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# A Review on Fuzzy Differential Equations

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**ABSTRACT** Since the term ''Fuzzy differential equations'' (FDEs) emerged in the literature in 1978, prevailing research effort has been dedicated not only to the development of the concepts concerning the topic, but also to its potential applications. This paper presents a chronological survey on fuzzy differential equations of integer and fractional orders. Attention is concentrated on the FDEs in which a definition of fuzzy derivative of a fuzzy number-valued function has been taken into account. The chronological rationale behind considering FDEs under each concept of fuzzy derivative is highlighted. The pros and cons of each approach dealing with FDEs are also discussed. Moreover, some of the proposed FDEs applications and methods for solving them are investigated. Finally, some of the future perspectives and challenges of fuzzy differential equations are discussed based on our personal view point.

**INDEX TERMS** Fuzzy numbers, fuzzy derivatives, Hukuhra difference, granular differentiability, fuzzy mathematics, fuzzy number-valued functions.

## I. INTRODUCTION

The advent of fuzzy sets and fuzzy logic has had a significant impact on the evolution of many concepts and relationships in various fields of science. As a matter of fact, one of the principle contributions of fuzzy logic referred to as FL-generalization is the generalization of the defined concepts and obtained relationships based on crisp sets to those that are based on fuzzy sets. What is referred to as fuzzy differential equations (FDEs) may be viewed as a case of FL-generalization of differential equations. FDEs underlie a branch of FL-generalization of mathematics which may be called mathematics of fuzziness (or fuzzy mathematics). A fuzzy differential equation is a differential equation in which some coefficients and/or parameters and/or boundary conditions are assumed to be a class of fuzzy sets. The class of fuzzy sets is mainly regarded as the class of fuzzy numbers consisting of sets that are normal, fuzzy convex, upper semicontinuous and compactly supported fuzzy subsets of the real numbers.

FDEs may be viewed as a type of uncertain differential equations in which the uncertain values of parameters, coefficients, and/or boundary conditions are taken into account as fuzzy numbers. A fuzzy number may be viewed as a result of the granular precisiation of a precisiend that describes an imprecise value assigned to a variable. As such, a fuzzy differential equation may be viewed as a differential equation whose parameters, coefficients, and/or conditions are imprecise values precisiated as granular values. In this perspective, a fuzzy differential equation may be also considered as a class of granular differential equations.

As the granular precisiation of a precisiend is reduced to the singular precisiation of the precisiend as gradually as the uncertainty is reduced, a fuzzy differential equation is reduced to a differential equation whose uncertain parameters, coefficients, and/or conditions are treated as the degranulation of the precisiends. In the context of FDEs, the result of granular precisiation is of the type of possibility distribution. Thus, a fuzzy differential equation may be also termed as a possibilistic differential equation.

In this paper, our attention is confined on the sets of research carried out in the context of FDEs where a definition of fuzzy derivative has been considered. As such, in order to make a clarified survey, the integer order FDEs and fractional order FDEs are reviewed, chronologically, in two separate sections, and then the proposed methods for solving FDEs and potential applications of FDEs come in the other sections. Moreover, in the last section, a discussion on the challenges and future perspectives of FDEs, based on our personal viewpoint, is provided.

*Note 1:* Hereafter, we denote the set of all real numbers by $\mathbb{R}$, the respective set of all type-1 and type-2 fuzzy numbers

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on $\mathbb{R}$ by $E_1$ and $E_2$, the well-known $\alpha$-level sets of a fuzzy set $\tilde{A}$ by $[\tilde{A}]^\alpha$ whose left and right end-points (or lower and upper $\alpha$-level cuts) are $\underline{A}^\alpha$ and $\overline{A}^\alpha$, respectively. In this paper the meaning of a type-1 fuzzy function is associated with a fuzzy number-valued function $\tilde{f} : (a, b) \subseteq \mathbb{R} \to E_1$, and a type-2 fuzzy function with $\tilde{\tilde{f}} : (a, b) \subseteq \mathbb{R} \to E_2$. Unless stated to the contrary, the meaning of a fuzzy function will be understood to be the type-1 fuzzy function. The diameter (or width of fuzziness) of a fuzzy function, $\tilde{f}$ mapping $t \to \tilde{f}(t)$, denoted by $\mathcal{D}(\tilde{f}(t))$ is defined as $\mathcal{D}(\tilde{f}(t)) = \overline{f}^\alpha(t) - \underline{f}^\alpha(t)$.

## II. FUZZY DIFFERENTIAL EQUATIONS

This section presents various approaches dealing with a definition of a derivative of type-1 or type-2 fuzzy functions. The most important part of history of FDEs is formed by different definitions of fuzzy derivatives. As a matter of fact, since the concept of derivative is the fundamental element of a differential equation, the evolution of fuzzy derivatives plays a key role in the evolution of FDEs. The fuzzy derivatives may be classified as: integer order and fractional order fuzzy derivatives. Integer order fuzzy derivatives are sub-classified as integer order fuzzy derivatives of type-1 fuzzy functions (or type-1 fuzzy derivatives), and integer order fuzzy derivatives of type-2 fuzzy functions (or type-2 fuzzy derivatives). Similarly, there are type-1 and type-2 fuzzy fractional derivatives. It should be noted that corresponding to each class or sub-class of fuzzy derivatives, FDEs may be classified. For example, what may be referred to as type-1 fuzzy fractional differential equations is associated with FDEs in which the derivative is of the kind of type-1 fuzzy fractional derivatives.

### A. INTEGER ORDER FUZZY DIFFERENTIAL EQUATIONS

Although the term fuzzy differential equations for the first time emerged in the literature in 1978 [1], FDEs, as they are known nowadays, was initiated in 1982 based on a definition of a fuzzy derivative which may be called Dubois-Prade derivative [2]. Thereafter, different definitions of fuzzy derivatives were proposed among which were Hukuhara derivative (or Puri–Ralescu derivative) presented in 1983 [3], Goetschel-Voxman derivative in 1986 [4], Seikkala derivative in 1987 [5], and Friedman-Ming-Kandel derivative introduced in 1996 [6], respectively. In spite of the fact that all these fuzzy derivatives have been presented in different forms, it has been proved that they are equivalent provided that the subjected fuzzy function lower and upper $\alpha$-level cuts are continuous functions, for more details see [7].

Among the mentioned fuzzy derivatives, Hukuhara and Seikkala derivatives are more widely known. The difference between the definitions of Hukuhara and Seikkala derivatives is that Hukuhara derivative (H-derivative) is, in essence, defined based on what is called Hukuhara difference (H-difference) [8], but Seikkala derivative is defined based on derivatives of the lower and upper $\alpha$-level cuts of the fuzzy function in question. The existence and uniqueness of the solution for FDEs under H-derivative and Seikkala derivative have been investigated in [5], [9].

A large number of studies conducted on FDEs, for instance see [10]–[16], demonstrate that Hukuhara and Seikkala derivatives, despite being equivalent, are more palatable definitions. However, the research results have revealed that these derivatives suffer from a number of major limitations among which the most serious is that the diameter of the fuzzy function under study needs to be necessarily non-decreasing. Such a limitation causes the obtained solution of an FDE, in a great number of cases, to differ from what is realized intuitively from the nature of the system or phenomenon modeled by the FDE. As an illustration, the diameter of the obtained solution of an FDE in the form of $\dot{\tilde{x}}(t) = -\tilde{x}(t)$ whose initial condition is a fuzzy number, increases as time goes by. This is while we intuitively expect that the natural behavior of such a differential equation show that $\tilde{x}$ decreases as time passes. As a conclusion, considering such definitions in an FDE necessitates that the fuzziness of the solution be non-decreasing which imposes a great restriction on their real case applications.

To overcome this issue, in 1990 [17] and then with more details in 1997 [18], it was suggested that FDEs should be considered as fuzzy differential inclusions. Almost simultaneously, an alternative approach based on the use of Zadeh's extension principle (ZEP) for dealing with FDEs was introduced in 1999 [19]. Although these approaches have attracted considerable attention and led to many remarkable studies on FDEs, for instance see [20]–[30], they do not come with a definition of fuzzy derivative. In plain words, the concept of a fuzzy derivative is in effect lost in the proposed approaches. Another effort made in order to overcome the issue coming from applying Hukuhara (or equivalently Seikkala) derivative, was the presentation of the same-order and reverse-order derivatives [31]–[34] that were made based on Seikkala derivative. It should be underscored that this approach has a close relationship with the ones presented for fuzzy derivatives called generalized Hukuhara derivative and generalized Seikkala derivative which will be explained in the sequel.

The year 2004 came with a point of departure for making an evolution in dealing with FDEs by introducing the concept of strongly generalized Hukuhara (SGH) derivative [35] that was presented in a more comprehensive way in 2005 [36]. The structure of SGH-derivative, in general and under some conditions, presents two forms of differentiability of a fuzzy function which may be called the first form and the second form of differentiability [37]. The first form coincides with the Hukuhara derivative. But it is the second form of differentiability, if it exists, that addresses the issue of non-decreasing diameter of a differentiable fuzzy function. Simply put, if the fuzzy function $\tilde{f}$ mapping $t \to \tilde{f}(t)$ is SGH-differentiable in the second form, then its diameter is non-increasing, i.e. $\frac{d\mathcal{D}(\tilde{f}(t))}{dt} \leq 0$.

Thus, solving the FDE $\dot{\tilde{x}}(t) = -\tilde{x}(t)$ whose initial condition is a fuzzy number, in the sense of SGH derivative second form, results in a solution which satisfies what is intuitively expected from the nature of the equation structure.

This approach also comes with an interesting concept called switching points which are the points in an interval where the switch between the first form and the second form of differentiability occurs. This concept has opened a gate to the study of periodic behavior of some phenomena whose mathematical models may be considered as FDEs in which a definition of fuzzy derivative has been presented, for instance see [38], [39]. The result concerning existence and uniqueness of the solutions of an FDE under SGH-derivative given in [36], [40] shows that a first order FDE, under some conditions, has two solutions which may be called the first form and the second form solutions. The first and second forms solutions are associated with the concepts of the first and second forms of the differentiability.

Although SGH-derivative has made a turning point in the analysis of FDEs and a considerable amount of research on FDEs has been performed based on such a derivative, e.g. see [41]–[49], it suffers from some shortcomings the most important of which are as outlined below. First, since SGH-derivative, in essence, has been introduced based on Hukuhara difference, the existence of such derivative depends on the existence of H-difference. Nonetheless, in most cases the H-difference does not exist and the conditions for the existence of such a difference would significantly restrict the applicability of the SGH-derivative. Second, SGH-derivative would be applied on the fuzzy functions with monotonic diameters. More precisely, for taking the derivative of a fuzzy function in the sense of the first form of differentiability, the diameter of the fuzzy function needs to be necessarily non-decreasing. Analogously, that the diameter of a fuzzy function is non-increasing is one of the necessary conditions of the differentiability of the fuzzy function in the second form.

The concept of $\pi$-derivative is another alternative approach that was introduced in 2009 [50]. In [51], it has been stated that under the conditions of the Representation Theorem [52], the $\pi$-derivative of a fuzzy function exists. Moreover, the obtained solution of a fuzzy differential equation under the concept of $\pi$-derivative coincides with that obtained under the concept of SGH-derivative, under certain conditions, see more details in [51].

To overcome the limitations of SGH-derivative, generalized Hukuhara (gH) derivative of a fuzzy function was presented in 2013 [53]. The gH-derivative was defined based on generalized Hukuhara difference (gH-difference) [53], [54] which is a more general concept than H-difference. Although the existence of gH-difference comes with less restrictions in comparison with H-difference, it is possible that the gH-difference of two fuzzy numbers does not exist, see Example 16 in [53]. Accordingly, one fails to guarantee the existence of gH-derivative of a fuzzy function. In spite of this fact, gH-derivative addresses the second limitation of SGH-derivative. That is, the diameter of a gH-differentiable fuzzy function does not need to be monotonic. In addition, the concept of switching points has been better clarified based on gH-derivative, see Definition 38 in [53].

As a result, the investigation of FDEs under the concept of gH-differentiability comes with much fewer restrictions in comparison with other concepts. This may be one of the reasons that a large body of studies has been conducted on gH-differentiability of a fuzzy function and FDEs equipped with such a concept, e.g. see [55]–[69]. Concomitantly, in order to address the issue of the existence of gH-derivative of a fuzzy function, the concept of generalized derivative (g-derivative) based on the generalized difference (g-difference) was introduced in 2013 [53]. Although it had been initially claimed that the g-difference of two fuzzy numbers always exists, with a counter example presented in 2015, it was shown that it is not the case. By a little modification in the definition of g-difference, however, the existence of such a difference of fuzzy numbers can be guaranteed, see more details in [70]. It is noteworthy that under some conditions, g-differentiability, gH-differentiability and SGH-differentiability become equivalent concepts, see Theorem 41 in [53]. Note that on the basis of level-wise gH-difference of two fuzzy numbers, the concept of level-wise gH-derivative (LgH-derivative) has also been defined in [53] and investigated further in [66], [67]. Although the LgH-differentiability of a fuzzy function is a more general concept than gH-differentiability and less general than g-differentiability concept, the existence of LgH-derivative of a fuzzy function is not guaranteed. Some remarkable results regarding the differentiability of a fuzzy function in the sense of gH-derivative and LgH-derivative have been presented in [55], [64], [66], [67]. One of the points regarding the characteristic of gH-derivative and LgH-derivative that should be highlighted is that the existence of such derivatives for a fuzzy function does not necessarily imply that the end-points of the fuzzy function are differentiable, see Example 1 in [66], [67].

The development of the theory of FDEs has been continued, based on integer order derivatives, by other approaches on fuzzy derivatives, namely $\hat{D}$-derivative [71], $H_2$-derivative [72], interactive derivative [73], the $gr$-derivative [74], and so on that will be discussed in the sequel. The year 2013 came with an approach for studying FDEs in which the definition of a fuzzy derivative was drawn from the fuzzification of the classical derivative operator by the use of Zadeh's extension principle [71]. There is a close relationship between this approach and that introduced for fuzzy differential inclusions, and under some hypotheses, the results obtained by this approach are reduced to those obtained by fuzzy differential inclusions, see [71], [75], [76] for more details. Furthermore, under the conditions expressed in Theorem 3.17 in [75], the $\hat{D}$-derivative is equivalent to the gH-derivative. Thus, it is possible that a first order FDE has more than one solution based on the concept of $\hat{D}$-derivative, see Example 6.4 in [71].

In 2014, type-2 fuzzy differential equations (T2FDEs) involving type-2 fuzzy functions and type-2 fuzzy numbers were introduced, for the first time, in [72]. The concept of $H_2$-derivative has been presented for dealing with T2FDEs. This derivative is in the form of SGH-derivative and is defined based on H-difference of type-2 fuzzy numbers, i.e. $H_2$-difference. The main reason for the presentation of T2FDEs came from the fact that an exact form of a type-1 fuzzy number may not be always determined.

With the notions of the joint possibility distribution and the interactive arithmetic operation considered, the interactive derivative of a fuzzy function was introduced in 2017 [73]. In the interactive derivative, the difference, that is called interactive difference, has been defined based on the so-called sup-$J$ extension principle [77]. One of the reasons for the analysis of FDEs using this approach has been stated due to the existence of possible interactivities (or dependencies) between variables in a process. Recently, it has been proved that H-derivative, gH-derivative, g-derivative and $\pi$-derivative are particular cases of the interactive derivative, for more details see [78]. Some of the research work elaborating on the concept of interactive derivative and interactive arithmetic operation can be found in [79]–[85].

The year 2018 came with a new approach for the analysis of FDEs by introducing a new concept of fuzzy derivative that is called granular derivative ($gr$-derivative) [74]. The $gr$-derivative has been defined based on the notion of $gr$-difference. The main difference between this approach and the others is that it employs relative-distance-measure fuzzy interval arithmetic (RDM-FIA) [86]–[88] to deal with FDEs. A key concept in RDM-FIA is the horizontal membership function (HMF) [89]–[91] based on which operations on fuzzy numbers are defined. The main reason why $gr$-derivative has been proposed is to overcome drawbacks of the approaches - namely H-derivative, SGH-derivative, gH-derivative, g-derivative, and $\pi$-derivative - that employ fuzzy standard interval arithmetic (FSIA) to handle FDEs. The drawbacks would be outlined as: the existence of the derivative, monotonic uncertainty, multiplicity of solutions, doubling property, symmetric uncertainty around zero problem (SUAZ problem) and unnatural behavior in modeling (UBM) phenomenon, for more details see [74], [92]–[97]. Some of the drawbacks will be explained briefly in Section V. Further research work inspired by the concept of $gr$-derivative can be found in [98]–[111]. It should be underscored that solving a first order FDE under the concept of the $gr$-derivative leads to just one solution.

In the same year, the notion of generalized Seikkala derivative (gS-derivative) was put forward in [112]. This notion is, in essence, a combination of same-order and reverse-order derivatives by the use of minimum and maximum operators. Moreover, it has been proved that gS-derivative is equivalent to SGH-derivative, see Theorem 3.1 in [112]. Inspired by quantum calculus and q-derivative, fuzzy generalized Hukuhara q-derivative was proposed in 2019 [113] as a combination of q-derivative and gH-difference. It should be noted that based on such a derivative, a first order FDE, under certain conditions, could have two solutions.

Table 1 in the Appendix shows some of the fuzzy derivatives explained in this section.

### B. FRACTIONAL ORDER FUZZY DIFFERENTIAL EQUATIONS

The idea for studying fuzzy differential equations of fractional order was first presented in 2010 [114]. There are different definitions of classical fractional derivatives, namely in the sense of Riemann-Liouville, Caputo, Modified Riemann-Liouville, conformable fractional derivative, Caputo-Fabrizio fractional derivative, to name but a few. Possible combinations of such derivatives with the concepts of fuzzy derivatives and/or fuzzy differences have resulted in the introduction of different definitions of fuzzy fractional derivatives based on which fuzzy fractional differential equations (FFDEs) have been examined.

The Riemann-Liouville fuzzy fractional derivative in the sense of H-derivative; and the existence and uniqueness of the solution for a class of FFDEs with infinite delay were presented in 2010 [115]. In 2011, the Riemann-Liouville fuzzy fractional derivative in the sense of Seikkala derivative was proposed in [116]; and the existence and uniqueness of the solution for FFDEs with fuzzy initial conditions under such a derivative have been shown in [116], [117].

A combination of Riemann-Liouville fuzzy fractional derivative with SGH-derivative (Riemann-Liouville H-derivative) was introduced in 2012 [118]. It is noteworthy that although such a derivative has been denoted as Riemann-Liouville H-derivative in the literature, the derivative has been defined in the form of the SGH-derivative. Due to this fact, similar to SGH-derivative, in general, Riemann-Liouville H-derivative, under some conditions, presents two forms of differentiability of a fuzzy function. Thus, an FFDE of order $\beta \in (0, 1)$ under such a derivative may have two solutions, i.e. a solution that comes from the first form of differentiability, and the other from the second form of differentiability. The existence and uniqueness of the solution for FFDEs using Krasnoselskii-Krein-type condition and Nagumo-type condition have been presented in [119], [120].

In 2012, fuzzy fractional differential equations under the concept of Caputo's derivative in combination with SGH-derivative, i.e. Caputo-type H-derivative, were investigated in two different representations in [121] and [122]. The existence and uniqueness theory of the solution for FFDEs under such a derivative presented in [122] shows that an FFDE of order $\beta \in (0, 1)$, under some conditions, may have two solutions corresponding to the first and second forms of the Caputo-type H-derivative.

It should be underscored that there are some differences between the Caputo's derivative and Riemann-Liouville derivative. What follows is two more important differences. First, Caputo's derivative of a constant function is zero which is not the case with Riemann-Liouville derivative. The second difference has to do with the initial conditions of a fractional differential equation. A fractional differential equation under the concept of Riemann-Liouville derivative involves initial conditions of fractional order that does not happen under the concept of Caputo's derivative.

With a combination of Riemann-Liouville derivative and Goetschel-Voxman derivative considered, the existence and uniqueness of the solution for FFDEs were demonstrated in 2013 [123]. Simultaneously, the Riemann-Liouville fuzzy fractional derivative in the sense of H-derivative was defined in [124]; and using Schauder fixed point theorem, the existence of the solution of an FFDE under such derivative was investigated in [124], [125].

In 2014, type-2 fuzzy fractional differential equations (T2FFDEs) under the concept of type-2 fuzzy fractional derivatives were established in [126]. T2FFDEs are FFDEs in which type-2 fuzzy numbers and type-2 fuzzy functions are involved. Type-2 fuzzy fractional derivatives have been defined in the form of Caputo's and Riemann-Liouville derivatives in a combination with $H_2$-derivative, i.e. Caputo-type $H_2$-derivative and Riemann-Liouville $H_2$-derivative. These derivatives, in general, are in the form of SGH-derivative, and due to this fact, the existence and uniqueness theory of the solution for T2FFDEs given in [126] shows that a T2FFDE of order $\beta \in (0, 1)$ under such derivatives may have two solutions which may be called the first form and the second form solutions. In the same year, FFDEs under the concept of generalized fuzzy Caputo derivative (Caputo gH-derivative) with the existence and uniqueness theory of their solutions by the use of Krasnoselskii-Krein condition were presented in [127]. The Caputo gH-derivative has been constituted by a combination of Caputo's derivative and $gH$-derivative. So far, extensive research has been carried out on FFDEs based on the mentioned fuzzy fractional derivatives, for instance see [128]–[138]. Some of such results presented in [139] are remarkable; only under specific conditions, is an FFDE equivalent to a fractional fuzzy integral equation, for more details see [139].

A combination of modified Riemann-Liouville derivative [140] and $g$-derivative for type-1 and type-2 fuzzy fractional differential equations was introduced in 2016 [141]. The main reason for presenting the modified Riemann-Liouville fuzzy fractional derivative comes from the fact that, unlike the Caputo gH-derivative, it does not require that the function in question be differentiable of higher order. As a case in point, for the fractional derivative of order $\beta \in (0, 1)$, the modified Riemann-Liouville fuzzy fractional derivative does not necessitate the function in question be first order differentiable. In addition, unlike fuzzy fractional derivative in the sense of Riemann-Liouville, the initial conditions appear in the same way as they do in an integer order differential equation.

FFDEs under the concept of Caputo-Fabrizio fractional derivative in combination with SGH-derivative (Caputo-Fabrizio SGH-derivative) were studied in 2018 [142]. The main reason why such a derivative was introduced is that the kernel in Caputo-Fabrizio fractional derivative, unlike Riemann-Liouville and Caputo's derivatives, is non-singular. It should be also noted that the idea for applying Caputo-Fabrizio fractional derivative on uncertain fractional differential equations where interval-valued functions are involved, was first reported in [143]. At the same time, by combining Riemann-Liouville and Caputo's derivatives with $gr$-derivative, i.e. granular Riemann-Liouville derivative and granular Caputo derivative, granular fuzzy fractional derivatives emerged in [94]. Granular fuzzy fractional integral has also been presented in this work. The main reason for the analysis of FFDEs under granular fuzzy fractional derivatives is the fact that the investigation of FFDEs under the notions of fuzzy fractional derivatives that are defined based on SGH-derivative, gH-derivative and, in general, FSIA-based approaches comes with some restrictions. Such restrictions like multiplicity of solutions and UBM phenomenon can be overcome by the use of granular fuzzy fractional derivatives that are based on RDM-FIA approach. It has been also shown that an FFDE of order $\beta \in (0, 1)$ under the concept of granular fuzzy fractional derivatives has only one solution. Inspired by granular fuzzy fractional derivatives, granular Riemann-Liouville q-fractional integral and granular Caputo q-fractional derivative have been presented in [144] for the investigation of FFDEs on a time scale.

Caputo-Katugampola gH-derivative and Riemann-Liouville-Katugampola gH-derivative as generalizations of Caputo gH-derivative and Riemann-Liouville H-derivative by utilizing Katugampola concept [145] and gH-derivative were introduced in 2019 [146]. The existence and uniqueness of solutions for FFDEs under Caputo-Katugampola gH-derivative by the use of successive approximations under generalized Lipschitz condition have been shown in [146]. Specifically, similar to other definitions of fuzzy derivatives established by gH-derivative or gH-difference, it has been proved that, under some conditions, an FFDE of order $\beta \in (0, 1)$ has two solutions, i.e. solutions corresponding to the first and second forms of differentiability.

By a combination of Caputo q-fractional derivative, that is a fractional derivative coming from quantum calculus, and gH-derivative, Caputo q-fractional gH-derivative was introduced in 2019 [113] and the existence and uniqueness of solutions for FFDEs were demonstrated by Krasnoselskii-Krein-type conditions. Since this approach also underlies FSIA-based approaches, solving an FFDE, under some conditions, brings us more than one solution.

In 2020, the concept of Atangana-Baleanu gH-derivative as a combination of Atangana-Baleanu fractional derivative [147] and gH-derivative was reported in [148]. One of the reasons for proposing such a derivative has been based on non-locality and non-singularity property of the new kernel defined in Atangana-Baleanu fractional derivative. In the same year, by employing conformable fractional derivative [149] and SGH-derivative, FFDEs have been studied under the concept of conformable SGH-derivative (or fuzzy conformable derivative) in [150], [151] and, almost simultaneously, in [152]. It has been stated that, unlike some other definitions of fractional derivatives, conformable fractional derivative enables us to have a definition of the differentiability of a function in the same way as the definition of a derivative comes from the limit of the function.

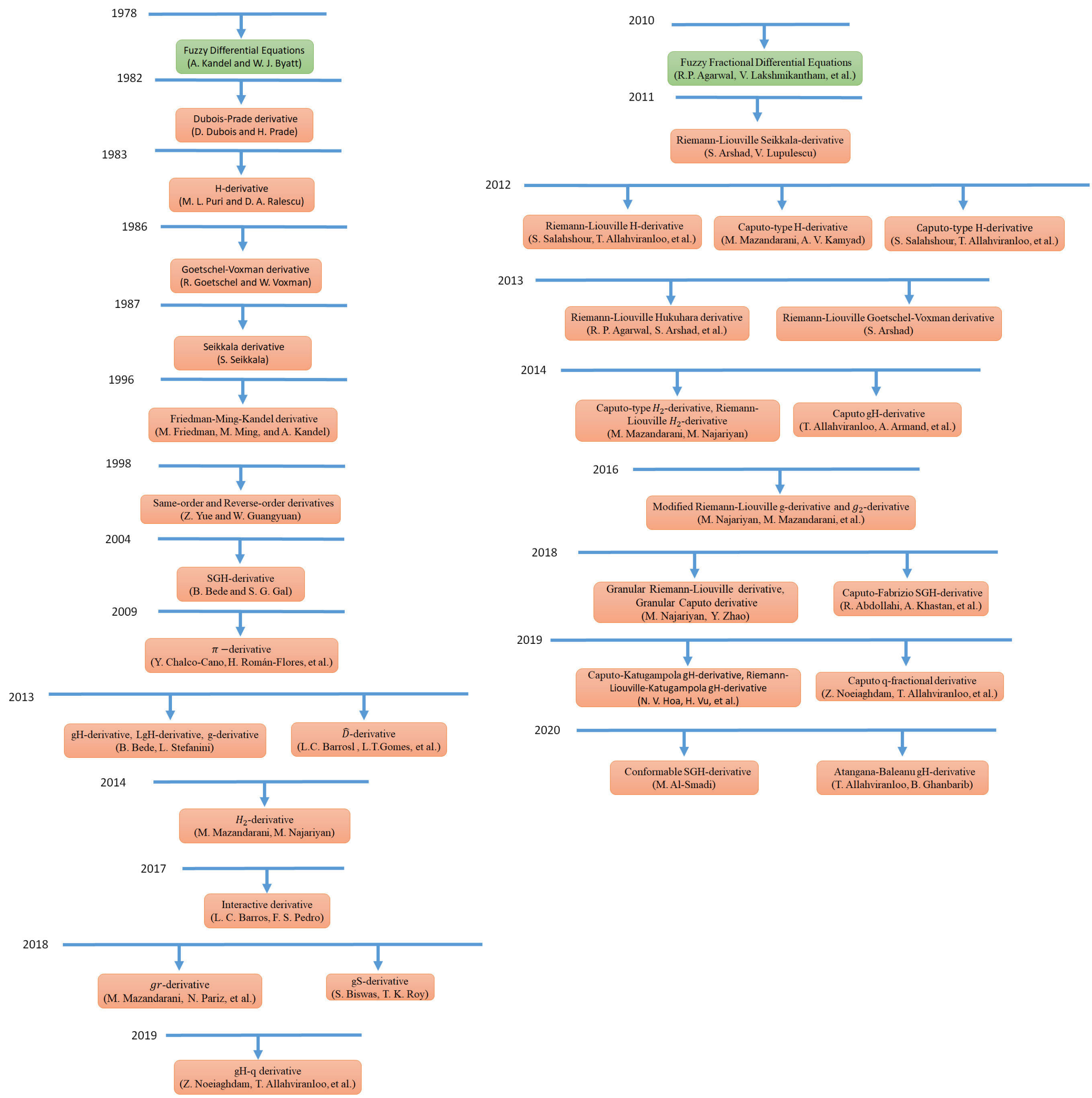


**FIGURE 1.** **A history of the evolution of FDEs.**

The characterization theorem, and existence and uniqueness of solutions for FFDEs under the concept of fuzzy conformable differentiability have been also given in [151].

Fig. 1 illustrates the chronological evolution of FDEs based on various definitions of fuzzy derivatives.

## III. SOLUTION METHODS OF FDEs

In this section, out of a large number of methods proposed for obtaining the solutions of FDEs, a few are mentioned. Almost all the methods have translated an FDE to a system of crisp differential equations by the aid of a characterization theorem corresponding to each fuzzy derivative.

The most well-known and important characterization theorem is what has been given in [153]. This theorem states that, under some certain hypotheses, a first order fuzzy initial value problem (FIVP) under Hukuhara differentiability concept is equivalent to a system of two crisp first order differential equations characterizing the FIVP level-wise. In an analogous way, under some conditions, a first order FIVP under the concept of SGH-derivative is equivalent level-wise to two systems of crisp first order differential equations. Each system including two crisp differential equations characterizing the FIVP level-wise corresponds to one of the forms of differentiability in the sense of SGH-derivative, i.e. the first and second forms of differentiability, see [40] and Theorem 9.11 in [154]. Thus, almost all the proposed methods for solving FDEs have been drawn from those proposed for obtaining the solutions of crisp differential equations. This is one of the reasons for the existence of a huge volume of literature in which a wide variety of methods have been proposed for solving FDEs. The differences between the characteristics of such methods, i.e. accuracy, convergence, and etc., correspond to those explained well in the literature concerning numerical or analytical solution methods of ordinary or partial differential equations. This is one of the reasons that comparison between such methods is excluded in this section.

It is expedient to remark that translating a first order FDE to a system of first order differential equations corresponds to the approaches in which the FDE is taken into account under the concepts related to H-derivative, SGH-derivative, gH-derivative, $\pi$-derivative, etc. However, the approaches in which an FDE deals with the concept of $gr$-derivative (or $\hat{D}$-derivative under some hypotheses), translate a first order FDE to a first order differential equation.

The proposed methods for solving FDEs are categorized based on each concept of differentiability under which FDEs have been considered. Moreover, solving an FDE, here, is to be understood as solving a fuzzy initial value problem, or fuzzy boundary value problem. In the following, some of the solution methods are mentioned, first for integer order FDEs and then for fractional order FDEs.

Runge-Kutta method [155], Taylor method [156], Nystrom method [157], artificial neural network [158], and F-transform [159] have been proposed for solving FDEs under the concept of H-differentiability; and under Seikkala-differentiability, the Runge-Kutta method has been presented in [160]. There have been many papers dedicated to finding solutions of FDEs under the concept of SGH-derivative among which are Runge-Kutta method [161], [162], reproducing kernel theory [163], extended Runge-Kutta [164], Euler method [165], differential transform method [166], fuzzy Sumudu transforms [167], [168], diameter-based method of a fuzzy function [169], fuzzy Fourier transform [170], Picard method [171], Laplace transform [172], [173], quasi-level-wise-system [174], and shooting method [175]. In addition, the variation of constant formula for a linear first order fuzzy differential equation with crisp coefficients and fuzzy initial condition has been introduced in [176], [177].

Since gH-derivative of a fuzzy function under the assumption of a set of finite switching points is equivalent to SGH-derivative of the fuzzy function, the proposed methods for finding the solutions of FDEs equipped with SGH-derivative concept can also be utilized for FDEs where the derivative is taken into account as gH-derivative. Moreover, almost all the effort made to solve FDEs under SGH-derivative or gH-derivative is restricted to finding solutions corresponding to the first form and the second form of differentiability, i.e. two solutions for a first order FDE are obtained. One of the reasons might be the fact that determining switching points, if they exist, is not an easy task and becomes complex for a first order nonlinear FDE and even more complicated for higher order FDEs. Indeed, there are a few reports in which a unique solution has been obtained for the problem under study, for instance see [38], [39], [178]. It should be noted that obtaining the solution of an FDE, if it exists, does not depend on determining switching points on condition that the concept of $gr$-derivative (or $\hat{D}$-derivative under some hypotheses) is employed, for example see [98].

Since in the recent years, fuzzy fractional calculus has attracted much attention and its development is continuing. Plenty of methods for solving fuzzy fractional differential equations have been proposed that some of which are as follows.

Although solving FFDEs under the concept of Riemann-Liouville H-derivative has been investigated by some proposed methods such as fuzzy Laplace transform method [118] and Mittag-Leffler functions [179], Caputo H-derivative and Caputo gH-derivative have been more palatable concepts of fuzzy fractional derivatives that have triggered much research work. Hence, by the consideration of such fuzzy fractional derivatives, some of the methods dedicated to obtain solutions of FFDEs are modified fractional Euler method [121], spline collocation method [180], Chebyshev polynomials [181], differential transform method [182], spectral method [183], [184], residual power series [185], perturbation-iteration algorithm [186].

Under the concept of Caputo-type $H_2$-derivative, approximate solutions of type-2 fuzzy fractional differential equations have been obtained using the predictor-evaluate-corrector-evaluate (PECE) method in [126]. By presenting an approximation of the granular fuzzy fractional integral and the granular Caputo fuzzy fractional derivative, an approximate solution for FFDEs under the concept of gr-Caputo fuzzy fractional derivative has been given in [94]. Finally, FFDEs under the concept of conformable SGH-derivative have been solved by the aid of Taylor series expansion in [150] and reproducing kernel Hilbert space method in [151].

## IV. APPLICATIONS OF FDEs

This section presents some of the proposed applications of FDEs in various fields of science. The point that to be highlighted is that in spite of so many proposals, there is no report on the real experiments of the applications of FDEs. One of the important applications of FDEs proposed in recent years has fallen in the scope of the control theory. In this regard, optimal control of a fuzzy linear dynamical system based on gH-differentiability and SGH-differentiability has been studied in [178], [187], [188]. The design of an optimal feedback control for regulating a fuzzy linear dynamical system with a proposed application on a Boeing 747 was presented in [93] where FDEs were considered under the concept of $gr$-differentiability. In addition, in [92], the problem of fuzzy time optimal control by the use of $gr$-differentiability has been investigated. A deep analysis of the stability of fuzzy linear dynamical systems under the notion of $gr$-differentiability has been reported in [102]. The performance criteria of second-order fuzzy linear dynamical systems and fuzzy tracking control of fuzzy linear dynamical systems under the concept of $gr$-differentiability have been investigated, respectively, in [95], [101]. By the aid of $gr$-Caputo derivative concept, the fuzzy fractional quadratic regulator problem was studied in [94] where a fuzzy feedback control was designed. For further study about the applications of FDEs of fractional order in optimal control and controllability refer to [189].

Indeed, the applications of FDEs are not limited to the theory of control, but they have been proposed in studying biomathematics [190], diabetes mellitus [191], [192], cerebrospinal fluid pressure [126], hydrologic process [193], water movement in a horizontal column [194], epidemic model [195], population growth model [82], [98], Newton's law of cooling [72], electric circuits [72], [196], and the brain tumor [197]. Moreover, some of the other proposed applications of FDEs can be found in [198] for a predator-prey model, in [56] for production inventory model, in [72] for an economical investment, and in [199] for oil palm frond. For further applications of FDEs refer to [200]–[202].

*Note 2:* Since by the aid of a definition of first order derivative of a fuzzy function the higher order derivatives of the fuzzy function can be determined, higher order FDEs under various notions of fuzzy derivatives have been examined in the literature, for instance see [203]–[206]. Moreover, fuzzy partial differential equations (FPDEs) under some concepts of fuzzy derivatives explained in Section II-A have been investigated in many studies, e.g. see [207]–[209]. The higher order FDEs and FPDEs have been excluded in this survey. Furthermore, there are some other types of uncertain differential equations such as random fuzzy differential equations [210]–[212], fuzzy integro differential equations [213], [214], and fuzzy fractional integro differential equations [215], [216] which have not been included in this paper.

*Note 3:* Interval-valued differential equations may be viewed as a special case of fuzzy differential equations where the uncertainty is considered as an interval in which each member has a full grade of membership to the interval. Such a differential equations have been dealt with in the literature, for instance see [217]–[219]. Although some sections of the present paper - particularly some of the challenges presented in next section - might well be stated for the cases dealing with interval-valued differential equations, such a differential equations have not been pronounced explicitly in this paper.

## V. CONCLUSION

This paper has presented a brief survey on the evolution of fuzzy differential equations. The particular attention has been given to FDEs in which a definition of fuzzy derivative of fuzzy number-valued functions has been considered. Through a selective list of papers, the historical motivations and current research progress of FDEs have been outlined. Great advances on both fundamental aspects and applications of FDEs have been made with a multitude of available publications on the topic. Nonetheless, some issues still remain challenging which provide opportunities for further research on FDEs in the future. What follows presents our personal perspectives on the issues and FDEs.

### A. CHALLENGES OF FUZZY DIFFERENTIAL EQUATIONS

The challenges given in here are substantially associated with FSIA-based approaches and those which are based on or equivalent with concepts of H-derivative, SGH-derivative, gH-derivative, and g-derivative. Thus, the challenges have also to do with $\pi$-derivative, Caputo-type H-derivative, Caputo gH-derivative, gH-q derivative, Caputo-q fractional derivative, Atangana-Baleanu gH-derivative, conformable SGH-derivative, Caputo-Katugampola gH-derivative, Riemann-Liouville H-derivative and etc.

**1.** Even though a large body of research dedicated to FDEs are those in which the fuzzy derivative is a member of Hukuhara derivatives family such as H-derivative, SGH-derivative, gH-derivative, Caputo H-derivative, and etc., the existence and obtaining the solution of FDEs under such concepts of derivatives are challengeable. As a matter of fact, in dealing with FDEs, in almost all cases, FDEs are solved based on the characterization theorem which helps find the solutions related to the first and second form of the differentiability. These solutions correspond to the cases where the diameter of fuzzy function is non-decreasing and non-increasing, respectively. In some papers, it has been stated that having more than one solution enables us to choose from a set of solutions. Such solutions whose fuzziness is monotonic may be acceptable in very special cases of linear first order FDEs; however, in a general setting, for the analysis or prediction of behavior of a phenomenon or a dynamical system, a unique solution is needed based on which a decision is made. It should be understood that by the "unique solution" we mean a single fuzzy solution whose diameter is not necessarily monotonic. As an illustration, assume a model of DC electrical motor with some uncertain parameters. Additionally, suppose that the goal is to investigate the position of the shaft, i.e. $\tilde{x}$. The simplest model of such dynamical system, formally, would be shown as follows

$$\tilde{a}\dddot{\tilde{x}}(t)+\tilde{b}\ddot{\tilde{x}}(t)+\tilde{c}\dot{\tilde{x}}(t)=\tilde{d}\tilde{V} \tag{1}$$

where the coefficients $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and $\tilde{d}$ are assumed to be fuzzy numbers and $\tilde{V}$ denotes the input voltage. To achieve the goal, the FDE shown in (1) needs to be solved. Considering the coefficients as fuzzy numbers means that an elastic constraint on the values that may be assigned to each coefficient has been taken into account. The elastic constraint equates to the possibility distribution corresponding to each coefficient. Thus, each coefficient in the application assumes a single value with a degree of possibility. As a result, in the real case, the shaft of the electrical motor is in a determined unique position with a degree of possibility. Therefore, unquestionably, the electrical motor, as a dynamical system, shows a unique behavior once the coefficients are assigned a value. Hence, a unique solution of the FDE must be obtained by employing an approach to show the possible positions of the shaft. Consequently, any approach that brings us more than one solution to "choose" from is fundamentally challengeable and might be in contrast with what happens in the reality. In other words, in the context of the solutions of FDEs, we are not allowed to choose what happens in the reality, but we need to show what possibly happens in the reality which is in essence unique. It should be noted that the electrical motor is the simplest example of a dynamical system, a little more complex system may be chaotic systems or nonlinear systems with unknown uncertain control inputs. Furthermore, since the behavior of a phenomenon in the presence of uncertainty depends, in essence, on its natural intrinsic properties and dynamics, any attempt to necessarily obtain a solution with predetermined uncertainty form might lead to a contrast with the natural behavior of the phenomenon. Thus, the first and second forms of the solutions which necessitate the monotonicity of the fuzziness might not be solely applicable for the analysis of natural behavior of phenomena. One of the alternative approaches adopted in a few papers is to find switching points and presenting a solution made of the first and second forms of solutions, for instance see Example 4.1. in [38], [39]. Nevertheless, such an alternative needs a complex computation process even for the cases in the form of linear FDEs and becomes even more complex for the nonlinear FDEs such that one may fail to obtain the unique solution. It should be also noted that the unique solution must be differentiable in the sense of the concept of differentiability under study, e.g. gH-differentiability, for any intended time interval, and not just for a particular time interval. This is one of the reasons that the existence of a solution to FDEs based on the family of Hukuhara derivatives is challengeable. Indeed, there are also some other challenges that will be discussed in the sequel. What should be remarked is that applying the concepts of $gr$-derivative or $\hat{D}$-derivative may be suitable alternatives to overcome the issue or at least alleviate it.

**2.** Determining the first form solution of a first order FDE equates to solving a system of two differential equations. This is also the case with the second form solution. Hence, for obtaining the solution of a first order FDE, a system of four differential equations is to be solved. Specifically, for each order of the derivative, there are two alternatives of differentiability, i.e. the first form of differentiability and the second form of differentiability, to each of which an FDE is associated. This fact raises a challenge for higher order FDEs. For the higher order FDEs, a set of all the combinations of differentiability alternatives needs to be considered. As an illustration, let us assume the third order FDE shown in (1). In this case, there is a set of 8 or $2^3$ alternatives to each of which an FDE is associated. To solve each FDE, the solution of a system of two differential equations is to be obtained. Thus, totally, 16 differential equations are to be solved. Subsequently, we need to examine each solution to recognize which one (or ones) would satisfy the concept of differentiability taken into account in the FDE from which the solution has been obtained. Obviously, in this perspective, an $n$th order FDE may have maximum $2^n$ solutions. This feature that belongs to the family of generalized Hukuhara derivatives may be called multiplicity of solutions property. Due to this fact, based on the earlier explanations about the unique solution, the analysis of even the simplest dynamical systems on the basis of the family of Hukuhara derivatives is challengeable. It should be stressed that in the sequel, we show that an $n$th order FDE may have even more than $2^n$ solutions.

**3.** In order to find solutions of FDEs under a concept of fuzzy derivative by the use of the characterization theorem, one needs to determine the lower and upper $\alpha$-level cuts of functions in question involved in the FDEs. Whereas characterizing such $\alpha$-level cuts in simple cases such as linear FDEs may be feasible, it becomes a complicated task for a general setting such as nonlinear FDEs including unknown functions. As an illustration, consider the following simple problem:

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \dot{\tilde{y}}(t) \\ \dot{\tilde{z}}(t) \end{bmatrix} = \begin{bmatrix} \tilde{x}(t)\tilde{y}(t) \\ \tilde{y}(t)\tilde{z}(t) + \tilde{a}\tilde{x}(t) \\ \tilde{b}\tilde{z}(t) + \tilde{u}(t) \end{bmatrix} \tag{2}$$

where the initial conditions and coefficients are assumed to be fuzzy numbers. Suppose that the goal is to find the fuzzy control function $\tilde{u}(t)$ under a certain criterion. Under the concept of the first form of differentiability, system of FDEs (2) is translated into the following system of differential equations:

$$\begin{bmatrix} \dot{\underline{x}}^{\alpha}(t) \\ \dot{\overline{x}}^{\alpha}(t) \\ \dot{\underline{y}}^{\alpha}(t) \\ \dot{\overline{y}}^{\alpha}(t) \\ \dot{\underline{z}}^{\alpha}(t) \\ \dot{\overline{z}}^{\alpha}(t) \end{bmatrix} = \begin{bmatrix} \underline{\left[\tilde{x}(t)\tilde{y}(t)\right]}^{\alpha} \\ \overline{\left[\tilde{x}(t)\tilde{y}(t)\right]}^{\alpha} \\ \underline{\left[\tilde{y}(t)\tilde{z}(t) + \tilde{a}\tilde{x}(t)\right]}^{\alpha} \\ \overline{\left[\tilde{y}(t)\tilde{z}(t) + \tilde{a}\tilde{x}(t)\right]}^{\alpha} \\ \underline{\left[\tilde{b}\tilde{z}(t) + \tilde{u}(t)\right]}^{\alpha} \\ \overline{\left[\tilde{b}\tilde{z}(t) + \tilde{u}(t)\right]}^{\alpha} \end{bmatrix} \tag{3}$$

where based on FSIA we have

$$\begin{cases} \underline{\left[\tilde{x}(t)\tilde{y}(t)\right]}^{\alpha} = \min\left\{\overline{x}^{\alpha}(t)\overline{y}^{\alpha}(t), \overline{x}^{\alpha}(t)\underline{y}^{\alpha}(t), \underline{x}^{\alpha}(t)\underline{y}^{\alpha}(t), \underline{x}^{\alpha}(t)\overline{y}^{\alpha}(t)\right\} \\ \overline{\left[\tilde{x}(t)\tilde{y}(t)\right]}^{\alpha} = \max\left\{\overline{x}^{\alpha}(t)\overline{y}^{\alpha}(t), \overline{x}^{\alpha}(t)\underline{y}^{\alpha}(t), \underline{x}^{\alpha}(t)\underline{y}^{\alpha}(t), \underline{x}^{\alpha}(t)\overline{y}^{\alpha}(t)\right\} \\ \underline{\left[\tilde{y}\tilde{z}(t)+\tilde{a}\tilde{x}(t)\right]}^{\alpha} = \min\left\{\overline{y}^{\alpha}(t)\overline{z}^{\alpha}(t), \overline{y}^{\alpha}(t)\underline{z}^{\alpha}(t), \underline{y}^{\alpha}(t)\underline{z}^{\alpha}(t), \underline{y}^{\alpha}(t)\overline{z}^{\alpha}(t)\right\} \\ \qquad + \min\left\{\overline{a}^{\alpha}\overline{x}^{\alpha}(t), \overline{a}^{\alpha}\underline{x}^{\alpha}(t), \underline{a}^{\alpha}\underline{x}^{\alpha}(t), \underline{a}^{\alpha}\overline{x}^{\alpha}(t)\right\} \\ \overline{\left[\tilde{y}\tilde{z}(t)+\tilde{a}\tilde{x}(t)\right]}^{\alpha} = \max\left\{\overline{y}^{\alpha}(t)\overline{z}^{\alpha}(t), \overline{y}^{\alpha}(t)\underline{z}^{\alpha}(t), \underline{y}^{\alpha}(t)\underline{z}^{\alpha}(t), \underline{y}^{\alpha}(t)\overline{z}^{\alpha}(t)\right\} \\ \qquad + \max\left\{\overline{a}^{\alpha}\overline{x}^{\alpha}(t), \overline{a}^{\alpha}\underline{x}^{\alpha}(t), \underline{a}^{\alpha}\underline{x}^{\alpha}(t), \underline{a}^{\alpha}\overline{x}^{\alpha}(t)\right\} \\ \underline{\left[\tilde{b}\tilde{z}(t)+\tilde{u}(t)\right]}^{\alpha} = \min\{\overline{b}^{\alpha}\overline{z}^{\alpha}(t), \overline{b}^{\alpha}\underline{z}^{\alpha}(t), \underline{b}^{\alpha}\underline{z}^{\alpha}(t), \underline{b}^{\alpha}\overline{z}^{\alpha}(t)\} + \underline{u}^{\alpha}(t) \\ \overline{\left[\tilde{b}\tilde{z}(t)+\tilde{u}(t)\right]}^{\alpha} = \max\{\overline{b}^{\alpha}\overline{z}^{\alpha}(t), \overline{b}^{\alpha}\underline{z}^{\alpha}(t), \underline{b}^{\alpha}\underline{z}^{\alpha}(t), \underline{b}^{\alpha}\overline{z}^{\alpha}(t)\} + \overline{u}^{\alpha}(t) \end{cases} \tag{4}$$

Due to the fact that, $\tilde{x}(t)$, $\tilde{y}(t)$, $\tilde{z}(t)$, and $\tilde{u}(t)$, as a whole, are unknown fuzzy functions, determining the expressions in the right hand side of relation (3) as explicit terms may not be possible, as a whole. Thus, finding the solutions of even such simple problems might involve a complicated task whose feasibility is reduced by an increase in the number of uncertain parameters or functions involved in the problem. This is another challenge that emerges when the family of Hukuhara derivatives, or FSIA-based approaches, are employed to find the solutions of FDEs.

**4.** Unnatural behavior in modelling (UBM) phenomenon is the other challenge concerning the FSIA-based approaches including the family of Hukuhara derivatives. The UBM phenomenon is a phenomenon representing that different guises of a same structure of a system model may show different behaviors of the system. As an illustration, let us consider the model of the DC electrical motor which can be presented in different forms among which are:

$$\tilde{a}\dddot{\tilde{x}}(t) + \tilde{b}\ddot{\tilde{x}}(t) + \tilde{c}\dot{\tilde{x}}(t) = \tilde{d}\tilde{V} \tag{5}$$
$$\tilde{a}\dddot{\tilde{x}}(t) + \tilde{b}\ddot{\tilde{x}}(t) = \tilde{d}\tilde{V} - \tilde{c}\dot{\tilde{x}}(t) \tag{6}$$
$$\tilde{a}\dddot{\tilde{x}}(t) + \tilde{c}\dot{\tilde{x}}(t) = \tilde{d}\tilde{V} - \tilde{b}\ddot{\tilde{x}}(t) \tag{7}$$
$$\tilde{b}\ddot{\tilde{x}}(t) + \tilde{c}\dot{\tilde{x}}(t) = \tilde{d}\tilde{V} - \tilde{a}\dddot{\tilde{x}}(t) \tag{8}$$
$$\tilde{a}\dddot{\tilde{x}}(t) = \tilde{d}\tilde{V} - \tilde{b}\ddot{\tilde{x}}(t) - \tilde{c}\dot{\tilde{x}}(t) \tag{9}$$
$$\tilde{a}\dddot{\tilde{x}}(t) - \tilde{d}\tilde{V} = -\tilde{b}\ddot{\tilde{x}}(t) - \tilde{c}\dot{\tilde{x}}(t) \tag{10}$$
$$\tilde{a}\dddot{\tilde{x}}(t) - \tilde{d}\tilde{V} + \tilde{b}\ddot{\tilde{x}}(t) = -\tilde{c}\dot{\tilde{x}}(t) \tag{11}$$
$$\tilde{a}\dddot{\tilde{x}}(t) - \tilde{d}\tilde{V} + \tilde{c}\dot{\tilde{x}}(t) = -\tilde{b}\ddot{\tilde{x}}(t) \tag{12}$$

Unquestionably, all the FDEs shown above have the same structure and correspond to the same system. However, solving FDEs (5) to (12) under each concept of Hukuhara derivatives family, e.g. SGH-derivative and gH-derivative, may result in different solutions. Specifically, such FSIA-based approaches are unable to recognize the structure of the model, and are heavily dependent on the guises of the model. As explained before, based on FSIA-based approaches, each $n$th order FDE may have up to $2^n$ solutions. The dependency on the different forms of a same model causes the obtained solutions of the model of a system of $n$th order to exceed $2^n$. It should be underscored that approaches corresponding to $gr$-derivative (and $\hat{D}$-derivative under some hypotheses) are not sensitive to different forms of a model and can accordingly recognize the structure of the model.

**5.** The other challenge arising in dealing with FDEs based on FSIA-based approaches corresponds to what may be called the zero form. The zero form of a system model is a form of the model structure in which all the constants and functions involved in the structure take place in one side of the equation and only zero remains in the other side. The zero form is similar to the homogeneous form of differential or algebraic equations. As an illustration, the zero forms of the model of the DC electrical motor are expressed as follows:

$$\tilde{a}\dddot{\tilde{x}}(t) + \tilde{b}\ddot{\tilde{x}}(t) + \tilde{c}\dot{\tilde{x}}(t) - \tilde{d}\tilde{V} = 0 \tag{13}$$
$$\tilde{d}\tilde{V} - \tilde{a}\dddot{\tilde{x}}(t) - \tilde{b}\ddot{\tilde{x}}(t) - \tilde{c}\dot{\tilde{x}}(t) = 0 \tag{14}$$

Although FSIA-based approaches (i.e. SGH-derivative, gH-derivative, g-derivative, etc.) are sensitive to different guises of a same structure of a system model and they may yield different solutions, they are unable to give a solution to the zero forms. Specifically, one fails to obtain a fuzzy solution for the FDEs (13) and (14) based on any concept belonging to Hukuhara derivatives family. However, it is possible to obtain the solution based on $gr$-derivative and the solution is the same as those obtained from FDEs 5 to 12.

**6.** Assume $\tilde{k}$ is a fuzzy number and $\tilde{x}(t)$ is a fuzzy function. Let $(f(\tilde{x}(t))$ and $g(\tilde{x}(t))$ be two possibly different functions of $\tilde{x}(t)$. Then, on the basis of H-derivative, SGH-derivative, gH-derivative, g-derivative, or any concept that has been defined based on (or equivalent with) such derivatives, the following two FDEs, in a general setting, are not equivalent:

$$\dot{\tilde{x}}(t) = \tilde{k}\left(f\left(\tilde{x}(t)\right) + g\left(\tilde{x}(t)\right)\right) \tag{15}$$
$$\dot{\tilde{x}}(t) = \tilde{k}f\left(\tilde{x}(t)\right) + \tilde{k}g\left(\tilde{x}(t)\right) \tag{16}$$

where the initial condition is a fuzzy number. In other words, the factorization cannot be applied in fuzzy differential equations if the FSIA-based approaches (i.e. H-derivative, SGH-derivative, gH-derivative, g-derivative, or any concept that has been defined based on (or equivalent with) such derivatives) are employed.

**TABLE 1.** Some of the fuzzy derivatives.

| Fuzzy derivative | Definition | Comments |
|---|---|---|
| $H$-derivative | $\tilde{f}'(t)=\lim\limits_{h\to 0^+}\frac{\tilde{f}(t+h)\overset{H}{-}\tilde{f}(t)}{h}=\lim\limits_{h\to 0^+}\frac{\tilde{f}(t)\overset{H}{-}\tilde{f}(t-h)}{h}$ | $\tilde{u}\overset{H}{-}\tilde{v}=\tilde{w}\Leftrightarrow \tilde{u}=\tilde{v}+\tilde{w}$ |
| Seikkala derivative | $\left[\tilde{f}'(t)\right]^{\alpha}=\left[\underline{f}'^{\alpha}(t),\overline{f}'^{\alpha}(t)\right]$ | |
| Same-order and reverse-order derivatives | (a) $\left[\tilde{f}'(t)\right]^{\alpha}=\left[\underline{f}'^{\alpha}(t),\overline{f}'^{\alpha}(t)\right]$<br>(b) $\left[\tilde{f}'(t)\right]^{\alpha}=\left[\overline{f}'^{\alpha}(t),\underline{f}'^{\alpha}(t)\right]$ | (a) Same-order<br>(b) Reverse-order |
| $SGH$-derivative | $\tilde{f}'(t)=\lim\limits_{h\to 0^+}\frac{\tilde{f}(t+h)\overset{H}{-}\tilde{f}(t)}{h}=\lim\limits_{h\to 0^+}\frac{\tilde{f}(t)\overset{H}{-}\tilde{f}(t-h)}{h}$<br>$\tilde{f}'(t)=\lim\limits_{h\to 0^+}\frac{\tilde{f}(t)\overset{H}{-}\tilde{f}(t+h)}{-h}=\lim\limits_{h\to 0^+}\frac{\tilde{f}(t-h)\overset{H}{-}\tilde{f}(t)}{-h}$ | |
| $\pi$-derivative | $\tilde{f}'(t)=\lim\limits_{h\to 0}\frac{\tilde{f}(t+h)\overset{\pi}{-}\tilde{f}(t)}{h}$ | $\tilde{u}\overset{\pi}{-}\tilde{v}=\tilde{w}\Leftrightarrow[\tilde{w}]^{\alpha}=[\min\{\underline{u}^{\alpha}-\underline{v}^{\alpha}\},\max\{\overline{u}^{\alpha}-\overline{v}^{\alpha}\}]$ |
| $gH$-derivative | $\tilde{f}'(t)=\lim\limits_{h\to 0}\frac{\tilde{f}(t+h)\overset{gH}{-}\tilde{f}(t)}{h}$ | $\tilde{u}\overset{gH}{-}\tilde{v}=\tilde{w}\Leftrightarrow(\tilde{u}=\tilde{v}+\tilde{w}\ \text{ or }\ \tilde{v}=\tilde{u}-\tilde{w})$ |
| $g$-derivative | $\tilde{f}'(t)=\lim\limits_{h\to 0}\frac{\tilde{f}(t+h)\overset{g}{-}\tilde{f}(t)}{h}$ | $\tilde{u}\overset{g}{-}\tilde{v}=\tilde{w}\Leftrightarrow[\tilde{w}]^{\alpha}=cl\left(conv\bigcup\limits_{\beta\geq\alpha}([\tilde{u}]^{\beta}\overset{gH}{-}[\tilde{v}]^{\beta})\right)$ |
| $\hat{D}$-derivative | $\tilde{f}'(t)=\hat{D}\tilde{f}(t)$ | $\mu_{\hat{D}\tilde{f}(t)}(y)=\begin{cases}\sup_{f(t)\in D^{-1}y}\mu_{\tilde{f}(t)}(f(t)) & \text{if } D^{-1}y\neq\emptyset\\ 0 & \text{if } D^{-1}y=\emptyset\end{cases}$ |
| Interactive derivative | $\tilde{f}'(t)=\lim\limits_{h\to 0}\frac{\tilde{f}(t+h)\overset{J}{-}\tilde{f}(t)}{h}$ | $\mu_{(\tilde{u}\overset{J}{-}\tilde{v})}(z)=\sup\limits_{x-y=z}\mu_J(x,y)$ |
| $gS$-derivative | $\left[\tilde{f}'(t)\right]^{\alpha}=\left[\min\left\{\underline{f}'^{\alpha}(t),\overline{f}'^{\alpha}(t)\right\},\max\left\{\underline{f}'^{\alpha}(t),\overline{f}'^{\alpha}(t)\right\}\right]$ | |
| $gr$-derivative | $\tilde{f}'(t)=\lim\limits_{h\to 0}\frac{\tilde{f}(t+h)\overset{gr}{-}\tilde{f}(t)}{h}$ | $\mathcal{H}(\tilde{u})\triangleq\underline{u}^{\alpha}+(\overline{u}^{\alpha}-\underline{u}^{\alpha})\beta_u$<br>$\tilde{u}\overset{gr}{-}\tilde{v}=\tilde{w}\Leftrightarrow\mathcal{H}(\tilde{u})-\mathcal{H}(\tilde{v})=\mathcal{H}(\tilde{w})$ |

### B. FUTURE PERSPECTIVES OF FUZZY DIFFERENTIAL EQUATIONS

Although addressing each challenge expressed in the preceding section has the potential to be considered as future studies, future perspectives of FDEs, in a broad outlook, are outlined as follows.

**1.** In order to overcome some of the challenges concerning FSIA-based approaches including Hukuhara derivatives family, new theories are to be put forward with the capacity to solve FDEs and tackle their corresponding problems in a practical and direct way. The direct way means one is able to solve the intended problems, e.g. finding the solutions of FDEs, solving an optimal control problem, etc., without using the characterization theorem. The practical way refers to the way through which a solution to the problem may be characterized in the sense of the unique solution, that is the solution whose diameter is not necessarily monotonic.

**2.** In order to justify the effectiveness and applicability of each approach dealing with FDEs, e.g. the family of Hukuhara derivatives, $gr$-derivative, $\hat{D}$-derivative and interactive derivative, some models of real cases should be examined rigorously by such approaches and the results must be compared with what happens in the reality.

**3.** The solution of an FDE is a fuzzy function whose value in any point of its domain expresses a fuzzy number. These fuzzy numbers, from a point of view, equate to the possibility distributions of the values of fuzzy function. Specifically, they convey information about the possibility degrees of the values of the fuzzy function which may be expressed as linguistic values. The question arising here is: How valuable or applicable such information is in the real cases? As an illustration, let us consider a model of cerebrospinal fluid (CSF) pressure that is a medical disorder. The model may be considered as

$$\dot{\tilde{p}}(t)=\frac{-k}{r}\tilde{p}^2(t)+k\left(I_f(t)+\frac{p_d}{r}\right)\tilde{p}(t),\quad \tilde{p}(t_0)\in E_1 \tag{17}$$

where $\tilde{p}(t)$ denotes the CSF pressure and the initial condition is assumed to be a fuzzy number. By solving the FDE (17), the CSF pressure might be predictable at any time. The value of pressure, at any time, is a fuzzy number which may be expressed as a linguistic value, e.g. about 157 at $t=3$. Although in this way the prediction of CSF pressure is explainable, the decision maker might only consider the support of the solutions. One of the reasons is that the values even with the smallest degree of possibility are possible. Therefore, they are important, and these values take place in the closest neighborhood of the supports. Thus, what often occurs is that the decision makers might only pay attention to the minimum and maximum values of the fuzzy numbers which are in fact the supports. Simply put, the decision makers treat the solution as if it is the solution of an interval valued differential equation. In these situations, the obtained possibility distributions may not be as productive as they should. Employing the so-called Z-differential equations [220] can prove the effectiveness and applicability of possibility distributions obtained through solving FDEs. The Z-differential equations (ZDE) have been established based on Z-numbers, $Z^+$-numbers, and Z-number-valued functions that play the similar role of fuzzy numbers and fuzzy number-valued functions in FDEs. The concept of Z-differential equations is more general than FDEs. Based on the conceptual unity presented in [220], it has been demonstrated that a Z-differential equation may be expressed as a bimodal differential equation combining an FDE with a random differential equation. In this setting, the solutions of FDEs in the form of possibility distributions play a pivotal role in presenting explainable information in the form of linguistic value coming with the concepts of acceptable time, acceptable information area, and most importantly the sureness. Informally, in the case of CSF pressure, one may be able to express how sure they are that the pressure is about 157 at $t = 3$. Consequently, employing and extending the advanced obtained results of FDEs in the setting of ZDEs might be the other perspective of FDEs.

## APPENDIX

See Table 1.

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