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A Numerical Comparison of Iterative Algorithms for Inconsistency Reduction in Pairwise Comparisons

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ABSTRACT The aim of this paper is to compare selected iterative algorithms for inconsistency reduction in pairwise comparisons by Monte Carlo simulations. We perform simulations for pairwise comparison matrices of the order $n = 4$ and $n = 8$ with the initial inconsistency $0.10 < CR < 0.80$ and entries drawn from Saaty's fundamental scale. Subsequently, we evaluate the algorithms' performance with respect to four measures that express the degree of original preference preservation. Our results indicate that no algorithm outperforms all other algorithms with respect to every measure of preference preservation. The Xu and Wei's algorithm is the best with regard to the preservation of an original priority vector and the ranking of objects, the Step-by-Step algorithm best preserves the original preferences expressed in the form of a pairwise comparison matrix, and the algorithm of Szybowski keeps the most matrix entries unchanged during inconsistency reduction.

INDEX TERMS Algorithm, consistency, inconsistency reduction, pairwise comparisons.

I. INTRODUCTION

Pairwise comparisons constitute an inherent part of many popular and successful multiple-criteria decision-making methods (MCDM), such as the AHP/ANP (the analytic hierarchy/network process), PROMETHEE (Preference Ranking Organization Method for Enrichment of Evaluations), ELECTRE (Elimination Et Choice Translating Reality), PAPRIKA (Potentially All Pairwise Rankings of all Possible Alternatives), or the BWM (the Best-Worst method), see [14], [31], [41], [42], [43], and in the construction of [14], [31], [41]–[43] and in the construction of 3D models, see [16], for the end-to-end pairwise attentive adversarial spatiotemporal network, or [17], where the pairwise discrimination loss function is proposed to improve the feature discrimination of the model.

One of the most often discussed properties of pairwise comparisons is their cardinal or ordinal inconsistency. Human experts are rarely fully consistent in their judgements, which is especially true for larger numbers of compared objects.

The inconsistency of pairwise comparisons is evaluated by measures (functions) called inconsistency indices. Perhaps the most well-known indices are Saaty's consistency index (CI) and consistency ratio (CR) [42], and Koczkodaj's inconsistency index (KI) [22].

Since pairwise comparisons, or pairwise comparison matrices, arising from solutions of real-world problems are seldom consistent, a low degree of inconsistency is usually tolerated. Saaty suggested in his analytic hierarchy process that the inconsistency $CR < 0.10$ is acceptable. Similarly, pairwise comparisons are deemed to be acceptably consistent if $KI < 0.33$ (other thresholds of inconsistency also exist for other indices).

When the inconsistency of pairwise comparisons is unacceptably high, a decision-maker has two options: to ask an expert to revise his/her judgements, or to find a pairwise comparison matrix that is consistent enough while being as close as possible to the original matrix expressing the expert's preferences.

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In the latter case, many approaches or algorithms of inconsistency reduction have been proposed in the literature in recent decades. Perhaps the first algorithm for inconsistency reduction in pairwise comparisons was proposed in [19] in 1996. The distance-based algorithm searched for the most inconsistent triad (in terms of Koczkodaj's inconsistency index) and replaced it with a consistent one. Later, the same idea was followed in [23]. In general, algorithms for inconsistency reduction can be divided into two groups: iterative and non-iterative. The former algorithms adjust one or more (or all) matrix elements in each iteration so that the inconsistency of the whole pairwise comparison (PC) matrix gradually decreases, and stops when the level of inconsistency falls below a given threshold. The algorithms of Cao *et al.* [11], Kou *et al.* [25], Mazurek *et al.* [35], Szybowski [44], or Xu and Wei [47] fall into this category. Non-iterative algorithms are represented, for instance, by the INSITE algorithm by Abel *et al.* [1] that applies a multiple-objective linear programming method, or algorithms by Bozoki *et al.* [6], [7] which are based on non-linear mixed-integer optimization. Similarly, Negahban [36] proposed three mixed-integer non-linear programming models for minimizing the sum of adjustments, maximum adjustment, and the number of adjusted elements. The approaches of Gao *et al.* [15] and Girsang [18] applied a genetic ant algorithm for inconsistency reduction. A segment tree approach was adopted by Zhang [49], while the application of Gower plots were used by Li and Ma [30] and orthogonal projections onto the space of consistent PC matrices by Benitez *et al.* (see [3]–[5]). Other approaches and algorithms can be found e.g. in [13], [27], [40], [45], [46], or [48].

As for a numerical comparison of the aforementioned methods, Cao *et al.* [11] compared their algorithm with the algorithm of Xu and Wei via one matrix of the order $n = 8$. Abel *et al.* [1] applied two PC matrices of the order $n = 6$ and $n = 8$ for a comparison of five algorithms, including INSITE. Girsang *et al.* [18] compared their algorithm with one other method via one PC matrix of the order $n = 4$ and with two other methods for a PC matrix of the order $n = 8$. Pereira and Costa [39] provided the most extensive comparison so far, with up to eight algorithms and one PC matrix of the order $n = 7$, $n = 8$ and $n = 9$ respectively.

However, a comparison of algorithms via two or three ad hoc examples does not enable any general conclusions to be drawn about the algorithms' performance with respect to preserving initial preferences or computational complexity (time consumption). Further on, the examination of differences among algorithms with respect to the degree of inconsistency and the size of the initial PC matrix is completely missing in the literature. This constitutes a large and serious gap in the current knowledge regarding the problem of numerical inconsistency reduction in pairwise comparisons.

The aim of the paper is to fill the aforementioned gap and provide a numerical comparison of selected iterative algorithms for inconsistent PC matrices of the order $n = 4$, representing 'small' PC matrices, and $n = 8$, representing 'large' PC matrices and also corresponding to the most common size of PC matrices used for comparisons in the previous studies. Iterative algorithms are more suitable for numerical simulations since non-iterative algorithms are usually based on the solution of non-linear programming problems, which are NP hard (non-deterministic polynomial acceptable problems) and therefore less suitable for modelling and simulation. In our study, we generate a large sample of random inconsistent PC matrices of a given order, and evaluate the algorithms' performance in terms of the preservation of original preferences. Moreover, we divide randomly generated inconsistent PC matrices into two categories: less inconsistent matrices with $0.10 < CR < 0.30$ and more inconsistent matrices with $0.30 \leq CR < 0.80$, and examine each case separately since the behaviour of algorithms might differ with different degrees of (input) matrix inconsistency.

The paper is organized as follows: Section II gives preliminaries and notation of a pairwise comparison method, in Section III measures of the preference preservation are provided, in Section IV the algorithms for inconsistency reduction applied in this study are briefly described, and Section V is devoted to Monte Carlo simulations. Sections VI Discussion and Section VII Conclusions close the article.

II. PRELIMINARIES

A PC matrix is a square matrix $\mathbf{A} = (a_{ij})$ of the order n, where $a_{ij} \in \mathbb{R}_+$ expresses the preference (or importance) of an object i over another object j. In most cases, the compared objects correspond to alternatives or criteria in a multiple-criteria decision-making problem. A PC matrix constitutes an input for the problem of finding objects' weights and/or their ranking.

Usually, it is assumed that a PC matrix satisfies the following property:

Definition 1: A PC matrix $\mathbf{A} = (a_{ij})$ is said to be reciprocal if $\forall i, j \in \{1, \ldots, n\} : a_{ij} = \frac{1}{a_{ji}}$ and $\mathbf{A} = (a_{ij})$ is said to be consistent if $\forall i, j, k \in \{1, \ldots, n\} : a_{ij} \cdot a_{jk} \cdot a_{ki} = 1$.

Since human judgments are subjective and imprecise, a pairwise comparison matrix is often inconsistent. To evaluate the degree of this inconsistency, many inconsistency indices have been proposed in the literature and their properties extensively studied, see e.g. [2], [8]–[10], [12], [24], [28], [32]–[34], or [37].

In our study, we use the consistency ratio CR proposed by Saaty [42], [43], see below. As for the derivation of the priority vector w (a vector of weights of compared objects), we use the eigenvalue method (EV) proposed by Saaty [42]. In the EV method, the vector w is determined as the rescaled principal eigenvector of A. Thus, assuming that $Aw = \lambda_{max} w$ the priority vector w is given as

$$w = \gamma \left[w_1, \ldots, w_n \right]^T ,$$

where γ is a scaling factor. Usually, it is assumed that:

$$\gamma = \left(\sum_{i=1}^{n} w_i \right)^{-1} .$$

Definition 2: Saaty's eigenvalue based consistency index CI and consistency ratio CR of $n \times n$ reciprocal matrix $\mathbf{A} = (a_{ij})$ are defined as follows [43]:

$$CI(A) = \frac{\lambda_{max} - n}{n - 1} \tag{1}$$

$$CR(A) = \frac{CI(A)}{RI_n} \tag{2}$$

where λ_{max} is the principal eigenvalue of A and RI_n is the random consistency index, see [43].

The value $\lambda_{max} \geq n$, and $\lambda_{max} = n$ if and only if A is consistent [43].

Since the algorithm of Mazurek *et al.* [35] implements Koczkodaj's inconsistency index and a notion of a triad inconsistency, here we provide the necessary notation.

Definition 3: Koczkodaj's inconsistency index [22], $KI(\mathbf{A})$, of an $n \times n$ PC matrix $\mathbf{A} = (a_{ij})$ is defined as

$$KI(\mathbf{A}) = \max\Big\{1 - \min\Big\{\frac{a_{ij}}{a_{ik}a_{kj}}, \frac{a_{ik}a_{kj}}{a_{ij}}\Big\} \; | i, j, k \in \{1, \ldots, n\}\Big\}. \tag{3}$$

Definition 4: Let $\mathbf{A} = (a_{ij})$ be a pairwise comparison matrix. A *triad* is every triple (a_{ij}, a_{jk}, a_{ik}), where $a_{ij}, a_{jk}, a_{ik} \in \mathbf{A}$, $i, j, k \in \mathbf{N}$ and $1 \leq i < j < k \leq n$.

Remark 1: For a pairwise comparison matrix of the order n there are $\frac{n(n-1)(n-2)}{6}$ triads, see e.g. [26].

Further, we define Koczkodaj's inconsistency index for one triad (TKI) as follows:

Definition 5: Let (a_{ij}, a_{jk}, a_{ik}) be a triad associated with a pairwise comparison matrix $\mathbf{A}$. *A triad Koczkodaj's inconsistency index TKI* is defined as follows:

$$TKI(a_{ij}, a_{jk}, a_{ik}) = 1 - min\{\frac{a_{ij} \cdot a_{jk}}{a_{ik}}, \frac{a_{ik}}{a_{ij} \cdot a_{jk}}\} \tag{4}$$

Definition 6 [35]: Let $\mathbf{A} = (a_{ij})$ be a pairwise comparison matrix of the order n. Let $T\{(a_{ij}, a_{jk}, a_{ik})\}$ be the set of all triads, $i < j < k, \forall i, j, k \in \{1, \ldots, n\}$. Let $TKI(a_{ij}, a_{jk}, a_{ik})$ denote Koczkodaj's inconsistency of a triad (a_{ij}, a_{jk}, a_{ik}). Further, let the set $S(a_{ij})$ be a set of all triads containing a_{ij}, clearly $S(a_{ij}) \subset T$.

Then $TEI(a_{ij})$ denotes the *total element inconsistency* of a matrix element a_{ij}:

$$TEI(a_{ij}) = \sum_S TKI(a_{ij}, a_{jk}, a_{ik}), \quad \forall i, j, k, \tag{5}$$

Definition 7 [35]: Let $\mathbf{A} = (a_{ij})$ be a pairwise comparison matrix, then $E(\mathbf{A})$ denotes the set of all $a_{ij} \in A$, $i < j$, $\forall i, j \in \{1, \ldots, n\}$.

III. MEASURES OF PREFERENCE PRESERVATION

It is generally agreed that, during the process of consistency improvement, the experts' original preferences should be preserved as much as possible. This section provides measures of the preservation of preferences (expressed in the form of pairwise comparisons) that are used in the numerical section of this study. Let us start with the definition of an algorithm for inconsistency reduction.

Definition 8 [35]: Let A be a set of pairwise comparison matrices of the order n, and let μ denote a measure of inconsistency such that: $\mu : A \rightarrow R_{0+}$. Then an algorithm whose main objective is to transform the PC matrix $\mathbf{A} \in A$ with $\mu(A) = m$ into the matrix $\mathbf{A}' \in A$ so that $\mu(\mathbf{A}') < \mu(\mathbf{A})$ and $\mu(\mathbf{A}') \leq \varepsilon$, $\varepsilon > 0$, is called an *algorithm for inconsistency reduction (AIR)*.

Various measures of preference preservation (pairwise comparisons) have recently been proposed in the literature. Xu and Wei [47] introduced two measures, δ and σ, where δ is the maximal difference between an element of an original matrix and a modified matrix, and σ is equal to the mean quadratic distance between all elements of an original matrix and a modified matrix. Pereira and Costa [39] proposed a new measure called the total number of deviation points (TND). Abel *et al.* [1] added several 'measures of compromise': the number of judgement violations (NJV) which corresponds to the number of matrix elements that were adjusted, total judgement deviation (TJD) that is equal to the L_1 distance of an original matrix and a modified matrix, squared total judgement deviation ($STJD$), a variant of TJD, and the number of judgement reversals (NJR).

After elaboration of the aforementioned measures of preference preservation in the pairwise comparison framework, we define the following measures of preference preservation to be applied in the numerical section.

Definition 9: Let $\mathbf{A} = (a_{ij})$ be an inconsistent pairwise comparison matrix of the order n, and let $w = (w_1, \ldots, w_n)$ be the priority vector associated with $\mathbf{A}$. Let $\mathbf{A}' = (a_{ij})$ denote a PC matrix derived from $\mathbf{A} = (a_{ij})$ via an AIR, and let $w' = (w'_1, \ldots, w'_n)$ be a corresponding priority vector. Then

$$d(w, w') = \frac{1}{n} \sum_{i=1}^{n} |w_i - w'_i| \tag{6}$$

describes the average change in the priority vector after the transformation from $\mathbf{A}$ to $\mathbf{A}'$.

Remark 2: For instance, $d = 0.03$ can be interpreted in the way that each weight w_i of a priority vector w changed by 3% on average.

The priority vector provides not only the weights of all compared objects but also their ranking from the best (with the highest weight) to the worst, (with the lowest weight), possibly including ties. The next measure, Kendall's *tau* distance [21], evaluates how much this ranking changes after the transformation.

Definition 10: Let r_1 and r_2 be two rankings (permutations) of n objects. Then $\tau(r_1, r_2)$ is equal to the least number of swaps of two adjacent objects in the ranking r_1 necessary to obtain the ranking r_2.

The following definition of the measure D expressing a distance of two matrices has a form of a standard L_1 matrix norm.

Definition 11: Let $\mathbf{A} = (a_{ij})$ and $\mathbf{A}' = (a'_{ij})$ be the original matrix and an AIR transformed matrix, respectively. Then the distance D between $\mathbf{A}$ and $\mathbf{A}'$ is defined as follows:

$$D(A, A') = \|A - A'\| = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij} - a'_{ij}| \qquad (7)$$

Notice, that D closely relates to *TJD* of Abel *et al.* [1], since $\frac{D}{n^2} = TJD$.

Definition 12: Let $\mathbf{A} = (a_{ij})$ and $\mathbf{A}' = (a'_{ij})$ be the original matrix and an AIR transformed matrix, respectively. Then $N = |a_{ij}; a_{ij} \neq a'_{ij}, \forall i, j|$ denotes the number of matrix elements which were changed (adjusted) in the inconsistency reduction.

The lower the values of d, Kendall's *tau*, D and N, the better the AIR preserves the original preferences.

IV. ITERATIVE ALGORITHMS FOR INCONSISTENCY REDUCTION

In this section, we introduce selected iterative algorithms for inconsistency reduction applied in the numerical section of our study (in alphabetical order). In the literature, several other iterative algorithms have been proposed, however, they require the assistance of a decision maker in each iteration, see e.g. [13], [19], or [20], a feature that makes them unsuitable for extensive simulations.

We use the consistency ratio *CR* (and the threshold value $CR = 0.10$) for the evaluation of inconsistency of PC matrices during inconsistency reduction procedures in accord with previous studies, see [1], [11], [13], [25], [36], [39], [40], or [47], though other inconsistency indices, such as Koczkodaj's inconsistency index *KI*, could be, in principle, used as well.

A. CAO et al.'s ALGORITHM

Let $A = (a_{ij})$ be an inconsistent $n \times n$ pairwise comparisons matrix with an associated priority vector (an eigenvector corresponding to a principal right eigenvalue of $A = (a_{ij})$) $w = (w_1, \ldots, w_n)$. Let $W = (\frac{w_i}{w_j})$ and let $D = (d_{ij})$ be a positive reciprocal matrix such that $A = W \otimes D$, where "$\otimes$" is the Hadamard product of two matrices. The matrix D is called a deviation matrix. If the matrix A is consistent, then $d_{ij} = 1, \forall i, j$, and the corresponding matrix is denoted $D1$.

The AIR of Cao *et al.* [11] proceeds as follows:

Step 1: Let $A^{(0)} = (a_{ij}^{(0)}) = (a_{ij})$; $CR^* = 0.10$; $k = 0$.

Step 2: Calculate the maximum eigenvalue $\lambda_{max}(A^{(k)})$ of $A^{(k)}$, the corresponding priority vector $w^{(k)} = (w_1^{(k)}, \ldots, w_n^{(k)})^T$, and the deviation matrix $D^{(k)} = (d_{ij}^{(k)}) = (a_{ij}^{(k)} / \frac{w_i^{(k)}}{w_j^{(k)}})$.

Step 3: Calculate $CR(A^{(k)})$.

Step 4: If $CR(A^{(k)}) \leq CR^*$, go to Step 6. Otherwise proceed.

Step 5: Let $A^{(k+1)} = [\frac{w^{(k)}_i}{w^{(k)}_j}] \otimes D^{(k)}$, where $D^{(k)} = (d_{ij}^{(k)}) = \gamma D^{(k)} + (1 - \gamma) D1$. Let $k = k + 1$ and go to Step 2.

Step 6: $A^{(k)}$ is a modified pairwise comparison matrix with acceptable consistency, $w^{(k)}$ is a priority vector.

Step 7: Print the last value of A, w, $CR(A)$ and k, and end.

In Step 5 only matrix elements above the main diagonal ($a_{ij}, i > j$) are modified via a corresponding formula, while reciprocal elements are calculated as $a_{ij} = 1/a_{ji}$. We used $\gamma = 0.5$ and $\gamma = 0.98$ for simulations, the same values that were applied in the original paper.

B. KOU et al.'s ALGORITHM

Let $A = (a_{ij})$ be an inconsistent $n \times n$ pairwise comparison matrix. Let $k = 0$.

The algorithm of Kou *et al.* [25] proceeds as follows:

Step 1: Form the matrix $C = \frac{1}{n} A^2 \otimes A^T$, where "$\otimes$" denotes the Hadamard product and A^T is the transpose of A.

Step 2: Set $k = k + 1$. Find k^{th} maximal $c_{ij} \in C$.

Step 3: Form a new matrix A' such that $a'_{ij} = \dfrac{n \cdot c_{ij} - 2}{n - 2} \cdot a_{ij}$, and $a'_{ji} = 1/a'_{ij}$.

Step 4: Calculate $CR(A')$.

Step 5: If $CR(A') < CR^*$, print the last value of A', w, $CR(A')$ and k, and end. Otherwise go to Step 2.

C. MAZUREK et al.'s ALGORITHM

The Step-by-step algorithm, see [35], was slightly modified: Koczkodaj's inconsistency index was substituted by Saaty's *CR* index. The variables *TKI* and *TEI* are explained in Definitions 5 and 6.

Let $A = (a_{ij})$ be an inconsistent $n \times n$ pairwise comparison matrix, let $E(A)$ be the set from Definition 7, and let k be the number of iterations.

The SBS algorithm [35] proceeds as follows:

Step 1: Let $A = (a_{ij})$; $CR^* = 0.10$; $N = 1$, $k = 0$.

Step 2: Set $k = k + 1$. Calculate $CR(A)$. If $CR(A) < CR^*$, go to Step 9. Otherwise proceed.

Step 3: Calculate *TKI* for all triads in A, and *TEI* for all $a_{ij} \in E$.

Step 4: If $N = \dfrac{n(n-1)}{2} + 1$, print "A consistent enough matrix could not be found", A and $CR(A)$. Otherwise, find $a_{pq} \in E$ with the N^{th} maximal value of *TEI*.

Step 5: If $a_{pq} \geq 1$, set $r = p, s = q$, otherwise set $r = q, s = p$. Form a matrix A' such that $a'_{rs} = a_{rs} + 1$, and $a'_{sr} = 1/a'_{rs}$, keep other matrix elements unchanged.

Step 6: Calculate $CR(A')$. If $CR(A') < CR(A)$, set $N = 1$, $A = A'$ and proceed to Step 3. Otherwise, go to Step 7.

Step 7: Form a matrix A'' such that $a''_{rs} = a_{rs} - 1$, and $a''_{sr} = 1/a''_{rs}$, keep other matrix elements unchanged.

Step 8: Calculate $CR(A'')$. If $CR(A'') < CR(A)$, set $N = 1$, $A = A''$ and proceed to Step 3. Otherwise, set $N = N + 1$ and go to Step 4.

Step 9: Print the last value of A, w, $CR(A)$ and k and end.

D. SZYBOWSKI's ALGORITHM

Let $A = (a_{ij})$ be an inconsistent $n \times n$ pairwise comparison matrix, let $CR^* = 0.10$; $N = 1$, and $k = 0$.

The algorithm of Szybowski [44] proceeds as follows:

Step 1: Set $k = k + 1$. Calculate $CR(A)$. If $CR(A) < CR^*$, go to Step 5. Otherwise proceed.

Step 2: Calculate w and $e_{ij} = |ln(a_{ij}\frac{w_j}{w_i})|$ for all i, j.

Step 3: Find $e_{pq} = max\{e_{ij}\}$.

Step 4: Set $a_{pq} = \frac{w_p}{w_j}$, keep other matrix elements unchanged, and go to Step 1.

Step 5: Print the last value of A, w, $CR(A)$ and k and end.

Note: In the original formulation of the algorithm, the geometric consistency index GCI was used instead of CR.

E. XU AND WEI's ALGORITHM

Let $A = (a_{ij})$ be an inconsistent $n \times n$ pairwise comparison matrix, let k be the number of iterations, and let $0 < \lambda < 1$.

The Xu and Wei's algorithm proceeds as follows [47]:

Step 1: Let $A^{(0)} = (a_{ij}^{(0)}) = (a_{ij})$; $CR^* = 0.10$; $k = 0$.

Step 2: Calculate the maximal eigenvalue $\lambda_{max}(A^{(k)})$ of $A^{(k)}$ and the normalized principal right eigenvector $w^{(k)} = (w_1^{(k)}, \ldots, w_n^{(k)})^T$.

Step 3: Calculate the consistency index, $CI^{(k)} = (\lambda_{max}(A^{(k)}) - n)/(n-1)$ and the consistency ratio $CR^{(k)} = CI^{(k)}/RI$, where RI is given in [43].

Step 4: If $CR^{(k)} < CR^*$, then go to Step 6. Otherwise, continue to the next step.

Step 5: Let $A^{(k+1)} = (a_{ij}^{(k+1)})$, where $a_{ij}^{(k+1)} = (a_{ij}^{(k)})^{\lambda}(\frac{w_i^{(k)}}{w_j^{(k)}})^{1-\lambda}$.

Let $k = k + 1$ and return to Step 2.

Step 6: Print the last value of A, w, $CR(A)$ and k.

Step 7: End.

We used $\lambda = 0.5$ and $\lambda = 0.9$ for simulations, the same values that were applied in the original paper.

V. MONTE CARLO SIMULATIONS

At the beginning, we randomly generated a large number (more than 10,000 cases) of PC matrices of the order $n = \{4, 8\}$ with $CR \geq 0.10$, where matrix entries were drawn from Saaty's fundamental scale to examine the distribution of the consistency ratio CR. We found that in the case of $n = 4$, the mode was in the interval (0.20, 0.30), the median was 0.86 and the arithmetic mean (by definition of CR) equal to 1. In the case of $n = 8$, the arithmetic mean, mode and median were all close to 1. Since pairwise comparisons are usually provided by someone with suitable knowledge (called an 'expert'), it can be safely assumed that the expert's preferences would be less inconsistent than random preferences. That is why we set the upper limit for inconsistency in our study at $CR = 0.80$. To distinguish between less inconsistent and more inconsistent matrices, we used the mode value $CR = 0.30$ for $n = 4$.

After this preparation phase, we randomly generated large samples (over 10,000 cases) of PC matrices of the order $n = 4$ and $n = 8$ for actual simulations via the same procedure. Subsequently, we filtered out PC matrices with

TABLE 1. AIR performance, average values for $n = 4$ and initial $0.10 \leq CR < 0.30$, 630 matrices (the best values are in bold).

Algorithm	d	D	tau	k	N
Cao et al. ($\gamma = 0.98$) [11]	0.582	4.74	0.024	16.02	12
Cao et al. II ($\gamma = 0.50$) [11]	0.568	7.94	0.027	**1**	12
Kou et al. [25]	5.86	10.97	0.57	**1**	2
Mazurek et al. [35]	2.33	**3.94**	0.132	4.69	3.29
Szybowski [44]	3.22	5.34	0.185	1.24	**2.47**
Xu and Wei ($\lambda = 0.5$) [47]	**0.212**	7.96	**0.008**	**1**	12
Xu and Wei II ($\lambda = 0.9$) [47]	0.217	5.20	**0.008**	3.60	12

TABLE 2. AIR performance, average values for $n = 4$ and initial $0.30 \leq CR < 0.80$, 1193 matrices (the best values are in bold).

Algorithm	d	D	tau	k	N
Cao et al. [11]	2.132	11.67	0.191	35.50	12
Cao et al. II [11]	1.552	14.44	0.147	**1.64**	12
Kou et al. [25]	-	-	-	-	-
Mazurek et al. [35]	5.341	**8.74**	0.595	8.89	4.62
Szybowski [44]	5.827	9.97	0.638	2.19	**3.84**
Xu and Wei [47]	**0.567**	14.47	**0.039**	1.65	12
Xu and Wei II [47]	0.723	12.12	0.049	7.60	12

TABLE 3. AIR performance, average values for $n = 8$ and initial $0.10 < CR < 0.30$, 491 matrices (the best values are in bold).

Algorithm	d	D	tau	k	N
Cao et al. [11]	0.818	37.27	0.727	21.3	56
Cao et al. II [11]	0.629	47.83	0.529	**1**	56
Mazurek et al. [35]	1.780	**29.25**	1.610	23.4	17.13
Szybowski [44]	2.357	31.18	2.218	4.8	**8.78**
Xu and Wei [47]	**0.241**	48.38	**0.188**	**1**	56
Xu and Wei II [47]	0.289	39.73	0.216	4.73	56

$0.10 < CR < 0.80$. Then, each and every PC matrix was used as an input for all selected algorithms for inconsistency reduction. The algorithms stopped when the consistency ratio CR of a modified matrix decreased below the 0.10 threshold. The output consisted of a final modified matrix, maximal eigenvalue and a priority vector w. Finally, the measures of preference preservation, d, D, N, and tau, were evaluated for each modified matrix.

Simulation results – the AIR performance with respect to matrix size and an initial matrix inconsistency – are provided in Tables 1–4. The numbers in tables' captions express how many matrices from the original set of 10,000 generated PC matrices fell into aforementioned CR intervals (according to Lerch and Mudford [29] these numbers are sufficient).

The Kou *et al.*'s algorithm failed to find a consistent enough matrix in 3% of cases for $n = 4$ and $0.10 < CR < 0.30$, and in 45% of cases for $n = 4$ and $0.30 \leq CR < 0.80$. In the case of $n = 8$, the algorithm failed in more than 60% of cases. That is why we report its results only for $n = 4$ and $0.10 < CR < 0.30$ (read more on the Kou *et al.* algorithm in Discussion).

VI. DISCUSSION

During simulations, we found that the Kou *et al.* [25] algorithm did not work properly when the main procedure described in Section IV.B was performed more than once.

TABLE 4. AIR performance, average values for $n = 8$ and initial $0.30 \leq CR < 0.80$, 4082 matrices (the best values are in bold).

Algorithm	d	D	tau	k	N
Cao et al. [11]	1.980	62.47	2.764	36.30	56
Cao et al. II [11]	1.426	76.03	2.049	1.77	56
Mazurek et al. [35]	3.198	49.52	3.762	41.92	24.94
Szybowski [44]	3.659	**49.18**	4.048	8.18	**16.18**
Xu and Wei [47]	**0.581**	76.79	**0.665**	**1.75**	56
Xu and Wei II [47]	0.727	65.03	0.798	7.70	56

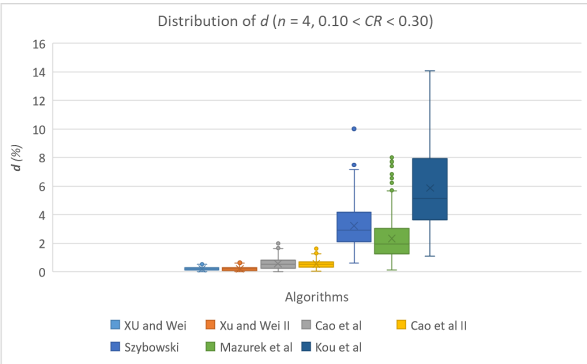

FIGURE 1. Distribution of d(%), $n = 4$, $0.10 < CR < 0.30$.

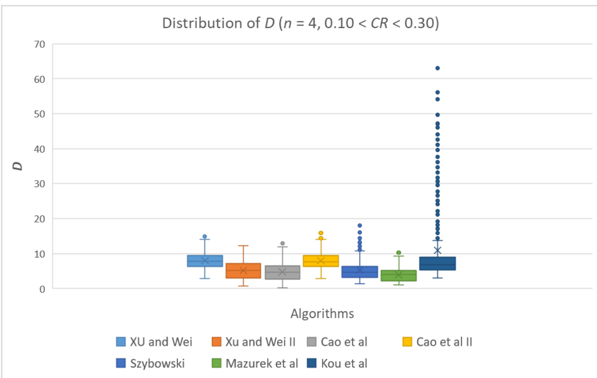

FIGURE 2. Distribution of D, $n = 4$, $0.10 < CR < 0.30$.

In such a case, the algorithm diverged. As a consequence, the algorithm failed in over 60% of cases for $n = 8$, making the comparison with other algorithms infeasible. It is worth noting that, in the original paper [25], proof of the algorithm's convergence is missing. The likely cause of the divergence is that, after the modification of the most inconsistent element in the first iteration, the matrix C is not updated. To fix the problem, we suggest updating the matrix C in each iteration.

As could be expected, inconsistency reduction of larger matrices ($n = 8$) and/or more inconsistent matrices ($0.30 \leq CR < 0.80$) led to larger values, hence the lower preservation of initial preferences, of the variables d, D and tau, see also Figures 1-8. These matrices also required iterations of more algorithms, and in the case of the Szybowski's and Mazurek *et al.*'s algorithms also a larger number of adjusted elements (algorithms of Xu and Wei, Cao *et al.* and Kou *et al.* are designed to modify all non-diagonal elements). From a computational point of view, the Cao *et al.* II and Xu and Wei algorithms displayed the fastest convergence.

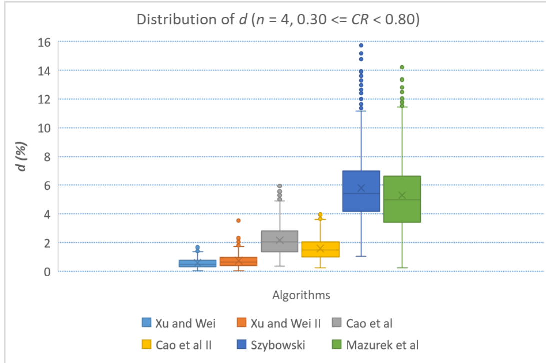

FIGURE 3. Distribution of d(%), $n = 4$, $0.30 \leq CR < 0.80$.

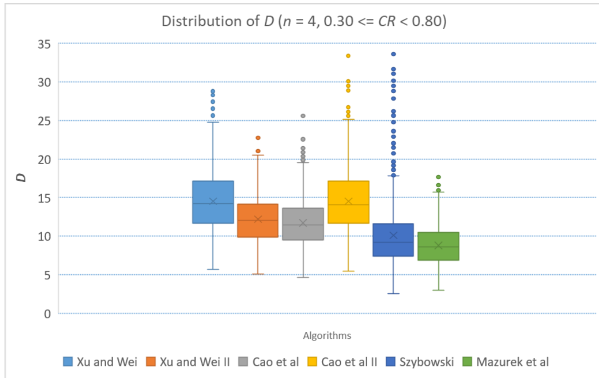

FIGURE 4. Distribution of D, $n = 4$, $0.30 \leq CR < 0.80$.

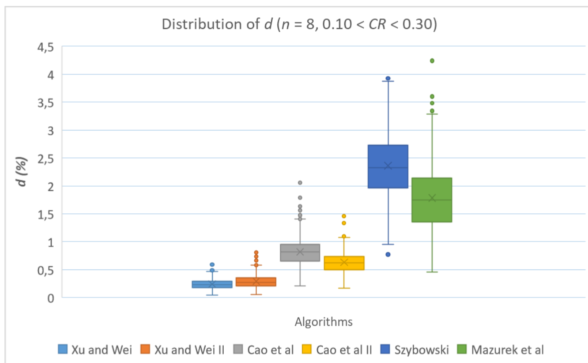

FIGURE 5. Distribution of d(%), $n = 8$, $0.10 < CR < 0.30$.

As for the impact of parameter values in the Cao *et al.*'s and Xu and Wei's algorithms respectively, in the case of the Cao *et al.*'s algorithm and $\gamma = 0.98$, matrix modifications were more refined, which required more steps to achieve the threshold inconsistency than the case with $\gamma = 0.50$. The same observation applies to the Xu and Wei's algorithm, the case with $\lambda = 0.90$ proceeded by smaller steps in more iterations.

The ANOVA (analysis of variance) test confirmed that the differences in the mean values of d, D and tau for all algo-

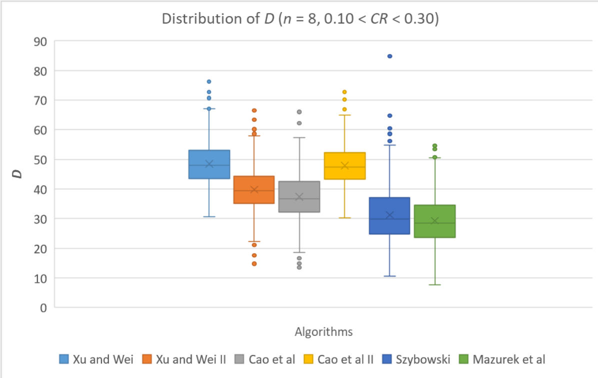

FIGURE 6. Distribution of D, $n = 8$, $0.10 < CR < 0.30$.

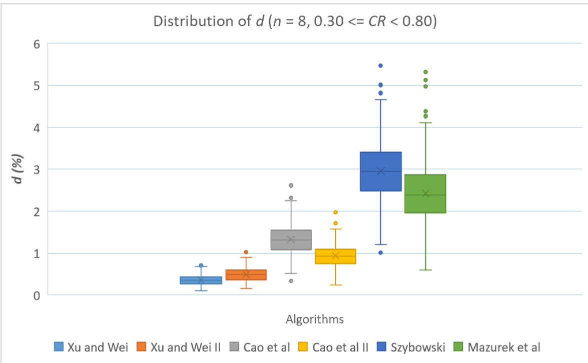

FIGURE 7. Distribution of d(%), $n = 8$, $0.30 \leq CR < 0.80$.

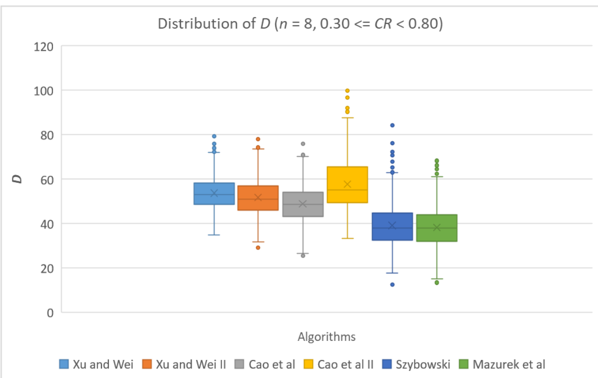

FIGURE 8. Distribution of D, $n = 8$, $0.30 \leq CR < 0.80$.

rithms (with the exception of the Kou *et al.*'s algorithm) were statistically significant at least at the $p = 10^{-10}$ level. As for the algorithms' similarities, the Xu and Wei's and Cao *et al.*'s algorithms were highly correlated in the variable D, with the maximal Pearson correlation coefficient $\varrho = 0.973$ for $n = 4$ and $0.10 < CR < 0.30$. On the other hand, the Szybowski's algorithm correlated most of all with the Mazurek *et al.*'s algorithm, specifically in the variable D, with the maximal Pearson correlation coefficient $\varrho = 0.909$ for $n = 8$ and $0.30 \leq CR < 0.80$.

The algorithms' outputs were processed with the accuracy of four decimal places. The majority of the programming was performed in Python, the rest in R and C#. All the data, algorithms and a technical report can be found at a free access GitHub repository [50].

VII. CONCLUSION

The aim of this paper was to perform Monte Carlo simulations to compare selected iterative algorithms for inconsistency reduction with respect to the preservation of original preferences in the pairwise comparison framework. Our results indicate that no algorithm outperformed all other algorithms with respect to every measure of original preference preservation. The Xu and Wei's algorithm was the best algorithm with regard to the preservation of a priority vector and the ranking of objects. The algorithm of Mazurek *et al.*'s was the best algorithm with respect to the preservation of preferences expressed by entries of the original PC matrix, while the algorithm of Szybowski's kept the most matrix entries unchanged during inconsistency reduction. Therefore, the choice of the most suitable AIR depends on the decision maker's needs. The Kou *et al.* 'salgorithm appeared to be divergent in most cases, hence it could not be compared with other algorithms.

Further research may focus on the comparison of non-iterative algorithms for inconsistency reduction, or may aim towards the framework of interval, fuzzy or fuzzy hesitant pairwise comparisons.

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