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Asymmetrical Barrier Function-Based Adaptive Vibration Control for Nonlinear Flexible Cantilever Beam With Obstacle Restriction

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ABSTRACT In this paper, a novel adaptive asymmetrical barrier function-based vibration control law is proposed for the nonlinear flexible cantilever beam system with obstacle restriction, model uncertainties, and distributed disturbances. Firstly, by employing the Hamilton's principle and the Galerkin projection method, the dynamic of the nonlinear flexible cantilever beam with the piezoelectric actuator is constructed in partial differential equations and simplified to nonlinear ordinary differential equations for the control law design. Then, by introducing the fast nonsingular terminal sliding mode surface, a novel asymmetrical barrier function based sliding mode control law is proposed, in which by means of a novel asymmetrical barrier Lyapunov function is used to guarantee the finite time stability with the obstacle restriction. Further, to deal with the model uncertainties, an adaptive updating law is incorporated with the fast nonsingular terminal sliding mode control law based on asymmetrical barrier function, stability proof shows that the proposed control law can ensure the distributed disturbance rejection and the model uncertainties compensation simultaneously. Finally, the effectiveness of the proposed control laws is demonstrated by the numerical simulations.

INDEX TERMS Vibration control, asymmetrical barrier Lyapunov function, fast nonsingular terminal sliding mode surface, adaptive control, obstacle restriction.

I. INTRODUCTION

With the development of precision science, recent years have witnessed significant requirements on the researches of spatial structure, which can be applied in solar panels, robotic arms, and optical motion detection systems. Whereas in practice, as an important focus for flexible large lightweight space structure, the resulting vibration phenomenon needs to be overcome. In view of this, a tremendous amount of interest has been generated regarding the researches of vibration suppression that can improve system performance [1]–[3].

In [4], backstepping-boundary iterative learning control was presented to tackle the vibration problem for an Euler–Bernoulli beam system with boundary disturbance. By combining with a backstepping technique, a robust sliding mode boundary control method was proposed to suppress vibration for a pinned–pinned Euler– Bernoulli beam in [5]. Distinguished from the lumped parameter systems, the flexible space structure vibration system is a nonlinear distributed parameter system, which is consists of the coupled partial differential equations (PDEs) and ordinary differential equations (ODEs). Owing to the characteristics of infinite dimension system, it is a challenge to design the control law [6]–[8]. In the past few years, many methods and theories have been proposed for distributed parameter system, including the boundary control method with which is constructed based on the original partial differential equations [9]–[11]. In [12], both the Lyapunov redesign and active disturbance rejection control approach were employed for boundary control of a flexible rectangular plate when existing exogenous disturbances. In [13], robust boundary control was proposed to stabilize the beam and reject the vibration. The disturbance observer was put forward for attenuating the effect of the external disturbances. In [14], a flexible hose system with varying length and input constraint based on original PDEs was considered, backstepping method based boundary control scheme was proposed to regulate the hose's vibration and handle the effect of the input constraint.

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However, it can be seen that the dynamic model is complicated when the nonlinearity caused by the mid-plane stretching is considered [15]–[17]. Hence, the characteristics of the actuator [18]–[21] play an important role in practical applications. In response to this challenge, by using the Galerkin projection method [22], [23], the original infinite-dimensional PDE model can be simplified to finite-dimensional ODE [24]–[27]. In [28]–[30], for the case of considering the nonlinear deformation and the actuator simultaneously, the partial differential equations of micro-scale beam system were constructed according to Hamilton's theorem, in which the PDEs were transformed into ordinary differential equations by the Galerkin projection method, and the mathematical model was simplified effectively. While the model simplification may make correspondingly system uncertainties, robust control and adaptive sliding mode control are widely used due to their robustness and disturbance rejection [31]–[36]. In [37], mechanical servo systems with mismatched time-variant uncertainties are controlled by recursive robust integral of the sign of the error control law, which achieve asymptotic tracking performance. In [38], in order to deal with mismatched disturbance and time-invariant modeling uncertainties, an extended state observer was constructed, whose unknown parameters estimates were updated by a novel adaptive law. Further, in [31], a flexible beam with piezoelectric sensors and actuators was considered. Besides, a novel robust sliding mode control was presented to reject the unexpected vibration. The command input shaping based sliding mode output feedback control was introduced to deal with the mismatched uncertainties in the flexible systems in [32]. However, the improved sliding mode control law still exists singularity problems that will cause the chattering phenomenon. In [33], to avoid singularity achieve good control performance for nonlinear systems, the nonsingular fast terminal sliding mode control (NFTSMC) was investigated.

The deformation constraints should also be considered due to the complicated environment, such as the possible obstacles in practical application. Hence, it is necessary to drive attention to solve the control problem of constraint systems [39], [40]. In [41], a barrier Lyapunov function based control method was employed to prevent constraint violation for single-input single-output strict feedback nonlinear system. In [42], the novel symmetric and asymmetric barrier Lyapunov functions based gain technique was proposed to guarantee that all the states do not violate their constraints and the tracking error of the systems can be controlled to a small neighborhood around zero.

Motivated by the aforementioned literature, this paper is concerned to investigate vibration control problems of the nonlinear flexible cantilever beam system under obstacle restriction with model uncertainties and distributed disturbances. We consider a flexible cantilever beam system consisting of nonlinear mid-plane stretching, obstacle restriction, and time-varying disturbance scattered throughout the beam. All the terms mentioned above complicate the system model. Compared with the existing researches, the main contributions are summarized as follows.

(1) It proposes a novel asymmetrical barrier function-based sliding mode control law for vibration suppression of the nonlinear flexible cantilever beam system, where guarantees the convergence speed while constraint the partial maximum displacement of the flexible beam.
(2) It considers system uncertainty brought by the process of system model simplification and approximation of partial differential equations. In terms of solving this problem, an adaptive updating law is presented, by which the model uncertainty and distributed disturbance can be estimated and compensated.
(3) It develops a novel piecewise Lyapunov function which combines the fast nonsingular terminal sliding mode surface with asymmetrical barrier function, ensuring the closed-loop system with obstacle restriction finite-time stabilization.

The remainder of the paper is arranged as follows, Section II introduces the PDEs dynamic model of the nonlinear flexible cantilever beam system, followed by the proposed control laws and stability analysis in Section III. Numerical simulations and comparative analysis results are provided for the proposed control law in Section IV. Some conclusions are given for this paper in Section V.

Notations. The following notation given in this paper is the unified standard form. $(\cdot)' = \frac{\partial(\cdot)}{\partial x}$, $(\cdot)'' = \frac{\partial^2(\cdot)}{\partial x^2}$, $(\cdot)'''' = \frac{\partial^4(\cdot)}{\partial x^4}$, $\dot{(\cdot)} = \frac{\partial(\cdot)}{\partial t}$. And δ denotes the variational operator.

II. PROBLEM FORMULATIONS AND PRELIMINARIES

The dynamic analysis for the nonlinear flexible cantilever beam system is given in this section. And some assumptions are assumed afterward. Some necessary lemmas are presented for further derivation.

The vibration suppression system which is shown in Figure 1 consists of a cantilever beam of length L with a payload at the end and a piezoelectric actuator. The distributed disturbance $d(x, t)$ along the beam is considered time-varying and unknown. Consider that some particular circumstances cannot be installed at the end of the beam, the piezoelectric actuator is placed at the upper surface of the flexible cantilever beam. Distance from the left edge of the actuator to the left boundary of the beam is $x = l_1$, and the right edge of the actuator to the right boundary of the beam is $x = l_2$.

In the paper, the flexible cantilever beam is assumed to be deformed only in the z-axis. The strain energy $W_s(t)$ of the nonlinear cantilever beam with mid-plane stretching is expressed by,

$$W_s(t) = \frac{1}{2}\int_0^L \Big(\sigma^0 + \frac{EA}{2L}\int_0^L \Big(\frac{\partial \omega(x,t)}{\partial x}\Big)^2 \mathrm{d}x\Big)\omega'(x,t)^2 \mathrm{d}x + \frac{1}{2}EI\int_0^L \Big(\frac{\partial^2 \omega(x,t)}{\partial x^2}\Big)^2 \mathrm{d}x \quad (1)$$

where E is Young's modulus. I and A are area moment of inertia and cross-section area of the flexible beam,

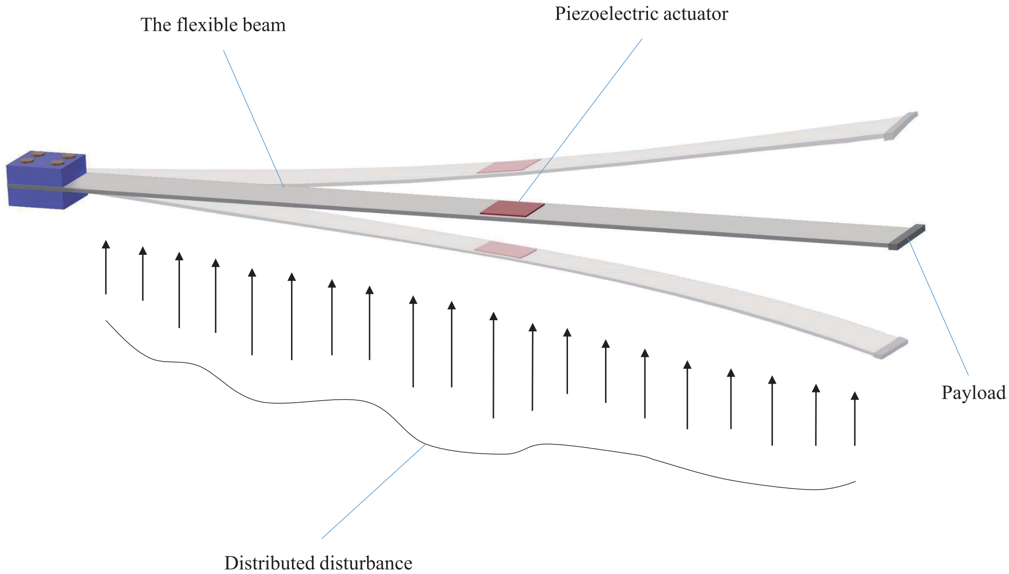

FIGURE 1. A schematic of the flexible cantilever beam with piezoelectric actuator.

respectively. σ^0 represents the initial axial force. $\omega(x, t)$ is the lateral deformation in the z-axis, where $x \in [0, L]$. The kinetic energy $W_k(t)$ is given by,

$$W_k(t) = \frac{1}{2}\rho A \int_0^L \left(\frac{\partial \omega(x,t)}{\partial t}\right)^2 \mathrm{d}x + \frac{1}{2}m\left(\frac{\partial \omega(L,t)}{\partial t}\right)^2 \mathrm{d}x \tag{2}$$

where ρ denotes the density of the flexible beam, m is the mass of the payload. The piezoelectric actuator is supposed to be thin enough that the nonlinear effects of the piezoelectric layer can be negligible. Notice that to obtain the dynamic model of motion, it can be considered that the dimensions of the piezoelectric actuator are negligible. Neglecting the longitude elongation of the beam, the variation of electrical energy $\delta W_a(t)$ of the piezoelectric actuator layer can be characterized as follows ([28]),

$$\begin{aligned}\delta W_a(t) = \int_0^L \Big(& Y_a e_{31} A_a z_m^a \frac{K(t)}{h_p}\frac{\partial \delta(x-l_2)}{\partial x} \\ & - Y_p e_{31} A_P z \frac{K(t)}{h_p}\frac{\partial \delta(x-l_1)}{\partial x}\Big)\delta\omega(x,t)\mathrm{d}x\mathrm{d}t\end{aligned} \tag{3}$$

where Y_a is the Young modulus of the piezoelectric actuator, e_{31} is the piezoelectric constant, z_m^a is the distance between the neutral axis of the beam and the middle line of the piezoelectric actuator. A_p and $K(t)$ are the cross-section area of the piezoelectric actuator and the control voltage, respectively. $\delta(x)$ is the Dirac delta function. z represents the distance from the neutral axis, h_p denotes the height of the actuator.

The virtual work $\delta W_d(t)$ done by the distributed disturbance $d(x, t)$ can be expressed by,

$$\delta W_d(t) = \int_0^L d(x,t)\delta\omega(x,t)\mathrm{d}x \tag{4}$$

Through Hamilton's principle, it has $t \in [t_1,\ t_2]$ which can make the variation of system energy equal to zero. t_1 and t_2 are two-time instants, and $t_1 < t < t_2$ is the operating interval, further for the system the following equation holds,

$$\int_{t_1}^{t_2} \Big(\delta W_k(t) - \delta W_s(t) - \delta W_p(t) + \delta W_d(t)\Big)\mathrm{d}t = 0 \tag{5}$$

Apply variation to (1) and (2), respectively. Substituting $\delta W_s(t)$, $\delta W_k(t)$, $\delta W_a(t)$, and $\delta W_d(t)$ into (5) and assuming the initial axial force σ^0 to be zero. Hence, the following PDEs of nonlinear cantilever beam system can be represented as,

$$\begin{aligned}& EI\frac{\partial^4\omega(x,t)}{\partial x^4} - \left(\frac{EA}{2L}\int_0^L \left(\frac{\partial\omega(x,t)}{\partial x}\right)^2\mathrm{d}x\right)\frac{\partial^2\omega(x,t)}{\partial x^2} \\ &\quad + \rho A\frac{\partial^2\omega(x,t)}{\partial t^2} - \left(\frac{\partial\delta(x-l_2)}{\partial x} - \frac{\partial\delta(x-l_1)}{\partial x}\right)\gamma K(t) \\ &\quad - d(x,t) = 0\end{aligned} \tag{6}$$

where

$$\gamma = -\frac{Y_a e_{31} A_a z}{h_a} \tag{7}$$

Accordingly, the boundary conditions can be concluded as the following,

$$\begin{cases} EI\frac{\partial^3\omega(L,t)}{\partial x^3} - \frac{EA}{2L}\frac{\partial^2\omega(L,t)}{\partial x^2}\int_0^L\left(\frac{\partial\omega(x,t)}{\partial x}\right)^2\mathrm{d}x \\ \qquad -m\frac{\partial^2\omega(L,t)}{\partial t^2} = 0 \\ \omega(x,t) = 0 \\ \frac{\partial\omega(x,t)}{\partial x} = 0 \\ \frac{\partial^2\omega(L,t)}{\partial x^2} = 0 \end{cases} \tag{8}$$

where $t \in [0, \infty)$. The Galerkin projection method is introduced to simplify the above PDEs (6) to the ODEs, which is

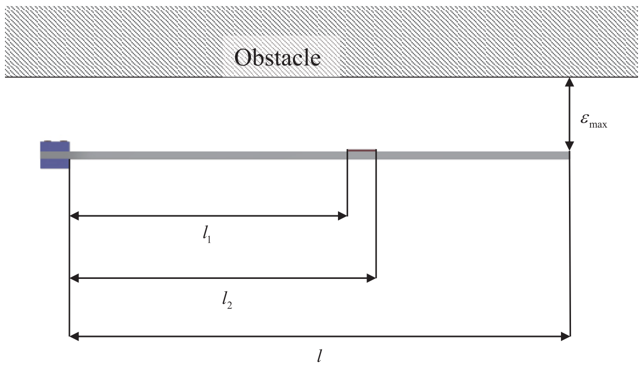

FIGURE 2. A schematic of the flexible cantilever beam with obstacle restriction.

given as below,

$$\omega(x,t)=\sum_{i=1}^{\infty}\upsilon_i(t)\phi_i(x)\approx\upsilon_1(t)\phi_1(x) \tag{9}$$

where $\upsilon_i(t)$ is the temporal part of the i-th mode of the solution, $\phi_i(x)$ is the i-th shape of the beam which satisfy the boundary conditions. Based on the most dominant first mode shape, and define the sate variables $z_1(t)=\upsilon_1(t)$, $z_2(t)=\dot{\upsilon}_1(t)$, respectively. The nonlinear differential equation of the flexible beam system with piezoelectric actuator can be simplified as [28],

$$\dot{z}_1(t)=z_2(t) \tag{10}$$
$$G(\phi_1)\dot{z}_2(t)=F(z_1(t),\phi_1)+u(t)+H(\phi_d)\tilde{d}(t) \tag{11}$$

where

$$G(\phi_1)=\frac{\rho A}{\gamma\left(\phi_1'(x)(l_1)-\phi_1'(l_2)\right)}\int_0^L\left(\phi_1(x)\right)^2\mathrm{d}x \tag{12}$$

$$F(z_1(t),\phi_1)=-EIz_1(t)\int_0^L\phi_1(x)\phi_1''''(x)\mathrm{d}x+\frac{EA}{2L}z_1^3(t)\int_0^L\phi_1(x)\left(\phi_1'(x)\right)^2\phi_1''(x)\mathrm{d}x \tag{13}$$

$$H(\phi_d)=\int_0^L\phi_1(x)\phi_d(x)\mathrm{d}x \tag{14}$$

In practical applications, due to the complexity of the test environments, it cannot precisely achieve the exact vibration shape $\phi_1(x)$, there still exist other vibration shapes which may lead to undesirable redundant vibration and direct effect the flexible beam system control performance. To explicitly consider the system modeling accuracy, the nonlinear matrices $G(\phi_1)$ and $F(z_1(t),\phi_1)$ are denoted as the following, respectively.

$$G(\phi_1)=G_0(\phi_1)+\widetilde{G}(\phi_1) \tag{15}$$
$$F(z_1(t),\phi_1)=F_0(z_1(t),\phi_1)+\widetilde{F}(z_1(t),\phi_1) \tag{16}$$

where $G_0(\phi_1)$ and $F_0(z_1(t),\phi_1)$ are known positive definite matrix, $\widetilde{G}(\phi_1)$ and $\widetilde{F}(z_1(t),\phi_1)$ represent the unknown nonlinear uncertainty parts. Then, rewrite (11) as follows,

$$G_0(\phi_1)\dot{z}_2(t)=F_0(z_1(t),\phi_1)+u(t)+\tau_d+H(\phi_d)\tilde{d}(t) \tag{17}$$

with

$$\tau_d=-\widetilde{G}(\phi_1)\dot{z}_2+\widetilde{F}(z_1(t),\phi_1) \tag{18}$$

where τ_d denotes the lumped system uncertainty defining with the system uncertainties and the unknown disturbances.

To facilitate the analysis of the following sections, several assumed conditions are further considered as below,

Assumption 1: The time-varying distributed disturbance $d(x,t)$ and its decomposition $H(\phi_d)\tilde{d}(t)$ are assumed to be bounded such that $|d(x,t)|\le d_{m1}$ and $|H(\phi_d)\tilde{d}(t)|\le d_{m2}$ with positive constants $d_{m1}\in \mathrm{R}^+$, $d_{m2}\in \mathrm{R}^+$. This is a rational assumption, for the reason that the distributed disturbance $d(x,t)$ has finite energy and therefore bounded.

Assumption 2: The initial condition of the flexible cantilever beam satisfies the prescribed constraint which is rational. That is the displacement $z_1(0)$ satisfies $|z_1(0)|<\varepsilon$, where ε denotes the positive constant.

Assumption 3 [43]: In the paper, we consider the weak initial condition with $\omega(x,0)$ and distributed external disturbances simultaneously. Besides, the piezoelectric actuator for the system has no serious saturation problem. Based on the above discussions, one can conclude that the lumped system uncertainty is bounded,

$$|\tau_d|<\xi_0+\xi_1|z_1(t)|+\xi_2|\dot{z}_1(t)|^2 \tag{19}$$

where ξ_0, ξ_1, and ξ_2 denote the unknown positive constants.

Lemma 1 [44]: Assume that there exist scalars $a>0$, $0<l<1$, and $\psi>0$, for the nonlinear system $\dot{x}=f(x,u)$ of the controlled systems, if the smooth positive definite function $V(x)$ satisfies the following inequality,

$$\dot{V}(x)\le -aV^l(x)+\psi,\quad t\ge 0 \tag{20}$$

then the system $\dot{x}=f(x,u)$ is semi-global practical finite-time stable.

Lemma 2 [45]: For the nonlinear system $\dot{x}=f(x,u)$, $x=0$ is the equilibrium value of the system. If there exists $\epsilon>0$ and the settling time $T(\epsilon,x_0)<\infty$ to satisfy $|x(t)|<\epsilon$, the nonlinear system is semi-global practical finite time stable for all $t\ge t_0+T$.

III. MAIN RESULTS

A fast nonsingular terminal sliding mode surface is first employed in this section, which can both accelerate the convergence rate and avoid the singular phenomenon. Second, we propose the asymmetrical barrier function based sliding mode control law for the flexible beam system. Third, based on the obtained results, the asymmetrical barrier function based adaptive fast nonsingular terminal sliding mode control law is proposed via an adaptive updating law, estimating the unknown nonlinear uncertainty parts in (15)-(16).

A. THE FAST NONSINGULAR TERMINAL SLIDING MODE CONTROL LAW BASED ON ASYMMETRICAL BARRIER FUNCTION

For the later design of the asymmetrical barrier function based sliding mode control law for flexible beam system (10)-(11), the following sliding mode surface, namely the fast nonsingular terminal sliding mode is presented as,

$$s = z_1(t) + \kappa_1 |z_1(t)|^{\alpha+1} + \kappa_2 z_2^{m/n}(t) \tag{21}$$

where κ_1, α, and κ_2 are all the designed positive constants; m and n are the positive odd integers satisfying $1 < m/n < 2$ and $\alpha + 1 > m/n$, respectively.

If the state variables reach the sliding mode surface $s = 0$, the following equation holds,

$$\begin{aligned}\dot{z}_1(t) &= -\left(\frac{1}{\kappa_2}\right)^{n/m}\left(z_1(t) + \kappa_1|z_1(t)|^{\alpha+1}\right)^{n/m} \\ &= -\left(\frac{1}{\kappa_2}\right)^{n/m}\left(z_1(t) + \kappa_1 z_1^{\alpha+1}(t)\text{sign}\big(z_1(t)\big)^{\alpha+1}\right)^{n/m} \\ &= -z_1^{n/m}(t)\left[\frac{1}{\kappa_2}\left(1 + \kappa_1 z_1^{\alpha}(t)\text{sign}\big(z_1(t)\big)^{\alpha+1}\right)\right]^{n/m} \end{aligned} \tag{22}$$

To analyze the convergence time of the proposed sliding mode surface, the time variable t_c is assumed to be the time from the initial state variable $z_1(0) \neq 0$ to $z_1(t_c) = 0$, then integrating the time along both sides of (22),

$$\begin{aligned}&\int_{z_1(0)}^{z_1(t_c)} \frac{1}{z_1^{n/m}(t)} \mathrm{d}z_1(t) \\ &= -\int_0^{t_c}\left[\frac{1}{\kappa_2}\left(1 + \kappa_1 z_1^{\alpha}(t)\text{sign}\big(z_1(t)\big)^{\alpha+1}\right)\right]^{n/m}\mathrm{d}\tau \\ &\leq -\int_0^{t_c}\left(\frac{1}{\kappa_2}\right)^{n/m}\mathrm{d}\tau \end{aligned} \tag{23}$$

Therefore, it can be concluded from (23) that the convergence time t_c is given as follows,

$$t_c \leq \frac{m}{(1/\kappa_2)^{n/m}(m-n)} z_1^{(1-n/m)}(0) \tag{24}$$

Remark 1: As can be seen from the proposed sliding mode surface (21), when the state variables are far from the equilibrium point, the term $z_1(t)$ in sliding mode surface s mainly affect the convergence performance, which can make the system trajectory converge at a fast rate. On the contrary, if the state variables are closed to the equilibrium point, the term $\kappa_1|z_1(t)|^{\alpha+1}$ plays a major role which can also make fast convergence of the state trajectory. Hence, the proposed sliding mode surface (21) can not only satisfy the fast trajectory convergence in the region adjacent to the equilibrium point but also meet the global state variables in the system.

Theorem 1: Consider the flexible beam system (10)-(11) *under Assumption 1 with the proposed fast nonsingular terminal sliding mode surface* (21)*, then by selecting appropriate control parameters, the asymmetrical barrier function based sliding mode control law is designed as,*

$$u(t) = u_1(t) + u_2(t) + u_s(t) \tag{25}$$

with

$$\begin{aligned}u_1(t) = &-F(z_1(t), \phi_1) \\ &-\frac{n}{\kappa_2 m}G(\phi_1)|z_2(t)|^{2-m/n}sign\big(z_2(t)\big) \\ &-\frac{(\alpha+1)\kappa_1}{m\kappa_2}G(\phi_1)|z_1(t)|^{\alpha}|z_2(t)|^{2-m/n}sign\big(z_2(t)\big)\end{aligned} \tag{26}$$

$$\begin{aligned}u_2(t) = &-\frac{n}{\kappa_2 m}G(\phi_1)|z_2(t)|^{1-m/n} \cdot \\ &\left[\frac{M(i)sz_1(t)z_2(t) + c_1 M(i)\big(\varepsilon^2 - z_1^2(t)\big)^{\frac{3}{2}}}{M(i)\big(\varepsilon^2 - z_1^2(t)\big) + \big(1-M(i)\big)\big(\varepsilon^2 - z_1^2(t)\big)^2}\right. \\ &\left.+\frac{c_2\big(1-M(i)\big)\big(\varepsilon^2 - z_1^2(t)\big)^2}{M(i)\big(\varepsilon^2 - z_1^2(t)\big) + \big(1-M(i)\big)\big(\varepsilon^2 - z_1^2(t)\big)^2}\right]\end{aligned} \tag{27}$$

$$\begin{aligned}u_3(t) = &-\frac{n}{\kappa_2 m}G(\phi_1)|z_2(t)|^{1-m/n}\Big(\frac{\kappa_3}{\Phi(s)}sign(s) \\ &+\kappa_4|s|^{\theta+1}sign(s)\Big)\end{aligned} \tag{28}$$

where $M(i)$ is the switching term that can be described as,

$$M(i) = \begin{cases} 1 \ , z_1(t) > 0 \\ 0 \ , z_1(t) \leq 0 \end{cases} \tag{29}$$

$\Phi(s) = \vartheta_0 + (1-\vartheta_0)e^{-a|s|^q}$*, in which ϑ_0 denotes the positive offset which is satisfied $\vartheta_0 < 1$, a and q denote positive constant and positive integer, respectively. Then, the proposed control law can guarantee that $z_1(t)$ and $z_2(t)$ remain uniformly in finite time with semi-global practical finite-time stable. Further, the displacement $z_1(t)$ can converge to the origin without violating the output constraint ε.*

Remark 2: In the paper, the deformation $\omega(x,t)$ is not allowed reach its upper limit ε_{max} which is a positive constant. From (9), define the constraint $\varepsilon \in \mathrm{R}^+$ of the displacement $z_1(t)$ at location $x = l_0$ with the following form,

$$\varepsilon \leq \varepsilon_{max} \cdot \frac{\phi_1(l_0)}{\phi_1(L)} \tag{30}$$

Proof: From (21), one can conclude that the derivative of the proposed sliding mode surface s is given as follows,

$$\begin{aligned}\dot{s} &= \dot{z}_1(t) + \kappa_1(\alpha+1)|z_1(t)|^{\alpha}z_2(t) + \kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}\dot{z}_2(t) \\ &= \dot{z}_1(t) + \kappa_1(\alpha+1)|z_1(t)|^{\alpha}z_2(t) + \kappa_2\frac{m}{n}|z_2(t)|^{m/n-1} \cdot \\ &\quad \times G^{-1}(\phi_1)\big(F(z_1(t),\phi_1) + u(t) + H(\phi_d)\tilde{d}(t)\big)\end{aligned} \tag{31}$$

Choose the asymmetrical barrier based Lyapunov function as

$$V_1(t) = \frac{1}{2}M(i)\frac{s^2}{\varepsilon^2 - z_1^2(t)} + \frac{1}{2}\big(1-M(i)\big)s^2 \tag{32}$$

Then taking the derivative of (32), then it follows that

$$\begin{aligned}\dot{V}_1(t) = &M(i)\frac{s^2 z_1(t)\dot{z}_1(t)}{\big(\varepsilon^2 - z_1^2(t)\big)^2} + M(i)\frac{s\dot{s}}{\varepsilon^2 - z_1^2(t)} \\ &+ \big(1-M(i)\big)s\dot{s}\end{aligned} \tag{33}$$

Substituting the proposed control law (25)-(29) and (31) into the derivative of the Lyapunov function (32), respectively, that leads to,

$$\begin{aligned}\dot{V}_1(t) &= M(i)\frac{s^2 z_1(t)\dot{z}_1(t)}{\big(\varepsilon^2 - z_1^2(t)\big)^2} + \Big(M(i)\frac{s}{\varepsilon^2 - z_1^2(t)} \\ &\quad + \big(1 - M(i)\big)s\Big)\Big[\dot{z}_1(t) + \kappa_1(\alpha+1)|z_1(t)|^{\alpha}\dot{z}_1(t) \\ &\quad + \kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}G^{-1}(\phi_1)\big(F(z_1(t),\phi_1) \\ &\quad + u(t) + \tau_d\big)\Big] \qquad (34)\end{aligned}$$

In what follows, to simplify the writing, define the function Γ as follows,

$$\Gamma = M(i)\frac{1}{\varepsilon^2 - z_1^2(t)} + \big(1 - M(i)\big) \qquad (35)$$

One can further infer that there exist a positive constant τ_{dm} satisfying that $|\tau| \leq \tau_{dm}$. Then (34) can be derived that

$$\begin{aligned}\dot{V}_1(t) &\leq -M(i)\frac{c_1 s}{\big(\varepsilon^2 - z_1^2(t)\big)^{1/2}} - c_2\big(1 - M(i)\big)s \\ &\quad + \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|G(\phi_1)|^{-1}\Big(\tau_{dm}s \\ &\quad - \frac{\kappa_3}{\Phi(s)}|s| - \kappa_4|s|^{\theta+1}\Big) \qquad (36)\end{aligned}$$

It implies that there exist a positive constant Q, together with (36), one has that,

$$\tau_{dm}s - \frac{\kappa_3}{\Phi(s)}|s| - \kappa_4|s|^{\theta+1} \leq Q \qquad (37)$$

Afterwards, by defining $C_{min} = \min\big(\sqrt{2}c_1, \sqrt{2}c_2\big)$ and $Q_m = \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|G(\phi_1)|^{-1}Q$, respectively, there holds,

$$\dot{V}_1(t) \leq -C_{min}V_1^{1/2}(t) + Q_m \qquad (38)$$

The convergent time T^* obtains as follows,

$$T^* = \frac{2}{\mu C_{min}}\Big[V_1^{1/2}(0) - \frac{Q_m}{(1-\mu)C_{min}}\Big] \qquad (39)$$

with $V_1(0)$ denotes the initial value of $V_1(t)$. On the other hand, based on Lemma 1, for $\forall t \geq T^\star$, one has $V_1^{1/2}(t) \leq (Q_m/[(1-\mu)C_{min}])$, then it can further obtain that the whole closed-loop system is semi-global practical finite-time stable. Considering that for the case of $M(i) = 1$, it yields that

$$|z_1(t)| \leq \varepsilon\Big[1 - 2\Big(\frac{Q_m}{(1-\mu)C_{min}}\Big)^2\Big]^{\frac{1}{2}} \qquad (40)$$

Consequently, it can be derived that $|z_1(t)| < \varepsilon$ after finite time $T^\star$, and further the displacement of the flexible beam at location $x = l_0$ can be maintained in small neighborhoods near zero. Thus, the constraint of the beam deformation are not violated.

Remark 3. In view of the proposed control law (25)-(29), one can conclude, the equivalent control laws $u_1(t)$ and $u_2(t)$ achieve desired trajectory tracking alone without external disturbances and parameter uncertainties. Moreover, when the state variables are far from the proposed sliding mode surface (21), the reaching control law $u_s(t)$ can accelerate the convergence rate and enhance the robustness property of the flexible beam system.

Remark 4. For the proposed control law $u_2(t)$, when the displacement $z_1(t)$ is far from the equilibrium point and $z_1(t) > 0$, the gain of barrier function affects the strength of the barrier function. However, when the displacement $z_1(t)$ is far from the equilibrium and $z_1(t) < 0$, the gain of sliding mode surface c_2 affects the convergence efficiency. In order to guarantee the vibration displacement of the flexible beam converge to zero in finite time and minimize the chattering, the choosing of the parameters c_1 and c_2 should be well-balanced.

Remark 5. Notice that the term $\Phi(s)$ in the exponential reaching law (28) is a positive one, the stability of the system cannot be affected. It should be emphasized that $\Phi(s)$ will approach to ϑ_0 if $|s|$ increases, and $\kappa_3/\Phi(s)$ is approximately equal to κ_3/ϑ_0. Therefore, $\kappa_3/\Phi(s)$ should be increased in the reaching phase which can accelerate the convergence rate of the sliding mode surface. On the contrary, as $|s|$ decreases, the term $\Phi(s)$ will approach one, which also make $\kappa_3/\Phi(s)$ convergence to κ_3. It is indicated that the speed of the term $\kappa_3/\Phi(s)$ is reduced which can alleviate the chattering phenomenon.

B. THE ADAPTIVE FAST NONSINGULAR TERMINAL SLIDING MODE CONTROL LAW BASED ON ASYMMETRICAL BARRIER FUNCTION

To estimate and compensate for the distributed external disturbances and the unknown uncertainties of the lumped system, an adaptive updating law is introduced in this section. In what follows, the stabilization analysis is given in the following theorem.

Theorem 2: Consider the flexible beam system (10)-(11) *under Assumptions 1 and 3 with the proposed fast nonsingular terminal sliding mode surface* (21)*, design the control law as*

$$u(t) = u_1(t) + u_2(t) + u_3(t) + u_s(t) \qquad (41)$$

where

$$\begin{aligned}u_1(t) &= -F_0(z_1(t),\phi_1) \\ &\quad - \frac{n}{\kappa_2 m}G_0(\phi_1)|z_2(t)|^{2-m/n}sign\big(z_2(t)\big) \\ &\quad - \frac{(\alpha+1)\kappa_1}{m\kappa_2}G_0(\phi_1)|z_1(t)|^{\alpha}|z_2(t)|^{2-m/n}sign\big(z_2(t)\big)\end{aligned} \qquad (42)$$

$$\begin{aligned}u_2(t) &= -\frac{n}{\kappa_2 m}G_0(\phi_1)|z_2(t)|^{1-m/n}\cdot \\ &\quad \left[\frac{M(i)sz_1(t)z_2(t) + c_a M(i)\big(\varepsilon^2 - z_1^2(t)\big)^{\frac{3}{2}}}{M(i)\big(\varepsilon^2 - z_1^2(t)\big) + \big(1-M(i)\big)\big(\varepsilon^2 - z_1^2(t)\big)^2}\right]\end{aligned} \qquad (43)$$

$$u_3(t) = \begin{cases}\frac{G_0(\phi_1)}{s}\Xi\ , |s| \neq 0 \\ 0\ , |s| = 0\end{cases} \qquad (44)$$

$$u_s(t) = -\frac{n}{m\kappa_2} G_0(\phi_1)|z_2(t)|^{1-m/n} \cdot \Big(\frac{\kappa_3}{\Phi(s)} sign(s) + \kappa_4 |s|^{\theta+1} sign(s)\Big) \tag{45}$$

with

$$\Xi = |s||G_0^{-1}(\phi_1)|\big(\hat{\xi}_0 + \hat{\xi}_1|z_1(t)| + \hat{\xi}_2|\dot{z}_1(t)|^2\big) \tag{46}$$

where $\hat{\xi}_j$ $(j = 0, 1, 2)$ *denote the estimation values of* ξ_j*. Further, with the following adaptive updating law as,*

$$\dot{\hat{\xi}}_0 = \lambda_0 \Gamma \kappa_2 \frac{m}{n} |z_2(t)|^{m/n-1} |s| G_0^{-1}(\phi_1) \tag{47}$$

$$\dot{\hat{\xi}}_1 = \lambda_1 \Gamma \kappa_2 \frac{m}{n} |z_2(t)|^{m/n-1} |s| G_0^{-1}(\phi_1)|z_1(t)| \tag{48}$$

$$\dot{\hat{\xi}}_2 = \lambda_2 \Gamma \kappa_2 \frac{m}{n} |z_2(t)|^{m/n-1} |s| G_0^{-1}(\phi_1)|\dot{z}_1(t)|^2 \tag{49}$$

Then the state variables $z_1(t)$ *and* $z_2(t)$ *of the flexible system can remain uniformly in finite time with semi-global practical finite-time stability, with which the displacement* $z_1(t)$ *is ultimately bounded and can converge to a small neighborhood of the origin.*

Proof: Similar to the above proof steps in Theorem 1, the following asymmetrical barrier based Lyapunov function is constructed. Notion that the following equation is given as: $\tilde{\xi}_j = \hat{\xi}_j - \breve{\xi}_j, j = 0, 1, 2,$

$$V_2(t) = \frac{1}{2} M(i) \frac{s^2}{\varepsilon^2 - z_1^2(t)} + \frac{1}{2}\big(1 - M(i)\big)s^2 + \frac{1}{2}\vartheta_0^{-1}\tilde{\xi}_0^2 + \frac{1}{2}\vartheta_1^{-1}\tilde{\xi}_1^2 + \frac{1}{2}\vartheta_2^{-1}\tilde{\xi}_2^2 \tag{50}$$

Then, taking the derivative of (50) with respect to time, substituting the proposed control law (41)-(46), (50) can be further derived as,

$$\begin{aligned}\dot{V}_2(t) &= \dot{V}_1(t) + \frac{1}{2}\vartheta_0^{-1}\tilde{\xi}_0^2 + \frac{1}{2}\vartheta_1^{-1}\tilde{\xi}_1^2 + \frac{1}{2}\vartheta_2^{-1}\tilde{\xi}_2^2 \\ &= -M(i)\frac{c_a s}{\big(\varepsilon^2 - z_1^2(t)\big)^{\frac{1}{2}}} + \Gamma\kappa_2 \frac{m}{n}|z_2(t)|^{m/n-1} s \cdot \\ &\quad \times G_0^{-1}(\phi_1)\Big(u_3(t) + \tau_d + H(\phi_d)\tilde{d}(t) - \frac{\kappa_3}{\Phi(s)}\text{sign}(s) \\ &\quad - \kappa_4|s|^{\theta}\text{sign}(s)\Big) + \frac{1}{2}\vartheta_0^{-1}\tilde{\xi}_0^2 + \frac{1}{2}\vartheta_1^{-1}\tilde{\xi}_1^2 + \frac{1}{2}\vartheta_2^{-1}\tilde{\xi}_2^2\end{aligned} \tag{51}$$

Therefore, based on (35) and Assumption 3, it can be inferred that $\Gamma > 0$. Further, rewrite (51) as follows,

$$\begin{aligned}\dot{V}_2(t) \le& -M(i)\frac{c_a s}{\big(\varepsilon^2 - z_1^2(t)\big)^{\frac{1}{2}}} + \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|s| \cdot \\ &\times |G_0(\phi_1)|^{-1}\big(\hat{\xi}_0 + \hat{\xi}_1|z_1(t)| + \hat{\xi}_2|\dot{z}_1(t)|^2\big) \\ &+ \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}s|G_0(\phi_1)|^{-1}\Big(\tau_d + H(\phi_d)\tilde{d}(t) \\ &- \frac{\kappa_3}{\Phi(s)}\text{sign}(s) - \kappa_4|s|^{\theta}\text{sign}(s)\Big) + \frac{1}{2}\vartheta_0^{-1}\tilde{\xi}_0^2 \\ &+ \frac{1}{2}\vartheta_1^{-1}\tilde{\xi}_1^2 + \frac{1}{2}\vartheta_2^{-1}\tilde{\xi}_2^2\end{aligned} \tag{52}$$

Since $\dot{\tilde{\xi}}_j = \dot{\hat{\xi}}_j$ with $j = 0, 1, 2$, then substituting the adaptive updating law (47)-(49) into (52), we have,

$$\begin{aligned}\dot{V}_2(t) \le& -M(i)\frac{c_a s}{\big(\varepsilon^2 - z_1^2(t)\big)^{\frac{1}{2}}} \\ &+ \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|s||G_0(\phi_1)|^{-1} \cdot \\ &\big(\hat{\xi}_0 + \hat{\xi}_1|z_1(t)| + \hat{\xi}_2|\dot{z}_1(t)|^2\big) \\ &+ \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|s||G_0(\phi_1)|^{-1} \cdot \\ &\big(\xi_0 + \xi_1|z_1(t)| + \xi_2|\dot{z}_1(t)|^2\big) \\ &+ \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|G_0(\phi_1)|^{-1} \cdot \\ &\Big(H(\phi_d)\tilde{d}(t) - \frac{\kappa_3}{\Phi(s)}|s| - \kappa_4|s|^{\theta+1}\Big) \\ &+ \frac{1}{2}\vartheta_0^{-1}\tilde{\xi}_0^2 + \frac{1}{2}\vartheta_1^{-1}\tilde{\xi}_1^2 + \frac{1}{2}\vartheta_2^{-1}\tilde{\xi}_2^2\end{aligned} \tag{53}$$

By introducing the positive constants $\breve{\xi}_j$ with $j = 0, 1, 2$, then (53) further becomes,

$$\begin{aligned}\dot{V}_2(t) \le& -M(i)\frac{c_a s}{\big(\varepsilon^2 - z_1^2(t)\big)^{\frac{1}{2}}} \\ &+ \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|G_0(\phi_1)|^{-1}\Big(H(\phi_d)\tilde{d}(t) \\ &- \frac{\kappa_3}{\Phi(s)}|s| - \kappa_4|s|^{\theta+1}\Big) \\ &+ \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|s||G_0(\phi_1)|^{-1} \cdot \\ &\times \big(\hat{\xi}_0 + \hat{\xi}_1|z_1(t)| + \hat{\xi}_2|\dot{z}_1(t)|^2\big) \\ &+ \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|s||G_0(\phi_1)|^{-1} \cdot \\ &\times \big(\xi_0 + \xi_1|z_1(t)| + \xi_2|\dot{z}_1(t)|^2\big) \\ &+ \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|s||G_0(\phi_1)|^{-1} \cdot \\ &\big(\breve{\xi}_0 + \breve{\xi}_1|z_1(t)| + \breve{\xi}_1|\dot{z}_1(t)|^2\big) \\ &- \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|s||G_0(\phi_1)|^{-1} \cdot \\ &\big(\breve{\xi}_0 + \breve{\xi}_1|z_1(t)| + \breve{\xi}_1|\dot{z}_1(t)|^2\big) \\ &- \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|s||G_0(\phi_1)|^{-1}\Big[\lambda_0\vartheta_0^{-1}(\hat{\xi}_0 - \breve{\xi}_0) \\ &+ \lambda_1\vartheta_1^{-1}|z_1(t)|(\hat{\xi}_1 - \breve{\xi}_1) + \lambda_2\vartheta_2^{-1}|\dot{z}_1(t)|^2(\hat{\xi}_2 - \breve{\xi}_2)\Big] \\ =& -M(i)\frac{c_a s}{\big(\varepsilon^2 - z_1^2(t)\big)^{\frac{1}{2}}} + \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1} \cdot \\ &\times |G_0(\phi_1)|^{-1}\Big(H(\phi_d)\tilde{d}(t) - \frac{\kappa_3}{\Phi(s)}|s| - \kappa_4|s|^{\theta+1}\Big) \\ &- \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|s||G_0(\phi_1)|^{-1} \cdot \\ &\times \Big[(\breve{\xi}_0 - \xi_0) + (\breve{\xi}_1 - \xi_1)|z_1(t)| + (\breve{\xi}_2 - \xi_2)|\dot{z}_1(t)|^2\Big] \\ &- \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|s||G_0(\phi_1)|^{-1} \cdot \\ &\times (\lambda_0\vartheta_0^{-1} - 1)(\breve{\xi}_0 - \hat{\xi}_0) \\ &- \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|s||G_0(\phi_1)|^{-1} \cdot \\ &\times (\lambda_1\vartheta_1^{-1} - 1)|z_1|(\breve{\xi}_1 - \hat{\xi}_1) \\ &- \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|s||G_0(\phi_1)|^{-1} \cdot \\ &(\lambda_2\vartheta_2^{-1} - 1)|\dot{z}_1(t)|^2(\breve{\xi}_2 - \hat{\xi}_2)\end{aligned} \tag{54}$$

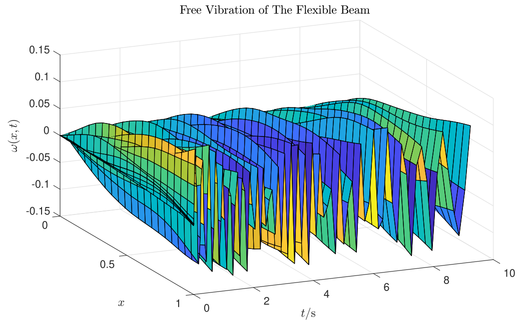

FIGURE 3. The free vibration of the flexible beam.

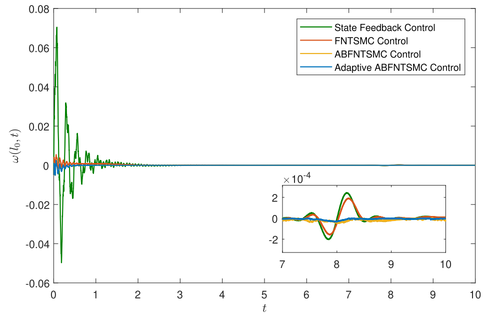

FIGURE 4. The displacement of the flexible beam at location $x = l_0$.

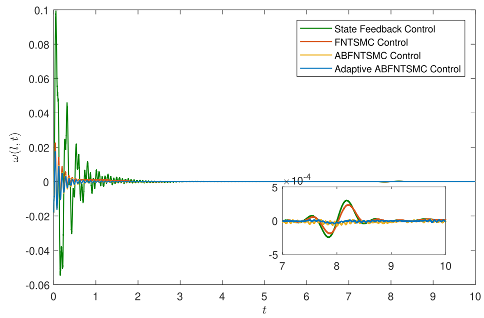

FIGURE 5. The displacement of the flexible beam at location $x = L$.

Define the state vectors φ_0, φ_1, φ_2, and φ_4 as follows.

$$\begin{aligned}\varphi_0 &= \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|G_0(\phi_1)|^{-1}\cdot\\&\quad\times\left[(\breve{\xi}_0-\xi_0)+(\breve{\xi}_1-\xi_1)|z_1(t)|+(\breve{\xi}_2-\xi_2)|\dot{z}_1(t)|^2\right]\\\varphi_1 &= \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|s||G_0(\phi_1)|^{-1}(\lambda_0\vartheta_0^{-1}-1)\\\varphi_2 &= \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|s||G_0(\phi_1)|^{-1}(\lambda_1\vartheta_1^{-1}-1)|z_1(t)|\\\varphi_3 &= \Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|s||G_0(\phi_1)|^{-1}(\lambda_2\vartheta_2^{-1}-1)|\dot{z}_1(t)|^2\end{aligned}\tag{55}$$

Finally, by combining with (54), the following can be concluded,

$$\begin{aligned}\dot{V}_2(t) &\le -M(i)\frac{c_a s}{\left(\varepsilon^2-z_1^2(t)\right)^{\frac{1}{2}}}+\Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}\cdot\\&\quad\times|G_0(\phi_1)|^{-1}\left(H(\phi_d)\tilde{d}(t)-\frac{\kappa_3}{\varPhi(s)}|s|-\kappa_4|s|^{\theta+1}\right)\\&\quad-\varphi_0|s|-\varphi_1(\breve{\xi}_0-\hat{\xi}_0)-\varphi_2(\breve{\xi}_1-\hat{\xi}_1)-\varphi_3(\breve{\xi}_2-\hat{\xi}_2)\end{aligned}\tag{56}$$

It can be observed from (56) that $\varphi_0 \ge 0$, hence, there exist a positive constant ϱ satisfying the following condition,

$$-\varphi_0|s| \le -\varrho\big(1-M(i)\big)|s| \tag{57}$$

Then, (56) can be further written as follows,

$$\begin{aligned}\dot{V}_2(t) &\le -M(i)\frac{c_a s}{\left(\varepsilon^2-z_1^2(t)\right)^{\frac{1}{2}}}-\varrho\big(1-M(i)\big)|s|\\&\quad+\Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|G_0(\phi_1)|^{-1}\Big(H(\phi_d)\tilde{d}(t)\\&\quad-\frac{\kappa_3}{\varPhi(s)}|s|-\kappa_4|s|^{\theta+1}\Big)-\varphi_1(\breve{\xi}_0-\hat{\xi}_0)\\&\quad-\varphi_2(\breve{\xi}_1-\hat{\xi}_1)-\varphi_3(\breve{\xi}_2-\hat{\xi}_2)\\&=-\sqrt{2}c_a\left(\frac{1}{2}M(i)\frac{s^2}{\varepsilon^2-z_1^2(t)}\right)^{\frac{1}{2}}\\&\quad-\sqrt{2}\varrho\left[\frac{1}{2}\big(1-M(i)\big)s^2\right]^{\frac{1}{2}}\\&\quad+\Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|G_0(\phi_1)|^{-1}\Big(H(\phi_d)\tilde{d}(t)\\&\quad-\frac{\kappa_3}{\varPhi(s)}|s|-\kappa_4|s|^{\theta+1}\Big)\\&\quad-\varphi_1\sqrt{2\vartheta_0}\left[\frac{1}{2}\vartheta_0{}^{-1}(\breve{\xi}_0-\hat{\xi}_0)^2\right]^{\frac{1}{2}}\\&\quad-\varphi_2\sqrt{2\vartheta_1}\left[\frac{1}{2}\vartheta_1{}^{-1}(\breve{\xi}_1-\hat{\xi}_1)^2\right]^{\frac{1}{2}}\\&\quad-\varphi_3\sqrt{2\vartheta_2}\left[\frac{1}{2}\vartheta_2{}^{-1}(\breve{\xi}_2-\hat{\xi}_2)^2\right]^{\frac{1}{2}}\end{aligned}\tag{58}$$

It implies that there exist a positive constant ι_m, together with (58), one has that,

$$\Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|G_0(\phi_1)|^{-1}\Big(H(\phi_d)\tilde{d}(t)-\frac{\kappa_3}{\varPhi(s)}|s|-\kappa_4|s|^{\theta+1}\Big)\le\iota_m \tag{59}$$

Define the state vector $\lambda_{min} > 0$ as the following form.

$$\lambda_{min} = \min\Big(\sqrt{2}c_a,\ \sqrt{2}\varrho,\ \varphi_1\sqrt{2\vartheta_0},\ \varphi_2\sqrt{2\vartheta_1},\ \varphi_3\sqrt{2\vartheta_2}\Big)$$

Further, applying the above definitions, (58) yields that,

$$\begin{aligned}&\dot{V}_2(t)\\&\le-\lambda_{min}\Bigg[\left(\frac{1}{2}M(i)\frac{s^2}{\varepsilon^2-z_1^2(t)}\right)^{\frac{1}{2}}+\left(\frac{1}{2}\big(1-M(i)\big)s^2\right)^{\frac{1}{2}}\\&\quad+\left(\frac{1}{2}\vartheta_0{}^{-1}(\breve{\xi}_0-\hat{\xi}_0)^2\right)^{\frac{1}{2}}+\left(\frac{1}{2}\vartheta_1{}^{-1}(\breve{\xi}_1-\hat{\xi}_1)^2\right)^{\frac{1}{2}}\end{aligned}$$

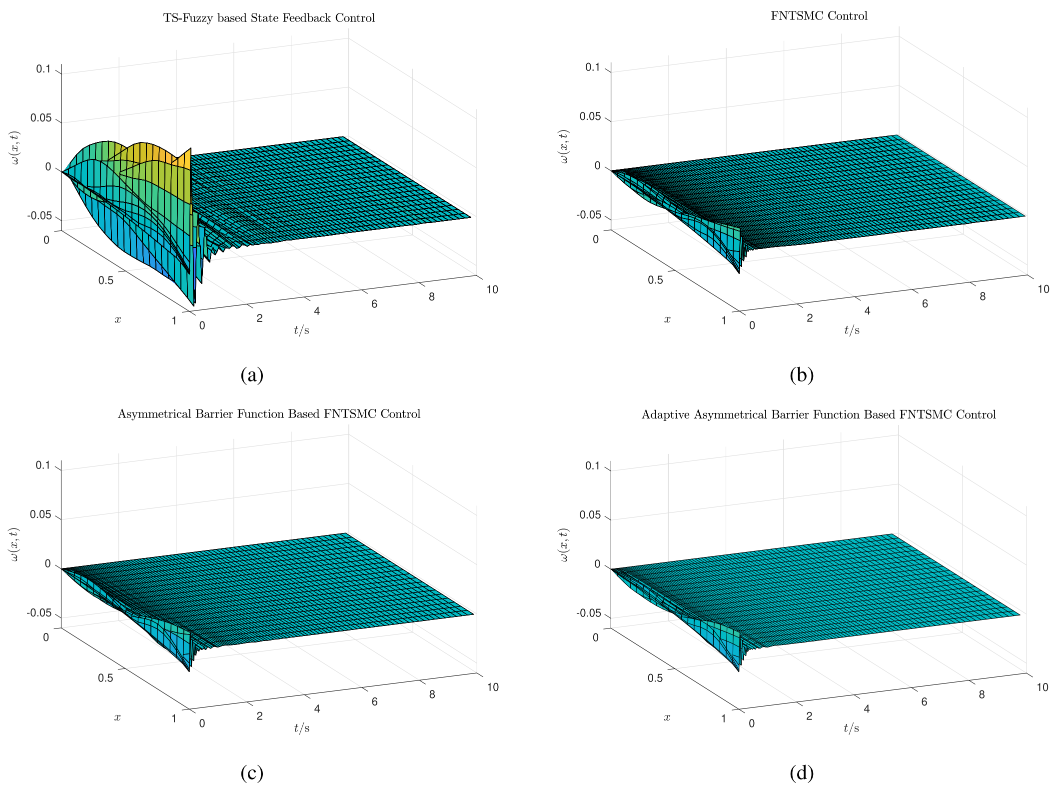

FIGURE 6. The deformation of the flexible beam (a) with T-S fuzzy based state feedback control. (b) with fast nonsingular terminal sliding mode control. (c) with asymmetrical barrier function based FNTSMC control. (d) with adaptive asymmetrical barrier function based FNTSMC control.

$$+\Big(\frac{1}{2}\vartheta_2{}^{-1}(\breve{\xi}_2-\hat{\xi}_2)^2\Big)^{\frac{1}{2}}\Bigg]+\Gamma\kappa_2\frac{m}{n}|z_2(t)|^{m/n-1}|G_0(\phi_1)|^{-1}$$
$$\times\Big(H(\phi_d)\tilde{d}(t)-\frac{\kappa_3}{\Phi(s)}|s|-\kappa_4|s|^{\theta+1}\Big)\leq-\lambda_{min}V_2^{\frac{1}{2}}(t)+\iota_m \tag{60}$$

In conclusion, it can be derived that the whole closed-loop states of the flexible beam system are semi-global practical finite-time stable, further the convergence time T_d is given as bellow,

$$T_d=\frac{2}{\mu\lambda_{min}}\left[V_2^{1/2}(0)-\frac{\iota_m}{(1-\mu)\lambda_{min}}\right] \tag{61}$$

where $V_2(0)$ denotes the initial value of $V_2(t)$, and $0<\mu\leq 1$. Then based on Lemma 1, for $\forall t\geq T^{\star}$, $V_2^{1/2}(t)\leq(\iota_m/[(1-\mu)\lambda_{min}])$. Furthermore, for $\forall t\geq T^{\star}$, $M(i)=1$ one has

$$|z_1(t)|\leq\varepsilon\left[1-2\Big(\frac{\iota_m}{(1-\mu)\lambda_{min}}\Big)^2\right]^{\frac{1}{2}} \tag{62}$$

Therefore, it can be concluded that the displacement of the flexible beam at location $x = l_0$ can be maintained in small neighborhoods near the origin, and the proof is completed. □

Remark 6. It can be observed that the upper bound of the unknown parameters is estimated by the adaptive updating law (47)-(49). The adaptive gains λ_i $(i=0,1,2)$ are positive constants which have direct effect on the adaptation rate. Further, the gains are closely related to the lumped uncertainty in the flexible beam system. In other words, the greater the uncertainty of the lumped system, the greater the value that ensures fast adaptive estimation. In addition, when the initial values $\hat{\xi}_i$ $(i=0,1,2)$ select too large, it will have an overlarge initial force.

IV. SIMULATION ANALYSIS

This section is devoted to demonstrate the effectiveness and validate the superiority of our designed control laws. It is necessary to note that, different from the previous researches, the simulation in this paper is completed on the original PDE model (8). The vibration of a flexible cantilever beam with mid-plane stretching is designed and simulated by the finite difference method, in which the proper temporal and spatial step size is chosen to approximate the solution of the PDE system (8). The obstacle restriction is considered in this paper, the distance between the obstacle and the beam is given by $0.02m$. The unknown disturbance distributed $d(x,t)$ across the flexible cantilever beam is given as follows,

$$d(x,t)=\frac{1}{2}\sin\Big(\frac{x}{L}\pi t\Big)+\frac{3}{10}\sin\Big(\frac{3x}{L}\pi t\Big)+\frac{1}{10}\sin\Big(\frac{5x}{L}\pi t\Big) \tag{63}$$

The first-order mode shape $\phi_1(x)$ is given approximately for control law design, for the simulation propose, the related

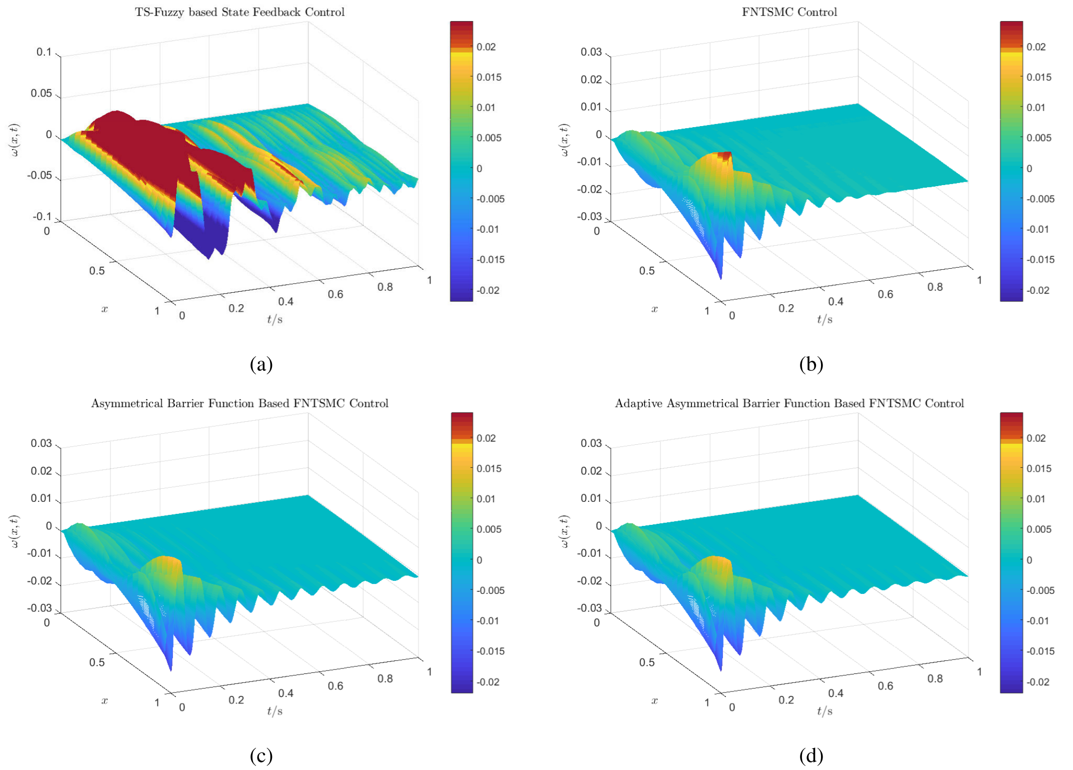

FIGURE 7. **The deformation of the flexible beam in $t \in [0, 1]$ (a) with T-S fuzzy based state feedback control. (b) with fast nonsingular terminal sliding mode control. (c) with asymmetrical barrier function based FNTSMC control. (d) with adaptive asymmetrical barrier function based FNTSMC control.**

parameters of the beam are given as follows: the bending rigidity EI and axial stiffness EA are set $15Nm^2$, $14Nm^2$, the density is simulated as $\rho = 2200kg/m^2$. The height h_b and the height L of the beam are set with $0.005m$ and $1m$, respectively. The mass of payload is denoted as $m = 5kg$. Further, the initial parameters for the piezoelectric actuator is also presented as follows, the Young modulus and piezoelectric constant are denoted as $Y_a = 71GPa$, $-175 \times 10^{-12}C/N$, respectively. The left disturbance and the right distance are given by $l_1 = 0.5m$ and $l_2 = 0.55m$. Notice that, in the paper, we set $l_0 = 1/2(l_1 + l_2)$.

Under initial conditions $\omega(x, 0) = 2.2 \times 10^{-2}x^2$ and $\dot{\omega}(x, 0) = 8.82x^2$, the flexible cantilever beam is generated for free vibration. The deformation of the beam in free vibration is depicted in Figure 3, from which it can be concluded that due to the interaction of distributed disturbance and boundary disturbance acting on the payload, significant and continuous vibration occurs along the beam.

Next, four simulation cases are given out to validate the effectiveness and rationality performances of the proposed control laws (25) and (41), respectively.

► Asymmetrical barrier function based fast nonsingular terminal sliding mode control law (ABFNTSMC)

The asymmetrical barrier function based FNTSMC control law is considered, in which the parameters are given as $\kappa_1 = 0.1$, $\kappa_2 = 0.5$, $\alpha = 8/19$, $m/n = 19/21$, $c_1 = 1 \times 10^3$, $c_2 = 1$. The coefficients for $\Phi(s)$ are given as $\kappa_3 = 50$, $\kappa_4 = 800$, $\vartheta_0 = 1 \times 10^{-5}$, $a = 8.5$, $q = 1$, respectively.

From Figure 6(c), it can be observed that under distributed interference and sensor noise, the proposed control law can suppress beam vibration exponentially within 1 second. Besides, the proposed control law (25) illustrate that the beam displacement can be effectively constrained not to exceed the bound 0.02.

► Adaptive asymmetrical barrier function based FNTSMC control law (Adaptive ABFNTSMC)

The coefficients of the adaptive asymmetrical barrier function based FNTSMC control law is simulated with the same parameters given in the asymmetrical barrier function based FNTSMC control law. In addition, the parameters of the adaptive law (47)-(49) are given as $\lambda_0 = \lambda_1 = \lambda_2 = 0.07$, the parameter of the asymmetrical barrier function is given as $c_a = 1 \times 10^3$. The initial values of the adaptive law are given as $\xi_i(0) = 0$ with $(i = 0, 1, 2)$.

The deformation response with the adaptive asymmetrical barrier function based FNTSMC control is shown in Figure 6(d), which implies better control performance with the model uncertainties than the asymmetrical barrier function based FNTSMC control law.

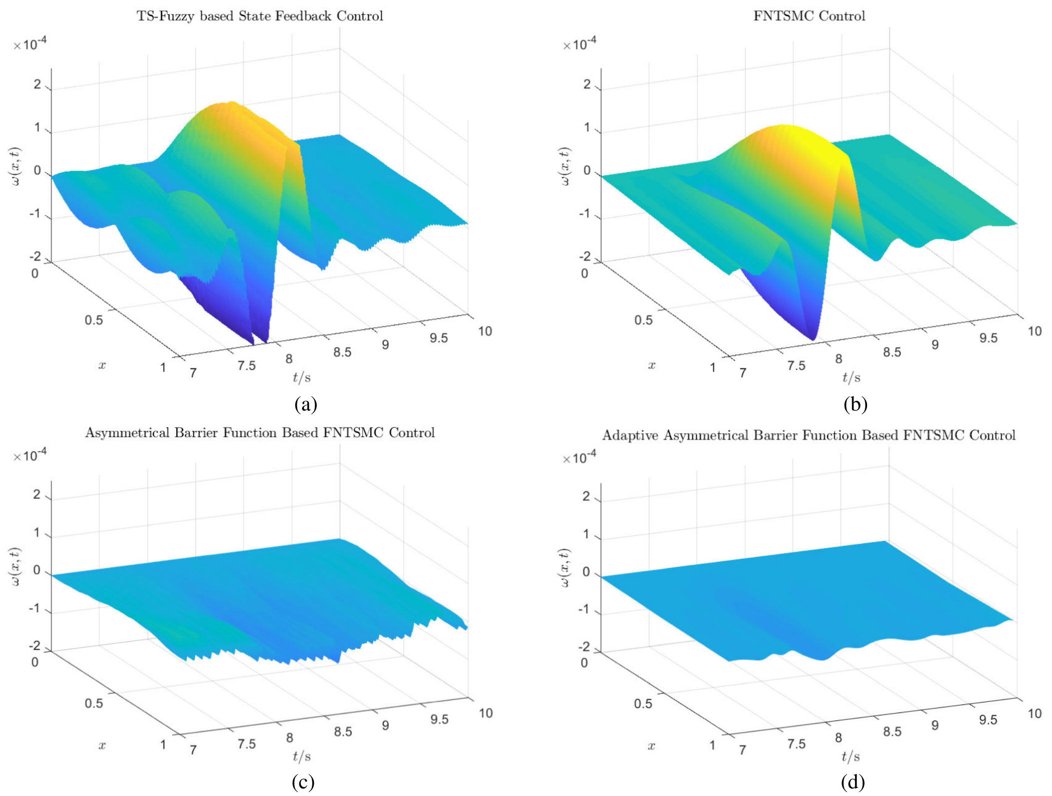

FIGURE 8. The deformation of the flexible beam in $t \in [7, 10]$ (a) with T-S fuzzy based state feedback control. (b) with fast nonsingular terminal sliding mode control. (c) with asymmetrical barrier function based FNTSMC control. (d) with adaptive asymmetrical barrier function based FNTSMC control.

► State feedback control law

To make a comparison, the flexible cantilever beam is simulated with T-S fuzzy-based state feedback control law.

$$u(t) = \Big(\frac{z_1^2(t)}{\chi^2}\Big)A_{\mu 1}Z(t) + \Big(\frac{\chi^2 - z_1^Z(t)}{\chi^2}\Big)A_{\mu 2}z(t) \quad (64)$$

where $Z(t) = [z_1(t)\ z_2(t)]^T$, $\chi = 0.1$. The corresponding coefficients are supposed to be assigned as ([28]),

$$A_{\mu 1} = [9.354, -0.772]$$
$$A_{\mu 2} = [9.39, -0.772] \quad (65)$$

► Fast nonsingular terminal sliding mode control law (FNTSMC)

The following fast nonsingular terminal sliding mode control law is given to make a comparison with our proposed control law,

$$u(t) = -F(z_1(t), \phi_1) - G(\phi_1)\frac{b}{k_1 a}|z_2(t)|^{1-a/b}\big(z_2(t) + k_2\alpha|z_1(t)|^{\alpha-1}z_2(t)\big) - k_3|s|^{\theta_1}\mathrm{sign}(s) - k_4|s|^{\theta_2}\mathrm{sign}(s) \quad (66)$$

Select the parameters $k_1 = 0.1$, $\alpha = 27/19$, $k_2 = 0.5$, $a/b = 19/21$, $k_3 = 1 \times 10^4$, $k_4 = 0.1$, $\theta_1 = 1.5$, $\theta_2 = 0.5$, respectively.

The deformations of the beam at location $x = l_0$ and $x = L$ under four different control laws are shown in Figure 4 and Figure 5. The four proposed control laws, including the asymmetrical barrier function based FNTSMC control law (25), the adaptive asymmetrical barrier function based FNTSMC control law (41), the state feedback control law (64), and the FNTSMC control law (66), which can all suppress the vibration of the flexible cantilever beam. Figure 7 shows the deformation of the beam in $t \in [0, 1]$, we can see that the deformation of the beam under the state feedback control law (64) and the FNTSMC control law (66) has exceeded our constraint $\varepsilon_{max} = 0.02$ which may lead the beam hitting the obstacle. The deformation of the beam in $t \in [7, 10]$ is shown in Figure 8, it can be seen that the vibration suppression of the beam under uncertainties and distributed disturbance with the adaptive asymmetrical barrier function based FNTSMC control law (41) have better flatness than the other three control laws.

Compared with the control law (25) and (41) proposed in this paper, the state feedback control law and the FNTSMC control law cannot guarantee that the flexible cantilever beam will not touch the obstacle. Besides, stabilization control accuracy and the speed of convergence are not as good as the proposed control law in this paper. It can be observed that, with the proposed adaptive asymmetrical barrier function based FNTSMC control law, the deflection response $\omega(x, t)$

is constrained and one may notice that it can converge to near to zero with the faster time also fewer vibrations.

V. CONCLUSION

This paper is concerned with the problem of stability control analysis for a class of nonlinear flexible cantilever beam vibration systems with obstacle restriction, model uncertainties, and distributed disturbance. In the aspect of model analysis, employed the Galerkin projection method, the partial differential equations of flexible beams are simplified to nonlinear ordinary differential equations. Then, the asymmetrical barrier function based FNTSMC control law is given in the light of a novel asymmetrical barrier Lyapunov function, guarantee that the convergence speed while constraint the maximum displacement of the flexible beam. By means of the compensation of the uncertainties caused by model simplification and distributed disturbance, together with an adaptive updating law, the adaptive FNTSMC control law based on asymmetrical barrier function is proposed. Several comparative simulation results are provided to illustrate the superiority of the presented control laws. In our future work, the noise effect in the velocity measurement should be considered. In addition, we will focus on the simplification and application in future research.

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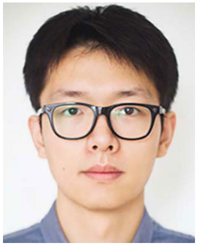

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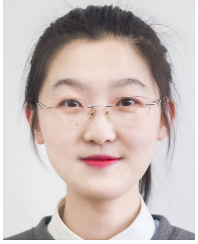

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