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On the Partition Dimension of Tri-Hexagonal α-Boron Nanotube

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ABSTRACT The production of low-cost, small in size, and high in efficiency objects is the topic of research in almost all scientific fields, especially of engineering. In this scenario, nanotechnology becomes of great importance. To achieve these tasks, one needs to study the different physical and chemical aspects of a chemical compound. In mathematical graph theory, chemical compounds are transformed in a unique mathematical representation and then examined under various parameters for these purposes. The Partition Dimension is also one of these parametric tools, which are used to identify each atom or vertex of a chemical structure under some chosen conditions. In the present paper, we use this tool to show the unique identification of each atom of a tri-hexagonal lattice of the α-boron nanotube.

INDEX TERMS α-boron nanotube, tri-hexagonal boron nanotube, molecular graph, resolving partition set, partition dimension.

I. INTRODUCTION AND SOME BASIC DEFINITIONS

Boron nanotubes are becoming highly attractive due to their extraordinary features, such as electronic structure, transport property, work function, and structural stability [6], [12]. The α-boron nanotube is constructed from a α-sheet. The first boron nanotube was made from the buckled triangular lattice [6] in 2004. Although the era of nanotubes was started in early 1990s, just after the discovery of fullerene [24] in 1985 and of carbon nanotubes [19] in 1991. But, recent research stamped on the worth and utilization of carbon nanotubes by proving that they are used in immobilization and enhancement of the duty of respective biomolecule [8]. In biosensors, architecture nanotubes worked for biological recognition. They also help in the optimization and development of biosensors. Such biochemical sensors obtained from nanotube are used to determine the enormous analytes that can analyze alcohols, glucose, lactate, cholesterol pyrophosphate [36], proteins and DNA [1]. Further, various electrochemical sensors are made up with nanotubes [7], and the near future work due to the electron transfer and material property of nanotube and biorecognition mixture yield to develop more high-performance electrochemical sensors [7].

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The concept of resolving partition set and partition dimension extensively appeared in the literature. For example, in [4] graphs with partition dimension $n-3$ are discussed. The graphs obtained by sum operation of cycle and path graphs are studied in [41]. [18] provided bounds of partition dimension of wheel related graphs. [15] and [27] discussed the partition dimension of circulant graph. The partition dimension of complete multipartite graphs is studied in [39]. The topic of strong partition dimension has discussed in [25], [35], while its local version dealt in [2]. In [28], the partition dimension of (4, 6) fullerene has studied and proved that this family of graph has bounded partition dimension. Results on the bounded partition dimension of the Cartesian product of graphs are given in [42]. [3] provoides bounds for the subdivision of different graphs. The bounds for tree structures graphs are given in [37]. In [38] bounds of uni-cyclic graphs are presented. For more recent literature and results, we refer to [13], [17], [20], [26], [28], [30]–[32], [34], [37], [38].

Applications of resolving partitions can be found in various fields such as, network verification [5], robot navigation [23], Djokovic-Winkler relation [9] and strategies of the mastermind games [14]. For applications of resolving sets see, [21], [22], [29]. Also see, [10], [16], [33] for applications in networking.

Definition 1: Let $V(G)$ and $E(G)$ be the vertex set and edge set (respectively) of a simple, connected graph (G).

The count of the edges of the shortest path between two vertices $a, b \in V(G)$ is called the distance $d(a, b)$, between a and b.

Definition 2: Let $Q = \{b_1, b_2, \ldots, b_\rho\}$ be an ordered set of vertices of G. For a vertex $b \in V(G)$, the representations denoted by $r(b|Q)$ is the ρ-tuple distances $(d(b, b_1), d(b, b_2), \ldots, d(b, b_\rho))$. If each vertex of the graph G has different representation with respect to Q, then Q is called the resolving set of G.

Definition 3: The minimum number of vertices in the resolving set is known as the metric dimension or $dim(G)$ of G.

Definition 4: Let $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_\rho\}$ be an ρ-ordered subset of $V(G)$. The distance between a vertex $b \in V(G)$ and a $\lambda_j \subset V(G)$ is defined as; $d(b, \lambda_j) = min_{x \in \lambda_j}\{d(b, x)\}$, $1 \leq j \leq \rho$. Let λ is the k-ordered partition set and $r(b|\lambda) = \{d(b, \lambda_1), d(b, \lambda_2), \ldots, d(b, \lambda_k)\}$, is the k-tuple distance representations of a vertex b with respect to λ. If the representation of b with respect to λ is unique, then λ is the resolving partition set of the vertex set of the graph G.

Definition 5: The minimum count of subsets in the resolving partition set of $V(G)$ is defined as the partition dimension $(pd(G))$ of G [11].

The relation of $dim(G)$ and $pd(G)$ for any non-trivial connected graph G has presented in [11];

$$pd(G) \leq dim(G) + 1. \tag{1}$$

Following theorems are very helpful in finding the partition dimension of a graph.

Theorem 1 [11]: Let λ be a resolving partition of $V(G)$ and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all vertices $w \in V(G) \backslash (u, v)$, then u, v belong to different classes of λ.

Theorem 2 [11]: Let G be a simple and connected graph. Then

- $pd(G)$ is 2 iff G is a path graph,
- $pd(G)$ is n iff G is a complete graph.

This article is devoted to compute partition dimension (a generalized concept of the well known metric dimension) of a tri-hexagonal α-boron nanotube whose molecular graph is shown in Figure 1. In mathematical chemistry for study purposes, a chemical structure is represented by a mathematical model called molecular graph, in which vertices specify atoms and edges are bonds between atoms. The concept of the partition dimension is based on the selection of a specific partition of the vertex set of the under study graph for which each vertex of the graph has a unique identification under defined partition.

Upcoming section II, is about our computed results while, in Section III, conclusion is drawn. In the end, sufficient references are stated in the reference section.

II. PARTITION DIMENSION OF α-BORON NANOTUBE ($\alpha_{a,b}$)

We start with some notations which are being used in upcoming results. We use the symbol $\alpha_{a,b}$ for the tri-hexagonal lattice of the α-boron nanotube of dimensions a and b. We use the term lattice only to denote the tri-hexagonal sheet of the α-boron nanotube. In the representation of a vertex $\chi_{\nu\mu}$, ν is the row number and μ is the column number. For the sake of simplicity, we are using following labeling of vertices in $V(\alpha_{a,b})$, shown in Figure 2.

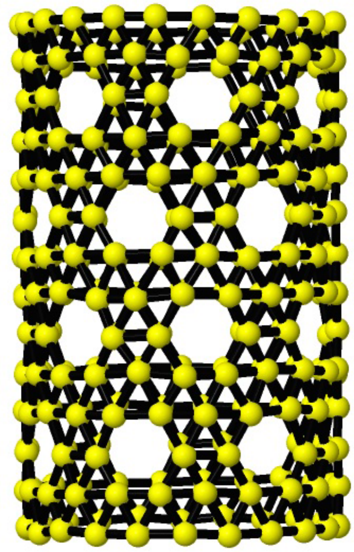

FIGURE 1. 3D view of α-Boron nanotube.

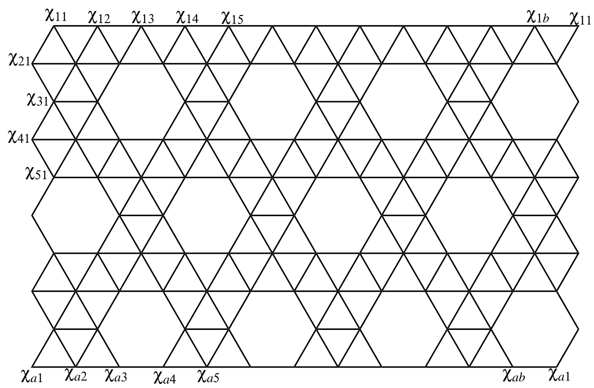

FIGURE 2. The vertex labeling of $\alpha_{a,b}$.

Theorem 3: For every $a < b$, the α-boron nanotube $\alpha_{a,b}$ has $pd(\alpha_{a,b}) = 3$.

Proof: Let $\lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ be a resolving partition set where $\lambda_1 = \{\chi_{11}\}$, $\lambda_2 = \{\chi_{1b}\}$, $\lambda_3 = \{v \in V(G)| v \notin \{\lambda_1, \lambda_2\}\}$. The representations of a vertex in $V(\alpha_{a,b})$ w.r.t resolving partition set λ is as follows;

Case 1: For $\nu = 1, 2$.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\mu - 1, b - \mu, z), & \text{if } \nu = 1, \text{ and } 1 \leq \mu \leq b; \\ (1, b, 0), & \text{if } \nu = 2, \text{ and } 2 \leq \mu \leq b. \end{cases} \tag{2}$$

If $\nu, \mu = 1$ and $\nu = 1$, $\mu = b$, then $z = 1$, otherwise $z = 0$.

Case 2: For $3 \leq \nu \leq a$, $\nu \neq 3\kappa$ and ν odd;

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu-1, b-\mu+\frac{\nu-1}{2}, 0), & \text{if } 1 \leq \mu \leq \frac{\nu-1}{2}; \\ (\mu+\frac{\nu-3}{2}, b-\mu+\frac{\nu-1}{2}, 0), & \text{if } \frac{\nu+1}{2} \leq \mu \leq b-\frac{\nu-1}{2}; \\ (\mu+\frac{\nu-3}{2}, \nu-1, 0), & \text{if } b-\frac{\nu-3}{2} \leq \mu \leq b. \end{cases} \tag{3}$$

Case 3: For $4 \leq \nu \leq a$, $\nu \neq 3\kappa$ and ν even;

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu-1, b-\mu+\frac{\nu}{2}, 0), & \text{if } 1 \leq \mu \leq \frac{\nu}{2}; \\ (\mu+\frac{\nu-4}{2}, b-\mu+\frac{\nu}{2}, 0), & \text{if } \frac{\nu+2}{2} \leq \mu \leq \frac{2b-\nu+2}{2}; \\ (\mu+\frac{\nu-4}{2}, \nu-1, 0), & \text{if } b-\frac{\nu-4}{2} \leq \mu \leq b. \end{cases} \tag{4}$$

Case 4: For $\nu = 3\kappa$, $\mu \neq 3p$ and 3κ odd;

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu-1, b-\mu+\frac{\nu-2}{2}, 0), & \text{if } 1 \leq \mu \leq \frac{\nu-1}{2}; \\ (\mu+\frac{\nu-3}{2}, b-\mu+\frac{\nu-1}{2}, 0), & \text{if } \frac{\nu+1}{2} \leq \mu \leq b-\frac{\nu-1}{2}; \\ (\mu+\frac{\nu-3}{2}, \nu-1, 0), & \text{if } b-\frac{\nu-3}{2} \leq \mu \leq b. \end{cases} \tag{5}$$

Case 5: For $\nu = 3\kappa$, $\mu \neq 3p-1$ and 3κ even;

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu-1, b-\mu+\frac{\nu}{2}, 0), & \text{if } 1 \leq \mu \leq \frac{\nu}{2}; \\ (\mu+\frac{\nu-4}{2}, b-\mu+\frac{\nu}{2}, 0), & \text{if } \frac{\nu+2}{2} \leq \mu \leq b-\frac{\nu-2}{2}; \\ (\mu+\frac{\nu-4}{2}, \nu-1, 0), & \text{if } b-\frac{\nu-4}{2} \leq \mu \leq b. \end{cases} \tag{6}$$

As, all the vertices have different representations with respect to λ. Hence,

$$pd(\alpha_{a,b}) \leq 3. \tag{7}$$

Converse: Now, we will prove that $pd(\alpha_{a,b}) \geq 3$. Suppose on contrary that $pd(\alpha_{a,b}) = 2$, which is not possible as graph showing in Figure 2, is not a path graph. Thus

$$pd(\alpha_{a,b}) \geq 3. \tag{8}$$

Hence, from Inequalities (7) and (8), we can conclude that

$$pd(\alpha_{a,b}) = 3.$$

□

Theorem 4: For every $a \geq b$, the α-boron nanotube $\alpha_{a,b}$ has $pd(\alpha_{a,b}) \leq 4$.

Proof: Let $\lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ be a resolving partition set, where $\lambda_1 = \{\chi_{11}\}$, $\lambda_2 = \{\chi_{1b}\}$, $\lambda_3 = \{\chi_{a1}\}$ and $\lambda_4 = \{v \in V(G) | v \notin \{\lambda_1, \lambda_2, \lambda_3\}\}$. The representations of vertices in $V(\alpha_{a,b})$ w.r.t resolving partition set λ are given below;

Case 1: For $b \leq a \leq 2b$, $a \neq 6\kappa+1$ and a is odd;

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\mu-1, b-\mu, a-1, z), & \text{if } \nu = 1, \text{ and } 1 \leq \mu \leq \frac{a+1}{2}; \\ (\mu-1, b-\mu, \mu+\frac{a-3}{2}, z), & \text{if } \nu = 1, \text{ and } \frac{a+3}{2} \leq \mu \leq b; \\ (1, b, a-2, 0), & \text{if } \nu = 2, \text{ and } \mu = 1; \\ (\mu-1, b-\mu+1, a-2, 0), & \text{if } \nu = 2, \text{ and } 2 \leq \mu \leq \frac{a+1}{2}; \\ (\mu-1, b-\mu+1, \mu+\frac{a-5}{2}, 0), & \text{if } \nu = 2, \text{ and } \frac{a+3}{2} \leq \mu \leq b. \end{cases} \tag{9}$$

If $\nu, \mu = 1$ and $\nu = 1, \mu = b$, then $z = 1$ otherwise, $z = 0$.

If ν is odd, $\nu \neq 3\kappa$ and $4 \leq \nu \leq \lceil \frac{a}{2} \rceil$.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu-1, b-\mu+\lfloor\frac{\nu}{2}\rfloor, a-\nu, 0), & \text{if } 1 \leq \mu \leq \frac{\nu-1}{2}; \\ (\mu+\lfloor\frac{\nu-3}{2}\rfloor, b-\mu+\lfloor\frac{\nu}{2}\rfloor, a-\nu, 0), & \text{if } \frac{\nu+1}{2} \leq \mu \leq \frac{a-\nu+2}{2}; \\ (\mu+\lfloor\frac{\nu-3}{2}\rfloor, b-\mu+\lfloor\frac{\nu}{2}\rfloor, \mu+\frac{a-\nu-2}{2}, 0), & \text{if } \frac{a-\nu+4}{2} \leq \mu \leq b-\frac{\nu-1}{2}; \\ (\mu+\lfloor\frac{\nu-3}{2}\rfloor, \nu-1, \mu+\frac{a-\nu-2}{2}, 0), & \text{if } b-\frac{\nu-3}{2} \leq \mu \leq b. \end{cases} \tag{10}$$

For $\lceil \frac{a}{2} \rceil + 1 \leq \nu \leq a$.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu-1, b-\mu+\left\lfloor \frac{\nu}{2} \right\rfloor, a-\nu, z), & \text{if } 1 \leq \mu \leq \frac{a-\nu+2}{2}; \\ (\nu-1, b-\mu+\left\lfloor \frac{\nu}{2} \right\rfloor, \mu+\frac{a-\nu-2}{2}, 0), & \text{if } \frac{a-\nu-4}{2} \leq \mu \leq \frac{\nu-1}{2}; \\ (\mu+\left\lfloor \frac{\nu-3}{2} \right\rfloor, b-\mu+\left\lfloor \frac{\nu}{2} \right\rfloor, \mu+\frac{a-\nu-2}{2}, 0), & \text{if } \frac{\nu+1}{2} \leq \mu \leq b-\frac{\nu-1}{2}; \\ (\mu+\left\lfloor \frac{\nu-3}{2} \right\rfloor, \nu-1, \mu+\frac{a-\nu-2}{2}, 0), & \text{if } b-\frac{\nu-3}{2} \leq \mu \leq b. \end{cases} \tag{11}$$

where $z = 1$, when $\nu = a$, $\mu = 1$ otherwise, $z = 0$.

If ν is even, $\nu \neq 3\kappa$ and $3 \leq \nu \leq \lceil \frac{a}{2} \rceil$.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu-1, b-\mu+\left\lfloor \frac{\nu}{2} \right\rfloor, a-\nu, z), & \text{if } 1 \leq \mu \leq \frac{\nu}{2}; \\ (\mu+\left\lfloor \frac{\nu-3}{2} \right\rfloor, b-\mu+\left\lfloor \frac{\nu}{2} \right\rfloor, a-\nu, 0), & \text{if } \frac{\nu+2}{2} \leq \mu \leq \frac{a-\nu+3}{2}; \\ (\mu+\left\lfloor \frac{\nu-3}{2} \right\rfloor, b-\mu+\left\lfloor \frac{\nu}{2} \right\rfloor, \mu+\frac{a-\nu-3}{2}, 0), & \text{if } \frac{a-\nu+5}{2} \leq \mu \leq b-\frac{\nu-2}{2}; \\ (\mu+\left\lfloor \frac{\nu-3}{2} \right\rfloor, \nu-1, \mu+\frac{a-\nu-3}{2}, 0), & \text{if } b-\frac{\nu-4}{2} \leq \mu \leq b. \end{cases} \tag{12}$$

For $\lceil \frac{a}{2} \rceil + 1 \leq \nu \leq a$.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu-1, b-\mu+\left\lfloor \frac{\nu}{2} \right\rfloor, a-\nu, z), & \text{if } 1 \leq \mu \leq \frac{a-\nu+3}{2}; \\ (\nu-1, b-\mu+\left\lfloor \frac{\nu}{2} \right\rfloor, \mu+\frac{a-\nu-3}{2}, 0), & \text{if } \frac{a-\nu+5}{2} \leq \mu \leq \frac{\nu}{2}; \\ (\mu+\left\lfloor \frac{\nu-3}{2} \right\rfloor, b-\mu+\left\lfloor \frac{\nu}{2} \right\rfloor, \mu+\frac{a-\nu-3}{2}, 0), & \text{if } \frac{\nu+1}{2} \leq \mu \leq b-\frac{\nu-2}{2}; \\ (\mu+\left\lfloor \frac{\nu-3}{2} \right\rfloor, \nu-1, \mu+\frac{a-\nu-3}{2}, 0), & \text{if } b-\frac{\nu-4}{2} \leq \mu \leq b. \end{cases} \tag{13}$$

where $z = 1$, when $\nu = a$, $\mu = 1$ otherwise, $z = 0$.

If $a = 6\kappa + 1$, ν is odd, $\nu \neq 3\kappa$ and $3 \leq \nu \leq \lceil \frac{a}{2} \rceil$.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu-1, b-\mu+\left\lfloor \frac{\nu}{2} \right\rfloor, a-\nu, 0), & \text{if } 1 \leq \mu \leq \frac{\nu-1}{2}; \\ (\mu+\left\lfloor \frac{\nu-3}{2} \right\rfloor, b-\mu+\left\lfloor \frac{\nu}{2} \right\rfloor, a-\nu, 0), & \text{if } \frac{\nu+2}{2} \leq \mu \leq \frac{a-\nu}{2}; \\ (\mu+\left\lfloor \frac{\nu-3}{2} \right\rfloor, b-\mu+\left\lfloor \frac{\nu}{2} \right\rfloor, a-\nu+1, 0), & \text{if } \mu = \frac{a-\nu+2}{2}; \\ (\mu+\left\lfloor \frac{\nu-3}{2} \right\rfloor, b-\mu+\left\lfloor \frac{\nu}{2} \right\rfloor, \mu+\frac{a-\nu-2}{2}, 0), & \text{if } \frac{a-\nu-4}{2} \leq \mu \leq b-\frac{\nu-1}{2}; \\ (\mu+\left\lfloor \frac{\nu-3}{2} \right\rfloor, \nu-1, \mu+\frac{a-\nu-2}{2}, 0), & \text{if } b-\frac{\nu-3}{2} \leq \mu \leq b. \end{cases} \tag{14}$$

For $\lceil \frac{a}{2} \rceil + 1 \leq \nu \leq a$.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu-1, b-\mu+\left\lfloor \frac{\nu}{2} \right\rfloor, a-\nu, z), & \text{if } 1 \leq \mu \leq \frac{a-\nu}{2}; \\ (\nu-1, b-\mu+\left\lfloor \frac{\nu}{2} \right\rfloor, a-\nu+1, 0), & \text{if } \mu = \frac{a-\nu+2}{2}; \\ (\nu-1, b-\mu+\left\lfloor \frac{\nu}{2} \right\rfloor, \mu+\frac{a-\nu-2}{2}, 0), & \text{if } \frac{a-\nu+4}{2} \leq \mu \leq \frac{\nu-1}{2}; \\ (\mu+\left\lfloor \frac{\nu-3}{2} \right\rfloor, b-\mu+\left\lfloor \frac{\nu}{2} \right\rfloor, \mu+\frac{a-\nu-2}{2}, 0), & \text{if } \frac{\nu+1}{2} \leq \mu \leq b-\frac{\nu-1}{2}; \\ (\mu+\left\lfloor \frac{\nu-3}{2} \right\rfloor, \nu-1, \mu+\frac{a-\nu-2}{2}, 0), & \text{if } b-\frac{\nu-3}{2} \leq \mu \leq b. \end{cases} \tag{15}$$

where $z = 1$, when $\nu = a$, $\mu = 1$ otherwise, $z = 0$.

If $a = 6\kappa + 1$, ν is even, $\nu \neq 3\kappa$ and $3 \leq \nu \leq \left\lceil \frac{a}{2} \right\rceil$.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu - 1, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, a - \nu, 0), & \text{if } 1 \leq \mu \leq \frac{\nu}{2}; \\ (\mu + \left\lfloor \frac{\nu-3}{2} \right\rfloor, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, a - \nu, 0), & \text{if } \frac{\nu+2}{2} \leq \mu \leq \frac{a-\nu+1}{2}; \\ (\mu + \left\lfloor \frac{\nu-3}{2} \right\rfloor, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, a - \nu + 1, 0), & \text{if } \mu = \frac{a-\nu+3}{2}; \\ (\mu + \left\lfloor \frac{\nu-3}{2} \right\rfloor, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, \mu + \frac{a-\nu-2}{2}, 0), & \text{if } \frac{a-\nu+5}{2} \leq \mu \leq b - \frac{\nu-1}{2}; \\ (\mu + \left\lfloor \frac{\nu-3}{2} \right\rfloor, \nu - 1, \mu + \frac{a-\nu-2}{2}, 0), & \text{if } b - \frac{\nu-4}{2} \leq \mu \leq b. \end{cases} \tag{16}$$

For $\left\lceil \frac{a}{2} \right\rceil + 1 \leq \nu \leq a$.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu - 1, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, a - \nu, z), & \text{if } 1 \leq \mu \leq \frac{a-\nu+1}{2}; \\ (\nu - 1, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, a - \nu + 1, 0), & \text{if } \mu = \frac{a-\nu+3}{2}; \\ (\nu - 1, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, \mu + \frac{a-\nu-2}{2}, 0), & \text{if } \frac{a-\nu+5}{2} \leq \mu \leq \frac{\nu}{2}; \\ (\mu + \left\lfloor \frac{\nu-3}{2} \right\rfloor, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, \mu + \frac{a-\nu-2}{2}, 0), & \text{if } \frac{\nu+2}{2} \leq \mu \leq b - \frac{\nu-2}{2}; \\ (\mu + \left\lfloor \frac{\nu-3}{2} \right\rfloor, \nu - 1, \mu + \frac{a-\nu-2}{2}, 0), & \text{if } b - \frac{\nu-4}{2} \leq \mu \leq b. \end{cases} \tag{17}$$

where $z = 1$, when $\nu = a$, $\mu = 1$ otherwise, $z = 0$.

If ν is odd, $\nu = 3\kappa$ and $3 \leq \nu \leq \left\lceil \frac{a}{2} \right\rceil$ also $\mu \neq 3l$.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu - 1, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, a - \nu, 0), & \text{if } 1 \leq \mu \leq \frac{\nu-1}{2}; \\ (\mu + \left\lfloor \frac{\nu-3}{2} \right\rfloor, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, a - \nu, 0), & \text{if } \frac{\nu+1}{2} \leq \mu \leq \frac{a-\nu+2}{2}; \\ (\mu + \left\lfloor \frac{\nu-3}{2} \right\rfloor, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, \mu + \frac{a-\nu-2}{2}, 0), & \text{if } \frac{a-\nu+4}{2} \leq \mu \leq b - \frac{\nu-1}{2}; \\ (\mu + \left\lfloor \frac{\nu-3}{2} \right\rfloor, \nu - 1, \mu + \frac{a-\nu-2}{2}, 0), & \text{if } b - \frac{\nu-3}{2} \leq \mu \leq b. \end{cases} \tag{18}$$

For $\left\lceil \frac{a}{2} \right\rceil + 1 \leq \nu \leq a$ also $\mu \neq 3l$.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu - 1, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, a - \nu, z), & \text{if } 1 \leq \mu \leq \frac{a-\nu+2}{2}; \\ (\nu - 1, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, \mu + \frac{a-\nu-2}{2}, 0), & \text{if } \frac{a-\nu+4}{2} \leq \mu \leq \frac{\nu-1}{2}; \\ (\mu + \left\lfloor \frac{\nu-3}{2} \right\rfloor, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, \mu + \frac{a-\nu-2}{2}, 0), & \text{if } \frac{\nu+1}{2} \leq \mu \leq b - \frac{\nu-1}{2}; \\ (\mu + \left\lfloor \frac{\nu-3}{2} \right\rfloor, \nu - 1, \mu + \frac{a-\nu-2}{2}, 0), & \text{if } b - \frac{\nu-3}{2} \leq \mu \leq b. \end{cases} \tag{19}$$

where $z = 1$, when $\nu = a$, $\mu = 1$ otherwise, $z = 0$.

If ν is even, $\nu = 3\kappa$ and $3 \leq \nu \leq \left\lceil \frac{a}{2} \right\rceil$ also $\mu \neq 3l - 1$.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu - 1, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, a - \nu, 0), & \text{if } 1 \leq \mu \leq \frac{\nu}{2}; \\ (\mu + \left\lfloor \frac{\nu-3}{2} \right\rfloor, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, a - \nu, 0), & \text{if } \frac{\nu+2}{2} \leq \mu \leq \frac{a-\nu+3}{2}; \\ (\mu + \left\lfloor \frac{\nu-3}{2} \right\rfloor, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, \mu + \frac{a-\nu-3}{2}, 0), & \text{if } \frac{a-\nu+5}{2} \leq \mu \leq b - \frac{\nu-2}{2}; \\ (\mu + \left\lfloor \frac{\nu-3}{2} \right\rfloor, \nu - 1, \mu + \frac{a-\nu-3}{2}, 0), & \text{if } b - \frac{\nu-4}{2} \leq \mu \leq b. \end{cases} \tag{20}$$

For $\lceil \frac{a}{2} \rceil + 1 \leq \nu \leq a$ also $\mu \neq 3l - 1$.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu - 1, b - \mu + \lfloor \frac{\nu}{2} \rfloor, a - \nu, z), \\ \quad \text{if } 1 \leq \mu \leq \frac{a - \nu + 3}{2}; \\ (\nu - 1, b - \mu + \lfloor \frac{\nu}{2} \rfloor, \mu + \frac{a - \nu - 3}{2}, 0), \\ \quad \text{if } \frac{a - \nu + 5}{2} \leq \mu \leq \frac{\nu}{2}; \\ (\mu + \lfloor \frac{\nu - 3}{2} \rfloor, b - \mu + \lfloor \frac{\nu}{2} \rfloor, \mu \\ \quad + \frac{a - \nu - 3}{2}, 0), \\ \quad \text{if } \frac{\nu + 1}{2} \leq \mu \leq b - \frac{\nu - 2}{2}; \\ (\mu + \lfloor \frac{\nu - 3}{2} \rfloor, \nu - 1, \mu + \frac{a - \nu - 3}{2}, 0), \\ \quad \text{if } b - \frac{\nu - 4}{2} \leq \mu \leq b. \end{cases} \quad (21)$$

where $z = 1$, when $\nu = a$, $\mu = 1$ otherwise, $z = 0$.

Case 2: For a is even, $a \neq 6\kappa + 2$, $\nu \neq 3\kappa$, ν is odd and $4 \leq \nu \leq \lceil \frac{a}{2} \rceil$;

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu - 1, b - \mu + \lfloor \frac{\nu}{2} \rfloor, a - \nu, 0), \\ \quad \text{if } 1 \leq \mu \leq \frac{\nu - 1}{2}; \\ (\mu + \lfloor \frac{\nu - 3}{2} \rfloor, b - \mu + \lfloor \frac{\nu}{2} \rfloor, a - \nu, 0), \\ \quad \text{if } \frac{\nu + 2}{2} \leq \mu \leq \frac{a - \nu + 1}{2}; \\ (\mu + \lfloor \frac{\nu - 3}{2} \rfloor, b - \mu + \lfloor \frac{\nu}{2} \rfloor, \mu \\ \quad + \frac{a - \nu - 1}{2}, 0), \\ \quad \text{if } \frac{a - \nu + 3}{2} \leq \mu \leq b - \frac{\nu - 1}{2}; \\ (\mu + \lfloor \frac{\nu - 3}{2} \rfloor, \nu - 1, \mu + \frac{a - \nu - 1}{2}, 0), \\ \quad \text{if } b - \frac{\nu - 3}{2} \leq \mu \leq b. \end{cases} \quad (22)$$

For $\lceil \frac{a}{2} \rceil + 1 \leq \nu \leq a$.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu - 1, b - \mu + \lfloor \frac{\nu}{2} \rfloor, a - \nu, z), \\ \quad \text{if } 1 \leq \mu \leq \frac{a - \nu + 1}{2}; \\ (\nu - 1, b - \mu + \lfloor \frac{\nu}{2} \rfloor, \mu + \frac{a - \nu - 1}{2}, 0), \\ \quad \text{if } \frac{a - \nu + 3}{2} \leq \mu \leq \frac{\nu - 1}{2}; \\ (\mu + \lfloor \frac{\nu - 3}{2} \rfloor, b - \mu + \lfloor \frac{\nu}{2} \rfloor, \mu \\ \quad + \frac{a - \nu - 1}{2}, 0), \\ \quad \text{if } \frac{\nu + 1}{2} \leq \mu \leq b - \frac{\nu - 1}{2}; \\ (\mu + \lfloor \frac{\nu - 3}{2} \rfloor, \nu - 1, \mu + \frac{a - \nu - 1}{2}, 0), \\ \quad \text{if } b - \frac{\nu - 3}{2} \leq \mu \leq b. \end{cases} \quad (23)$$

where $z = 1$, when $\nu = a$, $\mu = 1$ otherwise, $z = 0$.

If ν is even, $\nu \neq 3\kappa$ and $4 \leq \nu \leq \lceil \frac{a}{2} \rceil$;

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu - 1, b - \mu + \lfloor \frac{\nu}{2} \rfloor, a - \nu, 0), \\ \quad \text{if } 1 \leq \mu \leq \frac{\nu}{2}; \\ (\mu + \lfloor \frac{\nu - 3}{2} \rfloor, b - \mu + \lfloor \frac{\nu}{2} \rfloor, a - \nu, 0), \\ \quad \text{if } \frac{\nu + 2}{2} \leq \mu \leq \frac{a - \nu + 2}{2}; \\ (\mu + \lfloor \frac{\nu - 3}{2} \rfloor, b - \mu + \lfloor \frac{\nu}{2} \rfloor, \mu \\ \quad + \frac{a - \nu - 2}{2}, 0), \\ \quad \text{if } \frac{a - \nu + 4}{2} \leq \mu \leq b - \frac{\nu - 2}{2}; \\ (\mu + \lfloor \frac{\nu - 3}{2} \rfloor, \nu - 1, \mu + \frac{a - \nu - 2}{2}, 0), \\ \quad \text{if } b - \frac{\nu - 4}{2} \leq \mu \leq b. \end{cases} \quad (24)$$

For $\lceil \frac{a}{2} \rceil + 1 \leq \nu \leq a$;

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu - 1, b - \mu + \lfloor \frac{\nu}{2} \rfloor, a - \nu, z), \\ \quad \text{if } 1 \leq \mu \leq \frac{a - \nu + 2}{2}; \\ (\nu - 1, b - \mu + \lfloor \frac{\nu}{2} \rfloor, \mu + \frac{a - \nu - 2}{2}, 0), \\ \quad \text{if } \frac{a - \nu + 4}{2} \leq \mu \leq \frac{\nu}{2}; \\ (\mu + \lfloor \frac{\nu - 3}{2} \rfloor, b - \mu + \lfloor \frac{\nu}{2} \rfloor, \mu \\ \quad + \frac{a - \nu - 2}{2}, 0), \\ \quad \text{if } \frac{\nu + 2}{2} \leq \mu \leq b - \frac{\nu - 2}{2}; \\ (\mu + \lfloor \frac{\nu - 3}{2} \rfloor, \nu - 1, \mu + \frac{a - \nu - 2}{2}, 0), \\ \quad \text{if } b - \frac{\nu - 4}{2} \leq \mu \leq b. \end{cases} \quad (25)$$

where $z = 1$, when $\nu = a$, $\mu = 1$ otherwise, $z = 0$.

Case 3: For a is even, $a = 6\kappa + 2$, $\nu \neq 3\kappa$, ν is odd and $4 \leq \nu \leq \lceil \frac{a}{2} \rceil$;

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu - 1, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, a - \nu, 0), & \text{if } 1 \leq \mu \leq \frac{\nu - 3}{2}; \\ (\mu + \left\lfloor \frac{\nu - 3}{2} \right\rfloor, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, a - \nu, 0), & \text{if } \frac{\nu - 1}{2} \leq \mu \leq \frac{a - \nu - 1}{2}; \\ (\mu + \left\lfloor \frac{\nu - 3}{2} \right\rfloor, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, a - \nu + 1, 0), & \text{if } \mu = \frac{a - \nu + 1}{2}; \\ (\mu + \left\lfloor \frac{\nu - 3}{2} \right\rfloor, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, \mu + \frac{a - \nu - 2}{2}, 0), & \text{if } \frac{a - \nu + 3}{2} \leq \mu \leq b - \frac{\nu - 1}{2}; \\ (\mu + \left\lfloor \frac{\nu - 3}{2} \right\rfloor, \nu - 1, \mu + \frac{a - \nu - 2}{2}, 0), & \text{if } b - \frac{\nu - 3}{2} \leq \mu \leq b. \end{cases} \tag{26}$$

For $\lceil \frac{a}{2} \rceil + 1 \leq \nu \leq a$.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu - 1, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, a - \nu, z), & \text{if } 1 \leq \mu \leq \frac{a - \nu - 1}{2}; \\ (\nu - 1, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, a - \nu + 1, 0), & \text{if } \mu = \frac{a - \nu + 1}{2}; \\ (\nu - 1, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, \mu + \frac{a - \nu - 2}{2}, 0), & \text{if } \frac{a - \nu + 3}{2} \leq \mu \leq \frac{\nu - 3}{2}; \\ (\mu + \left\lfloor \frac{\nu - 3}{2} \right\rfloor, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, \mu + \frac{a - \nu - 2}{2}, 0), & \text{if } \frac{\nu - 1}{2} \leq \mu \leq b - \frac{\nu - 1}{2}; \\ (\mu + \left\lfloor \frac{\nu - 3}{2} \right\rfloor, \nu - 1, \mu + \frac{a - \nu - 2}{2}, 0), & \text{if } b - \frac{\nu - 3}{2} \leq \mu \leq b. \end{cases} \tag{27}$$

where $z = 1$, when $\nu = a$, $\mu = 1$ otherwise, $z = 0$.

Case 4: For a is even, $a = 6\kappa + 2$, $\nu \neq 3\kappa$, ν is even and $4 \leq \nu \leq \lceil \frac{a}{2} \rceil$;

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu - 1, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, a - \nu, 0), & \text{if } 1 \leq \mu \leq \frac{\nu}{2}; \\ (\mu + \left\lfloor \frac{\nu - 3}{2} \right\rfloor, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, a - \nu, 0), & \text{if } \frac{\nu + 2}{2} \leq \mu \leq \frac{a - \nu}{2}; \\ (\mu + \left\lfloor \frac{\nu - 3}{2} \right\rfloor, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, a - \nu + 1, 0), & \text{if } \mu = \frac{a - \nu + 2}{2}; \\ (\mu + \left\lfloor \frac{\nu - 3}{2} \right\rfloor, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, \mu + \frac{a - \nu - 2}{2}, 0), & \text{if } \frac{a - \nu + 4}{2} \leq \mu \leq b - \frac{\nu - 2}{2}; \\ (\mu + \left\lfloor \frac{\nu - 3}{2} \right\rfloor, \nu - 1, \mu + \frac{a - \nu - 2}{2}, 0), & \text{if } b - \frac{\nu - 4}{2} \leq \mu \leq b. \end{cases} \tag{28}$$

For $\lceil \frac{a}{2} \rceil + 1 \leq \nu \leq a$.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu - 1, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, a - \nu, z), & \text{if } 1 \leq \mu \leq \frac{a - \nu}{2}; \\ (\nu - 1, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, a - \nu + 1, 0), & \text{if } \mu = \frac{a - \nu + 2}{2}; \\ (\nu - 1, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, \mu + \frac{a - \nu - 2}{2}, 0), & \text{if } \frac{a - \nu + 4}{2} \leq \mu \leq \frac{\nu - 2}{2}; \\ (\mu + \left\lfloor \frac{\nu - 3}{2} \right\rfloor, b - \mu + \left\lfloor \frac{\nu}{2} \right\rfloor, \mu + \frac{a - \nu - 2}{2}, 0), & \text{if } \frac{\nu}{2} \leq \mu \leq b - \frac{\nu - 2}{2}; \\ (\mu + \left\lfloor \frac{\nu - 3}{2} \right\rfloor, \nu - 1, \mu + \frac{a - \nu - 2}{2}, 0), & \text{if } b - \frac{\nu - 4}{2} \leq \mu \leq b. \end{cases} \tag{29}$$

where $z = 1$, when $\nu = a, \mu = 1$ otherwise, $z = 0$.

For $\nu = 3\kappa = odd$ and $4 \le \nu \le \lceil \frac{a}{2} \rceil$ along $\mu \ne 3\kappa$;

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu-1, b-\mu+\lfloor \frac{\nu}{2} \rfloor, a-\nu, 0), \\ \quad \text{if } 1 \le \mu \le \frac{\nu-1}{2}; \\ (\mu+\lfloor \frac{\nu-3}{2} \rfloor, b-\mu+\lfloor \frac{\nu}{2} \rfloor, a-\nu, 0), \\ \quad \text{if } \frac{\nu+1}{2} \le \mu \le \frac{a-\nu+2}{2}; \\ (\mu+\lfloor \frac{\nu-3}{2} \rfloor, b-\mu+\lfloor \frac{\nu}{2} \rfloor, \mu \\ \quad +\frac{a-\nu-2}{2}, 0), \\ \quad \text{if } \frac{a-\nu+4}{2} \le \mu \le b-\frac{\nu-1}{2}; \\ (\mu+\lfloor \frac{\nu-3}{2} \rfloor, \nu-1, \mu+\frac{a-\nu-2}{2}, 0), \\ \quad \text{if } b-\frac{\nu-3}{2} \le \mu \le b. \end{cases} \tag{30}$$

For $\lceil \frac{a}{2} \rceil + 1 \le \nu \le a$ along $\mu \ne 3\kappa$;

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu-1, b-\mu+\lfloor \frac{\nu}{2} \rfloor, a-\nu, z), \\ \quad \text{if } 1 \le \mu \le \frac{a-\nu+2}{2}; \\ (\nu-1, b-\mu+\lfloor \frac{\nu}{2} \rfloor, \mu+\frac{a-\nu-2}{2}, 0), \\ \quad \text{if } \frac{a-\nu+4}{2} \le \mu \le \frac{\nu-1}{2}; \\ (\mu+\lfloor \frac{\nu-3}{2} \rfloor, b-\mu+\lfloor \frac{\nu}{2} \rfloor, \mu \\ \quad +\frac{a-\nu-2}{2}, 0), \\ \quad \text{if } \frac{\nu+1}{2} \le \mu \le b-\frac{\nu-1}{2}; \\ (\mu+\lfloor \frac{\nu-3}{2} \rfloor, \nu-1, \mu+\frac{a-\nu-2}{2}, 0), \\ \quad \text{if } b-\frac{\nu-3}{2} \le \mu \le b. \end{cases} \tag{31}$$

where $z = 1$, when $\nu = a$, $\mu = 1$ otherwise, $z = 0$.

For $\nu = 3\kappa = even$ and $4 \le \nu \le \lceil \frac{a}{2} \rceil$ along $\mu \ne 3\kappa - 1$;

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu-1, b-\mu+\lfloor \frac{\nu}{2} \rfloor, a-\nu, 0), \\ \quad \text{if } 1 \le \mu \le \frac{\nu}{2}; \\ (\mu+\lfloor \frac{\nu-3}{2} \rfloor, b-\mu+\lfloor \frac{\nu}{2} \rfloor, a-\nu, 0), \\ \quad \text{if } \frac{\nu+2}{2} \le \mu \le \frac{a-\nu+3}{2}; \\ (\mu+\lfloor \frac{\nu-3}{2} \rfloor, b-\mu+\lfloor \frac{\nu}{2} \rfloor, \mu \\ \quad +\frac{a-\nu-3}{2}, 0), \\ \quad \text{if } \frac{a-\nu+5}{2} \le \mu \le b-\frac{\nu-2}{2}; \\ (\mu+\lfloor \frac{\nu-3}{2} \rfloor, \nu-1, \mu+\frac{a-\nu-3}{2}, 0), \\ \quad \text{if } b-\frac{\nu-4}{2} \le \mu \le b. \end{cases} \tag{32}$$

For $\lceil \frac{a}{2} \rceil + 1 \le \nu \le a$ along $\mu \ne 3\kappa$;

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (\nu-1, b-\mu+\lfloor \frac{\nu}{2} \rfloor, a-\nu, z), \\ \quad \text{if } 1 \le \mu \le \frac{a-\nu+3}{2}; \\ (\nu-1, b-\mu+\lfloor \frac{\nu}{2} \rfloor, \mu+\frac{a-\nu-3}{2}, 0), \\ \quad \text{if } \frac{a-\nu+5}{2} \le \mu \le \frac{\nu}{2}; \\ (\mu+\lfloor \frac{\nu-3}{2} \rfloor, b-\mu+\lfloor \frac{\nu}{2} \rfloor, \mu \\ \quad +\frac{a-\nu-3}{2}, 0), \\ \quad \text{if } \frac{\nu+2}{2} \le \mu \le b-\frac{\nu-2}{2}; \\ (\mu+\lfloor \frac{\nu-3}{2} \rfloor, \nu-1, \mu+\frac{a-\nu-3}{2}, 0), \\ \quad \text{if } b-\frac{\nu-4}{2} \le \mu \le b. \end{cases} \tag{33}$$

where $z = 1$, when $\nu = a$, $\mu = 1$ otherwise, $z = 0$.

For $\nu = 2b - 1$;

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (2b-2, 2b-2, \mu-1, 0), & \text{if } 1 \le \mu \le b. \end{cases} \tag{34}$$

$\nu = 2b - 2$;

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (2b-3, 2b-2, 1, 0), & \text{if } \mu = 1; \\ (2b-3, 2b-3, \mu-1, 0), & \text{if } 2 \le \mu \le b. \end{cases} \tag{35}$$

Case 5: For a and ν are odd, $rb \le a \le (r+1)b-1$, $\nu = rb+\kappa$, ν where $0 \le \kappa \le b-1$ and $r \ge 2$;

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (rb+\kappa-1, rb+\kappa-1, a-\nu, 0), \\ \quad \text{if } 1 \le \mu \le \frac{a-rb-\kappa+2}{2}; \\ (rb+\kappa-1, rb+\kappa-1, \mu+\frac{a-rb-\kappa-2}{2}, 0), \\ \quad \text{if } \frac{a-rb-\kappa+4}{2} \le \mu \le b. \end{cases} \tag{36}$$

If ν is even.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (rb+\kappa-1, rb+\kappa-1, a-\nu, 0), \\ \quad \text{if } 1 \le \mu \le \frac{a-rb-\kappa+4}{2}; \\ (rb+\kappa-1, rb+\kappa-1, \mu+\frac{a-rb-\kappa-3}{2}, 0), \\ \quad \text{if } \frac{a-rb-\kappa+5}{2} \le \mu \le b. \end{cases} \tag{37}$$

If $\nu = 3\kappa$, ν is odd and $\mu \neq 3p$.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (rb+\kappa-1, rb+\kappa-1, a-\nu, 0), \\ \quad \text{if } 1 \leq \mu \leq \dfrac{a-rb-\kappa+2}{2}; \\ (rb+\kappa-1, rb+\kappa-1, \mu+\dfrac{a-rb-\kappa-2}{2}, 0), \\ \quad \text{if } \dfrac{a-rb-\kappa+4}{2} \leq \mu \leq b. \end{cases} \tag{38}$$

If $\nu = 3\kappa$, ν is even and $\mu \neq 3p-1$.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (rb+\kappa-1, rb+\kappa-1, a-\nu, 0), \\ \quad \text{if } 1 \leq \mu \leq \dfrac{a-rb-\kappa+3}{2}; \\ (rb+\kappa-1, rb+\kappa-1, \mu+\dfrac{a-rb-\kappa-3}{2}, 0), \\ \quad \text{if } \dfrac{a-rb-\kappa+5}{2} \leq \mu \leq b. \end{cases} \tag{39}$$

Case 6: For a is even, ν is odd, $rb \leq a \leq (r+1)b-1$, $\nu = rb+\kappa$, where $0 \leq \kappa \leq b-1$ and $r \geq 2$;

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (rb+\kappa-1, rb+\kappa-1, a-\nu, 0), \\ \quad \text{if } 1 \leq \mu \leq \dfrac{a-rb-\kappa+1}{2}; \\ (rb+\kappa-1, rb+\kappa-1, \mu+\dfrac{a-rb-\kappa-1}{2} 0), \\ \quad \text{if } \dfrac{a-rb-\kappa+3}{2} \leq \mu \leq b. \end{cases} \tag{40}$$

If ν is even.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (rb+\kappa-1, rb+\kappa-1, a-\nu, 0), \\ \quad \text{if } 1 \leq \mu \leq \dfrac{a-rb-\kappa+2}{2}; \\ (rb+\kappa-1, rn+\kappa-1, \mu+\dfrac{a-rb-\kappa-2}{2}, 0), \\ \quad \text{if } \dfrac{a-rb-\kappa+4}{2} \leq \mu \leq b. \end{cases} \tag{41}$$

If $\nu = 3\kappa$, ν is odd and $\mu \neq 3p$.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (rb+\kappa-1, rb+\kappa-1, a-\nu, 0), \\ \quad \text{if } 1 \leq \mu \leq \dfrac{a-rb-\kappa+1}{2}; \\ (rb+\kappa-1, rb+\kappa-1, \mu+\dfrac{a-rb-\kappa-1}{2}, 0), \\ \quad \text{if } \dfrac{a-rb-\kappa+3}{2} \leq \mu \leq b. \end{cases} \tag{42}$$

If $\nu = 3\kappa$, ν is even and $\mu \neq 3p-1$.

$$r(\chi_{\nu\mu}|\lambda) = \begin{cases} (rb+\kappa-1, rb+\kappa-1, a-\nu, 0), \\ \quad \text{if } 1 \leq \mu \leq \dfrac{a-rb-\kappa+2}{2}; \\ (rb+\kappa-1, rb+\kappa-1, \mu+\dfrac{a-rb-\kappa-2}{2}, 0), \\ \quad \text{if } \dfrac{a-rb-\kappa+4}{2} \leq \mu \leq b. \end{cases} \tag{43}$$

As, all the vertices have different representations with respect to λ. Hence,

$$pd(\alpha_{a,b}) \leq 4. \tag{44}$$

□

III. CONCLUSION AND DISCUSSION

In this work, we determined the partition dimension of α-boron nanotube ($\alpha_{a,b}$) and following are conclusion of our findings:

$$pd(\alpha_{a,b}) = 3, \quad \text{if } a < b,$$
$$pd(\alpha_{a,b}) \leq 4, \quad \text{if } a \geq b.$$

For every possible values of $a, b \in \mathbb{Z}^+$. It is clear to see that the partition dimension is free from the order of structure/graph, but it depends upon the vertical length of the nanotube. When the vertical size (a) is less than the horizontal size (b), the partition dimension is three, while for the remaining possibilities it is four or less. We already have discussed the applications and usage of partition dimension in chemical structures. Now our results states that the entire cluster of atoms (vertex set) of α-boron nanotube can be divide into three ($a < b$) and four ($a \geq b$) subsets. And, for these divisions each atom of α-boron nanotube has its unique identification and accessed through resolving partition set. These results are helpful to hardware designers that use α-boron nanotube in specific required industries as well as engineers.

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