\right)$ in Example 1, and $\widetilde{\mho}' = \widetilde{\mho} \cup \{\mho_6\}$, where

$$\mho_6 = \frac{0.41e^{i2\pi(0.42)}}{\S_1} + \frac{0.31^{i2\pi(0.32)}}{\S_2} + \frac{0.31^{i2\pi(0.32)}}{\S_3} + \frac{0.41e^{i2\pi(0.42)}}{\S_4} + \frac{0.21e^{i2\pi(0.22)}}{\S_5} + \frac{0.11e^{i2\pi(0.12)}}{\S_6},$$

From the above analysis, it is clear that the $\widetilde{\mho}'$ is a CF$\beta-$C on fix set $\acute{U}$, $(0 < \beta \leq 0.73)$, then $\widetilde{\mathcal{M}}d^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho}'}(\S_i)$, $i = 1, 2, 3, 4, 5, 6$, are discussed in the form of Table 4.

We easily find that, $\widetilde{\mathcal{M}}d^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho}'}(\S_i) = \widetilde{\mathcal{M}}d^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho}}(\S_i)$, for any $i = 1, 2, 3, 4, 5, 6$, but $\widetilde{\mho}' \neq \widetilde{\mho}$. Additionally, we present some ideas of $\beta-$reduct of a CF$\beta-$C.

Definition 14: For any CF$\beta-$CAS $\left(\acute{U}, \widetilde{\mho}\right)$ and $\mho \in \widetilde{\mho}$, if one of the following rules holds:

1. $\mho$ is the $\beta-$independent element of $\widetilde{\mho}$.
2. For $(\mho)(\S) \geq \beta$ for each $\S \in \acute{U}$ implies there exists $\mho' \in \widetilde{\mho} - \{\mho\}$ such that $\mho' \subseteq \mho$ and $(\mho')(\S) \geq \beta$, then $\mho$ is a $\beta-$reducable element of $\widetilde{\mho}$, otherwise, $\mho$ is a $\beta-$irreducible element of $\widetilde{\mho}$

Example 7: Let $\acute{U} = \{\S_1, \S_2, \S_3, \S_4, \S_5\}$ with $\widetilde{\mho} = \{\mho_1, \mho_2, \mho_3, \mho_4\}$ are discussed below for examining $\widetilde{\mho}$ expresses CF$\beta-$C of fix set $\acute{U}$, $(0 < \beta \leq 0.53)$.

$$\mho_1 = \frac{0.61e^{i2\pi(0.62)}}{\S_1} + \frac{0.81e^{i2\pi(0.82)}}{\S_2} + \frac{0.41e^{i2\pi(0.42)}}{\S_3} + \frac{0.61e^{i2\pi(0.62)}}{\S_4} + \frac{0.31e^{i2\pi(0.32)}}{\S_5},$$

$$\mho_2 = \frac{0.41e^{i2\pi(0.42)}}{\S_1} + \frac{0.81^{\,i2\pi(0.82)}}{\S_2} + \frac{0.41e^{i2\pi(0.42)}}{\S_3} + \frac{0.61^{\,i2\pi(0.62)}}{\S_4} + \frac{0.11e^{i2\pi(0.12)}}{\S_5},$$

$$\mho_3 = \frac{0.51e^{i2\pi(0.52)}}{\S_1} + \frac{0.31e^{i2\pi(0.32)}}{\S_2} + \frac{0.51e^{i2\pi(0.52)}}{\S_3} + \frac{0.21e^{i2\pi(0.22)}}{\S_4} + \frac{0.71e^{i2\pi(0.72)}}{\S_5},$$

$$\mho_4 = \frac{0.61e^{i2\pi(0.62)}}{\S_1} + \frac{0.61e^{i2\pi(0.62)}}{\S_2} + \frac{0.11e^{i2\pi(0.12)}}{\S_3} + \frac{0.31e^{i2\pi(0.32)}}{\S_4} + \frac{0.31e^{i2\pi(0.32)}}{\S_5},$$

where $\beta = 0.51e^{i2\pi(0.52)}$ where $0 < \beta_{RP}, \beta_{IP} \leq 0.53$, then $\mho_1(\S_i) \geq 0.51e^{i2\pi(0.52)}$, $(i = 1, 2, 4)$, $\mho_j \subseteq \mho_1, j = 2, 4$, and $\mho_4(\S_1) \geq 0.51e^{i2\pi(0.52)}$, $\mho_2(\S_2) \geq 0.51e^{i2\pi(0.52)}$ and $\mho_2(\S_4) \geq 0.51e^{i2\pi(0.52)}$. Then $\mho_1$ is a $0.51e^{i2\pi(0.52)}-$reducable element of $\widetilde{\mho}$. $\widetilde{\mho} - \{\mho_1\}$ is also CF$\beta-$C of fix set $\acute{U}$, $(0 < \beta \leq 0.53)$.

Proposition 4: For any CF$\beta-$CAS $\left(\acute{U}, \widetilde{\mho}\right)$, if $\mho$ is a $\beta-$reducable element of $\widetilde{\mho}$, then $\widetilde{\mho} - \{\mho\}$ is also a CF$\beta-$C of fix set $\acute{U}$.

Proof: Consider $\widetilde{\mho} = \{\mho_1, \mho_2, \mho_3, \ldots, \mho_m\}$, where $\mho, \mho_i \in \mathcal{F}\left(\acute{U}\right)$, $(i = 1, 2, 3, \ldots, m)$, if $\widetilde{\mho}$ is a $\beta-$reducable element of $\widetilde{\mho}$, then we discussed the following two cases:

Case 1: $\mho$ is $\beta-$independent element of $\widetilde{\mho}$.

Case 2: For $(\mho)(\S) \geq \beta$ for each $\S \in \acute{U}$, then there exists $\mho_r \in \{\mho_1, \mho_2, \mho_3, \ldots, \mho_m\}$ such that $\mho_r \subseteq \mho$ and $(\mho_r)(\S) \geq \beta$.

For case 1, we choose for any CF$\beta-$CAS$\left(\acute{U}, \widetilde{\mho}\right)$, if $\mho$ expresses the $\beta-$independent element of $\widetilde{\mho}$, then $\widetilde{\mho} - \{\mho\}$ is also CF$\beta-$C of fix set $\acute{U}$.

For case 2, for $(\mho)(\S) \geq \beta$ for each $\S \in \acute{U}$, then $(\mho_r)(\S) \geq \beta$ i.e., $\coprod_{j=1}^{m}(\mho_j)(\S) \geq (\mho_r)(\S) \geq \beta$. Therefore, $\widetilde{\mho} - \{\mho\}$ is also a CF$\beta-$C of fix set $\acute{U}$.

Proposition 5: For any CF$\beta-$CAS $\left(\acute{U}, \widetilde{\mho}\right)$, if $\mho$ is a $\beta-$reducable element of $\widetilde{\mho}$, then $\mho_1 \in \widetilde{\mho} - \{\mho\}$, then $\mho_1$ is a $\beta-$reducable element of $\widetilde{\mho}$ iff it's $\beta-$reducable element of $\widetilde{\mho} - \{\mho\}$.

Proof: We assume that $\mho_1$ is a $\beta-$reducable element of $\widetilde{\mho}$, then to prove that it's $\beta-$reducible element of $\widetilde{\mho} - \{\mho\}$. Consider $\widetilde{\mho} = \{\mho_1, \mho_2, \mho_3, \ldots, \mho_m\}$, where $\mho, \mho_i \in \mathcal{F}\left(\acute{U}\right)$, $(i = 1, 2, 3, \ldots, m)$, if $\mho$ is a $\beta-$reducable element of $\widetilde{\mho}$, then we discussed the following two cases:

Case 1: $\mho$ is $\beta-$independent element of $\widetilde{\mho}$.

Case 2: For $(\mho)(\S) \geq \beta$ for each $\S \in \acute{U}$, then there exists $\mho_r \in \{\mho_1, \mho_2, \mho_3, \ldots, \mho_m\}$ such that $\mho_r \subseteq \mho$ and $(\mho_r)(\S) \geq \beta$.

For case 1, we choose for any CF$\beta-$CAS $\left(\acute{U}, \widetilde{\mho}\right)$, if $\mho$ expresses the $\beta-$independent element of $\widetilde{\mho}$, and $\mho_1 \in \widetilde{\mho} - \{\mho\}$, then $\mho_1$ expresses the $\beta-$independent element iff it's $\beta-$independent element of $\widetilde{\mho} - \{\mho\}$.

For case 2, if $\mho_1$ is $\beta-$reduceable element of $\widetilde{\mho}$, then $(\mho_1)(\S') < \beta$ for any $\S' \in \acute{U}$. If $(\mho_1)(\S') \geq \beta$, then there exists $\mho' \in \widetilde{\mho}$ such that $\mho' \subseteq \mho_1$ and $(\mho')(\S') \geq \beta$. If $(\mho_1)(\S') < \beta$

for any $\mathfrak{k}' \in \acute{U}$, then obviously $\mho_1$ is $\beta-$reducecable element of $\widetilde{\mho} - \{\mho\}$. For any $\mathfrak{k}' \in \acute{U}$, if $(\mho')(\mathfrak{k}') \geq \beta$ and $\mho' = \mho$, then there exists $\mho_r \in \{\mho_1, \mho_2, \mho_3, \ldots, \mho_m\}$ such that $\mho_r \subseteq \mho' \subseteq \mho_1$ and $(\mho_r)(\mho') \geq \beta$, then $\mho_1$ is $\beta-$reducecable element of $\widetilde{\mho} - \{\mho\}$. If $\mho' \neq \mho$, then it's also obviously $\mho_1$ is $\beta-$reducecable element of $\widetilde{\mho} - \{\mho\}$. We assume that it's $\beta-$reducible element of $\widetilde{\mho} - \{\mho\}$, then to prove that $\mho_1$ is a $\beta-$reducecable element of $\widetilde{\mho}$, which is straightforward, hence the proof of the result is completed.

Definition 15: For any CF$\beta-$CAS $\left(\acute{U}, \widetilde{\mho}\right)$ and $\tilde{\mathfrak{D}} \subseteq \widetilde{\mho}$, if $\widetilde{\mho} - \tilde{\mathfrak{D}}$ is the set of all $\beta-$reducible element of $\widetilde{\mho}$, then $\tilde{\mathfrak{D}}$ is called $\beta-$reduet of $\widetilde{\mho}$, and it is expressed by $\mathcal{R}^{\beta}\left(\widetilde{\mho}\right)$.

Example 8: For any CF$\beta-$CAS $\left(\acute{U}, \widetilde{\mho}\right)$ in Example 7, then $\mathcal{R}^{0.51e^{i2\pi(0.52)}}\left(\widetilde{\mho}\right) = \{\mho_2, \mho_3, \mho_4\}$.

Definition 16: For any CF$\beta-$CAS$\left(\acute{U}, \widetilde{\mho}\right)$, if every element of $\widetilde{\mho}$ is $\beta-$irreducecable element i.e., $\mathcal{R}^{\beta}\left(\widetilde{\mho}\right) = \widetilde{\mho}$, then $\widetilde{\mho}$ is called $\beta-$irreducible; otherwise $\widetilde{\mho}\beta-$reducible.

The following proposition state that, when we deleting the $\beta-$reducible element from CF$\beta-$C in CF$\beta-$MND, so it has no influence.

Proposition 6: For any CF$\beta-$CAS $\left(\acute{U}, \widetilde{\mho}\right)$ and for any $\mathfrak{k} \in \acute{U}$, then

$$\widetilde{\mathfrak{M}}d^{\beta}_{\widetilde{\mho}}(\mathfrak{k}) = \widetilde{\mathfrak{M}}d^{\beta}_{\mathcal{R}^{\beta}\left(\widetilde{\mho}\right)}(\mathfrak{k}) \tag{35}$$

Proof: Consider $\widetilde{\mho} = \{\mho_1, \mho_2, \ldots, \mho_m\}$ with $\mho, \mho_i \in \mathcal{F}\left(\acute{U}\right)(i = 1, 2, \ldots, m)$ and $\mho$ is a $\beta-$reducecable element $\widetilde{\mho}$. Then by using the Proposition 4 state that $\widetilde{\mho} - \{\mho\}$ is also a CF$\beta-$C of fix set $\acute{U}$, then we discussed the following two cases:

Case 1: $\mho$ is $\beta-$independent element of $\widetilde{\mho}$.

Case 2: For $(\mathfrak{k})(\mathfrak{k}) \geq \beta$ for each $\mathfrak{k} \in \acute{U}$, then there exists $\mho_r \in \{\mho_1, \mho_2, \mho_3, \ldots, \mho_m\}$ such that $\mho_r \subseteq \mho$ and $(\mho_r)(\mathfrak{k}) \geq \beta$.

For each $\mathfrak{k} \in \acute{U}$ and by using the Eq. (33), $\widetilde{\mathfrak{M}}d^{\beta}_{\widetilde{\mho}}(\mathfrak{k}) = \widetilde{\mathfrak{M}}d^{\beta}_{\mathcal{R}^{\beta}\left(\widetilde{\mho}\right)}(\mathfrak{k})$ is holds obviously. Additionally, for each $\mathfrak{k} \in \acute{U}$, we express the CF$\beta-$MND of $\mathfrak{k}$ generates by CF$\beta-$C $\widetilde{\mho}$ as $\widetilde{\mathfrak{M}}d^{\beta}_{\widetilde{\mho}}(\mathfrak{k})$ express the CF$\beta-$MND of $\mathfrak{k}$ generates by CF$\beta-$C $\widetilde{\mho}-\{\mho\}$ as $\widetilde{\mathfrak{M}}d^{\beta}_{\widetilde{\mho}-\{\mho\}}(\mathfrak{k})$. If $(\mho)(\mathfrak{k}) \geq \beta$ by **Case 2**, there exists $\mho_r \in \{\mho_1, \mho_2, \mho_3, \ldots, \mho_m\}$ such that $\mho_r \subseteq \mho$ and $(\mho_r)(\mathfrak{k}) \geq \beta$. It's clear that $\mho \notin \widetilde{\mathfrak{M}}d^{\beta}_{\widetilde{\mho}}(\mathfrak{k})$, thus $\widetilde{\mathfrak{M}}d^{\beta}_{\widetilde{\mho}}(\mathfrak{k}) = \widetilde{\mathfrak{M}}d^{\beta}_{\widetilde{\mho}-\{\mho\}}(\mathfrak{k})$ holds for each $\mathfrak{k} \in \acute{U}$. Then we choose the following two steps:

Step 1: If $\widetilde{\mho} - \{\mho\}$ is $\beta-$irreducible, then $\mathcal{R}^{\beta}\left(\widetilde{\mho}\right) = \widetilde{\mho} - \{\mho\} = \mathcal{R}^{\beta}\left(\widetilde{\mho} - \mho\right)$ and $\widetilde{\mathfrak{M}}d^{\beta}_{\widetilde{\mho}}(\mathfrak{k}) = \widetilde{\mathfrak{M}}d^{\beta}_{\widetilde{\mho}-\{\mho\}}(\mathfrak{k}) = \widetilde{\mathfrak{M}}d^{\beta}_{\mathcal{R}^{\beta}\left(\widetilde{\mho}\right)}(\mathfrak{k}) = \widetilde{\mathfrak{M}}d^{\beta}_{\mathcal{R}^{\beta}\left(\widetilde{\mho}-\mho\right)}(\mathfrak{k})$ for each $\mathfrak{k} \in \acute{U}$.

Step 2: If $\widetilde{\mho} - \{\mho\}$ is $\beta-$reduceable, then there exists $\mho_{i_1}, \mho_{i_2}, \ldots, \mho_{i_s} \in \widetilde{\mho} - \{\mho\}, (i_1, i_2, \ldots, i_s \in \{1, 2, \ldots, m\})$ such that $\mho_{i_1}, \mho_{i_2}, \ldots, \mho_{i_s}$ is $\beta-$reduceable element of $\widetilde{\mho} - \{\mho\}$, then $\widetilde{\mho} - \left\{\mho_{i_1}, \mho_{i_2}, \ldots, \mho_{i_s}\right\}$ is $\beta-$irreducible. Therefore, $\mathcal{R}^{\beta}\left(\widetilde{\mho} - \mho\right) = \widetilde{\mho} - \left\{\mho_{i_1}, \mho_{i_2}, \ldots, \mho_{i_s}\right\}$, then by using Proposition 5, $\left\{\mho, \mho_{i_1}, \mho_{i_2}, \ldots, \mho_{i_s}\right\}$ is $\beta-$reduceable element of $\widetilde{\mho}$. Therefore, $\mathcal{R}^{\beta}\left(\widetilde{\mho}\right) = \widetilde{\mho} - \left\{\mho, \mho_{i_1}, \mho_{i_2}, \ldots, \mho_{i_s}\right\} = \mathcal{R}^{\beta}\left(\widetilde{\mho} - \mho\right)$, in other hand, we have $\widetilde{\mathfrak{M}}d^{\beta}_{\widetilde{\mho}}(\mathfrak{k}) = \widetilde{\mathfrak{M}}d^{\beta}_{\widetilde{\mho}-\{\mho\}}(\mathfrak{k}) = \widetilde{\mathfrak{M}}d^{\beta}_{\widetilde{\mho}-\{\mho,\mho_{i_1}\}}(\mathfrak{k}) = \widetilde{\mathfrak{M}}d^{\beta}_{\widetilde{\mho}-\{\mho,\mho_{i_1},\mho_{i_2}\}}(\mathfrak{k}) = \widetilde{\mathfrak{M}}d^{\beta}_{\widetilde{\mho}-\{\mho,\mho_{i_1},\mho_{i_2},\ldots,\mho_{i_s}\}}(\mathfrak{k}) = \widetilde{\mathfrak{M}}d^{\beta}_{\mathcal{R}^{\beta}\left(\widetilde{\mho}\right)}(\mathfrak{k}) = \widetilde{\mathfrak{M}}d^{\beta}_{\mathcal{R}^{\beta}\left(\widetilde{\mho}-\mho\right)}(\mathfrak{k})$ for each $\mathfrak{k} \in \acute{U}$. Hence, $\widetilde{\mathfrak{M}}d^{\beta}_{\widetilde{\mho}}(\mathfrak{k}) = \widetilde{\mathfrak{M}}d^{\beta}_{\mathcal{R}^{\beta}\left(\widetilde{\mho}\right)}(\mathfrak{k})$ holds for each $\mathfrak{k} \in \acute{U}$.

From the above analysis, we get the result, for any $\widetilde{\mho}_1, \widetilde{\mho}_2$ are two CF$\beta-$Cs of fix set $\acute{U}$, then $\widetilde{\mathfrak{M}}d^{\beta}_{\widetilde{\mho}_1}(\mathfrak{k}) = \widetilde{\mathfrak{M}}d^{\beta}_{\widetilde{\mho}_2}(\mathfrak{k})$ iff $\mathcal{R}^{\beta}\left(\widetilde{\mho}_1\right) = \mathcal{R}^{\beta}\left(\widetilde{\mho}_2\right)$. Additionally, for any $\widetilde{\mho}_1, \widetilde{\mho}_2$ are two irreducible CF$\beta-$Cs of fix set $\acute{U}$, then $\tilde{M}d^{\beta}_{\widetilde{\mho}_1}(\mathfrak{k}) = \tilde{M}d^{\beta}_{\widetilde{\mho}_2}(\mathfrak{k})$ iff $\widetilde{\mho}_1 = \widetilde{\mho}_2$.

D. INTERDEPENDENC OF COMPLEX FUZZY $\beta-$ MAXIMAL DESCRIPTION

For any two CF$\beta-$Cs $\widetilde{\mho}_1, \widetilde{\mho}_2$ on fix set $\acute{U}$ by using the value of parameters $\beta = \beta_{RP}e^{i2\pi(\beta_{IP})}$ where $\beta_{RP}, \beta_{IP} \in (0, 1]$. If $\widetilde{\mathfrak{M}}\mathfrak{D}^{\beta}_{\widetilde{\mho}_1}(\mathfrak{k}) = \widetilde{\mathfrak{M}}\mathfrak{D}^{\beta}_{\widetilde{\mho}_2}(\mathfrak{k})$, for any $\mathfrak{k} \in \acute{U}$, then $\widetilde{\mho}_1$ is not necessarily equal to $\widetilde{\mho}_2$. These laws are also verified with the help of Example 6, which is discussed below.

Example 9: For any CF$\beta-$CAS $\left(\acute{U}, \widetilde{\mho}\right)$ in Example 1, and $\widetilde{\mho'} = \widetilde{\mho} \cup \{\mho_6\}$, where

$$\mho_6 = \frac{0.41\ i2\pi\ (0.42)}{\mathfrak{k}_1} + \frac{0.31e^{i2\pi(0.32)}}{\mathfrak{k}_2} + \frac{0.31e^{i2\pi(0.32)}}{\mathfrak{k}_3} + \frac{0.41e^{i2\pi(0.42)}}{\mathfrak{k}_4} + \frac{0.21e^{i2\pi(0.22)}}{\mathfrak{k}_5} + \frac{0.11e^{i2\pi(0.12)}}{\mathfrak{k}_6},$$

From the above analysis, it is clear that the $\widetilde{\mho'}$ is a CF$\beta-$C on fix set $\acute{U}$, $(0 < \beta \leq 0.73)$, then $\widetilde{\mathfrak{M}}\mathfrak{D}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho'}}(\mathfrak{k}_i), i = 1, 2, 3, 4, 5, 6$, are discussed in the form of Table 5.

We easily find that, $\widetilde{\mathfrak{M}}\mathfrak{D}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho'}}(\mathfrak{k}_i) = \widetilde{\mathfrak{M}}\mathfrak{D}^{0.5e^{i2\pi(0.51)}}_{\widetilde{\mho}}(\mathfrak{k}_i)$, for any $i = 1, 2, 3, 4, 5, 6$, but $\widetilde{\mho'} \neq \widetilde{\mho}$.

Definition 17: For any CF$\beta-$CAS $\left(\acute{U}, \widetilde{\mho}\right)$ and $\mho \in \widetilde{\mho}$, if one of the following rules holds:

3. $\mho$ is the $\beta-$independent element of $\widetilde{\mho}$.
4. For $(\mho)(\mathfrak{k}) \geq \beta$ for each $\mathfrak{k} \in \acute{U}$ implies there exists $\mho' \in \widetilde{\mho} - \{\mho\}$ such that $\mho' \subseteq \mho$, then $\mho$ is a $\beta-$dispensable element of $\widetilde{\mho}$, otherwise, $\mho$ is a $\beta-$indispensable element of $\widetilde{\mho}$

TABLE 5. Representation of the values of the $\widetilde{\mathcal{M}\mathcal{D}}^{0.5e^{i2\pi(0.51)}}_{\overset{\frown}{\mho}'}\left(\S_i\right)$, I = 1,2,3,4,5,6 .

u	u_1	u_2	u_3	u_4	u_5	u_6
$\widetilde{\mathcal{MD}}^{0.5e^{i2\pi(0.51)}}_{c'}(u_i)$	$\{\mho_2,\mho_5\}$	$\{\mho_3,\mho_4\}$	$\{\mho_2,\mho_5\}$	$\{\mho_2,\mho_4,\mho_5\}$	$\{\mho_4,\mho_5\}$	$\{\mho_2,\mho_5\}$

Example 10: Let $\acute{\ddot{U}} = \{\S_1,\S_2,\S_3,\S_4,\S_5\}$ with $\overset{\frown}{\mho} = \{\mho_1,\mho_2,\mho_3,\mho_4\}$ are discussed below for examining $\overset{\frown}{\mho}$ expresses CF$\beta-$C of fix set $\acute{\ddot{U}}$, $(0<\beta\leq 0.53)$.

$$\mho_1 = \frac{0.61e^{i2\pi(0.62)}}{\S_1}+\frac{0.81e^{i2\pi(0.82)}}{\S_2}+\frac{0.41e^{i2\pi(0.42)}}{\S_3}+\frac{0.61e^{i2\pi(0.62)}}{\S_4}+\frac{0.31e^{i2\pi(0.32)}}{\S_5},$$

$$\mho_2 = \frac{0.41e^{i2\pi(0.42)}}{\S_1}+\frac{0.81\ {}^{i2\pi(0.82)}}{\S_2}+\frac{0.41e^{i2\pi(0.42)}}{\S_3}+\frac{0.61e^{i2\pi(0.62)}}{\S_4}+\frac{0.11e^{i2\pi(0.12)}}{\S_5},$$

$$\mho_3 = \frac{0.51e^{i2\pi(0.52)}}{\S_1}+\frac{0.31e^{i2\pi(0.32)}}{\S_2}+\frac{0.51\ {}^{i2\pi(0.52)}}{\S_3}+\frac{0.21e^{i2\pi(0.22)}}{\S_4}+\frac{0.71\ {}^{i2\pi(0.72)}}{\S_5},$$

$$\mho_4 = \frac{0.61e^{i2\pi(0.62)}}{\S_1}+\frac{0.61\ {}^{i2\pi(0.62)}}{\S_2}+\frac{0.11\ {}^{i2\pi(0.12)}}{\S_3}+\frac{0.31e^{i2\pi(0.32)}}{\S_4}+\frac{0.31e^{i2\pi(0.32)}}{\S_5},$$

where $\beta = 0.51e^{i2\pi(0.52)}$ where $0<\beta_{RP},\beta_{IP}\leq 0.53$, then $\mho_2(\S_i)\geq 0.51e^{i2\pi(0.52)}$, $(i=2,4)$, $\mho_2\subseteq\mho_j, j=1,4,$. Then $\mho_2$ is a $0.51e^{i2\pi(0.52)}-$dispensable element of $\overset{\frown}{\mho}$. $\overset{\frown}{\mho}-\{\mho_2\}$ is also CF$\beta-$C of fix set $\acute{\ddot{U}}$, $(0<\beta_{RP},\beta_{IP}\leq 0.53)$.

Proposition 7: For any CF$\beta-$CAS $\left(\acute{\ddot{U}},\overset{\frown}{\mho}\right)$, if $\mho$ is a $\beta-$dispensable element of $\overset{\frown}{\mho}$, then $\overset{\frown}{\mho}-\{\mho\}$ is also a CF$\beta-$C of fix set $\acute{\ddot{U}}$.

Proof: Consider $\overset{\frown}{\mho}=\{\mho_1,\mho_2,\mho_3,\ldots,\mho_m\}$, where $\mho,\mho_i\in\mathcal{F}\left(\acute{\ddot{U}}\right)$, $(i=1,2,3,\ldots,m)$, if $\mho$ is a $\beta-$dispensable element of $\overset{\frown}{\mho}$, then we discussed the following two cases:

Case 1: $\mho$ is $\beta-$independent element of $\overset{\frown}{\mho}$.

Case 2: For $(\mho)(\S)\geq\beta$ for each $\S\in\acute{\ddot{U}}$, then there exists $\mho_r\in\{\mho_1,\mho_2,\mho_3,\ldots,\mho_m\}$ such that $\mho_r\subseteq\mho$.

For case 1, we choose for any CF$\beta-$CAS $\left(\acute{\ddot{U}},\overset{\frown}{\mho}\right)$, if $\mho$ expresses the $\beta-$independent element of $\overset{\frown}{\mho}$, then $\overset{\frown}{\mho}-\{\mho\}$ is also CF$\beta-$C of fix set $\acute{\ddot{U}}$.

For case 2, for $(\mho)(\S)\geq\beta$ for each $\S\in\acute{\ddot{U}}$, then $(\mho_r)(\S)\geq\beta$ i.e., $\coprod_{j=1}^{m}\left(\mho_j\right)(\S)\geq(\mho_r)(\S)\geq\beta$. Therefore, $\overset{\frown}{\mho}-\{\mho\}$ is also a CF$\beta-$C of fix set $\acute{\ddot{U}}$.

Proposition 8: For any CF$\beta-$CAS $\left(\acute{\ddot{U}},\overset{\frown}{\mho}\right)$, if $\mho$ is a $\beta-$dispensable element of $\overset{\frown}{\mho}$, then $\mho_1\in\overset{\frown}{\mho}-\{\mho\}$, then $\mho_1$ is a $\beta-$dispensable element of $\overset{\frown}{\mho}$ iff it's $\beta-$dispensable element of $\overset{\frown}{\mho}-\{\mho\}$.

Proof: We assume that $\mho_1$ is a $\beta-$dispensable element of $\overset{\frown}{\mho}$, then to prove that it's $\beta-$dispensable element of $\overset{\frown}{\mho}-\{\mho\}$. Consider $\overset{\frown}{\mho}=\{\mho_1,\mho_2,\mho_3,\ldots,\mho_m\}$, where $\mho,\mho_i\in\mathcal{F}\left(\acute{\ddot{U}}\right)$, $(i=1,2,3,\ldots,m)$, if $\mho$ is a $\beta-$dispensable element of $\overset{\frown}{\mho}$, then we discussed the following two cases:

Case 1: $\mho$ is $\beta-$independent element of $\overset{\frown}{\mho}$.

Case 2: For $(\mho)(\S)\geq\beta$ for each $\S\in\acute{\ddot{U}}$, then there exists $\mho_r\in\{\mho_1,\mho_2,\mho_3,\ldots,\mho_m\}$ such that $\mho_r\subseteq\mho$.

For case 1, we choose for any CF$\beta-$CAS $\left(\acute{\ddot{U}},\overset{\frown}{\mho}\right)$, if $\mho$ expresses the $\beta-$independent element of $\overset{\frown}{\mho}$, and $\mho_1\in\overset{\frown}{\mho}-\{\mho\}$, then $\mho_1$ expresses the $\beta-$independent element iff it's $\beta-$independent element of $\overset{\frown}{\mho}-\{\mho\}$.

For case 2, if $\mho_1$ is $\beta-$dispensable element of $\overset{\frown}{\mho}$, then $(\mho_1)\left(\S'\right)<\beta$ for any $\S'\in\acute{\ddot{U}}$. If $(\mho_1)\left(\S'\right)\geq\beta$, then there exists $\mho'\in\overset{\frown}{\mho}$ such that $\mho'\subseteq\mho_1$ and $\left(\mho'\right)\left(\S'\right)\geq\beta$. If $(\mho_1)\left(\S'\right)<\beta$ for any $\S'\in\acute{\ddot{U}}$, then obviously $\mho_1$ is $\beta-$dispensable element of $\overset{\frown}{\mho}-\{\mho\}$. For any $\S'\in\acute{\ddot{U}}$, if $\left(\mho'\right)\left(\S'\right)\geq\beta$ and $\mho'=\mho$, then there exists $\mho_r\in\{\mho_1,\mho_2,\mho_3,\ldots,\mho_m\}$ such that $\mho_r\subseteq\mho'\subseteq\mho_1$ and $(\mho_r)\left(\S'\right)\geq\beta$, then $\mho_1$ is $\beta-$dispensable element of $\overset{\frown}{\mho}-\{\mho\}$. If $\mho'\neq\mho$, then it's also obviously $\mho_1$ is $\beta-$dispensable element of $\overset{\frown}{\mho}-\{\mho\}$. We assume that it's $\beta-$dispensable element of $\overset{\frown}{\mho}-\{\mho\}$, then to prove that $\mho_1$ is a $\beta-$dispensable element of $\overset{\frown}{\mho}$, which is straightforward, hence the proof of the result is completed.

Definition 18: For any CF$\beta-$CAS $\left(\acute{\ddot{U}},\overset{\frown}{\mho}\right)$ and $\tilde{\mathfrak{D}}\subseteq\overset{\frown}{\mho}$, if $\overset{\frown}{\mho}-\tilde{\mathfrak{D}}$ is the set of all $\beta-$dispensable element of $\overset{\frown}{\mho}$, then $\tilde{\mathfrak{D}}$ is called the kernel of $\overset{\frown}{\mho}$, and it is expressed by $\ddot{\eth}^{\beta}\left(\overset{\frown}{\mho}\right)$.

Example 11: For any CF$\beta-$CAS $\left(\acute{\ddot{U}},\overset{\frown}{\mho}\right)$ in Example 10, then $\ddot{\eth}^{0.51e^{i2\pi(0.52)}}\left(\overset{\frown}{\mho}\right)=\{\mho_1,\mho_3,\mho_4\}$.

Definition 19: For any CF$\beta-$CAS $\left(\acute{\ddot{U}},\overset{\frown}{\mho}\right)$, if every element of $\overset{\frown}{\mho}$ is $\beta-$indispensable element i.e. $\ddot{\eth}^{\beta}\left(\overset{\frown}{\mho}\right)=\overset{\frown}{\mho}$, then $\overset{\frown}{\mho}$ is called $\beta-$indispensable; otherwise $\overset{\frown}{\mho}\beta-$dispensable.

Example 12: For any CF$\beta-$CAS $\left(\acute{\ddot{U}},\overset{\frown}{\mho}\right)$ in Example 1, then $\ddot{\eth}^{0.5e^{i2\pi(0.51)}}\left(\overset{\frown}{\mho}\right)=\overset{\frown}{\mho}$, i.e. $\overset{\frown}{\mho}$ is $0.5e^{i2\pi(0.51)}-$indispensable.

The following proposition state that, when we deleting the $\beta-$dispensable element from CF$\beta-$C in CF$\beta-$MXD, so it has no influence.

Proposition 9: For any CF$\beta-$CAS $\left(\acute{\ddot{U}},\overset{\frown}{\mho}\right)$ and for any $\S\in\acute{\ddot{U}}$, then

$$\widetilde{\mathcal{M}\mathcal{D}}^{\beta}_{\overset{\frown}{\mho}}(\S)=\widetilde{\mathcal{M}\mathcal{D}}^{\beta}_{\ddot{\eth}^{\beta}\left(\overset{\frown}{\mho}\right)}(\S) \tag{36}$$

Proof: Consider $\widetilde{\mho} = \{\mho_1, \mho_2, \ldots, \mho_m\}$ with $\mho, \mho_i \in \mathcal{F}\left(\acute{U}\right) (i = 1, 2, \ldots, m)$ and $\mho$ is a $\beta-$dispensable element $\widetilde{\mho}$. Then by using the Proposition 7 state that $\widetilde{\mho} - \{\mho\}$ is also a CF$\beta-$C of fix set $\acute{U}$, then we discussed the following two cases:

Case 1: $\mho$ is $\beta-$independent element of $\widetilde{\mho}$.

Case 2: For $(\mho)(\mathfrak{S}) \geq \beta$ for each $\mathfrak{S} \in \acute{U}$, then there exists $\mho_r \in \{\mho_1, \mho_2, \mho_3, \ldots, \mho_m\}$ such that $\mho_r \subseteq \mho$.

For each $\mathfrak{S} \in \acute{U}$ and by using the Eq. (33), $\widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S}) = \widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\bar{\bar{\mho}}^{\beta}\left(\widetilde{\mho}\right)}(\mathfrak{S})$ is holds obviously. Additionally, for each $\mathfrak{S} \in \acute{U}$, we express the CF$\beta-$MXD of $\mathfrak{S}$ generates by CF$\beta-$C $\widetilde{\mho}$ as $\widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S})$ express the CF$\beta-$MXD of $\mathfrak{S}$ generates by CF$\beta-$C $\widetilde{\mho}-\{\mho\}$ as $\widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\widetilde{\mho}-\{\mho\}}(\mathfrak{S})$. If $(\mho)(\mathfrak{S}) \geq \beta$ by **Case 2**, there exists $\mho_r \in \{\mho_1, \mho_2, \mho_3, \ldots, \mho_m\}$ such that $\mho_r \subseteq \mho$ and $(\mho_r)(\mathfrak{S}) \geq \beta$. It's clear that $\mho \notin \widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S})$, thus $\widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S}) = \widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\widetilde{\mho}-\{\mho\}}(\mathfrak{S})$ holds for each $\mathfrak{S} \in \acute{U}$. Then we choose the following two steps:

Step 1: If $\widetilde{\mho} - \{\mho\}$ is $\beta-$indispensable, then $\bar{\bar{\mho}}^{\beta}\left(\widetilde{\mho}\right) = \widetilde{\mho} - \{\mho\} = \bar{\bar{\mho}}^{\beta}\left(\widetilde{\mho} - \mho\right)$ and $\widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S}) = \widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\widetilde{\mho}-\{\mho\}}(\mathfrak{S}) = \widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\bar{\bar{\mho}}^{\beta}\left(\widetilde{\mho}\right)}(\mathfrak{S}) = \widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\bar{\bar{\mho}}^{\beta}\left(\widetilde{\mho}-\mho\right)}(\mathfrak{S})$ for each $\mathfrak{S} \in \acute{U}$.

Step 2: If $\widetilde{\mho} - \{\mho\}$ is $\beta-$dispensable, then there exists $\mho_{i_1}, \mho_{i_2}, \ldots, \mho_{i_s} \in \widetilde{\mho} - \{\mho\}, (i_1, i_2, \ldots, i_s \in \{1, 2, \ldots, m\})$ such that $\mho_{i_1}, \mho_{i_2}, \ldots, \mho_{i_s}$ is $\beta-$dispensable element of $\widetilde{\mho} - \{\mho\}$, then $\widetilde{\mho} - \left\{\mho_{i_1}, \mho_{i_2}, \ldots, \mho_{i_s}\right\}$ is $\beta-$indispensable. Therefore, $\bar{\bar{\mho}}^{\beta}\left(\widetilde{\mho} - \mho\right) = \widetilde{\mho} - \{\mho_{i_1}, \mho_{i_2}, \ldots, \mho_{i_s}\}$, then by using Proposition 8, $\left\{\mho, \mho_{i_1}, \mho_{i_2}, \ldots, \mho_{i_s}\right\}$ is $\beta-$dispensable element of $\widetilde{\mho}$. Therefore, $\bar{\bar{\mho}}^{\beta}\left(\widetilde{\mho}\right) = \widetilde{\mho} - \left\{\mho, \mho_{i_1}, \mho_{i_2}, \ldots, \mho_{i_s}\right\} = \bar{\bar{\mho}}^{\beta}\left(\widetilde{\mho} - \mho\right)$, in the other hand, we have $\widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S}) = \widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\widetilde{\mho}-\{\mho\}}(\mathfrak{S}) = \widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\widetilde{\mho}-\{\mho,\mho_{i_1}\}}(\mathfrak{S}) = \widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\widetilde{\mho}-\{\mho,\mho_{i_1},\mho_{i_2}\}}(\mathfrak{S}) = \widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\widetilde{\mho}-\{\mho,\mho_{i_1},\mho_{i_2},\ldots,\mho_{i_s}\}}(\mathfrak{S}) = \widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\bar{\bar{\mho}}^{\beta}\left(\widetilde{\mho}\right)}(\mathfrak{S}) = \widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\bar{\bar{\mho}}^{\beta}\left(\widetilde{\mho}-\mho\right)}(\mathfrak{S})$ for each $\mathfrak{S} \in \acute{U}$. Hence, $\widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S}) = \widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\bar{\bar{\mho}}^{\beta}\left(\widetilde{\mho}\right)}(\mathfrak{S})$ holds for each $\mathfrak{S} \in \acute{U}$.

From the above analysis, we get the result, for any $\widetilde{\mho_1}, \widetilde{\mho_2}$ are two CF$\beta-$Cs of fix set $\acute{U}$, then $\widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\widetilde{\mho_1}}(\mathfrak{S}) = \widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\widetilde{\mho_2}}(\mathfrak{S})$ iff $\bar{\bar{\mho}}^{\beta}\left(\widetilde{\mho_1}\right) = \bar{\bar{\mho}}^{\beta}\left(\widetilde{\mho_2}\right)$. Additionally, for any $\widetilde{\mho_1}, \widetilde{\mho_2}$ are two indispensable CF$\beta-$Cs of fix set $\acute{U}$, then $\widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\widetilde{\mho_1}}(\mathfrak{S}) = \widetilde{\mathfrak{M}\mathfrak{D}}^{\beta}_{\widetilde{\mho_2}}(\mathfrak{S})$ iff $\widetilde{\mho_1} = \widetilde{\mho_2}$.

IV. COMPLEX FUZZY NEIGHBORHOOD OPERATORS B ASED ON A COMPLEX FUZZY β COVERING

This study aims to present the idea of complex fuzzy neighborhood operators in the environment of rough sets theory based on the modifications such as $\beta-$neighborhood system, CF$\beta-$MND, CF$\beta-$MXD. Additionally, and the complex fuzzy neighborhood operators (CFNO) are discussed below.

Definition 19: A CFNO is elaborated by:

$$\widetilde{\mathcal{N}} : \acute{U} \to \mathcal{F}\left(\acute{U}\right) \tag{37}$$

where $\mathfrak{S} \in \acute{U}$ such that $\widetilde{\mathcal{N}}(\mathfrak{S}) \in \mathcal{F}\left(\acute{U}\right)$.

Definition 20: For any CF$\beta-$CAS $\left(\acute{U}, \widetilde{\mho}\right)$ and for any $\mathfrak{S} \in \acute{U}$ with $\widetilde{\mho} = \{\mho_1, \mho_2, \ldots, \mho_m\}$, then the CF$\beta-$N $\widetilde{\mathcal{N}}^{\beta}(\mathfrak{S})$ is elaborated by:

$$\widetilde{\mathcal{N}}^{\beta}(\mathfrak{S}) = \cap\left\{\widetilde{\mho}_i \in \widetilde{\mho} : \widetilde{\mho}_i(\mathfrak{S}) \geq \beta\right\} = \cap\widetilde{\mathcal{N}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S}) \tag{38}$$

Definition 21: For any CF$\beta-$CAS $\left(\acute{U}, \widetilde{\mho}\right)$ and for any $\mathfrak{S} \in \acute{U}$ with $\widetilde{\mho} = \{\mho_1, \mho_2, \ldots, \mho_m\}$, then the complex fuzzy complementary $\beta-$neighborhood (CFC$\beta-$N) $\tilde{M}^{\beta}(\mathfrak{S})$ is elaborated by:

$$\tilde{M}^{\beta}(\mathfrak{S})(\mathfrak{S}') = \widetilde{\mathcal{N}}^{\beta}(\mathfrak{S}')(\mathfrak{S}) \tag{39}$$

For all $\mathfrak{S}' \in \acute{U}$. Furthermore, the relation between CF$\beta-$MND and CF$\beta-$N is discussed below. For any CF$\beta-$CAS $\left(\acute{U}, \widetilde{\mho}\right)$ and for any $\mathfrak{S} \in \acute{U}$, then $\widetilde{\mathcal{N}}^{\beta}(\mathfrak{S}) = \cap\tilde{M}d^{\beta}_{\widetilde{\mho}}(\mathfrak{S})$. The complex fuzzy neighborhoods operators (CFNO) are discussed below.

A. COMPLEX FUZZY NEIGHBORHOODS OPERATOR

To explore the complex fuzzy modification for the operator $\tilde{\mathfrak{N}}^{1}_{\mho}$, where $\tilde{\mathfrak{N}}^{1}_{\mho} = \cap\mathfrak{N}_{\mho}(\mathfrak{S}) = \cap Md_{\mho}(\mathfrak{S})$ for any $\mathfrak{S} \in \acute{U}$. The original idea is replacing by $\mathfrak{N}_{\mho}(\mathfrak{S})$ with $\widetilde{\mathcal{N}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S})$ and $Md_{\mho}(\mathfrak{S})$ with $\tilde{M}d^{\beta}_{\widetilde{\mho}}(\mathfrak{S})$, where the $\mathcal{F}\left(\acute{U}\right)$ is contains the family of complex fuzzy numbers. The first kind of CFNO $\widetilde{\mathcal{FN}}^{\beta}_{\widetilde{\mho}}$ is elaborated below.

Definition 22: For any CF$\beta-$CAS $\left(\acute{U}, \widetilde{\mho}\right)$, then $\widetilde{\mathcal{FN}}^{\beta}_{\widetilde{\mho}} : \acute{U} \to \mathcal{F}\left(\acute{U}\right), \mathfrak{S} \to \widetilde{\mathcal{FN}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S})$ expresses the CFNO, for which the CFN $\widetilde{\mathcal{FN}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S})$ is elaborated by:

$$\widetilde{\mathcal{FN}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S}) = \cap\left\{\mho \in \widetilde{\mho} : \mho(\mathfrak{S}) \geq \beta\right\} = \cap\widetilde{\mathcal{N}}^{\beta}_{\widetilde{\mho}}(\mathfrak{S}) \tag{40}$$

where the value of parameters $\beta = \beta_{RP}e^{i2\pi(\beta_{IP})}$ with $\beta_{RP}, \beta_{IP} \in (0, 1]$. Moreover, based on the above analysis, we get the following rules.

1. For the CF$\beta-$C $\widetilde{\mho}$ is a crisp and complex covering, then the CFN of $\mathfrak{S}$ elaborated the Def. (22), coincides with

the complex and crisp neighborhood of $\mathfrak{s}$. If, for $\mathfrak{s} \in \acute{U}$, $\mho(\mathfrak{s}) = 0 \; or \; 0\, e^{i2\pi(0)}$ and $\mho(\mathfrak{s}) = 1 \; or \; 1\, e^{i2\pi(1)}$ i.e., $\mathfrak{s} \notin \acute{U}$ or $\mathfrak{s} \in \acute{U}$. Moreover, $\mathfrak{s} \in \acute{U}, \mho(\mathfrak{s}) \geq \beta \Leftrightarrow \mho(\mathfrak{s}) = 1e^{i2\pi(1)}$ for any $\mho \in \widetilde{\mho}$. Then

$$\begin{aligned} \overbrace{\mathcal{FN}}^{\beta}_{\widetilde{\mho}}(\mathfrak{s}) &= \cap\left\{\mho \in \widetilde{\mho} : \mho(\mathfrak{s}) \geq \beta\right\} \\ &= \cap\left\{\mho \in \widetilde{\mho} : \mho(\mathfrak{s}) = 1e^{i2\pi(1)}\right\} \\ &= \cap\left\{\mho \in \widetilde{\mho} : \mathfrak{s} \in \mho\right\} = \mathcal{N}^{1}_{\widetilde{\mho}}(\mathfrak{s}). \end{aligned}$$

2. Consider $\widetilde{\mho} = \{\mho_1, \mho_2, \ldots, \mho_m\}$ is a CF$\beta-$C of $\acute{U}$, then for any $\mho \in \mathcal{F}\left(\acute{U}\right)$, if $\mho^{\beta} = \left\{\mathfrak{s} \in \acute{U} : \mho(\mathfrak{s}) \geq \beta\right\}$ and $\widetilde{\mho_{\beta}} = \left\{\mho_1^{\beta}, \mho_2^{\beta}, \ldots, \mho_m^{\beta}\right\}$ is a covering of $\acute{U}$. Then $\left(\overbrace{\mathcal{N}}^{\beta}_{\widetilde{\mho}}(\mathfrak{s})\right)_{\beta} = \mathcal{N}_{\widetilde{\mho_{\beta}}}(\mathfrak{s})$ for any $\mathfrak{s} \in \acute{U}$. Hence, $\left(\overbrace{\mathcal{FN}}^{\beta}_{\widetilde{\mho}}(\mathfrak{s})\right)_{\beta} = \mathcal{N}^{1}_{\widetilde{\mho_{\beta}}}(\mathfrak{s})$ for any $\mathfrak{s} \in \acute{U}$.

For any complex fuzzy covering $\widetilde{\mho}$ with an implication $\mathcal{J}$, then $\mathcal{N}_1^{\widetilde{\mho}} : \acute{U} \to \mathcal{F}\left(\acute{U}\right)$, $\mathfrak{s} \to \mathcal{N}_1^{\widetilde{\mho}}(\mathfrak{s})$ expresses the CFNO, for which CFN $\mathcal{N}_1^{\widetilde{\mho}}(\mathfrak{s})$ is elaborated by:

$$\mathcal{N}_1^{\widetilde{\mho}}(\mathfrak{s}) : \acute{U} \to RI\,[0,1], \mathfrak{s} \to \inf_{\ddot{O} \in \widetilde{\mho}} \mathcal{J}\left(\ddot{O}(\mathfrak{s}), \ddot{O}(\mathfrak{s}')\right) \tag{41}$$

with $\inf_{\ddot{O} \in \widetilde{\mho}} \mathcal{J}\left(\ddot{O}(\mathfrak{s}), \ddot{O}(\mathfrak{s}')\right) = \inf_{\ddot{O} \in \widetilde{\mho}} \left(\mathcal{J}\left(\ddot{O}_{RP}(\mathfrak{s}), \ddot{O}_{RP}(\mathfrak{s}')\right) e^{i2\pi\left(\mathcal{J}\left(\ddot{O}_{IP}(\mathfrak{s}), \ddot{O}_{IP}(\mathfrak{s}')\right)\right)}\right)$, where $RI\,[0,1]$ expresses the family of complex numbers in a unit disc belonging to the complex plane. If, $\beta = 1e^{i2\pi(1)}$ and $\mathcal{J}$ satisfies:

NP: $\mathcal{J}(1, a)\, e^{i2\pi(\mathcal{J}(1,a))} = ae^{i2\pi(a)}$ for any $a \in [0,1]$, for any $\mathfrak{s} \in \acute{U}$, then

$$\begin{aligned} \overbrace{\mathcal{FN}}^{1}_{\widetilde{\mho}}(\mathfrak{s})(\mathfrak{s}') &= \inf_{\mho \in \widetilde{\mho}, \mho(\mathfrak{s}) = 1e^{i2\pi(1)}} \mho(\mathfrak{s}') \\ &= \inf_{\mho \in \widetilde{\mho}, \mho(\mathfrak{s}) = 1e^{i2\pi(1)}} \mathcal{J}\left(\mho(\mathfrak{s}), \mho(\mathfrak{s}')\right) \\ &\geq \inf_{\mho \in \widetilde{\mho}} \mathcal{J}\left(\mho(\mathfrak{s}), \mho(\mathfrak{s}')\right) = \mathcal{N}_1^{\widetilde{\mho}}(\mathfrak{s})(\mathfrak{s}') \end{aligned}$$

Hence, $\mathcal{N}_1^{\widetilde{\mho}}(\mathfrak{s}) \subseteq \overbrace{\mathcal{FN}}^{1}_{\widetilde{\mho}}(\mathfrak{s})$ for any $\mathfrak{s} \in \acute{U}$. For any CF$\beta-$CAS $\left(\acute{U}, \widetilde{\mho}\right)$, then

$$\overbrace{\mathcal{FN}}^{\beta}_{\widetilde{\mho}}(\mathfrak{s})(\mathfrak{s}') \geq \beta \text{ for each } \mathfrak{s} \in \acute{U}; \tag{42}$$

For any $\mathfrak{s}, \mathfrak{s}', \mathfrak{s}'' \in \acute{U}$, if $\overbrace{\mathcal{FN}}^{\beta}_{\widetilde{\mho}}(\mathfrak{s})(\mathfrak{s}') \geq \beta$ and

$$\overbrace{\mathcal{FN}}^{\beta}_{\widetilde{\mho}}(\mathfrak{s}')\left(\mathfrak{s}''\right) \geq \beta, \quad \text{then } \overbrace{\mathcal{FN}}^{\beta}_{\widetilde{\mho}}(\mathfrak{s})\left(\mathfrak{s}''\right) \geq \beta; \tag{43}$$

$$\text{If } 0 \leq \beta_1 \leq \beta_2 \leq \beta, \text{ then } \overbrace{\mathcal{FN}}^{\beta_1}_{\widetilde{\mho}}(\mathfrak{s}) \subseteq \overbrace{\mathcal{FN}}^{\beta_2}_{\widetilde{\mho}}(\mathfrak{s}) \text{ for each } \mathfrak{s} \in \acute{U}; \tag{44}$$

$$\begin{aligned} &\text{For any } \mathfrak{s}, \mathfrak{s}' \in \acute{U}, \overbrace{\mathcal{FN}}^{\beta}_{\widetilde{\mho}}(\mathfrak{s})(\mathfrak{s}') \\ &\quad \geq \beta \text{iff } \overbrace{\mathcal{FN}}^{\beta}_{\widetilde{\mho}}(\mathfrak{s}') \subseteq \overbrace{\mathcal{FN}}^{\beta}_{\widetilde{\mho}}(\mathfrak{s}), \\ &\text{similarly, } \overbrace{\mathcal{FN}}^{\beta}_{\widetilde{\mho}}(\mathfrak{s}')(\mathfrak{s}) \\ &\quad \geq \beta \text{iff } \overbrace{\mathcal{FN}}^{\beta}_{\widetilde{\mho}}(\mathfrak{s}') = \overbrace{\mathcal{FN}}^{\beta}_{\widetilde{\mho}}(\mathfrak{s}). \end{aligned} \tag{45}$$

Further, for any $\widetilde{\mho_1}, \widetilde{\mho_2}$ are two CF$\beta-$CS of $\acute{U}$, then for each $\mathfrak{s} \in \acute{U}$, we have

$$\text{If } \mathcal{B}^{\beta}\left(\widetilde{\mho_1}\right) = \mathcal{B}^{\beta}\left(\widetilde{\mho_2}\right), \quad \text{then } \overbrace{\mathcal{FN}}^{\beta}_{\widetilde{\mho_1}}(\mathfrak{s}) = \overbrace{\mathcal{FN}}^{\beta}_{\widetilde{\mho_2}}(\mathfrak{s}); \tag{46}$$

$$\text{If } \mathcal{R}^{\beta}\left(\widetilde{\mho_1}\right) = \mathcal{R}^{\beta}\left(\widetilde{\mho_2}\right), \quad \text{then } \overbrace{\mathcal{FN}}^{\beta}_{\widetilde{\mho_1}}(\mathfrak{s}) = \overbrace{\mathcal{FN}}^{\beta}_{\widetilde{\mho_2}}(\mathfrak{s}); \tag{47}$$

The reverse processes of Eq. (46) and Eq. (47) are not held. For this, we illustrate Example 13, which is discussed below.

Example 13: Let $\acute{U} = \{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4, \mathfrak{s}_5\}$ with $\widetilde{\mho_1} = \{\mho_1, \mho_2, \mho_3, \mho_4\}$ and $\widetilde{\mho_2} = \{\mho_1, \mho_2, \mho_3, \mho_4, \mho_5\}$ are discussed below for examining $\widetilde{\mho}$ expresses CF$\beta-$C of fix set $\acute{U}$, $(0 < \beta_{RP}, \beta_{IP} \leq 0.53)$.

$$\begin{aligned} \mho_1 &= \frac{0.51e^{i2\pi(0.52)}}{\mathfrak{s}_1} + \frac{0.21e^{i2\pi(0.22)}}{\mathfrak{s}_2} + \frac{0.71e^{i2\pi(0.72)}}{\mathfrak{s}_3} \\ &\quad + \frac{0.81e^{i2\pi(0.82)}}{\mathfrak{s}_4} + \frac{0.31^{i2\pi(0.32)}}{\mathfrak{s}_5}, \\ \mho_2 &= \frac{0.71e^{i2\pi(0.72)}}{\mathfrak{s}_1} + \frac{0.11e^{i2\pi(0.12)}}{\mathfrak{s}_2} + \frac{0.31^{i2\pi(0.32)}}{\mathfrak{s}_3} \\ &\quad + \frac{0.41e^{i2\pi(0.42)}}{\mathfrak{s}_4} + \frac{0.71e^{i2\pi(0.72)}}{\mathfrak{s}_5}, \\ \mho_3 &= \frac{0.61^{i2\pi(0.62)}}{\mathfrak{s}_1} + \frac{0.71^{i2\pi(0.72)}}{\mathfrak{s}_2} + \frac{0.61e^{i2\pi(0.62)}}{\mathfrak{s}_3} \\ &\quad + \frac{0.21e^{i2\pi(0.22)}}{\mathfrak{s}_4} + \frac{0.51e^{i2\pi(0.52)}}{\mathfrak{s}_5}, \\ \mho_4 &= \frac{0.31^{i2\pi(0.32)}}{\mathfrak{s}_1} + \frac{0.91^{i2\pi(0.92)}}{\mathfrak{s}_2} + \frac{0.11e^{i2\pi(0.12)}}{\mathfrak{s}_3} \\ &\quad + \frac{0.71e^{i2\pi(0.72)}}{\mathfrak{s}_4} + \frac{0.11^{i2\pi(0.12)}}{\mathfrak{s}_5}, \\ \mho_5 &= \frac{0.31e^{i2\pi(0.32)}}{\mathfrak{s}_1} + \frac{0.81^{i2\pi(0.82)}}{\mathfrak{s}_2} + \frac{0.11^{i2\pi(0.12)}}{\mathfrak{s}_3} \\ &\quad + \frac{0.71e^{i2\pi(0.72)}}{\mathfrak{s}_4} + \frac{0.21e^{i2\pi(0.22)}}{\mathfrak{s}_5}. \end{aligned}$$

The $\widetilde{\mho_1}, \widetilde{\mho_2}$ are two CF$\beta-$Cs of $\acute{U}$, $(0 < \beta_{RP}, \beta_{IP} \leq 0.53)$, for $\beta = 0.51e^{i2\pi(0.52)}$, then

$$\begin{aligned} &\overbrace{\mathcal{FN}}^{0.51e^{i2\pi(0.52)}}_{\widetilde{\mho_1}}(\mathfrak{s}_1) \\ &\quad = \frac{0.51^{i2\pi(0.52)}}{\mathfrak{s}_1} + \frac{0.11e^{i2\pi(0.12)}}{\mathfrak{s}_2} + \frac{0.31^{i2\pi(0.32)}}{\mathfrak{s}_3} \end{aligned}$$

$$+\frac{0.21e^{i2\pi(0.22)}}{\mathfrak{S}_4}+\frac{0.31^{i2\pi(0.32)}}{\mathfrak{S}_5};$$

$$\widetilde{\mathcal{FN}\widetilde{\mho}_1}^{0.51e^{i2\pi(0.52)}}(\mathfrak{S}_2)=\frac{0.31e^{i2\pi(0.32)}}{\mathfrak{S}_1}+\frac{0.71e^{i2\pi(0.72)}}{\mathfrak{S}_2}+\frac{0.11e^{i2\pi(0.12)}}{\mathfrak{S}_3}+\frac{0.21^{i2\pi(0.22)}}{\mathfrak{S}_4}+\frac{0.11e^{i2\pi(0.12)}}{\mathfrak{S}_5};$$

$$\widetilde{\mathcal{FN}\widetilde{\mho}_1}^{0.51e^{i2\pi(0.52)}}(\mathfrak{S}_3)=\frac{0.51e^{i2\pi(0.52)}}{\mathfrak{S}_1}+\frac{0.21e^{i2\pi(0.22)}}{\mathfrak{S}_2}+\frac{0.61\ ^{i2\pi(0.62)}}{\mathfrak{S}_3}+\frac{0.21\ ^{i2\pi(0.22)}}{\mathfrak{S}_4}+\frac{0.31e^{i2\pi(0.32)}}{\mathfrak{S}_5};$$

$$\widetilde{\mathcal{FN}\widetilde{\mho}_1}^{0.51e^{i2\pi(0.52)}}(\mathfrak{S}_4)=\frac{0.31e^{i2\pi(0.32)}}{\mathfrak{S}_1}+\frac{0.21e^{i2\pi(0.22)}}{\mathfrak{S}_2}+\frac{0.11e^{i2\pi(0.12)}}{\mathfrak{S}_3}+\frac{0.71e^{i2\pi(0.72)}}{\mathfrak{S}_4}+\frac{0.11e^{i2\pi(0.12)}}{\mathfrak{S}_5};$$

$$\widetilde{\mathcal{FN}\widetilde{\mho}_1}^{0.51e^{i2\pi(0.52)}}(\mathfrak{S}_5)=\frac{0.61e^{i2\pi(0.62)}}{\mathfrak{S}_1}+\frac{0.11e^{i2\pi(0.12)}}{\mathfrak{S}_2}+\frac{0.31e^{i2\pi(0.32)}}{\mathfrak{S}_3}+\frac{0.21e^{i2\pi(0.22)}}{\mathfrak{S}_4}+\frac{0.51e^{i2\pi(0.52)}}{\mathfrak{S}_5};$$

$$\widetilde{\mathcal{FN}\widetilde{\mho}_2}^{0.51e^{i2\pi(0.52)}}(\mathfrak{S}_1)=\frac{0.51e^{i2\pi(0.52)}}{\mathfrak{S}_1}+\frac{0.11e^{i2\pi(0.12)}}{\mathfrak{S}_2}+\frac{0.31e^{i2\pi(0.32)}}{\mathfrak{S}_3}+\frac{0.21e^{i2\pi(0.22)}}{\mathfrak{S}_4}+\frac{0.31^{i2\pi(0.32)}}{\mathfrak{S}_5};$$

$$\widetilde{\mathcal{FN}\widetilde{\mho}_2}^{0.51e^{i2\pi(0.52)}}(\mathfrak{S}_2)=\frac{0.31^{i2\pi(0.32)}}{\mathfrak{S}_1}+\frac{0.71^{i2\pi(0.72)}}{\mathfrak{S}_2}+\frac{0.11^{i2\pi(0.12)}}{\mathfrak{S}_3}+\frac{0.21^{i2\pi(0.22)}}{\mathfrak{S}_4}+\frac{0.11e^{i2\pi(0.12)}}{\mathfrak{S}_5};$$

$$\widetilde{\mathcal{FN}\widetilde{\mho}_2}^{0.51e^{i2\pi(0.52)}}(\mathfrak{S}_3)=\frac{0.51e^{i2\pi(0.52)}}{\mathfrak{S}_1}+\frac{0.21e^{i2\pi(0.22)}}{\mathfrak{S}_2}+\frac{0.61e^{i2\pi(0.62)}}{\mathfrak{S}_3}+\frac{0.21e^{i2\pi(0.22)}}{\mathfrak{S}_4}+\frac{0.31e^{i2\pi(0.32)}}{\mathfrak{S}_5};$$

$$\widetilde{\mathcal{FN}\widetilde{\mho}_2}^{0.51e^{i2\pi(0.52)}}(\mathfrak{S}_4)=\frac{0.31e^{i2\pi(0.32)}}{\mathfrak{S}_1}+\frac{0.21e^{i2\pi(0.22)}}{\mathfrak{S}_2}+\frac{0.11e^{i2\pi(0.12)}}{\mathfrak{S}_3}+\frac{0.71e^{i2\pi(0.72)}}{\mathfrak{S}_4}+\frac{0.11^{i2\pi(0.12)}}{\mathfrak{S}_5};$$

$$\widetilde{\mathcal{FN}\widetilde{\mho}_2}^{0.51e^{i2\pi(0.52)}}(\mathfrak{S}_5)=\frac{0.61e^{i2\pi(0.62)}}{\mathfrak{S}_1}+\frac{0.11e^{i2\pi(0.12)}}{\mathfrak{S}_2}+\frac{0.31e^{i2\pi(0.32)}}{\mathfrak{S}_3}+\frac{0.21e^{i2\pi(0.22)}}{\mathfrak{S}_4}+\frac{0.51e^{i2\pi(0.52)}}{\mathfrak{S}_5};$$

Additionally, we have $\mathcal{B}_{\widetilde{\mho}_1}^{0.51e^{i2\pi(0.52)}}=\{\mho_1,\mho_2,\mho_3,\mho_4\}$, $\mathcal{B}_{\widetilde{\mho}_2}^{0.51e^{i2\pi(0.52)}}=\{\mho_1,\mho_2,\mho_3,\mho_4,\mho_5\}$, $\mathcal{R}^{0.51e^{i2\pi(0.52)}}\left(\widetilde{\mho}_1\right)=\{\mho_1,\mho_2,\mho_3,\mho_4\}$, $\mathcal{R}^{0.51e^{i2\pi(0.52)}}\left(\widetilde{\mho}_2\right)=\{\mho_1,\mho_2,\mho_3,\mho_4,\mho_5\}$. By using the notion of Def. (22), the idea of complex fuzzy complementary neighborhood operator $\widetilde{\mathcal{FM}}_{\widetilde{\mho}}^{\beta}(\mathfrak{S})$ is elaborated by:

$$\widetilde{\mathcal{FM}}_{\widetilde{\mho}}^{\beta}(\mathfrak{S})\left(\mathfrak{S}'\right)=\widetilde{\mathcal{FN}}_{\widetilde{\mho}}^{\beta}\left(\mathfrak{S}'\right)(\mathfrak{S}) \tag{48}$$

By using the Eq. (48), we have, Let $\widetilde{\mathcal{FM}}_{\widetilde{\mho}}^{\beta}(\mathfrak{S})(\mathfrak{S}\prime)=\widetilde{\mathcal{FN}}_{\widetilde{\mho}}^{\beta}\left(\mathfrak{S}'\right)(\mathfrak{S})$ iff $\widetilde{\mathcal{FN}}_{\widetilde{\mho}}^{\beta}$ is symmetric iff $\widetilde{\mathcal{FM}}_{\widetilde{\mho}}^{\beta}$ is symmetric.

V. COMPLEX FUZZY $\beta-$ COVERING DERIVED FROM COMPLEX FUZZY NEIGHBORHOOD OPERATORS

The purpose of this study is to discover six types of coverings derived from $\mho$ based on CFSs. Additionally, modifications of derived coverings $\widetilde{\mho^1}$, $\widetilde{\mho^2}$, $\widetilde{\mho^3}$, $\widetilde{\mho^4}$, $\widetilde{\mho^5}$, and $\widetilde{\mho^6}$ are explored. The relationships among these operators are coverings are also discussed.

Definition 23: For any CF$\beta-$CAS $\left(\acute{\mho},\widetilde{\mho}\right)$, then

$$\widetilde{\mho^1}=\cup\left\{\tilde{M}d_{\widetilde{\mho}}^{\beta}(\mathfrak{S}):\mathfrak{S}\in\acute{\mho}\right\} \tag{49}$$

$$\widetilde{\mho^2}=\cup\left\{\widetilde{\mathcal{MD}}_{\widetilde{\mho}}^{\beta}(\mathfrak{S}):\mathfrak{S}\in\acute{\mho}\right\} \tag{50}$$

$$\widetilde{\mho^3}=\left\{\cap\tilde{M}d_{\widetilde{\mho}}^{\beta}(\mathfrak{S}):\mathfrak{S}\in\acute{\mho}\right\}=\left\{\cap\ \widetilde{\mathcal{N}}_{\widetilde{\mho}}^{\beta}(\mathfrak{S}):\mathfrak{S}\in\acute{\mho}\right\}=\left\{\widetilde{\mathcal{FN}}_{\widetilde{\mho}}^{\beta}(\mathfrak{S}):\mathfrak{S}\in\acute{\mho}\right\} \tag{51}$$

$$\widetilde{\mho^4}=\left\{\cup\widetilde{\mathcal{MD}}_{\widetilde{\mho}}^{\beta}(\mathfrak{S}):\mathfrak{S}\in\acute{\mho}\right\}=\left\{\cup\ \widetilde{\mathcal{N}}_{\widetilde{\mho}}^{\beta}(\mathfrak{S}):\mathfrak{S}\in\acute{\mho}\right\}=\left\{\widetilde{\mathcal{RN}}_{\widetilde{\mho}}^{\beta}(\mathfrak{S}):\mathfrak{S}\in\acute{\mho}\right\} \tag{52}$$

$$\widetilde{\mho^5}=\widetilde{\mho}-\left\{\bar{\tilde{o}}\in\widetilde{\mho}:\left(\exists\widetilde{\mho}'\subseteq\widetilde{\mho}-\left\{\bar{\tilde{o}}\right\}\right)\left(\bar{\tilde{o}}=\cap\widetilde{\mho}'\right)\right\} \tag{53}$$

$$\widetilde{\mho^6}=\widetilde{\mho}-\left\{\bar{\tilde{o}}\in\widetilde{\mho}:\left(\exists\widetilde{\mho}'\subseteq\widetilde{\mho}-\left\{\bar{\tilde{o}}\right\}\right)\left(\bar{\tilde{o}}=\cup\widetilde{\mho}'\right)\right\} \tag{54}$$

By using the above analysis, we can resolve the following example.

Example 14: By using the information of Example 1, with as shown at the bottom of the page, are discussed below for examining $\tilde{\mho}$ expresses CFβ−C of fix set $\acute{\mho}$, $(0 < \beta_{RP}, \beta_{IP} \leq 0.53)$.

$$\overbrace{\mathfrak{FN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_1) = \frac{0.51e^{i2\pi(0.52)}}{\mathfrak{s}_1} + \frac{0.11e^{i2\pi(0.12)}}{\mathfrak{s}_2} + \frac{0.31e^{i2\pi(0.32)}}{\mathfrak{s}_3} + \frac{0.51e^{i2\pi(0.52)}}{\mathfrak{s}_4} + \frac{0.31e^{i2\pi(0.32)}}{\mathfrak{s}_5} + \frac{0.61e^{i2\pi(0.62)}}{\mathfrak{s}_5},$$

$$\overbrace{\mathfrak{FN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_2) = \frac{0.21e^{i2\pi(0.52)}}{\mathfrak{s}_1} + \frac{0.61^{i2\pi(0.62)}}{\mathfrak{s}_2} + \frac{0.21e^{i2\pi(0.22)}}{\mathfrak{s}_3} + \frac{0.31e^{i2\pi(0.32)}}{\mathfrak{s}_4} + \frac{0.21e^{i2\pi(0.22)}}{\mathfrak{s}_5} + \frac{0.11\ ^{i2\pi(0.12)}}{\mathfrak{s}_5},$$

$$\overbrace{\mathfrak{FN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_3) = \frac{0.51^{i2\pi(0.52)}}{\mathfrak{s}_1} + \frac{0.11^{i2\pi(0.12)}}{\mathfrak{s}_2} + \frac{0.71e^{i2\pi(0.72)}}{\mathfrak{s}_3} + \frac{0.61e^{i2\pi(0.62)}}{\mathfrak{s}_4} + \frac{0.41e^{i2\pi(0.42)}}{\mathfrak{s}_5} + \frac{0.61e^{i2\pi(0.62)}}{\mathfrak{s}_5},$$

$$\overbrace{\mathfrak{FN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_4) = \frac{0.41e^{i2\pi(0.42)}}{\mathfrak{s}_1} + \frac{0.11e^{i2\pi(0.12)}}{\mathfrak{s}_2} + \frac{0.31e^{i2\pi(0.32)}}{\mathfrak{s}_3} + \frac{0.51e^{i2\pi(0.52)}}{\mathfrak{s}_4} + \frac{0.31e^{i2\pi(0.32)}}{\mathfrak{s}_5} + \frac{0.41e^{i2\pi(0.42)}}{\mathfrak{s}_5},$$

$$\overbrace{\mathfrak{FN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_5) = \frac{0.41^{i2\pi(0.42)}}{\mathfrak{s}_1} + \frac{0.41e^{i2\pi(0.42)}}{\mathfrak{s}_2} + \frac{0.41e^{i2\pi(0.42)}}{\mathfrak{s}_3} + \frac{0.81e^{i2\pi(0.82)}}{\mathfrak{s}_4} + \frac{0.61e^{i2\pi(0.62)}}{\mathfrak{s}_5} + \frac{0.41e^{i2\pi(0.42)}}{\mathfrak{s}_5},$$

$$\overbrace{\mathfrak{FN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_6) = \frac{0.51e^{i2\pi(0.52)}}{\mathfrak{s}_1} + \frac{0.11e^{i2\pi(0.12)}}{\mathfrak{s}_2} + \frac{0.31^{i2\pi(0.32)}}{\mathfrak{s}_3} + \frac{0.51e^{i2\pi(0.52)}}{\mathfrak{s}_4} + \frac{0.31^{i2\pi(0.32)}}{\mathfrak{s}_5} + \frac{0.61e^{i2\pi(0.62)}}{\mathfrak{s}_5},$$

$$\overbrace{\mathfrak{RN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_1) = \frac{0.91e^{i2\pi(0.92)}}{\mathfrak{s}_1} + \frac{0.41e^{i2\pi(0.42)}}{\mathfrak{s}_2} + \frac{0.81e^{i2\pi(0.82)}}{\mathfrak{s}_3} + \frac{0.91e^{i2\pi(0.92)}}{\mathfrak{s}_4} + \frac{0.61e^{i2\pi(0.62)}}{\mathfrak{s}_5} + \frac{0.71e^{i2\pi(0.72)}}{\mathfrak{s}_5},$$

$$\overbrace{\mathfrak{RN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_2) = \frac{0.41e^{i2\pi(0.42)}}{\mathfrak{s}_1} + \frac{0.71^{i2\pi(0.72)}}{\mathfrak{s}_2} + \frac{0.41e^{i2\pi(0.42)}}{\mathfrak{s}_3} + \frac{0.81e^{i2\pi(0.82)}}{\mathfrak{s}_4} + \frac{0.71e^{i2\pi(0.72)}}{\mathfrak{s}_5} + \frac{0.41^{i2\pi(0.42)}}{\mathfrak{s}_5},$$

$$\overbrace{\mathfrak{RN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_3) = \frac{0.91e^{i2\pi(0.92)}}{\mathfrak{s}_1} + \frac{0.41e^{i2\pi(0.14)}}{\mathfrak{s}_2} + \frac{0.81^{i2\pi(0.82)}}{\mathfrak{s}_3} + \frac{0.91e^{i2\pi(0.69)}}{\mathfrak{s}_4} + \frac{0.61e^{i2\pi(0.46)}}{\mathfrak{s}_5} + \frac{0.71e^{i2\pi(0.72)}}{\mathfrak{s}_5},$$

$$\overbrace{\mathfrak{RN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_4) = \frac{0.91\ ^{i2\pi(0.92)}}{\mathfrak{s}_1} + \frac{0.61e^{i2\pi(0.62)}}{\mathfrak{s}_2} + \frac{0.81\ ^{i2\pi(0.82)}}{\mathfrak{s}_3} + \frac{0.91e^{i2\pi(0.59)}}{\mathfrak{s}_4} + \frac{0.71e^{i2\pi(0.72)}}{\mathfrak{s}_5} + \frac{0.71e^{i2\pi(0.72)}}{\mathfrak{s}_5},$$

$$\overbrace{\mathfrak{RN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_5) = \frac{0.91\ ^{i2\pi(0.92)}}{\mathfrak{s}_1} + \frac{0.61\ ^{i2\pi(0.62)}}{\mathfrak{s}_2} + \frac{0.71e^{i2\pi(0.87)}}{\mathfrak{s}_3} + \frac{0.91e^{i2\pi(0.59)}}{\mathfrak{s}_4} + \frac{0.71e^{i2\pi(0.72)}}{\mathfrak{s}_5} + \frac{0.61e^{i2\pi(0.62)}}{\mathfrak{s}_5},$$

$$\overbrace{\mho_1} = \{\mho_1, \mho_2, \mho_3, \mho_4, \mho_5\} \quad \text{and} \ \overbrace{\mho_2} = \{\mho_2, \mho_3, \mho_4, \mho_5\},$$

$$\overbrace{\mho_5} = \{\mho_1, \mho_2, \mho_3, \mho_4, \mho_5\}, \quad \overbrace{\mho_6} = \{\mho_1, \mho_2, \mho_3, \mho_4, \mho_5\},$$

$$\overbrace{\mho_3} = \left\{ \begin{array}{l} \overbrace{\mathfrak{FN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_1), \overbrace{\mathfrak{FN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_2), \overbrace{\mathfrak{FN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_3), \\ \overbrace{\mathfrak{FN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_4), \overbrace{\mathfrak{FN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_5), \overbrace{\mathfrak{FN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_6) \end{array} \right\}$$

$$\overbrace{\mho_4} = \left\{ \begin{array}{l} \overbrace{\mathfrak{RN}}^{0.51^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_1), \overbrace{\mathfrak{RN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_2), \overbrace{\mathfrak{RN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_3), \\ \overbrace{\mathfrak{RN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_4), \overbrace{\mathfrak{RN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_5), \overbrace{\mathfrak{RN}}^{0.51e^{i2\pi(0.52)}}_{\tilde{\mho}}(\mathfrak{s}_6) \end{array} \right\}$$

$$\overbrace{\mathcal{RN}_{\tilde{\mho}}}^{0.51e^{i2\pi(0.52)}}(\mathfrak{S}_6)$$
$$=\frac{0.91\ ^{i2\pi(0.92)}}{\mathfrak{S}_1}+\frac{0.41^{i2\pi(0.42)}}{\mathfrak{S}_2}+\frac{0.81\ ^{i2\pi(0.82)}}{\mathfrak{S}_3}+\frac{0.91\ ^{i2\pi(0.59)}}{\mathfrak{S}_4}+\frac{0.61e^{i2\pi(0.62)}}{\mathfrak{S}_5}+\frac{0.71e^{i2\pi(0.72)}}{\mathfrak{S}_5}.$$

Theorem: For any CF$\beta-$CAS $\left(\acute{\mathrm{U}},\tilde{\mho}\right)$, then $\overbrace{\mho^{j}}$, $j=1,2,3,4,5,6$ are CF$\beta-$Cs $\acute{\mathrm{U}}$.

Proof: The proof of the $\overbrace{\mho^{j}}$ are stated below:

1. For any $\mathfrak{S}\in\acute{\mathrm{U}}$, then $(\mho)(\mathfrak{S})\geq\beta$ for all $\mho\in\tilde{M}d^{\beta}_{\tilde{\mho}}(\mathfrak{S})$ implies that $\left(\cup\overbrace{\mho^{1}}\right)(\mathfrak{S})\geq\left(\cup\tilde{M}d^{\beta}_{\tilde{\mho}}(\mathfrak{S})\right)(\mathfrak{S})\geq\beta$. Hence, $\overbrace{\mho^{1}}$ are CF$\beta-$Cs $\acute{\mathrm{U}}$.
2. For any $\mathfrak{S}\in\acute{\mathrm{U}}$, then $(\mho)(\mathfrak{S})\geq\beta$ for all $\mho\in\widetilde{\mathcal{MD}}^{\beta}_{\tilde{\mho}}(\mathfrak{S})$ implies that $\left(\cup\overbrace{\mho^{2}}\right)(\mathfrak{S})\geq\left(\cup\widetilde{\mathcal{MD}}^{\beta}_{\tilde{\mho}}(\mathfrak{S})\right)(\mathfrak{S})\geq\beta$. Hence, $\overbrace{\mho^{2}}$ are CF$\beta-$Cs $\acute{\mathrm{U}}$.
3. For any $\mathfrak{S}\in\acute{\mathrm{U}}$, then $\overbrace{\mathcal{FN}_{\tilde{\mho}}}^{\beta}(\mathfrak{S})(\mathfrak{S})\geq\beta$ implies that $\left(\cup\overbrace{\mho^{3}}\right)(\mathfrak{S})\geq\left(\overbrace{\mathcal{FN}_{\tilde{\mho}}}^{\beta}(\mathfrak{S})(\mathfrak{S})\right)\geq\beta$. Hence, $\overbrace{\mho^{3}}$ are CF$\beta-$Cs $\acute{\mathrm{U}}$.
4. For any $\mathfrak{S}\in\acute{\mathrm{U}}$, then $\overbrace{\mathcal{RN}_{\tilde{\mho}}}^{\beta}(\mathfrak{S})(\mathfrak{S})\geq\beta$ implies that $\left(\cup\overbrace{\mho^{4}}\right)(\mathfrak{S})\geq\left(\overbrace{\mathcal{RN}_{\tilde{\mho}}}^{\beta}(\mathfrak{S})(\mathfrak{S})\right)\geq\beta$. Hence, $\overbrace{\mho^{4}}$ are CF$\beta-$Cs $\acute{\mathrm{U}}$.
5. Let $\tilde{\mho}=\{\mho_1,\mho_2,\mho_3,\ldots,\mho_m\}$ where $\mho,\mho_i\in\mathcal{F}\left(\acute{\mathrm{U}}\right)$, then $\tilde{\mho}'\subseteq\tilde{\mho}-\{\mho\}$ such that $\cap\tilde{\mho}'=\mho$, then $\left(\left(\coprod_{j=1}^{m}(\mho_j)\right)\cup\mho\right)(\mathfrak{S})\geq\coprod_{j=1}^{m}(\mho_j)(\mathfrak{S})\geq\beta$. Therefore, $\tilde{\mho}-\{\mho\}$ is also a CF$\beta-$C of fix set $\acute{\mathrm{U}}$. Hence, $\overbrace{\mho^{5}}$ are CF$\beta-$Cs $\acute{\mathrm{U}}$.
6. Omitted.

VI. CONCLUSION

A complex fuzzy set is the modified version of the fuzzy set to cope with awkward and inconsistent information in guanine life troubles. The complex fuzzy set contains the grade of truth in a complex number that has real and unreal parts belong to the unit interval. Based on the advantages of the structure of the complex fuzzy set, in this manuscript, we compute the theory of CFCs are the natural mixture of the CFSs and coverings, which are the modified versions of the coverings by replacing crisp sets with CFSs. The goals of this paper are summarized in the following points:

1. We explore the CFNOs by introducing the notions such as β-NS, CFβ-MID, and CFβ-MXD.
2. First, we explore the CFβ-CAS, and then we propose the above notions and investigate their properties.
3. We construct the CFNOs based on the CFβ-Cs.
4. The CFβ-Cs were derived by using CFNOs, and their properties are considered. These all notions are also verified with suitable examples to show that the presented approaches are extensive, reliable, and proficient techniques.

In future work, we will extend these ideas into bi-polar soft sets [20], q-rung orthopair fuzzy sets [21]–[23], and complex q-rung orthopair fuzzy sets [24]–[29] to improve the quality of the research approaches.

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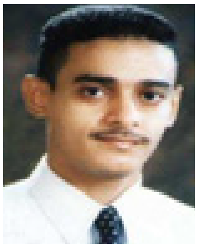

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