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Fractional Dynamics Based-Enhancing Control Scheme of a Delayed Predator-Prey Model

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ABSTRACT To retard the onset of undesired bifurcation, the bifurcation control has developed into a theme of centralized research activities in delayed fractional-order system. In this paper, the problem of bifurcation control for a delayed fractional-order predator-prey model is investigated by employing an enhancing feedback control technique. The bifurcation point is firstly established for controlled model by using delay as a bifurcation parameter. Then, a series of numerical comparative studies on the effects of bifurcation control are implemented covering the partial or total removal of the branch for feedback gains. It reveals that the stability performance of the proposed model can be overwhelmingly elevated via the devised approaches in comparison with the dislocated feedback ones. A numerical example with simulations is ultimately designed to confirm the merits of the proposed theoretical results.

INDEX TERMS Fractional order, time delay, enhancing feedback control, predator-prey model, Hopf bifurcation.

I. INTRODUCTION

In ecological systems, predator-prey system depicts the interactions between two or more species and their dynamics are affected by each other. Owing to the worldwide importance and existence, the dynamics of predator-prey systems is one of the basic topics in ecology, which constructs the complex food chains and food networks. The famous predator-prey model was established by [1], [2]. Afterwards, the dynamical behaviors of predator-prey models, such as chaos, stability, bifurcations and oscillations, usually depend on the system parameters. Basically, time delays are unavoidable and ubiquitous in prey-predator system due to gestation [3]. At present, many outstanding achievements have been made in the analysis of predator-prey model [4]–[8].

Fractional order dynamical systems have attracted numerous researchers' attraction in various branches, especially in science and engineering. Compared with the traditional integer order dynamical system, the fundamental difference of fractional order is that it has nonlocal and weakly singular kernel [9]–[11], so it has infinite memory and more degrees of freedom. Due to memory effect, most biological systems exhibit fractional dynamics. The memory in the model describes the history of the process involved and has an impact on the current and future development of the process. Therefore, the fractional differential equation can describe the actual phenomenon more accurately than the traditional integer differential equation. Generally, the modeling of fractional order population dynamics system can enrich dynamics, increase the complexity of the model and improve the performance of complex system. Recently, some scholars have incorporated fractional calculus into predator-prey models and developed fractional predator-prey ones, and obtained a large number of results related to fractional dynamics of delayed predator-prey without delays [12]–[15] or with delays [16]–[19].

Hopf bifurcation analysis is an effective tool to obtain more information of complex dynamical systems near the equilibrium point. In order to obtain more properties of nonlinear complex dynamical systems, bifurcation has been widely studied [20]–[24]. As we all know, in the traditional integer order delayed model, bifurcation has been widely studied, and some good results have been obtained. In recent years, more and more attention has been paid

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to the bifurcation of fractional order models with time delay [25]–[28]. In [27], the authors studied a delayed generalized fractional-order prey-predator model with interspecific competition, and derived the global asymptotic stability conditions and local bifurcation criterions of the equilibrium by choosing time delay as a bifurcation parameter.

Bifurcation control is a very necessary and effective method. Various bifurcation control methods have been proposed [29]–[31]. Using this method, a controller can be designed to suppress or reduce the bifurcation dynamics of a given nonlinear system, so as to expand the stability region of the system and obtain the ideal dynamic behavior. It is worth mentioning that the stability of fractional order dynamic system can be greatly improved by using active bifurcation control strategy. More and more attention has been paid to the bifurcation control of fractional order models with time delay [32]–[37]. In [32], the onset of bifurcation of a delayed fractional-order small-world networks was effectively controlled by using a fractional-order PD feedback controller. In [34], a state feedback controllers were implemented to suitably control the Hopf bifurcation for a fractional delayed predator-prey system. In [35], a parametric delay feedback control approach was further proposed to cope with bifurcation control for a delayed fractional dual congestion model, and it was found that the stability performance can be extremely heightened by adopting the parametric delay feedback controller. Generally, there exist many bifurcation control approaches including dislocated feedback control, speed feedback control and enhancing feedback control [38]–[40], *et al.* In [39], the author detected that the feedback coefficients were smaller than the ones of ordinary feedback control during controlling hyperchaotic Lorenz system, and the control cost were reduced. In [40], it revealed that the enhancing feedback control approach was the best choice of among the addressed four feedback control methods in controlling hyperchaotic Lorenz system involving relatively simple external inputs and relatively small necessary feedback coefficient. It should be pointed out that it is difficult to control a complex system with only one feedback variable. In this case, the feedback gain is always large. Therefore, in order to obtain high quality performance of fractional order dynamics system, it is necessary and urgent to use enhanced feedback control to control the occurrence of bifurcation. At present, the bifurcation control of fractional order predator-prey system with time delay based on enhanced feedback control tool has not been well studied.

Motivated by the aforemention discussions, we will use enhanced feedback control technology to conduct a theoretical analysis of the bifurcation control of the time-delay fractional predator-prey model. The key features of this paper are listed as follows:

1) Enhancing feedback control strategy is developed to deal with the bifurcation control in a fractional delayed predator-prey model.

2) The bifurcation point of the controlled model can be completely concluded by theoretical derivation.

3) The effects of fractional order on the bifurcation points are fully investigated by using enhancing feedback control strategy and dislocated feedback. It is found that the performance of control gradually becomes perfect with the decrement of fractional order.

4) We discover that enhancing feedback control strategy overmatches dislocated feedback ones in delaying the onset of bifurcation control for the considered controlled system for given fractional order.

The rest of the current paper is arranged as follows. Some mathematical preliminaries are presented in Section 2. In Section 3, the investigated model are addressed. Key bifurcation control results by using enhancing feedback control method are wholly determined in Section 4. The efficiency of the proposed control scheme is verified with the help of a simulation example in Section 5. Finally, the paper ends with a conclusion.

II. PRELIMINARIES

Many fractional derivative definitions are applied to deal with some practical issues including the Riemann-Liouville definition and the Caputo definition, *et al.* It is worth noting that the Caputo derivative has many advantages consisting of the consistence of given initial conditions with integer-order derivative, the description of well-understood features of physical situation. In this paper, Caputo derivatives are used to deal with the dynamic of fractional order systems.

Definition 1 ([9]): The Caputo fractional-order derivative is defined by

$$D_t^{\phi} f(t) = \frac{1}{\Gamma(l-\phi)} \int_{t_0}^{t} (t-s)^{l-\psi-1} f^{(l)}(s) ds,$$

where $l-1 \leq \phi < l \in Z^+$, $\Gamma(\cdot)$ is the Gamma function, $\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt$.

The Laplace transform of the Caputo fractional-order derivatives is

$$L\{D_t^{\phi} f(t); s\} = s^{\phi} F(s) - \sum_{k=0}^{l-1} s^{\phi-k-1} f^{(k)}(0),$$

where $l-1 \leq \phi < l \in Z^+$. If $f^{(k)}(0) = 0$, $k = 1, 2, \ldots, n$, then $L\{D_t^{\phi} f(t); s\} = s^{\phi} F(s)$.

Lemma 1 ([41]): Consider the following n-dimensional linear fractional-order system

$$\begin{cases} D^{\phi_1} \gamma_1(t) = \mathrm{k}_{11}\gamma_1(t) + \mathrm{k}_{12}\gamma_2(t) + \cdots + \mathrm{k}_{1n}\gamma_n(t), \\ D^{\phi_2} \gamma_2(t) = \mathrm{k}_{21}\gamma_1(t) + \mathrm{k}_{22}\gamma_2(t) + \cdots + \mathrm{k}_{2n}\gamma_n(t), \\ \vdots \\ D^{\phi_n} \gamma_n(t) = \mathrm{k}_{n1}\gamma_1(t) + \mathrm{k}_{n2}\gamma_2(t) + \cdots + \mathrm{k}_{nn}\gamma_n(t), \end{cases} \quad (1)$$

where $0 < \phi_i < 1 (i = 1, 2, \ldots, n)$. It is assumed that ϕ is the lowest common multiple of the denominators ψ_i of ϕ_i, where $\phi_i = \frac{\varphi_i}{\psi_i}$, $(\varphi_i, \psi_i) = 1$, $\varphi_i, \psi_i \in Z^+$, for $i = 1, 2, \ldots, n$.

Define

$$\triangle(s)=\begin{bmatrix} s^{\phi_1}-\mathrm{k}_{11} & -\mathrm{k}_{12} & \cdots & -\mathrm{k}_{1n} \\ -\mathrm{k}_{21} & s^{\phi_2}-\mathrm{k}_{22} & \cdots & -\mathrm{k}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\mathrm{k}_{n1} & -\mathrm{k}_{n2} & \cdots & s^{\phi_n}-\mathrm{k}_{nn} \end{bmatrix}$$

Then the zero solution of system (1) is globally asymptotically stable in the Lyapunov sense if all roots s of the equation $\det(\triangle(s))=0$ satisfy $|\arg(s)|>\phi_i\pi/2$.

Lemma 2 ([41]): Consider the following n-dimensional linear fractional-order delayed system

$$\begin{cases} D^{\phi_1}\gamma_1(t)=\mathrm{k}_{11}\gamma_1(t-\tau_{11})+\mathrm{k}_{12}\gamma_2(t-\tau_{12}) \\ \qquad\qquad +\cdots+\mathrm{k}_{1n}\gamma_n(t-\tau_{1n}), \\ D^{\phi_2}\gamma_2(t)=\mathrm{k}_{21}\gamma_1(t-\tau_{21})+\mathrm{k}_{22}\gamma_2(t-\tau_{22}) \\ \qquad\qquad +\cdots+\mathrm{k}_{2n}\gamma_n(t-\tau_{2n}), \\ \qquad \vdots \\ D^{\phi_n}\gamma_n(t)=\mathrm{k}_{n1}\gamma_1(t-\tau_{n1})+\mathrm{k}_{n2}\gamma_2(t-\tau_{n2}) \\ \qquad\qquad +\cdots+\mathrm{k}_{nn}\gamma_n(t-\tau_{nn}), \end{cases} \tag{2}$$

where $\phi_i\in(0,1)(i=1,2,\ldots,n)$, the initial values $\mathrm{V}_i(t)=\Psi_i(t)$ are given for $-\max_{i,j},\tau_{i,j}=-\max_{i,j}\le t\le 0$ and $i=1,2,\ldots,n$. For system (2), time-delay matrix $\tau=(\tau_{i,j})\in(R^+)_{n\times n}$, coefficient matrix $H=(\mathrm{k}_{i,j})_{n\times n}$, state variables $\gamma_i(t),\gamma_i(t-\tau_{i,j})\in R$, and initial values $\Psi_i(t)\in C^0[-\tau_{\max},0]$. Its fractional order is defined as $\phi=(\phi_1,\phi_2,\ldots,\phi_n)$. The characteristic equation $det(\triangle(s))$ is defined as

$$\begin{vmatrix} s^{\phi_1}-\mathrm{k}_{11}e^{-s\tau_{11}} & -\mathrm{k}_{12}e^{-s\tau_{12}} & \cdots & -\mathrm{k}_{1n}e^{-s\tau_{1n}} \\ -\mathrm{k}_{21}e^{-s\tau_{21}} & s^{\phi_2}-\mathrm{k}_{22}e^{-s\tau_{22}} & \cdots & -\mathrm{k}_{2n}e^{-s\tau_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -\mathrm{k}_{n1}e^{-s\tau_{n1}} & -\mathrm{k}_{n2}e^{-s\tau_{n2}} & \cdots & s^{\phi_n}-\mathrm{k}_{nn}e^{-s\tau_{nn}} \end{vmatrix}$$

Then the zero solution of system (2) is Lyapunov globally asymptotically stable if all the roots of the characteristic equation $\det(\triangle(s))=0$ have negative real parts.

III. THE MATHEMATICAL MODEL

In [42], the bifurcation of a ratio-dependent delayed predator-prey system with two delays was considered. The mathematical model was as follows:

$$\begin{cases} \dfrac{dN(t)}{dt}=r_1N(t)-\varepsilon P(t)N(t), \\ \dfrac{dP(t)}{dt}=P(t)\Big[r_2-\theta\dfrac{P(t-\tau_2)}{N(t-\tau_1)}\Big], \end{cases} \tag{3}$$

where the variables and parameters of system (3) are explained in Table.1.

For the sake of succinctness, we assume that $\tau_1=\tau_2=\tau$ in system (3), then the following system can be derived

$$\begin{cases} \dfrac{dN(t)}{dt}=r_1N(t)-\varepsilon P(t)N(t), \\ \dfrac{dP(t)}{dt}=P(t)\Big[r_2-\theta\dfrac{P(t-\tau)}{N(t-\tau)}\Big]. \end{cases} \tag{4}$$

TABLE 1. The instructions relevant variables and parameters of system (3).

Variables(Parameters)	Description
$N(t)$	Population densities of prey at time t
$P(t)$	Population densities of predator at time t
$N(t-\tau_1)$	Juveniles of prey who was born at time $t-\tau_1$ and survive at time t
$P(t-\tau_2)$	juveniles of prey and predator who were born at time $t-\tau_2$ and survive at time t
r_1	Predation rate of the mature predator
r_2	Conversion factor from the mature prey to the immature predator
ε	Death rates of the immature prey
θ	Death rates of the mature prey

In this paper, we add the enhancing feedback controllers $K_1[N(t)-N(t-\tau)]$, $K_2[P(t)-P(t-\tau)$ to the following fractional-order version predator-prey model

$$\begin{cases} D^{\phi}N(t)=r_1N(t)-\varepsilon P(t)N(t)+K_1[N(t)-N(t-\tau)], \\ D^{\phi}P(t)=P(t)\Big[r_2-\theta\dfrac{P(t-\tau)}{N(t-\tau)}\Big]+K_2[P(t)-P(t-\tau)], \end{cases} \tag{5}$$

where ϕ is fractional order, K_1,K_2 denote feedback gains. It is easy to see that the enhancing feedback controllers preserves the equilibrium point of the system (5).

Noting that system (5) degenerates into the uncontrolled integer-order version system (4) when $\phi=1,K_1=K_2=0$. It is not difficult to see that the positive equilibrium point $E^*=(N^*,P^*)$ of system (5) is consistent with system (3) and (4), which can be acquired by solving the following equations:

$$\begin{cases} r_1-\varepsilon P^*=0, \\ r_2\,N^*-\theta P^*=0. \end{cases}$$

It implies that $N^*=\dfrac{\theta r_1}{\varepsilon r_2}$, $P^*=\dfrac{r_1}{\varepsilon}$. Obviously, system (5) has a unique positive equilibrium point E^*.

In order to obtain better control effect, the following basic assumption is necessary:

$$(\mathbf{H}1)\ \ K_1\le 0,\quad K_2\le 0.$$

The core objective of this paper is to discuss the problem of bifurcation control for system (5) by taking time delay as a bifurcation parameter and the approach is from [41]. Then, some comparative investigations on bifurcation control are executed. It is found that the stability performance of the controlled system can be extremely improved by enhancing feedback control than the dislocated feedback control.

IV. THEORY ANALYSIS

In this section, time delay shall be selected as a bifurcation parameter to investigate the problem of bifurcation control for the predator-prey model (5). The existence bifurcation and bifurcation point for the proposed model shall be established.

Let $\rho(t) = N(t) - N^*$, $\varrho(t) = P(t) - P^*$, then the system (5) can be rewritten as:

$$\begin{cases} D^\phi \rho(t) = r_1(\rho(t)+N^*) - \varepsilon(\varrho(t)+P^*)(\rho(t)+N^*) \\ \qquad + K_1[\rho(t)-\rho(t-\tau)], \\ D^\phi \varrho(t) = (\varrho(t)+P^*)\Big[r_2 - \theta\dfrac{\varrho(t-\tau)+P^*}{\rho(t-\tau)+N^*}\Big] \\ \qquad + K_2[\varrho(t)-\varrho(t-\tau)]. \end{cases} \tag{6}$$

The linearized system of network (6) can be gained that

$$\begin{cases} D^\phi \rho(t) = -\varepsilon N^* \varrho(t) + K_1\rho(t) - K_1\rho(t-\tau), \\ D^\phi \varrho(t) = \theta\Big(\dfrac{P^*}{N^*}\Big)^2 \rho(t-\tau) - \theta\dfrac{P^*}{N^*}\varrho(t-\tau) \\ \qquad + K_2\varrho(t) - K_2\varrho(t-\tau). \end{cases} \tag{7}$$

The associated characteristic equation of (7) is as follows

$$\det\begin{bmatrix} s^\phi - K_1 + K_1e^{-s\tau} & \varepsilon N^* \\ -\theta\Big(\dfrac{P^*}{N^*}\Big)^2 e^{-s\tau} & s^\phi - K_2 + \Big(K_2 + \theta\dfrac{P^*}{N^*}\Big)e^{-s\tau} \end{bmatrix} = 0. \tag{8}$$

According to the theory of determinant, we conclude that

$$P_1(s) + P_2(s)e^{-s\tau} + P_3(s)e^{-2s\tau} = 0, \tag{9}$$

where

$$\begin{aligned} P_1(s) &= s^{2\phi} - (K_1+K_2)s^\phi + K_1K_2, \\ P_2(s) &= (K_1+K_2+\alpha)s^\phi + \beta - 2K_1K_2 - \alpha K_1, \\ P_3(s) &= K_1(K_2+\alpha), \\ \alpha &= \theta\frac{P^*}{N^*}, \\ \beta &= \alpha\varepsilon P^*. \end{aligned}$$

Multiply $e^{s\tau}$ on both sides of (9), then we get that

$$P_1(s)e^{s\tau} + P_2(s) + P_3(s)e^{-s\tau} = 0. \tag{10}$$

Assume that $s = \varpi(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})(\varpi > 0)$ is a purely imaginary root of (10), then we have

$$\begin{cases} (A_1+A_3)\cos\varpi\tau + (B_3-B_1)\sin\varpi\tau = -A_2, \\ (B_1+B_3)\cos\varpi\tau + (A_1-A_3)\sin\varpi\tau = -B_2, \end{cases} \tag{11}$$

where $A_i, B_i (i = 1, 2, 3)$ are the real parts and imaginary parts of $P_i(s)$. A_i, B_i can be described as follows:

$$\begin{aligned} A_1 &= \varpi^{2\phi}\cos\phi\pi - (K_1+K_2)\varpi^\phi\cos\frac{\phi\pi}{2} + K_1K_2, \\ B_1 &= \varpi^{2\phi}\sin\phi\pi - (K_1+K_2)\varpi^\phi\sin\frac{\phi\pi}{2}, \\ A_2 &= (K_1+K_2+\alpha)\varpi^\phi\cos\frac{\phi\pi}{2} + \beta - 2K_1K_2 - \alpha K_1, \\ B_2 &= (K_1+K_2+\alpha)\varpi^\phi\sin\frac{\phi\pi}{2}, \\ A_3 &= K_1(K_2+\alpha), \\ B_3 &= 0. \end{aligned}$$

We further label

$$\begin{aligned} F_1(\varpi) &= -A_2(A_1-A_3) - B_1B_2, \\ F_2(\varpi) &= -B_2(A_1+A_3) + B_1A_2, \\ G(\varpi) &= A_1^2 + B_1^2 - A_3^2. \end{aligned}$$

It follows from (11) that

$$\begin{cases} \cos\varpi\tau = \dfrac{F_1(\varpi)}{G(\varpi)}, \\ \sin\varpi\tau = \dfrac{F_2(\varpi)}{G(\varpi)}. \end{cases} \tag{12}$$

In terms of (12), we procure that

$$G^2(\varpi) = F_1^2(\varpi) + F_2^2(\varpi). \tag{13}$$

It can be defined from (13) that

$$H(\varpi) = G^2(\varpi) - F_1^2(\varpi) - F_2^2(\varpi) = 0. \tag{14}$$

Under Eq.(14), we can obtain that

$$\begin{aligned} H(\varpi) = \varpi^{8\phi} + l_1\varpi^{7\phi} + l_2\varpi^{6\phi} + l_3\varpi^{5\phi} + l_4\varpi^{4\phi} \\ + l_5\varpi^{3\phi} + l_6\varpi^{2\phi} + l_7\varpi^{\phi} + l_8 = 0, \end{aligned} \tag{15}$$

where $l_i (i = 1, 2, \ldots, 8)$ are computed in Appendix $A - B$.

In order to guarantee the occurrence of Hopf bifurcation for system (5), we further give the additional assumption:

(**H**2) (15) has at least positive real roots.

It should be noted that the assumption (**H**2) is only a necessary condition for the bifurcation of the system (5), not a sufficient condition.

According to $\cos\varpi\tau = \dfrac{F_1(\varpi)}{G(\varpi)}$, we can get

$$\tau^{(k)} = \frac{1}{\varpi}\Big[\arccos\frac{F_1(\varpi)}{G(\varpi)} + 2k\pi\Big], \quad k = 0, 1, 2, \ldots. \tag{16}$$

Define the bifurcation point

$$\tau_0 = \min\{\tau^{(k)}\}, \quad k = 0, 1, 2, \ldots,$$

where $\tau^{(k)}$ is defined by (16).

In what follows, we will consider the stability of system (5) when $\tau = 0$. If τ is removed, the characteristic (9) becomes

$$\lambda^{2\phi} + \alpha\lambda^\phi + \beta = 0. \tag{17}$$

It is obvious from $\alpha > 0, \beta > 0$ that the two roots of (17) have negative parts which satisfying Lemma 1. Hence, the positive equilibrium of the fractional system (5) is asymptotically stable.

In order to acquire the transversality condition of the occurrence for Hopf bifurcation, the following necessary assumption is needed for system (5):

$$(\mathbf{H3})\ \frac{\chi_1\upsilon_1 + \chi_2\upsilon_2}{\upsilon_1^2 + \upsilon_2^2} \neq 0,$$

where $\chi_1, \chi_2, \upsilon_1, \upsilon_2$ are defined by (20).

Lemma 3: Let $s(\tau) = \xi(\tau) + i\varpi(\tau)$ be the root of Eq.(9) near $\tau = \tau_j$ satisfying $\xi(\tau_j) = 0$, $\varpi(\tau_j) = \varpi_0$, then the following transversality condition holds

$$\mathrm{Re}\Big[\frac{ds}{d\tau}\Big]\Big|_{(\varpi=\varpi_0, \tau=\tau_0)} \neq 0,$$

where ϖ_0, τ_0 represent the critical frequency and bifurcation point of system (5).

Proof: By using implicit function theorem and differentiating (9) with respect to τ, we can get

$$P_1'(s)\frac{ds}{d\tau} + \Big[P_2'(s)\frac{ds}{d\tau}e^{-s\tau} + P_2(s)e^{-s\tau}\Big(-\tau\frac{ds}{d\tau} - s\Big)\Big] + \Big[P_3'(s)\frac{ds}{d\tau}e^{-2s\tau} + 2P_3(s)e^{-2s\tau}\Big(-\tau\frac{ds}{d\tau} - s\Big)\Big] = 0. \tag{18}$$

It is clear from Eq.(17) that $P_3'(s) = 0$. By mathematical derivation, it follows from Eq.(18) that

$$\frac{ds}{d\tau} = \frac{\chi(s)}{\upsilon(s)}, \tag{19}$$

where

$$\chi(s) = s[P_2(s)e^{-s\tau} + 2P_3(s)e^{-2s\tau}],$$
$$\upsilon(s) = P_1'(2) + [P_2'(s) - \tau P_2(s)]e^{-s\tau} - 2\tau P_3(s)e^{-2s\tau}.$$

Let P_i^R, P_i^I stand for the real parts and the imaginary parts of $P_i(s)$. Let $P_i'^R$, $P_i'^I$ denote the real parts and the imaginary parts of $P_i'(s)$. Then by some computation, it can be deduced from (19) that

$$\mathrm{Re}\Big[\frac{ds}{d\tau}\Big]\Big|_{(\varpi=\varpi_0,\tau=\tau_0)} = \frac{\chi_1\upsilon_1 + \chi_2\upsilon_2}{\upsilon_1^2 + \upsilon_2^2}, \tag{20}$$

where

$$\begin{aligned}
\chi_1 &= \varpi_0(P_2^R \sin\varpi_0\tau_0 - P_2^I\cos\varpi_0\tau_0 + 2P_3^R\sin 2\varpi_0\tau_0 \\
&\quad - 2P_3^I\cos 2\varpi_0\tau_0),\\
\chi_2 &= \varpi_0(P_2^R \cos\varpi_0\tau_0 + P_2^I\sin\varpi_0\tau_0 + 2P_3^R\cos 2\varpi_0\tau_0 \\
&\quad + 2P_3^I\sin 2\varpi_0\tau_0),\\
\upsilon_1 &= P_1'^R + (P_2'^R - \tau_0\, P_2^R)\cos\varpi_0\tau_0 + (P_2'^I - \tau_0\, P_2^I) \\
&\quad \cdot\sin\varpi_0\tau_0 - 2\tau_0(P_3^R\cos 2\varpi_0\tau_0 + P_3^I\sin 2\varpi_0\tau_0),\\
\upsilon_2 &= P_1'^I + (P_2'^I - \tau_0\, P_2^I)\cos\varpi_0\tau_0 - (P_2'^R - \tau_0\, P_2^R) \\
&\quad \cdot\sin\varpi_0\tau_0 - 2\tau_0(P_3^I\cos 2\varpi_0\tau_0 - P_3^R\sin 2\varpi_0\tau_0).
\end{aligned}$$

(**H**3) indicates that transversality condition hold. We accomplish the proof of Lemma 3.

Based on the assumptions (**H**1)-(**H**3) and previous analysis, the following theorem can be derived.

Theorem: Under (**H**1)-(**H**3), the following results hold:

1) E^* of the fractional system (5) is asymptotically stable when $\tau \in [0, \tau_0)$;

2) System (5) undergoes a Hopf bifurcation at E^* when $\tau = \tau_0$, i.e., it has a branch of periodic solutions bifurcating from E^* near $\tau = \tau_0$.

Remark 1: It is difficult to theoretically analyze all the positive real roots. Nevertheless, these positive real roots of (15) can be easily computed by using Maple numerical software. Hence, the critical frequency ϖ_0 and bifurcation point τ_0 can be accurately established.

Remark 2: Some analogous models were analyze in [43]–[46]. It is worth mentioning that these results only concentrated on the dynamics of integer-order predator-prey models. It is more realistic to explore the dynamics of delayed predator-prey models by fully considering the effects of fractional calculus for ecosystems in this paper.

Remark 3: In this paper, the effects of fractional order on the bifurcation point are adequately discussed by calculation. It implies that the better effects in delaying the onset of bifurcation can be achieved as fractional order decreases if feedback gain are established.

Remark 4: In [32]–[35], various bifurcation strategies were adopted to control the onset of bifurcation for delayed

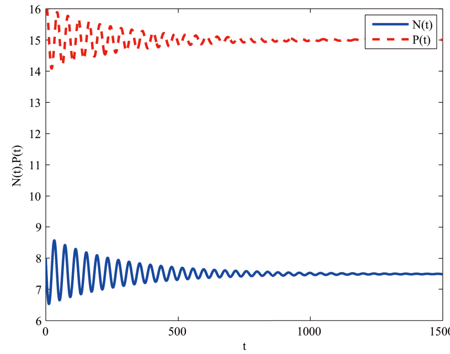

FIGURE 1. Time series of system (21) with $\phi = 0.98$, $K_1 = -0.08$, $K_2 = -0.15$, $\tau = 3.8 < \tau_0 = 4.4222$. The equilibrium E^* of the fractional system (21) is asymptotically stable.

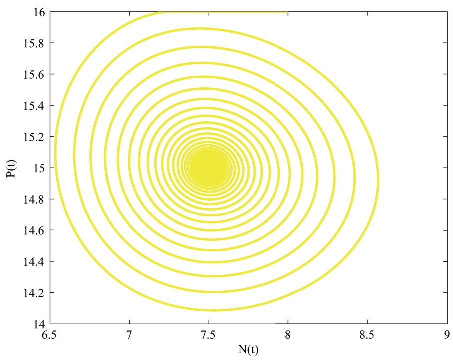

FIGURE 2. Portrait diagram of system (21) with $\phi = 0.98$, $K_1 = -0.08$, $K_2 = -0.15$, $\tau = 3.8 < \tau_0 = 4.4222$. The equilibrium E^* of the fractional system (21) is asymptotically stable.

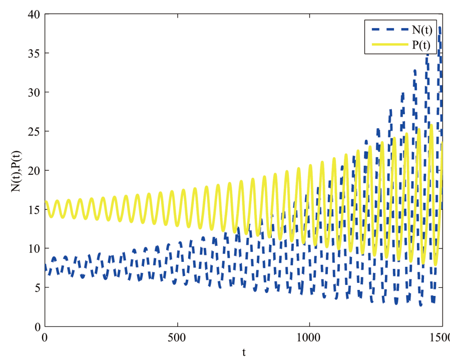

FIGURE 3. Time series of system (21) with $\phi = 0.98$, $K_1 = -0.08$, $K_2 = -0.15$, $\tau = 4.8 > \tau_0 = 4.4222$. A Hopf bifurcation bifurcates from the equilibrium E^* of the fractional system (21).

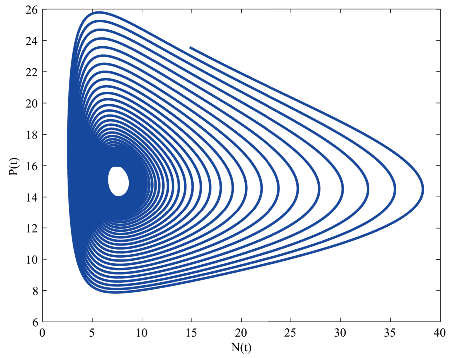

FIGURE 4. Portrait diagram of system (21) with $\phi = 0.98$, $K_1 = -0.08$, $K_2 = -0.15$, $\tau = 4.8 > \tau_0 = 4.4222$. A Hopf bifurcation bifurcates from the equilibrium E^* of the fractional system (21).

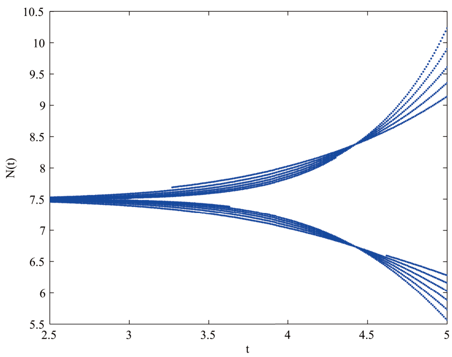

FIGURE 5. Bifurcation diagram of $N(t)$ for system (21).

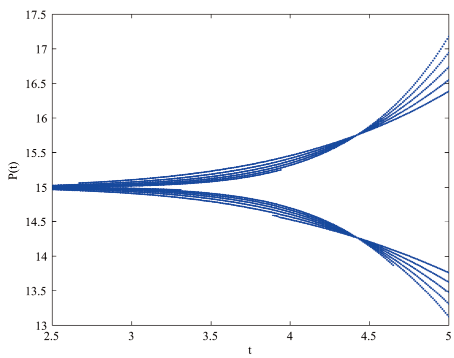

FIGURE 6. Bifurcation diagram of $P(t)$ for system (21).

fractional-order systems. Noting that these remarkable results only were obtained all based on the dislocated feedback approaches. Different from existing methods, the enhancing feedback control strategy is delay onset of the bifurcation for fractional delayed predator-prey system and satisfactory bifurcation control effects are realized compared with the dislocated feedback approaches in this paper. This hints that the proposed enhancing controllers possess a superior performance in controlling bifurcation in delayed fractional-order systems. The derived results can be extended to deal with others fractional-order systems with time delay.

V. NUMERICAL SIMULATIONS

In this section, a simulation example is exploited to exhibit the correctness of the addressed theory. In our simulations,

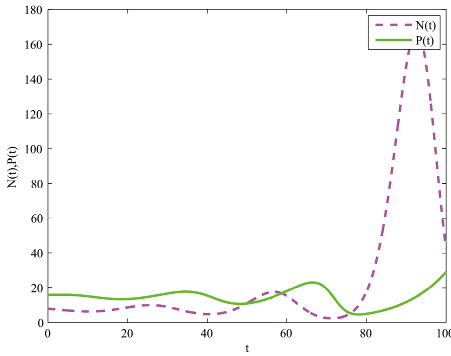

FIGURE 7. Time series of system (21) with $\phi = 0.98$, $K_1 = K_2 = 0$, $\tau = 3.8 > 3.0079$. System (21) becomes unstable.

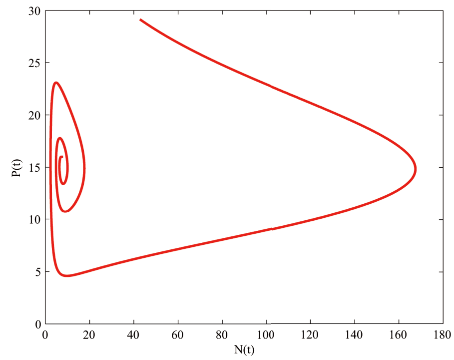

FIGURE 8. Portrait diagram of system (21) with $\phi = 0.98$, $K_1 = K_2 = 0$, $\tau = 3.8 > 3.0079$. System (21) becomes unstable.

Adama-Bashforth-Moulton predictor-corrector scheme is adopted in [47]. To aid consistent comparisons with the predecessor research progress, we consider model (5) with the same parameters which are used in [42]: $r_1 = 0.45$, $r_2 = 0.1$, $\varepsilon = 0.03$, $\theta = 0.05$. The positive equilibrium point E^* can be obtained as $(N^*, P^*) = (7.5, 15)$. Step-length is chosen as $h = 0.01$, and the initial values are taken as $(N(0), P(0)) = (8, 16)$. Consider the controlled system

$$\begin{cases} D^{\phi}N(t) = 0.45N(t) - 0.03\,P(t)N(t) \\ \qquad + K_1[(N(t) - N(t-\tau)], \\ D^{\phi}P(t) = P(t)\Big[0.1 - 0.05\dfrac{P(t-\tau)}{N(t-\tau)}\Big] \\ \qquad + K_2[(P(t) - P(t-\tau)]. \end{cases} \tag{21}$$

Selecting $\phi = 0.98$, $K_1 = -0.08$, $K_2 = -0.15$ in system (21), it is derived that $\varpi_0 = 0.1496$, then $\tau_0 = 4.4222$. In terms of Theorem 1, E^* of controlled system (21) is asymptotically stable when $\tau = 3.8 < \tau_0$, which, depicted in Figs.1-2, while Figs.3-4 display that E^* of controlled system (21) is unstable, Hopf bifurcation occurs when $\tau = 4.8 > \tau_0$. Bifurcation diagrams of system (21) are simulated in Figs. 5-6.

The same order is chosen as $\phi = 0.98$. We first select $K_1 = K_2 = 0$, which means that the controllers are removed, we derive $\tau_0 = 2.3807$. We further choose $\tau = 3.8 > \tau_0 = 2.3807$, it is clear that system (21) becomes unstable,

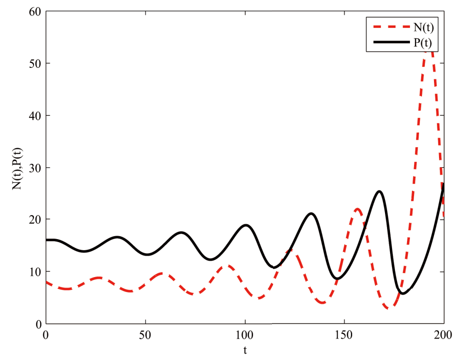

FIGURE 9. Time series of system (21) with $\phi = 0.98$, $K_1 = -0.08$, $K_2 = 0$, $\tau = 3.8 > 3.0079$. System (21) becomes unstable.

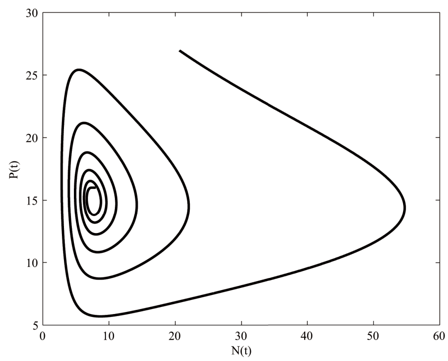

FIGURE 10. Portrait diagram of system (21) with $\phi = 0.98$, $K_1 = -0.08$, $K_2 = 0$, $\tau = 3.8 > 3.0079$. System (21) becomes unstable.

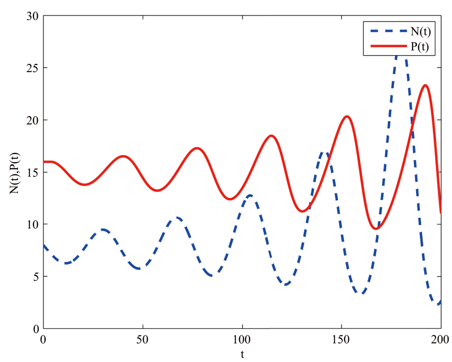

FIGURE 11. Time series of system (21) with $\phi = 0.98$, $K_1 = 0$, $K_2 = -0.15$, $\tau = 3.8 > 3.0079$. System (21) becomes unstable.

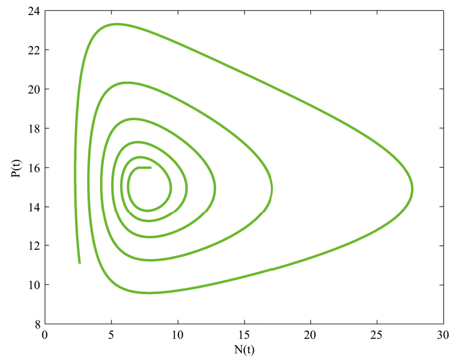

FIGURE 12. Portrait diagram of system (21) with $\phi = 0.98$, $K_1 = 0$, $K_2 = -0.15$, $\tau = 3.8 > 3.0079$. System (21) becomes unstable.

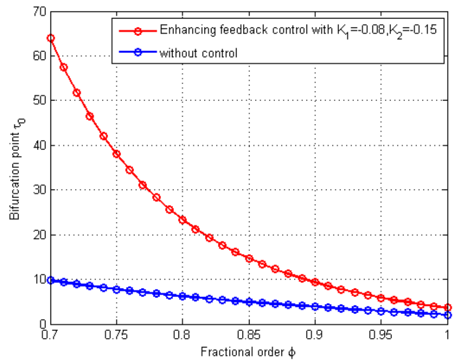

FIGURE 13. Comparison on the values of τ_0 versus ϕ for system (21). The bifurcation point is more larger with $K_1 = -0.08$, $K_2 = -0.15$ than that one with $K_1 = K_2 = 0$ for given ϕ.

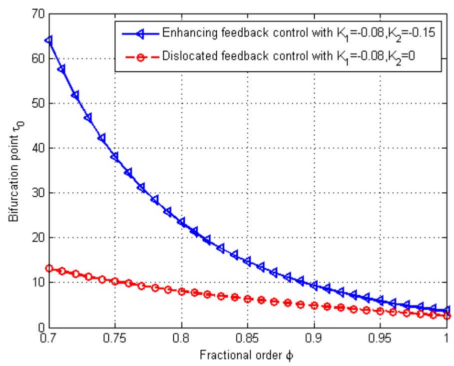

FIGURE 14. Comparison on the values of τ_0 versus ϕ for system (21). The performance of designed enhancing controllers overmatch the single controller.

which is depicted in Figs.7-8. Then we choose $K_2 = 0$, $K_1 = -0.08$. This indicates that dislocated feedback control emerges, then we have $\tau_0 = 3.0079$. We choose $\tau = 3.8 > \tau_0 = 3.0079$, it is obvious that system (21) becomes unstable, which is simulated in Figs. 9-10. If $K_1 = 0$, $K_2 = -0.15$, it suggests that dislocated feedback control occurs, then we have $\tau_0 = 2.8650$. We also choose $\tau = 3.8 > \tau_0 = 2.8650$, it is obvious that system (21) becomes unstable, which is simulated in Figs. 11-12.

In brief, system (21) will turn unstable once the controllers all are removed or dislocated feedback controller engenders. In what follows, we shall fully consider the effects of the proposed enhancing control scheme.

Case 1: Selecting three sets of parameters $K_1 = -0.08$, $K_2 = -0.15$; $K_1 = -0.08$, $K_2 = -0.15$; $K_1 = -0.08$, $K_2 = 0$; $K_1 = 0$, $K_2 = -0.15$, respectively. By varying ϕ, we derive the values of τ_0, the comparative results are addressed in Figs.13-15. Fig.13 describes that the bifurcation point is more larger with $K_1 = -0.08$, $K_2 = -0.15$ than that one with $K_1 = K_2 = 0$ for given ϕ. This verifies that the effectiveness of the devised controllers. Figs.14-15 reveal that the performance of designed enhancing controllers overmatch the single controller.

Case 2: Fixing $\phi = 0.98$ and select two sets of parameters $K_2 = -0.15$ and $K_2 = 0$, then we derive the values of

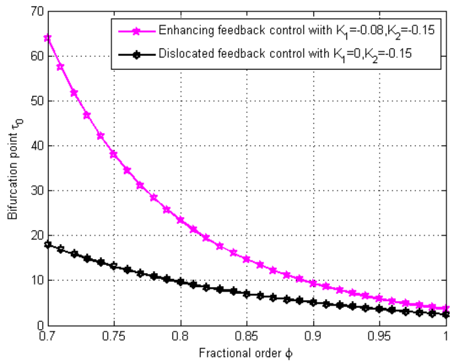

FIGURE 15. Comparison on the values of τ_0 versus ϕ for system (21). The performance of designed enhancing controllers overmatch the single controller.

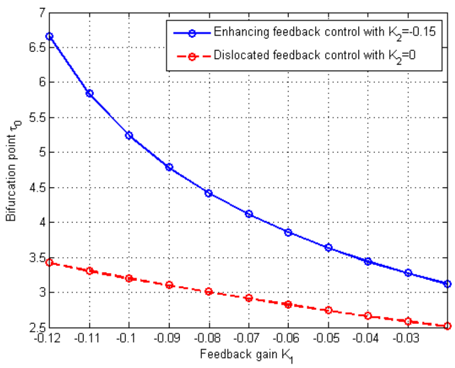

FIGURE 16. Comparison on the values of τ_0 versus K_1 for system (21) with $\phi = 0.98$. The control effects are more better with the present of feedback gain K_2 than the absence of it.

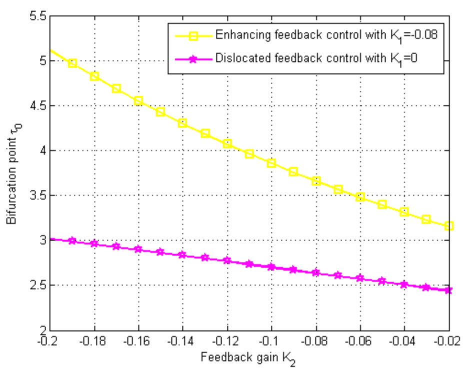

FIGURE 17. Comparison on the values of τ_0 versus K_2 for system (21) with $\phi = 0.98$. The control effects are more better with the present of feedback gain K_1 than the absence of it.

τ_0 with the change of K_1, which is demonstrated in Fig.16. Fig.16 indicates that the control effects are more better with the present of feedback gain K_2 than the absence of it.

Case 3: Taking $\phi = 0.98$ and select two sets of parameters $K_1 = -0.08$ and $K_1 = 0$, then we derive the values of τ_0 with the change of K_2, which is demonstrated in Fig.17. Fig.17 discloses that the control effects are more better with the present of feedback gain K_1 than the absence of it.

VI. CONCLUSION

Using the enhanced feedback control technique, we analyze the bifurcation control of a class of predator-prey model with fractional delay. An improved feedback control strategy is proposed for the bifurcation control of a fractional order predator-prey model with time delay. This shows that the proposed enhanced controller has superior performance in controlling the bifurcation of fractional order time-delay systems. Through theoretical derivation, the bifurcation point of the controlled model can be obtained. The influence of fractional order on bifurcation point is fully studied by using enhanced feedback control strategy and dislocation feedback. It is found that with the decrease of fractional order, the control performance tends to be perfect. This means that when the feedback gain is established, the reduction of fractional order can achieve better effect of delayed bifurcation. We find that the enhanced feedback control strategy is better than the dislocation feedback control strategy in delaying the start of bifurcation control for a given fractional order controlled system. Finally, the validity of the theoretical results is verified by numerical simulation.

APPENDIX A

$$
\begin{aligned}
l_1 =& -4(K_1+K_2)\cos\frac{\phi\pi}{2},\\
l_2 =& (K_1^2+K_2^2-\alpha^2-2K_1K_2-2\alpha K_1-2\alpha K_2)\\
&(\sin^2\phi\pi\cos^2\frac{\phi\pi}{2}+\cos^2\phi\pi\sin^2\frac{\phi\pi}{2})+(5K_1^2\\
&+5K_2^2-\alpha^2+10K_1K_2-2\alpha K_1-2\alpha K_2)\\
&(\sin^2\phi\pi\sin^2\frac{\phi\pi}{2}+\cos^2\phi\pi\cos^2\frac{\phi\pi}{2})+2(K_1+K_2)^2\\
&\sin\phi\pi\sin 2\phi\pi+4K_1K_2(\cos\phi\pi\sin^2\phi\pi+\cos^3\phi\pi\Big),\\
l_3 =& (-6K_1^2K_2-6K_1K_2^2+2\alpha^2K_1+2\alpha^2K_2+4\alpha K_1^2\\
&+4\alpha K_2^2+8\alpha K_1K_2-2K_1^3-2K_2^3)(\sin\phi\pi\sin^3\frac{\phi\pi}{2}\\
&+\cos\phi\pi\cos^3\frac{\phi\pi}{2})+(2\alpha^2K_1-2\beta K_1-2\beta K_2-2\alpha\beta\\
&-8K_2^2K_1-8K_1^2K_2+2\alpha K_1^2+6\alpha K_1K_2)\cos^2\phi\pi\cos\frac{\phi\pi}{2}\\
&+(-2\beta K_1-2\beta K_2-2\alpha\beta+2\alpha^2K_1+2\alpha K_1^2+6\alpha K_1K_2)\\
&\sin^2\phi\pi\cos\frac{\phi\pi}{2}+(-2K_1^3-2K_2^3+2\alpha^2K_1+2\alpha^2K_2\\
&+4\alpha K_1^2-6K_1^2K_2-6K_1K_2^2+4\alpha K_2^2+8\alpha K_1K_2)\\
&(\cos\phi\pi\cos\frac{\phi\pi}{2}\sin^2\frac{\phi\pi}{2}+\sin\phi\pi\sin\frac{\phi\pi}{2}\cos^2\frac{\phi\pi}{2})\\
&-8K_1K_2(K_1+K_2)\cdot\sin\phi\pi\cos\phi\pi\sin\frac{\phi\pi}{2},\\
l_4 =& -\alpha(2\alpha K_1K_2+6K_1^2K_2+6K_2^2K_1+2K_1^3+2K_2^3+\alpha K_2^2\\
&+\alpha K_1^2)(\sin^4\frac{\phi\pi}{2}+\cos^4\frac{\phi\pi}{2})+(4\beta K_1K_2+2\alpha\beta K_1\\
&-8\alpha K_1^2K_2-4K_1^2K_2^2-\beta^2-3\alpha^2K_1^2)\sin^2\phi\pi+(2\alpha\beta K_1\\
&-8\alpha K_1^2K_2+4\beta K_1K_2-\beta^2-3\alpha^2K_1^2)\cos^2\phi\pi+(4K_2^2K_1\\
&+4\beta K_2^2+4\beta K_1^2+2\alpha^3K_1-10\alpha K_2^2K_1+8\beta K_1K_2+4\alpha\beta K_1\\
&+4\alpha\beta K_2+4K_1^3K_2+8K_1^2K_2^2-2\alpha K_1^3-12\alpha K_1^3K_2)\\
&\cos\phi\pi\cos^2\frac{\phi\pi}{2}-\alpha(2\alpha K_2^2+2\alpha K_1^2+4K_2^3+4K_1^3
\end{aligned}
$$

$$
\begin{aligned}
&+4\alpha K_1K_2+12K_2^2K_1+12K_1^2K_2)\cos^2\frac{\phi\pi}{2}\sin^2\frac{\phi\pi}{2}\\
&-(2\alpha^3K_1+2\alpha K_1^3+4K_1^2+8\alpha^2K_1K_2+10K_1K_2^2\\
&+12\alpha K_1^2K_2)\cos\phi\pi\sin^2\frac{\phi\pi}{2}+4(\beta K_1^2+\beta K_2^2+K_2^2K_1\\
&+\alpha^2K_1+\alpha\beta K_1+\alpha\beta K_2+\alpha^2K_1^2+2\beta K_1K_2\\
&+2\alpha^2K_1K_2+2K_1^2K_2^2)\cdot\sin\phi\pi\sin\frac{\phi\pi}{2}\cos\frac{\phi\pi}{2}\\
&+4K_1^3K_2\sin\phi\pi\sin\frac{\phi\pi}{2}.
\end{aligned}
$$

APPENDIX B

$$
\begin{aligned}
l_5 = {} & (-4\alpha^2K_1^2K_2-4\alpha\beta K_1K_2+4\alpha K_1^3K_2+8\alpha K_1^2K_2^2\\
&+4\alpha K_2^3K_1-2\alpha^2K_1^3-6\beta K_1^2K_2-2\alpha^3K_1^2-2\alpha\beta K_1^2\\
&-2\alpha^3K_1K_2-2\alpha\beta K_2^2-2\alpha^2K_2^2K_1-6\beta K_2^2K_1-2\beta K_2^3\\
&-2\beta K_1^3)\cos^3\frac{\phi\pi}{2}+(4\alpha K_2^3K_1-2\alpha^2K_1^3-6\beta K_1^2K_2\\
&-2\alpha^3K_1-2\alpha\beta K_1^2-2\alpha\beta K_2^2-2\alpha^2K_2^2K_1-6\beta K_2^2K_1\\
&-2\beta K_2^3-2\beta K_1^3-4\alpha\beta K_1K_2+8\alpha K_1^2K_2^2+4\alpha K_1^3K_2\\
&-4\alpha^2K_1^2K_2-2\alpha^3K_1K_2)\cos\frac{\phi\pi}{2}\sin^2\frac{\phi\pi}{2}+(2\alpha^2K_1^3\\
&-4\alpha K_1^2K_2^2-10\alpha^2K_2K_1^2-4\beta K_1K_2^2+4\alpha^2\beta K_1+2\beta^2K_1\\
&+2\beta^2K_2-4\alpha^3K_1^2+4\alpha\beta K_1K_2-4\beta K_1^2K_2+4\alpha K_1^3K_2)\\
&\cdot\sin\phi\pi\sin\frac{\phi\pi}{2}+(-8\beta K_2^2K_1+4\alpha^2\beta K_1-6\alpha^2K_2K_1^2\\
&+8\alpha K_1^2K_2^2+2\alpha^2K_1^3+2\beta^2K_1+2\beta^2K_2-4\alpha^3K_1^2\\
&+8\alpha K_1^3K_2-8\beta K_1^2K_2)\cos\phi\pi\cos\frac{\phi\pi}{2},\\
l_6 = {} & (4\alpha^2K_1^3K_2-2\alpha\beta K_1^3-6\alpha\beta K_1K_2^2-4\alpha^2\beta K_1^2\\
&-4\alpha^2\beta K_1K_2+2\alpha^3K_1^3-2\alpha^3K_2K_1^2-2\beta^2K_1K_2\\
&-\beta^2K_1^2-\alpha^4K_1^2-\beta^2K_2^2-8\alpha\beta K_1^2K_2)\sin^2\frac{\phi\pi}{2}\\
&+(-2\alpha\beta K_2^2K_1-4\alpha^2\beta K_1K_2-2\alpha\beta K_1^3-4\alpha^2\beta K_1^2\\
&+8\alpha^2K_1^3K_2+4\beta K_1^3K_2+8K_1^2K_2^2+8\alpha^2K_1^2K_2^2\\
&+4\beta K_2^3K_1+2\alpha^3K_1^3+2\alpha^3K_2K_1^2-2\beta^2K_1K_2-\beta^2K_1^2\\
&-\alpha^4K_1^2-\beta^2K_2^2-4\alpha\beta K_1^2K_2)\cos^2\frac{\phi\pi}{2}+\beta K_1(-8K_1K_2\\
&+2\beta^2-4\alpha\beta K_1+4\alpha K_1^2K_2+2\alpha^2K_1^2)\cos\phi\pi,\\
l_7 = {} & \alpha K_1(2\alpha\beta K_1^2-2\alpha^2\beta K_1+8\beta K_1K_2^2+4\alpha^2K_1^2K_2\\
&-2\beta^2K_2+2\alpha^3K_1^2+2\alpha\beta K_1K_2+8\beta K_1^2K_2-2\beta^2K_1)\\
&\cdot\cos\frac{\phi\pi}{2},\\
l_8 = {} & \alpha^2\beta K_1^2(4K_1K_2-\beta+2\alpha K_1).
\end{aligned}
$$

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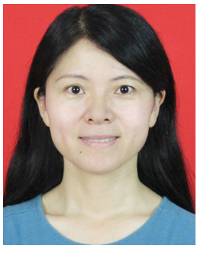

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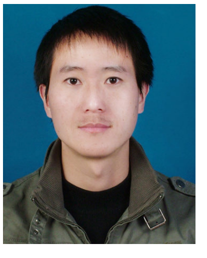

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