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A New Three Parameter Lifetime Model: The Complementary Poisson Generalized Half Logistic Distribution

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ABSTRACT We propose a new model with flexible failure rate called complementary Poisson generalized half logistic (CPGHL). Various properties of the model are explored and examine numerically such as the explicit expressions of the moments, mean deviations, Bonferroni and Lorenz curves, Shannon and Renyi entropy. The distribution of mixture of two CPGHL and some related models based on the log-transform of CPGHL are discussed. The asymptotic of moments of residual life and asymptotic distribution of order statistics are obtained. The characterization of Poisson half logistic (PHL) by truncated moments of a certain function of a random variable is discussed. Estimation of the model parameters was approached by maximum likelihood, least square, and percentile methods. Further, the estimation by maximum likelihood for right censored data of the model were considered. The proposed estimation techniques were assessed by simulation studies. Three data applications are provided one of them is a censored data to demonstrate how the new model outperforms some other existing distribution in practice.

INDEX TERMS Generalized (exponentiated) half logistic model, least square estimation, maximum likelihood estimation, moments, moments residual life, percentile method of estimation, Renyi entropy, Shannon entropy.

I. INTRODUCTION

Over decades, new families of probability distribution have been introduced to extend or generalize the classical distributions. This arises due to the vast progress in various fields of studies such as physics, engineering, computer science, insurance, biomedical sciences, public health, finance, communications, information theory, reliability, and life-testing, etc. Thus, resulting in new problems that required a high-dimensional data analysis as well as complex decision problems of real life phenomenon. Modeling and investigation of a lifetime data are fundamentals; whereas the statistical distributions are used to model the lifetime data drawn from such phenomenon in order to analyze its important properties effectively. Moreover, the inability of the classical distributions in accommodating various shapes of densities as well as non-monotonic failure rates are some of the important factors discussed in literature which encounter researchers to introduce more generalized and flexible distributions by introducing some additional parameter(s) to the classical ones through various techniques. These indicated a strong demand to propose more new flexible models for better exploration of real life phenomenon that occurs in practical applications.

For the past years, the technique of compounding continuous and discrete distributions has received extensive attention by practitioners in creating more flexible models. The most commonly used in this technique is to combine the discrete distribution with either the distribution of the maximum or minimum order statistics that follow a continuous distribution.

Here, we refer the reader to some distributions introduced by this technique: the complementary Weibull geometric [1], complementary exponentiated BurrXII Poisson [2], complementary generalized transmuted Poisson-G [3], complementary Poisson-Lindley [4], complementary Burr III Poisson [5], complementary geometric transmuted-G [6],

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complementary extended Weibull power series [7], complementary Lindley-geometric [8], complementary compound Lindley Power series [9], complementary exponential power series [10], exponentiated Weibull-logarithmic [11], Poisson odd-generalized exponential-G [12], Burr XII negative binomial [13], complementary exponential-geometric distribution based on maximum order statistics [14], and the complementary exponential geometric distribution based on generalized order statistics by [15], among others.

Generalized half logistic (GHL) distribution (or exponentiated half logistic) [16] has received a significant attention from various practitioners, the cumulative distribution function and probability density function are

$$G(w) = \left(\frac{1-e^{-\alpha w}}{1+e^{-\alpha w}}\right)^{\theta}, \quad w,\ \alpha,\ \theta > 0, \tag{1}$$

and

$$f(w) = \frac{2\alpha\theta e^{-\alpha w}}{(1+e^{-\alpha w})^2}\left(\frac{1-e^{-\alpha w}}{1+e^{-\alpha w}}\right)^{\theta-1}, \tag{2}$$

respectively. [17] discussed some important properties and application of the generalized half logistic distribution. [18] investigated the estimation of multicomponent stress-strength reliability based on generalized half logistic. [19] suggested more efficient technique for estimating shape and scale parameters of generalized half logistic based on record values from Bayesian and non-Bayesian perspectives. The maximum likelihood estimation of the scale parameter in an exponentiated half logistic distribution based on progressively Type-II censored samples have been analyzed by [20]. [21] discussed the maximum likelihood estimation, inverse moment estimation and modified inverse moment estimation for the generalized half logistic distribution and construct the joint confidence regions for the parameters.

The mixture of the Poisson distribution and generalized half logistic distribution based on minimum order statistic from GHL have been considered by [22]. Here, we are aiming at the convolution of the Poisson distribution and generalized half logistic distribution based on maximum order statistic that follow GHL, and we called the new model *complementary Poisson generalized half logistic* (CPGHL). Must of the two parameter classical models with simple closed form properties are incapable of accommodating non-monotone failure rates. The proposed three parameter model is capable of accommodating both monotone and non-monotone failure rates; its log transformation shows a strong relationship with other existing models, it also has closed form properties that can easily be computed numerically, thus, indicating wider applications in various fields of studies. We discuss some important properties of the CPHL, and various estimation techniques of the new model with some numerical results. Moreover, application of the new distribution to complete and censored data are provided for illustration.

The rest of the paper is arranged as follows. In section II, the CPGHL distribution is derived and some important mathematical and statistical properties are discussed. In section III, estimation of parameters for complete data is discussed based on maximum likelihood, least square and percentile methods. Also, the maximum likelihood for right censored data of the CPGHL is considered. Applications of the CPGHL to three real data is provided for illustration in section IV. Conclusions in section V.

II. THE NEW MODEL AND ITS PROPERTIES

In this section, we derive the new model and present some of its important properties. Given $k \in \mathbb{N}$, let $W_1, W_2, \ldots, W_K$, be independent and identically distributed (iid) random variable from GHL distribution, suppose K is discrete random variable distributed zero truncated Poisson with probability mass function given by $P(k;\lambda) = \frac{\lambda^k}{(\exp(\lambda)-1)\,k!}$, $\lambda > 0, k = 1, 2, 3, 4, \cdots$. Let $X = \max\{W_1, W_2, \cdots, W_K\}$, then, the conditional probability density function of X is $f_{X|K=k}(x) = k\, g(x)G(x)^{k-1}$, where, $G(.)$ and $g(.)$ are given by (1) and (2) respectively. The unconditional density function of X is $f(x) = \sum_{k=1}^{\infty} f_{X|K=k}(x)P(K = k)$, thus, the probability density function of X is obtained as

$$f(x) = \frac{2\alpha\theta\lambda e^{-\alpha x}}{(e^{\lambda}-1)(1+e^{-\alpha x})^2}\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\theta-1} e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\theta}}, \tag{3}$$

where $x,\ \alpha,\ \theta, > 0, \lambda \in \mathbb{R} - \{0\}$. When $x \to 0$, then the density $f(x) \to 0$ for $\theta > 1$; $f(x) \to \infty$ for $\theta < 1$, and $f(x) \to \frac{\alpha\lambda}{2(1-e^{-\lambda})}$ for $\theta = 1$. If $x \to \infty$, then $f(x) \to 0$ for all $\alpha, \theta, \lambda > 0$. The cumulative distribution function of the CPGHL distribution is given as

$$F(x) = \frac{e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\theta}} - 1}{e^{\lambda}-1}, \quad x,\ \alpha, \theta > 0, \lambda \in \mathbb{R} - \{0\}. \tag{4}$$

Interpretation 1: Let W be a random variable with pdf $v(y) = \lambda e^{\lambda y}/(e^{\lambda}-1)$, $y \in (0, 1)$, and $\lambda \in \mathbb{R} - \{0\}$. Let $G(x)$ be a valid cumulative distribution function of an absolutely continuous random variable X. A family of generalized cumulative distribution function of X can take the from

$$F(x) = \int_0^{G(x)} \frac{\lambda e^{\lambda y}}{e^{\lambda}-1} dy = \frac{e^{\lambda G(x)}-1}{e^{\lambda}-1}, \tag{5}$$

therefore, the cumulative distribution given in (4) can be a special case of (5) by taking $G(.)$ as the cdf of the GHL.

Proposition 2: For a very sufficiently small $\lambda > 0$ the limiting distribution of CPGHL is the GHL, i.e.

$$\lim_{\lambda\to 0^+} F(x) = \left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\theta}.$$

The survival function and hazard rate function of the CPGHL are given by

$$s(x) = \frac{e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\theta}} - e^{\lambda}}{1-e^{\lambda}}, \tag{6}$$

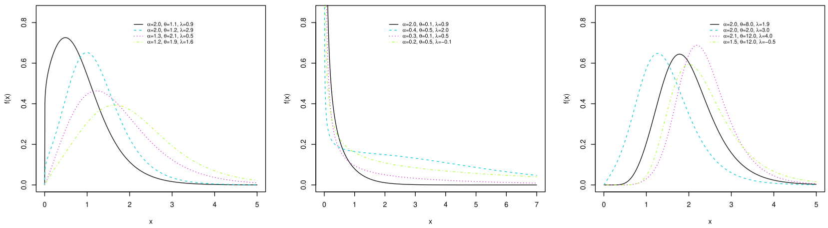

FIGURE 1. Plots of density function for some parameters values.

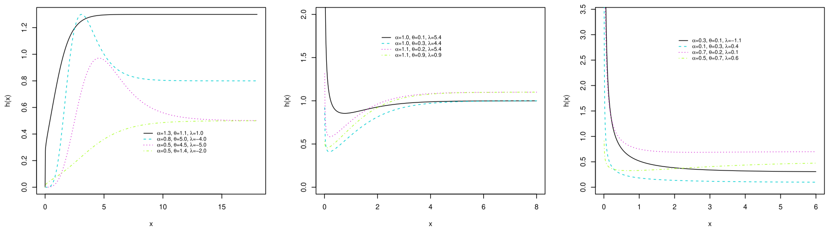

FIGURE 2. Plots of hazard function for some parameter values.

and

$$h(x) = \frac{2\alpha\theta\lambda e^{-\alpha x}\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\theta-1}}{(1+e^{-\alpha x})^2\left(e^{\lambda} - e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\theta}}\right)} e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\theta}}, \quad (7)$$

respectively, where $x,\ \alpha,\ \theta > 0,\ \lambda \in \mathbb{R} - \{0\}$. If $x \to 0$, then the hazard function $h(x) \to 0$ for $\theta > 1$; $h(x) \to \infty$ for $\theta < 1$, and $h(x) \to \frac{\alpha\lambda}{2(1-e^{-\lambda})}$ for $\theta = 1$, also as $x \to \infty$, $h(x) \to 0$ for all $\alpha, \theta, \lambda > 0$. Figure 1 and 2 illustrate the plots of the density and hazard function of the CPGHL for some various parameter values. Figure 2 shows that CPGHL can accommodate decreasing, increasing, upside-down bathtub and bathtub-shaped failure rates.

A. QUANTILE AND MOMENTS

In this subsection, we discuss the quantile and moments of the CPGHL. The quantile function in (8) can be used to analyze the skewness and kurtosis of the CPGHL; also for parameter estimation of CPGHL, and generating random data that follow the CPGHL by taking $W \sim U(0, 1)$, where $U(0, 1)$ is a uniform distribution. Quantile function also aid in computations of some distribution properties.

$$q(w) = \frac{1}{\alpha}\left[\log\left(1 - K_{\lambda}^{1/\theta}(w)\right) - \log\left(1 + K_{\lambda}^{1/\theta}(w)\right)\right], \quad (8)$$

where $K_{\lambda}(w) = \frac{\log(1+w(e^{\lambda}-1))}{\lambda}$. The median of X with CPGHL is $q(1/2)$.

The Moor's kurtosis (Mk) and MacGillivray's skewness (MGs) defined in [23] and [24] respectively, are measures based on quantile function used to analyze the kurtosis and skewness of a distribution respectively, defined by

$$Mk = \frac{q(\frac{7}{8}) - q(\frac{5}{8}) + q(\frac{3}{8}) - q(\frac{1}{8})}{q(\frac{6}{8}) - q(\frac{2}{8})},$$

and

$$MGs = \frac{m^{(1)}(u;\theta,\lambda) - m^{(2)}(u;\theta,\lambda)}{m^{(3)}(u;\theta,\lambda)},$$

where $q(.)$ is given by (8),

$$\begin{aligned} m^{(1)}(u;\theta,\lambda) &= \log\left(1 - K_{\lambda}^{1/\theta}(1-u)\right) + \log\left(1 - K_{\lambda}^{1/\theta}(u)\right) \\ &\quad - 2\log\left(1 - K_{\lambda}^{1/\theta}(1/2)\right), \\ m^{(2)}(u;\theta,\lambda) &= -\log\left(1+K_{\lambda}^{1/\theta}(1-u)\right) - \log\left(1+K_{\lambda}^{1/\theta}(u)\right) \\ &\quad + 2\log\left(1 + K_{\lambda}^{1/\theta}(1/2)\right), \\ m^{(3)}(u;\theta,\lambda) &= \log\left(1 - K_{\lambda}^{1/\theta}(1-u)\right) - \log\left(1 - K_{\lambda}^{1/\theta}(u)\right) \\ &\quad - \log\left(1+K_{\lambda}^{1/\theta}(1-u)\right) + \log\left(1+K_{\lambda}^{1/\theta}(u)\right), \end{aligned}$$

and $u \in \left(0, \frac{1}{2}\right)$. Notice that both Mk and MGs are independent of α. Figure 3 (i) shows that the skewness is decreasing then increasing as both θ and λ increases. Figure 3 (ii) shows for fixed λ the kurtosis is decreasing as θ increases; when the λ increases the kurtosis is decreasing then increasing as θ increases. Figure 3 (iii) shows for fixed θ the kurtosis is increasing as λ increases; when θ increases the kurtosis is increasing then decreasing as λ increases.

The r^{th} ordinary moment of the CPGHL distribution can be obtained by $\mu_r = E[X^r] = \int_0^{\infty} x^r f(x)dx$. After some algebraic simplification we get

$$\mu_r = b_r \int_0^1 \log^r\left(\frac{1 - u^{\frac{1}{\theta}}}{1 + u^{\frac{1}{\theta}}}\right) e^{\lambda u}\, du \quad (9)$$

where $b_r = \frac{(-1)^r \lambda}{\alpha^r (e^{\lambda}-1)}$. It is not sure that the integral in (9) has a closed form but it can be easily obtained numerically using `Mathematica` and `R` etc. Moreover, the expression of the r^{th} ordinary moment can be represented in a closed

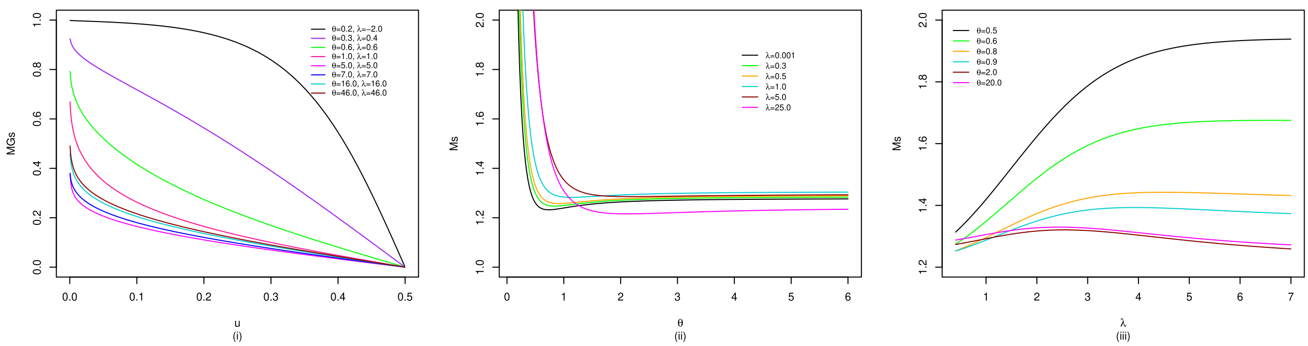

FIGURE 3. Plots of the Mk kurtosis and MGs skewness for some parameter values.

form in infinite series as follows. After the expansion of the $e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\theta}}$ in $f(x)$ we have

$$\mu_r = \frac{2\alpha\theta\lambda}{e^{\lambda}-1}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!}\int_0^{\infty} x^r e^{-\alpha x}\frac{\left(1-e^{-\alpha x}\right)^{\theta(i+1)-1}}{\left(1+e^{-\alpha x}\right)^{\theta(i+1)+1}}\,dx,$$

letting $v = 1-e^{-\alpha x}$, we have $\mu_r = \sum_{i=0}^{\infty} d_i \int_0^1 \frac{\log^r(1-v)v^{\theta(i+1)-1}}{(1+(1-v))^{\theta(i+1)+1}}\,dv$, where $d_i = \frac{2\theta\lambda^{i+1}(-1)^r}{(e^{\lambda}-1)i!\,\alpha^r}$. Applying the expansion of $(1+(1-v))^{-\theta(i+1)+1} = \sum_{j=0}^{\infty}(-1)^j\binom{\theta(i+1)+j}{j}(1-v)^j$ we get

$$\mu_r = \sum_{i=0}^{\infty} w_{i,j}\int_0^{\infty}\log^r(1-v)(1-v)^j v^{\theta(i+1)-1}\,dv,$$

the integral become the r^{th} partial derivative of beta function with respect to $j+1$, where $w_{i,j} = \sum_{j=0}^{\infty} d_i(-1)^j\binom{\theta(i+1)+j}{j}$, thus

$$\mu_r = \sum_{i=0}^{\infty} w_{i,j}\mathcal{B}_{0,r}(\theta(i+1), j+1), \tag{10}$$

where $\mathcal{B}_{p,r}(t,z) = \frac{\partial^{p+r}}{\partial t^p\,\partial z^r}\mathcal{B}(t,z)$ and $\mathcal{B}(t,z) = \int_0^1(1-v)^{z-1}v^{t-1}dv$ is a beta function, $t, z > 0$, the derivative $\mathcal{B}_{p,r}(t,z)$ can be found in [25]. In particular, the first four moments from (10) can be express as follows

$$\mu_1 = \sum_{j=0}^{\infty} w_{i,j}\left[\psi^{(0)}(j+1) - \psi^{(0)}(\theta(i+1)+j+1)\right] \times \mathcal{B}(\theta(i+1), j+1).$$

$$\begin{aligned}\mu_2 = \sum_{j=0}^{\infty} w_{i,j}\Big[&\left(\psi^{(0)}(j+1) - \psi^{(0)}(\theta(i+1)+j+1)\right)^2 \\ &- \psi^{(1)}(\theta(i+1)+j+1) + \psi^{(1)}(j+1)\Big] \\ &\times \mathcal{B}(\theta(i+1), j+1)\end{aligned}$$

$$\begin{aligned}\mu_3 = \sum_{j=0}^{\infty} w_{i,j}\Big[&\left(\psi^{(0)}(j+1) - \psi^{(0)}(\theta(i+1)+j+1)\right)^3 \\ &+3\left(\psi^{(1)}(j+1) - \psi^{(1)}(\theta(i+1)+j+1)\right) \\ &\times\left(\psi^{(0)}(j+1) - \psi^{(0)}(\theta(i+1)+j+1)\right) \\ &- \psi^{(2)}(\theta(i+1)+j+1) + \psi^{(2)}(j+1)\Big] \\ &\times \mathcal{B}(\theta(i+1), j+1).\end{aligned}$$

$$\begin{aligned}\mu_4 = \sum_{j=0}^{\infty} w_{i,j}\Big[&\left(\psi^{(0)}(j+1) - \psi^{(0)}(\theta(i+1)+j+1)\right)^4 \\ &+6\left(\psi^{(1)}(j+1) - \psi^{(1)}(\theta(i+1)+j+1)\right) \\ &\times\left(\psi^{(0)}(j+1) - \psi^{(0)}(\theta(i+1)+j+1)\right)^2 \\ &+4\left(\psi^{(2)}(j+1) - \psi^{(2)}(\theta(i+1)+j+1)\right) \\ &\times\left(\psi^{(0)}(j+1) - \psi^{(0)}(\theta(i+1)+j+1)\right) \\ &+3\left(\psi^{(1)}(j+1) - \psi^{(1)}(\theta(i+1)+j+1)\right)^2 \\ &- \psi^{(3)}(\theta(i+1)+j+1) + \psi^{(3)}(j+1)\Big] \\ &\times \mathcal{B}(\theta(i+1), j+1)\end{aligned}$$

where $\psi^{(m)}(t) = \frac{d^m}{dt^m}\psi(t) = \frac{d^{m+1}}{dt^{m+1}}\ln\Gamma(t)$, $t > 0$ is a polygamma function, and $\psi^{(0)}(t) = \frac{d}{dt}\ln\Gamma(t)$ is a digamma function.

Figure 4 shows that for $\alpha = 1$, the mean (μ_1) is increasing with increase in θ and $\lambda > 0$, while the variance $(\sigma^2 = \mu_2 - \mu_1^2)$ is increasing then decreasing with increase in θ and $\lambda > 0$.

The r^{th} incomplete moments $(\boldsymbol{J}_r(t))$ of CPGHL can be derived in similar way to (10) given as follows, its useful in computations of the mean deviations etc.

$$\begin{aligned}\boldsymbol{J}_r(t) &= \int_0^t x^r f(x)dx \\ &= \sum_{j=0}^{\infty} w_{i,j}\frac{\partial^r}{\partial u^r}\mathcal{B}(1-e^{-\alpha t};\theta(i+1), j+1),\end{aligned} \tag{11}$$

where $u = j+1$ and $\mathcal{B}(1-e^{-\alpha t};\theta(i+1), j+1)$ is incomplete beta function.

B. MEAN DEVIATIONS, BONFERRONI CURVE AND LORENZ CURVE

The mean deviation about the mean $\delta_1(X)$ and mean deviation about the median $\delta_2(X)$ of a random variable X with

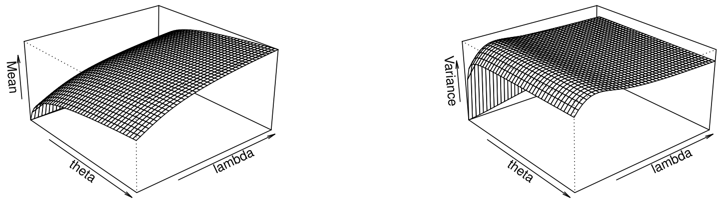

FIGURE 4. Plots of the mean (left) and variance (right) of CPGHL distribution for $\alpha = 1$ and $\lambda > 0$.

CPGHL are defined by $\delta_1(X) = \int_0^\infty \mid x-\mu \mid f(x)dx = 2\mu F(\mu) - 2\jmath_1(\mu)$ and $\delta_2(X) = \int_0^\infty \mid x - M \mid f(x)dx = \mu - 2\jmath_1(M)$, respectively, where $F(x)$ is the distribution function of X, $\mu_1 = E(X)$ is the mean of X and $M = Median(X)$. To obtain $\delta_1(X)$ and $\delta_2(X)$, it is enough to compute $\jmath_1(t)$ from (11) as.

$$\jmath_1(t) = \sum_{j=0}^{\infty} w_{i,j} \frac{\partial}{\partial u} \mathcal{B}(1-e^{-\alpha t}; \theta(i+1), j+1). \quad (12)$$

The Bonferroni and Lorenz curves are very important measures in econometrics, insurance, among others, so-called income inequalities. They are used to describes distribution of wealth among population. The Bonferroni and Lorenz curves are defined respectively by $B(p) = \frac{\jmath_1(q)}{p\mu_1}$ and $L(p) = \frac{\jmath_1(q)}{\mu_1}$, where $\jmath_1(q)$ is computed from (11), μ_1 can be computed from (10), $q = q(p)$ is derived from (8) and p is any given probability. Therefore, $\jmath_1(q)$ is give by

$$\jmath_1(q) = \sum_{j=0}^{\infty} w_{i,j} \frac{\partial}{\partial u} \mathcal{B}(1-e^{-\alpha q(p)}; \theta(i+1), j+1),$$

where $q(p) = \frac{1}{\alpha}\left[\log\left(1 - K_\lambda^{1/\theta}(p)\right) - \log\left(1 + K_\lambda^{1/\theta}(p)\right)\right]$. Figure 5 show the plots of the Bonferroni and Lorenz curves for some parameter values. The curves in figure 5 shows the ability of CPGHL in analyzing various populations overall income, wealth or social inequality. The upper curve in the L(p) is more closer to the line of perfect inequality i.e 45^o, also, for the B(p) curve the index is measured from the upper part of the curve and the line of perfect inequality is the top borderline, and the upper curve in B(p) is moving towards the perfect in equality indicating the flexibility of the CPGHL in various income analysis.

C. ENTROPY

Entropy is a measure of variation of uncertainty in a random variable. In this sub section, we compute the Renyi and Shannon entropies of the CPGHL. The following lemma aided in the computations of the Renyi and Shannon entropies.

Lemma 3: Let $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in \mathbb{R}, x > 0$, let

$$\boldsymbol{L}(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4) = \int_0^\infty \frac{e^{-\vartheta_1 x}\left(1-e^{-\alpha x}\right)^{\vartheta_2}}{\left(1+e^{-\alpha x}\right)^{\vartheta_3}} e^{\vartheta_4\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)^\theta} dx, \quad (13)$$

then,

$$\boldsymbol{L}(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j \vartheta_4^i}{\alpha i!}\binom{\vartheta_3+\theta i+j-1}{j} \times \mathcal{B}(\vartheta_2+\theta i+1, \frac{\vartheta_1}{\alpha}+j), \quad (14)$$

where $\mathcal{B}(.,.)$ is a beta function.

Proof: By expanding the exponential expression, then letting $v = 1-e^{-\alpha x}$ in equation (13) we get,

$$\boldsymbol{L}(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4) = \sum_{i=0}^{\infty} \frac{\vartheta_4^i}{\alpha i!} \int_0^1 \frac{(1-v)^{\frac{\vartheta_1}{\alpha}-1} v^{\vartheta_2+\theta i}}{(1+(1-v))^{\vartheta_3+\theta i}} dv,$$

we expand the expression $(1+(1-v))^{-(\vartheta_3+\theta i)} = \sum_{j=0}^{\infty}(-1)^j\binom{\vartheta_3+\theta i+j-1}{j}(1-v)^j$ and after some simplification the integral become beta function,

$$\boldsymbol{L}(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j \vartheta_4^i}{\alpha i!}\binom{\vartheta_3+\theta i+j-1}{j} \times \int_0^1 (1-v)^{\frac{\vartheta_1}{\alpha}+j-1} v^{\vartheta_2+\theta i} dv,$$

thus,

$$\boldsymbol{L}(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j \vartheta_4^i}{\alpha i!}\binom{\vartheta_3+\theta i+j-1}{j} \times \boldsymbol{B}(\vartheta_2+\theta i+1, \frac{\vartheta_1}{\alpha}+j).$$

□

The Renyi entropy of a random variable X with CPGHL can be obtained from $I_{R(\rho)} = (1-\rho)^{-1}\ln\int_{-\infty}^{\infty} f^\rho(x)dx$, for $\rho > 0$ and $\rho \neq 1$. We begin with computing $\int_0^\infty f^\rho(x)dx$.

$$\int_0^\infty f^\rho(x)dx = \frac{2^\rho\alpha^\rho\theta^\rho\lambda^\rho}{(e^\lambda-1)^\rho} \times \int_0^\infty \frac{e^{-\rho\alpha x}\left(1-e^{-\alpha x}\right)^{\rho(\theta-1)}}{\left(1+e^{-\alpha x}\right)^{\rho(\theta+1)}} e^{\lambda\rho\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)^\theta} dx = \frac{2^\rho\alpha^\rho\theta^\rho\lambda^\rho}{(e^\lambda-1)^\rho}\boldsymbol{L}(\alpha\rho, \rho(\theta-1), \rho(\theta+1), \lambda\rho).$$

Thus,

$$I_{R(\rho)} = (1-\rho)^{-1}\left(\frac{2^\rho\alpha^\rho\theta^\rho\lambda^\rho}{(e^\lambda-1)^\rho}\right) + (1-\rho)^{-1}\ln\boldsymbol{L}(\alpha\rho, \rho(\theta-1), \rho(\theta+1), \lambda\rho).$$

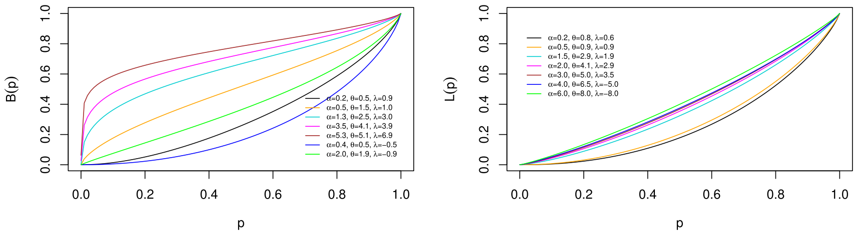

FIGURE 5. Plots of the Bonferroni curve *B*(*p*) and Lorenz curve *L*(*p*) of CPGHL distribution for some parameter values.

The Shannon entropy of X with CPGHL can be defined by $E[-\log f(X)]$, it is also a particular case of the Renyi entropy when $\rho \to 1$. We consider the following lemma 4 first.

Lemma 4: Let X follow CPGHL(α, θ, λ), then,

$$E\left[\log\left(1-e^{-\alpha X}\right)\right] = \frac{2\alpha\theta\lambda}{(e^{\lambda}-1)}\frac{\partial}{\partial t}\boldsymbol{L}(\alpha, \theta+t-1, \theta+1, \lambda)|_{t=0}, \tag{15}$$

$$E\left[\log\left(1+e^{-\alpha X}\right)\right] = \frac{2\alpha\theta\lambda}{(e^{\lambda}-1)}\frac{\partial}{\partial t}\boldsymbol{L}(\alpha, \theta-1, \theta-t+1, \lambda)|_{t=0}, \tag{16}$$

$$E\left[\left(\frac{1-e^{-\alpha X}}{1+e^{-\alpha X}}\right)^{\theta}\right] = \frac{2\alpha\theta\lambda}{(e^{\lambda}-1)}\boldsymbol{L}(\alpha, 2\theta-1, 2\theta+1, \lambda), \tag{17}$$

where $\boldsymbol{L}(., ., ., .)$ is given by (14).

Proof: For the first part, we have $E\left[\log\left(1-e^{-\alpha X}\right)\right] = \frac{\partial}{\partial t}E\left[\left(1-e^{-\alpha X}\right)^{t}\right]\Big|_{t=0} = \frac{2\alpha\theta\lambda}{(e^{\lambda}-1)}\frac{\partial}{\partial t}\int_0^{\infty} e^{-\alpha x}\frac{(1-e^{-\alpha x})^{\theta+t-1}}{(1+e^{-\alpha x})^{\theta+1}} e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\theta}} dx\,|_{t=0}$, thus, from lemma 3, $E\left[\log\left(1-e^{-\alpha X}\right)\right] = \frac{2\alpha\theta\lambda}{(e^{\lambda}-1)}\frac{\partial}{\partial t}\boldsymbol{L}(\alpha, \theta+t-1, \theta+1, \lambda)|_{t=0}$. Equation (16) follow similar to (15). Now,

$$E\left[\left(\frac{1-e^{-\alpha X}}{1+e^{-\alpha X}}\right)^{\theta}\right] = \frac{2\alpha\theta\lambda}{(e^{\lambda}-1)}\int_0^{\infty} e^{-\alpha x}\frac{\left(1-e^{-\alpha x}\right)^{2\theta-1}}{\left(1+e^{-\alpha x}\right)^{2\theta+1}} e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\theta}} dx = \frac{2\alpha\theta\lambda}{(e^{\lambda}-1)}\boldsymbol{L}(\alpha, 2\theta-1, 2\theta+1, \lambda).$$

□

Therefore, the Shannon entropy is given by

$$\begin{aligned} E\left[-\log f(X)\right] &= \log\left(\frac{e^{\lambda}-1}{2\alpha\theta\lambda}\right) - \alpha\mu_1 \\ &- \frac{2(\theta-1)\alpha\theta\lambda}{(e^{\lambda}-1)}\frac{\partial}{\partial t}\boldsymbol{L}(\alpha, \theta+t-1, \theta+1, \lambda)|_{t=0} \\ &+ \frac{2(\theta+1)\alpha\theta\lambda}{(e^{\lambda}-1)}\frac{\partial}{\partial t}\boldsymbol{L}(\alpha, \theta-1, \theta-t+1, \lambda)|_{t=0} \\ &- \frac{2\alpha\theta\lambda^2}{(e^{\lambda}-1)}\boldsymbol{L}(\alpha, 2\theta-1, 2\theta+1, \lambda). \end{aligned}$$

TABLE 1. Renyi and Shannon entropies of the CPGHL for some parameter values.

$(\rho, \alpha, \theta, \lambda)$	$I_{R(\rho)}$	$(\alpha, \theta, \lambda)$	$E[-\log f(X)]$
(0.1, 0.1, 0.1, -1.0)	4.6313	(0.2, 0.2, -2.0)	-2.2880
(0.1, 0.2, 0.2, -0.1)	4.0650	(0.3, 0.4, -0.5)	1.3455
(0.2, 0.3, 0.3, 0.1)	3.1062	(0.4, 0.5, 0.6)	1.8654
(0.3, 0.4, 0.4, 0.2)	2.5654	(0.5, 0.7, 0.9)	1.9902
(0.4, 0.8, 0.9, 0.9)	2.0034	(0.9, 0.8, 1.0)	1.4902
(0.6, 1.0, 1.0, 1.0)	1.6417	(1.2, 1.4, 1.1)	1.3818
(0.9, 1.6, 1.7, 1.9)	1.2053	(1.6, 1.5, 1.4)	1.1307
(1.5, 3.0, 2.0, 2.0)	0.4594	(1.9, 2.0, 2.1)	1.0244
(3.0, 4.0, 3.0, 5.0)	0.0180	(3.0, 4.0, 3.1)	0.6037
(3.0, 9.0, 6.0, 8.0)	-0.8287	(5.0, 7.0, 6.0)	0.0651
(6.0, 14.0, 15.0, 6.0)	-1.3593	(9.0, 8.0, 9.0)	-0.5554
(7.0, 15.0, 16.0, 8.0)	-1.4687	(19.0, 18.0, 19.0)	-1.3391

Table 1 indicated that the Renyi entropy is decreasing with increase in ρ and the parameters α, θ, and λ, while the Shannon entropy is increasing then decreasing with increase in α, θ, and λ.

D. MIXTURE OF TWO CPGHL AND LOG-CPGHL DISTRIBUTIONS

We present the mixture of two CPGHL distribution, also, the Log transform of CPGHL and some related distributions are derived.

The mixture of distributions have been studied in detail by many authors in literature. For more information about the application and estimation technique of mixture of distributions, see [27]–[31]. In computer sciences and engineering, probabilistic mixture models such as Gaussian mixture are used to resolve point set registration problems in image processing, computer vision fields, and reliability studies.

The density of the mixture of two CPGHL distributions (MixCPGHL) can be expressed as

$$f(x;\Theta) = \sum_{i=1}^{2} a_i f_i(x;\Theta_{\mathbf{i}}), \tag{18}$$

where $\sum_{i=1}^{2} a_i = 1$, $\Theta = (\Theta_{\mathbf{1}}, \Theta_{\mathbf{2}})^T$, $\Theta_{\mathbf{1}} = (\alpha_1, \theta_1, \lambda_1)^T$, $\Theta_{\mathbf{2}} = (\alpha_2, \theta_2, \lambda_2)^T$, and $f_i(x;\Theta_{\mathbf{i}})$, $i = 1, 2$, is the density of the CPGHL distribution, given by

$$f_i(x;\Theta_{\mathbf{i}}) = \frac{2\alpha_i\theta_i\lambda_i e^{-\alpha_i x}\left(\frac{1-e^{-\alpha_i x}}{1+e^{-\alpha_i x}}\right)^{\theta_i-1}}{(e^{\lambda_i}-1)(1+e^{-\alpha_i x})^2} e^{\lambda_i\left(\frac{1-e^{-\alpha_i x}}{1+e^{-\alpha_i x}}\right)^{\theta_i}}, \tag{19}$$

where $x, \alpha_i, \theta_i, \lambda_i > 0$. Figure 6 give some plots of the density functions of the MixCPGHL for some parameter values.

Despite the applications and important of log transformation in mathematics, it's a very vital tool in statistics and probability, for instances, log transformation can be used to transform a highly skewed distribution to a less skewed, it's also used to explore some characterizations of a distribution, it's also result in a given new useful model that is greatly used in solving various problems in practice. For example, log-normal from log transform of normal distribution, log logistic from log transform of logistic distribution, among others.

Let X be a random variable with CPGHL in (3), we can determine some related distributions as follows.

Proposition 5: Let X be a random variable having pdf in (3), let a random variable $T > 0$ take the form $T = \frac{1}{\gamma}\log\left(1+\frac{\gamma}{\beta}\log\left(\frac{1+e^{-\alpha X}}{2e^{-\alpha X}}\right)\right)$, then T has the generalized Gompertz Poisson (GGP) [32] with parameters $\beta, \theta, \gamma, \lambda > 0$, and if $\theta = 1$ we have Gompertz Poisson (GP) [32].

Proof: Let $T = \frac{1}{\gamma}\log\left(1+\frac{\gamma}{\beta}\log\left(\frac{1+e^{-\alpha X}}{2e^{-\alpha X}}\right)\right)$, then $X = \frac{1}{\alpha}\log\left(2e^{\frac{\beta}{\gamma}(e^{\gamma T}-1)}-1\right)$, where the Jacobian of the transformation is $J = \frac{2\beta e^{\gamma y}e^{\frac{\beta}{\gamma}(e^{\gamma y}-1)}}{\alpha\left(2e^{\frac{\beta}{\gamma}(e^{\gamma t}-1)}-1\right)}$, thus,

$$f(t) = \frac{\beta\theta\lambda e^{\gamma t}}{e^{\lambda}-1}e^{-\frac{\beta}{\gamma}(e^{\gamma t}-1)}\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma t}-1)}\right)^{\theta-1} \times e^{\lambda\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma t}-1)}\right)^{\theta}}$$

which is the pdf of the generalized Gompertz Poisson. □

Proposition 6: Let X be a random variable having CPGHL with pdf given by (3), let $T > 0$ be a random variable such that $T = \left(\frac{1}{\beta}\log\left(\frac{1+e^{-\alpha X}}{2\,e^{-\alpha X}}\right)\right)^{1/\gamma}$, then T has the complementary exponentiated Weibull Poisson distribution (EWP) [33] with parameters $\theta, \beta, \gamma, \lambda > 0$.

Proof: Let $T = \left(\frac{1}{\beta}\log\left(\frac{1+e^{-\alpha X}}{2\,e^{-\alpha X}}\right)\right)^{1/\gamma}$, this implies $X = \frac{1}{\alpha}\log(2e^{\beta T^{\gamma}}-1)$ and the Jacobian of the transformation is $J = \frac{2\beta\gamma e^{\beta t^{\gamma}}t^{\gamma-1}}{\alpha(2\,e^{\beta t^{\gamma}}-1)}$, thus T has the density of the form

$$f(t) = \frac{\beta\theta\gamma\lambda t^{\gamma-1}}{e^{\lambda}-1}e^{-\beta t^{\gamma}}(1-e^{-\beta t^{\gamma}})^{\theta-1}e^{\lambda(1-e^{-\beta t^{\gamma}})^{\theta}}$$

which is the pdf of the CEWP followed from [33]. □

Notice that other related distribution can be derived in similar way to proposition 6 using $T = \left(\frac{1}{\beta}\log\left(\frac{1+e^{-\alpha X}}{2\,e^{-\alpha X}}\right)\right)^{1/\gamma}$: If $\theta = 1$, T has the complementary Weibull Poisson (CWP) [33]; if $\gamma = 1$, T is generalized exponential Poisson (GEP) [33]; if $\theta = \gamma = 1$, T is Poisson exponential (PE) [34]; if $\beta = a(1-e^{-b})$, with $a,\ b > 0$, then T has complementary Poisson generalized new-weibull (CPGNW), with the baseline cdf $G(t) = (1-e^{-a(1-e^{-b})t^{\gamma}})^{\theta}$; if $\beta = a(1-e^{-b})$, with $a,\ b > 0$, and $\theta = 1$ then T has complementary Poisson new-weibull (CPNW); if $\beta = a(1-e^{-b})$, with $a,\ b > 0$, and $\gamma = 1$ then T has complementary Poisson exponentiated Erlang-truncated exponential (CPEETE); if $\beta = a(1-e^{-b})$, with $a,\ b > 0$, and $\theta = \gamma = 1$ then T has complementary Poisson Erlang-truncated exponential (CPETE).

E. MOMENTS OF RESIDUAL LIFE

Mean residual life and mean reverse residual life are important tools in quality control, engineering, and life testing, among others. In some cases they are used in the determination of the asymptotic distribution for sample minimum or maximum of order statistics. The mean residual life is the expected time beyond t until failure, given that a component has survived up to the time t. For a random variable X, the mean residual life is defined by $\mathcal{M}(t) = E(X-t|X>t)$, alternatively, $\mathcal{M}(t) = \int_0^{\infty}\frac{s(x+t)}{s(t)}dx$, where $s(t)$ is the survival function of X. The mean reversed residual life is defined to be the conditional random variable $t-X|X\le t$ which denotes the time elapsed from the failure of a component given that its life is less than or equal to t. The mean reverse residual life of X is defined by $\bar{\mathcal{M}}(t) = E(t-X|X\le t)$, in other form $\bar{\mathcal{M}}(t) = \int_0^t\frac{F(x)}{F(t)}dx$. Here, we derive the asymptotic of the two residual life.

Proposition 7: Let $X \sim$ CPGHL with survival function given by (6), then, for sufficiently large value of $t > 0$, i.e as $x\to\infty$,

$$\mathcal{M}(t) \sim \frac{1}{\alpha}.$$

Proof: We first determine the asymptotic of $s(x)$ in (6) as $x\to\infty$,

$$s(x) = \frac{e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)^{\theta}}-e^{\lambda}}{1-e^{\lambda}} \sim \frac{2\theta\lambda e^{\lambda}}{e^{\lambda}-1}e^{-\alpha x}\ as\ x\to\infty.$$

Notice that the survival function of CPGHL goes to exponential function for sufficiently large x. The exponential distribution with density $f(x) = \alpha e^{-\alpha x}$ has expectation $1/\alpha$, recall that exponential distribution as the limit of geometric distribution, and it is among the properties of geometric distribution "lack of memory" i.e whatever the present time the residual lifetime is unaffected by the past and has the same distribution as the lifetime itself. Now, for CPGHL distribution since as $x\to\infty$, $s(x)$ goes to exponential, indicating that when $t\to\infty$, $\frac{s(x+t)}{s(t)} \sim e^{-\alpha x}$. Therefore, as $t\to\infty$

$$\mathcal{M}(t) = \int_0^{\infty}\frac{s(x+t)}{s(t)}dx \sim \int_0^{\infty}e^{-\alpha x} = 1/\alpha.$$

□

Proposition 8: Let $X \sim$ CPGHL with cdf in (6), then, for sufficiently small value of $t > 0$,

$$\bar{\mathcal{M}}(t) \sim \frac{t}{\theta+1}$$

Proof: The asymptotic of the cdf of the CPGHL in (4) as $x\to 0$ is

$$F(x) \sim \frac{\lambda(1-e^{-\alpha x})^{\theta}}{2^{\theta}(e^{\lambda}-1)} \sim \frac{\lambda\alpha^{\theta}x^{\theta}}{2^{\theta}(e^{\lambda}-1)}.$$

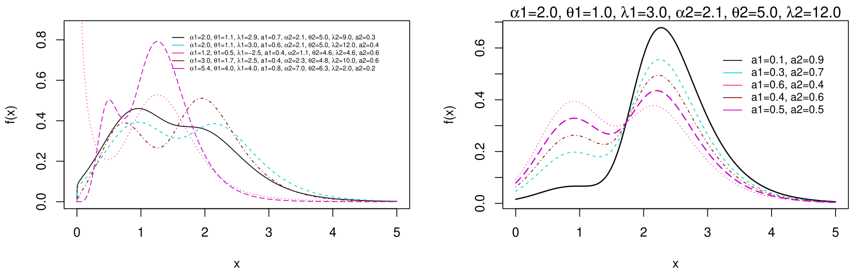

FIGURE 6. **Plots of the density function of the MixCPGHL for some parameter values.**

Thus, as $t \to 0$ we have

$$\bar{\mathcal{M}}(t) = \int_0^t \frac{F(x)}{F(t)}dx \sim \frac{1}{t^\theta}\int_0^t x^\theta dx = \frac{t}{\theta+1}.$$

□

F. ORDER STATISTICS AND ASYMPTOTIC

Order statistics are one of the essential tools for modeling random phenomena in life testing, and quality control among others. Let $X_1, X_2, \cdots, X_n, n \geq 1$, be an ordered sample from *CPGHL*, the probability density function of the j^{th}−order statistics denoted by $f_{j:n}(x)$ is given as follows.

$$f_{x_{j:n}}(x) = \frac{n!}{(j-1)!(n-j)!}f(x)(F(x))^{j-1}(1-F(x))^{n-j},$$
$$= \sum_{m=0}^{n-j}\frac{n!\,(-1)^m}{(j-1)!(n-j-m)!\,m!}f(x)F^{j+m-1}(x). \quad (20)$$

For $F(x)$ in (4) we have $F^{j+m-1}(x) = \sum_{k=0}^{j+m-1}(-1)^{j+m+k-1}\binom{j+m-1}{k}e^{\lambda k\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)^\theta}$. Substituting in the above together with $f(x)$ in (3) and after some algebra we get,

$$f_{X_{j:n}}(x) = \sum_{m=0}^{n-j}\sum_{k=0}^{j+m-1}\psi_{j,k,m,n}f(x;\alpha,\theta,\lambda(k+1)), \quad (21)$$

where $\psi_{j,k,m,n} = \frac{n!(-1)^{j+k+2m-1}\binom{j+m-1}{k}(e^{\lambda(k+1)}-1)}{(j-1)!\,(n-j-m)!m!(e^\lambda-1)^{j+m}}$ and $f(x;\alpha,\theta,\lambda(k+1))$ is the density of CPGHL with parameters $\alpha, \theta, \lambda(k+1)$.

The r^{th} moment of the j^{th}−order statistic is computed as follows,

$$E[X_{j:n}^r] = \sum_{m=0}^{n-j}\sum_{k=0}^{j+m-1}\psi_{j,k,m,n}\int_0^\infty x^r f(x;\alpha,\theta,\lambda(k+1))\,dx,$$

in similar way to (10) and some simplification we obtain

$$E[X_{j:n}^r] = \sum_{m=0}^{n-j}\sum_{k=0}^{j+m-1}\psi^*_{i,j,k,m,n} \times \int_0^\infty \frac{\alpha x^r e^{-\alpha x}(1-e^{-\alpha x})^{\theta(i+1)-1}}{(1-e^{-\alpha x})^{\theta(i+1)+1}}dx$$

where $\psi^*_{i,j,k,m,n} = \sum_{i=0}^\infty \frac{2\theta\lambda^{i+1}(k+1)^i n!(-1)^{j+k+2m-1}\binom{j+m-1}{k}}{(j-1)!\,(n-j-m)!(e^\lambda-1)^{j+m}m!i!}$. By letting $v = 1-e^{-\alpha x}$ we get,

$$E[X_{j:n}^r] = \sum_{m=0}^{n-j}\sum_{k=0}^{j+m-1}\psi^*_{i,j,k,m,n}b_{i,l} \times \int_0^1 \log^r(1-v)(1-v)^l v^{\theta(i+1)-1}dv,$$

where $b_{i,l} = \sum_{l=0}^\infty \frac{(-1)^{l+r}}{\alpha^r}\binom{\theta(i+1)+l}{l}$, thus,

$$E[X_{j:n}^r] = \sum_{m=0}^{n-j}\sum_{k=0}^{j+m-1}\psi^*_{i,j,k,m,n}b_{i,l}\mathcal{B}_{0,r}(\theta(i+1), l+1).$$

Here, we obtain the asymptotic distributions for the extreme order statistics ie. $X_{1:n}$ and $X_{n:n}$ from $X_1, X_2, X_3, \cdots, X_n$ follow CPGHL. Let $\xrightarrow{d}$ denote convergence in distribution, let W be a random variable with cdf G, then, saying that the cdf F is in the domain of maximal attraction of G is the same as saying $(X_{n:n}-a_n)/b_n \xrightarrow{d} W$, provided there exist a sequence $\{a_n\}$ and $\{b_n > 0\}$. Suppose that W^* be a random variable with cdf G^*, then, to say that the cdf F is in the domain of minimal attraction of G^* is the same as saying $(X_{1:n}-a_n^*)/b_n^* \xrightarrow{d} W^*$, provided there exist a sequence $\{a_n^*\}$ and $\{b_n^* > 0\}$. Fore detail information one can read [35], [36].

Theorem 9: Let $X_1, X_2, X_3, \cdots, X_n$ be a random sample from the CPGHL distribution, let $W_n = (X_{n:n}-a_n)/b_n$, then, $W_n \xrightarrow{d} W$ implies that

$$\lim_{n\to\infty} P(W_n \leq x) = G(x) = e^{-e^{-x}},$$

for every point $x \in \mathbb{R}$ of $G(x)$ for which $G(x)$ is continuous, where the normalizing constant can be derived from (8) according to the theorem 8.3.4 of [35], thus, $a_n = q(1-\frac{1}{n})$ and $b_n = q(1-\frac{1}{ne}) - q(1-\frac{1}{n})$.

Proof: According to the *theorem* 8.3.2 of [35], we consider the asymptotic of the mean residual life from *proposition* 7, thus, $\lim_{t\to q(1)}\frac{1-F(t+xE[X-t|X>t])}{1-F(t)} = \lim_{t\to\infty}\frac{s(t+xM(t))}{s(t)} \sim \lim_{t\to\infty}\frac{e^{-\alpha(t+x/\alpha)}}{e^{-\alpha t}} = e^{-x}$. □

Theorem 10: Let $X_1, X_2, X_3, \cdots, X_n$ be a random sample from the CPGHL distribution, let $W_n^* = (X_{1:n}-a_n^*)/b_n^*$, then,

$W_n^* \xrightarrow{d} W^*$ is equivalent to saying

$$\lim_{n\to\infty} P(W_n^* \leq x) = G^*(x;\theta) = 1 - e^{-x^\theta},$$

for every point $x \in \mathbb{R}^+$ of $G^*(x;\theta)$ for which $G^*(x;\theta)$ is continuous, where the normalizing constant can be derived from (8) by following the theorem 8.3.6 of [35], thus, $a_n^* = 0$ and $b_n^* = q(\frac{1}{n})$.

Proof: According to *theorem* 8.3.6 of [35] we can consider the asymptotic of $F(x)$ in the proof of *proposition* 8, thus, $\lim_{t\to q(0)} \frac{F(tx)}{F(t)} \sim \lim_{t\to 0} \frac{(tx)^\theta}{t^\theta} = x^\theta$. □

G. CHARACTERIZATION OF THE POISSON HALF LOGISTIC DISTRIBUTION (PHL) BY TRUNCATED MOMENTS

Poisson half logistic (PHL) distribution is a special case of the CGPHL when $\theta = 1$, its properties and application to right censored data was discussed by [37]. Characterizing a probability distributions based on certain statistics are very vital tools in statistical studies. [38] characterized distributions by truncated moments. [39] discussed the characterization of Lindley distribution based on conditional expectations. [40] provide characterization of the Marshall-Olkin-G family of distributions by truncated moments. The characterization of half logistic Poisson (HLP) [41] based on some conditional expectations of a certain function of random variable was discussed by [22]. In this subsection, we discuss the characterization of the PHL in similar way to [22]. The pdf and cdf of the PHL are given respectively by

$$f(x) = \frac{2\alpha\lambda e^{-\alpha x}\, e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}}{(e^\lambda - 1)(1+e^{-\alpha x})^2},$$
$$F(x) = \frac{e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)} - 1}{e^\lambda - 1}, \tag{22}$$

where $x, \alpha, \lambda > 0$. We consider lemma 11 and lemma 13 to describe the general conditions for the characterization of distribution by left and right truncated moments in this case, respectively.

Lemma 11: Suppose that the random variable X has an absolutely continuous c.d.f $F(x)$ such that $F(x) > 0\ \forall x > 0$ and $F(0) = 0$, with pdf $f(x) = F'(x)$ and hazard function $h(x) = f(x)[1 - F(x)]^{-1}$. Let $C(x)$ be a continuous function in $x > 0$ such that $E[C(X)] < \infty$. If $E[C(X)|X \geq x] = P(x)h(x), x > 0$, where $P(x)$ is a differentiable function in $x > 0$, then,

$$f(x) = D\exp\left[-\int_0^x \frac{C(y)+P'(y)}{P(y)}dy\right], \quad x > 0,$$

where $D > 0$ is a normalizing constant.

Proof: The proof can be found available in [22], [39], [40], therefore, omitted. □

Theorem 12: Suppose that the random variable X has an absolutely continuous c.d.f $F(x)$ with $F(x) > 0, \forall x > 0$, $F(0) = 0$, with pdf $f(x) = F'(x)$ and hazard rate $h(x) = f(x)/[1 - F(x)]$. Assume that $E\left[e^{\lambda\left(\frac{1-e^{-\alpha X}}{1+e^{-\alpha X}}\right)}\right] < \infty$ for all $\alpha, \lambda, x > 0$, then, X has PHL in (22) if and only if $E\left[e^{\lambda\left(\frac{1-e^{-\alpha X}}{1+e^{-\alpha X}}\right)}|X > x\right] = P(x)h(x)$, where $P(x) = \frac{\left(e^{2\lambda} - e^{2\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}\right)}{\frac{4\alpha\lambda e^{-\alpha x}}{(e^\lambda-1)(1+e^{-\alpha x})^2} e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}}$.

Proof: Sufficiently, let $C(x) = e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}$, then

$$E\left[e^{\lambda\left(\frac{1-e^{-\alpha X}}{1+e^{-\alpha X}}\right)}|X > x\right] = \frac{1}{1-F(t)} \times \int_x^\infty \frac{2\alpha\lambda e^{-\alpha t}}{(e^\lambda - 1)(1+e^{-\alpha t})^2} e^{2\lambda\left(\frac{1-e^{-\alpha t}}{1+e^{-\alpha t}}\right)} dt,$$

let $u = e^{\lambda\left(\frac{1-e^{-\alpha t}}{1+e^{-\alpha t}}\right)}$ we get

$$E\left[e^{\lambda\left(\frac{1-e^{-\alpha X}}{1+e^{-\alpha X}}\right)}|X > x\right] = \frac{h(x)}{(e^\lambda - 1)f(x)} \int_{e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}}^{e^\lambda} u\, du = \frac{h(x)\left(e^{2\lambda} - e^{2\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}\right)}{\frac{4\alpha\lambda e^{-\alpha x}}{(e^\lambda-1)(1+e^{-\alpha x})^2} e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}}$$

thus,

$$P(x) = \frac{\left(e^{2\lambda} - e^{2\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}\right)}{\frac{4\alpha\lambda e^{-\alpha x}}{(e^\lambda-1)(1+e^{-\alpha x})^2} e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}}.$$

Next, we compute $C(x)/P(x)$ and $P'(x)/P(x)$ as

$$\frac{C(x)}{P(x)} = \frac{4\alpha\lambda e^{-\alpha x} e^{2\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}}{(e^\lambda - 1)(1+e^{-\alpha x})^2 \left(e^{2\lambda} - e^{2\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}\right)}$$

and

$$\frac{P'(x)}{P(x)} = -\frac{4\alpha\lambda e^{-\alpha x} e^{2\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}}{(e^\lambda - 1)(1+e^{-\alpha x})^2 \left(e^{2\lambda} - e^{2\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}\right)} + \alpha - \frac{2\alpha e^{-\alpha x}}{(1+e^{-\alpha x})} - \frac{2\alpha\lambda e^{-\alpha x}}{(1+e^{-\alpha x})^2},$$

this implies

$$\frac{C(x)+P'(x)}{P(x)} = \alpha - \frac{2\alpha e^{-\alpha x}}{(1+e^{-\alpha x})} - \frac{2\alpha\lambda e^{-\alpha x}}{(1+e^{-\alpha x})^2},$$

by integrating the above expression we have

$$\int_0^x \frac{C(t)+P'(t)}{P(t)}dt = \alpha x + \log(1+e^{-\alpha x})^2 - \log 4 - \lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right),$$

finally the density function become

$$f(x) = De^{-\int_0^x \frac{C(t)+P'(t)}{P(t)}dx} = \frac{4De^{-\alpha x}}{(1+e^{-\alpha x})^2} e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)},$$

hence $D = \frac{\alpha\lambda}{2(e^\lambda - 1)}$. □

Lemma 13: Suppose that the random variable X has an absolutely continuous c.d.f $F(x)$ such that $F(x) > 0\ \forall x > 0$, $F(0) = 0$, with density function $f(x) = F'(x)$ and reverse failure rate $r(x) = f(x)/F(x)$. Let $C(x)$ be a continuous function in $x > 0$ such that $E[C(X)] < \infty$. If $E[C(X)|X \le x] = Z(x)r(x)$, $x > 0$, where $Z(x)$ is a differentiable function in $x > 0$, then,

$$f(x) = D\exp\left[-\int_0^x \frac{Z'(y)-C(y)}{Z(y)}dy\right], \quad x > 0,$$

where $D > 0$ is a normalizing constant.

Proof: The proof can be found available in [22], [39], [40], therefore omitted. □

Theorem 14: Suppose that the random variable X has an absolutely continuous c.d.f $F(x)$ with $F(0) = 0$, $F(x) > 0$, $\forall x > 0$, with density function $f(x) = F'(x)$ and reverse failure rate $r(x) = f(x)/[F(x)]$. Assume that $E\left[e^{\lambda\left(\frac{1-e^{-\alpha X}}{1+e^{-\alpha X}}\right)}\right] < \infty$ for all $\alpha, \lambda, x > 0$, then, X has PHL in (22) if and only if $E\left[e^{\lambda\left(\frac{1-e^{-\alpha X}}{1+e^{-\alpha X}}\right)}|X \le x\right] = Z(x)r(x)$ where $Z(x) = \frac{\left(e^{2\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}-1\right)}{\frac{4\alpha\lambda e^{-\alpha x}}{(e^{\lambda}-1)(1+e^{-\alpha x})^2}e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}}$.

Proof: Sufficiently, let $C(x) = e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}$, then,

$$E\left[e^{\lambda\left(\frac{1-e^{-\alpha X}}{1+e^{-\alpha X}}\right)}|X \le x\right] = \frac{1}{F(t)}\int_0^x \frac{2\alpha\lambda e^{-\alpha t}}{(e^{\lambda}-1)(1+e^{-\alpha t})^2}e^{2\lambda\left(\frac{1-e^{-\alpha t}}{1+e^{-\alpha t}}\right)}dt,$$

let $u = e^{\lambda\left(\frac{1-e^{-\alpha t}}{1+e^{-\alpha t}}\right)}$ we obtain

$$E\left[e^{\lambda\left(\frac{1-e^{-\alpha X}}{1+e^{-\alpha X}}\right)}|X \le x\right] = \frac{r(x)}{(e^{\lambda}-1)f(x)}\int_1^{e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}} u\,du = \frac{r(x)\left(e^{2\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}-1\right)}{\frac{4\alpha\lambda e^{-\alpha x}}{(e^{\lambda}-1)(1+e^{-\alpha x})^2}e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}},$$

this implies

$$Z(x) = \frac{\left(e^{2\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}-1\right)}{\frac{4\alpha\lambda e^{-\alpha x}}{(e^{\lambda}-1)(1+e^{-\alpha x})^2}e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}}.$$

Now, we compute the ratios $C(x)/Z(x)$ and $Z'(x)/P(x)$,

$$\frac{C(x)}{Z(x)} = \frac{4\alpha\lambda e^{-\alpha x}e^{2\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}}{(e^{\lambda}-1)(1+e^{-\alpha x})^2}\left(e^{2\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}-1\right)$$

and

$$\frac{Z'(x)}{Z(x)} = \frac{4\alpha\lambda e^{-\alpha x}e^{2\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}}{(e^{\lambda}-1)(1+e^{-\alpha x})^2}\left(e^{2\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)}-1\right) + \alpha - \frac{2\alpha\lambda e^{-\alpha x}}{(1+e^{-\alpha x})^2} - \frac{2\alpha e^{-\alpha x}}{(1+e^{-\alpha x})},$$

thus,

$$\frac{C(x)+Z'(x)}{Z(x)} = \alpha - \frac{2\alpha e^{-\alpha x}}{(1+e^{-\alpha x})} - \frac{2\alpha\lambda e^{-\alpha x}}{(1+e^{-\alpha x})^2},$$

and the integral of $\frac{C(x)+Z'(x)}{Z(x)}$ is obtain as

$$\int_0^x \frac{C(t)+Z'(t)}{Z(t)}dt = \alpha x + \log(1+e^{-\alpha x})^2 - \log 4 - \lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right),$$

finally we get

$$f(x) = De^{-\int_0^x \frac{C(t)+Z'(t)}{Z(t)}dx} = \frac{4De^{-\alpha x}}{(1+e^{-\alpha x})^2}e^{\lambda\left(\frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}\right)},$$

hence $D = \frac{\alpha\lambda}{2(e^{\lambda}-1)}$. □

III. ESTIMATION

In this section, the estimation of parameter of CPGHL for complete data and censored data are discuss. Three different techniques of parameter estimation of the CGHLP based on complete data set are proposed, namely, the maximum likelihood (MLE), Least square (LSE), and percentile (P) method. Moreover, the maximum likelihood estimation for censored data set is considered.

A. ESTIMATION BY MLE, LSE, AND P METHODS

Here, we study parameter estimation of CGHLP based on complete data set. The maximum likelihood, Least square, and percentile estimation methods are derived and their performance is examine by simulation study.

1) MAXIMUM LIKELIHOOD METHOD

Let $\boldsymbol{X} = (x_1, x_2, \ldots, x_n)$ be a random sample of size $n \ge 1$ from CPGHL, with unknown parameters $\varphi = (\alpha, \theta, \lambda)^T$. The log likelihood function $\log \ell(\varphi; \boldsymbol{X})$ can be written as

$$\begin{aligned}\log \ell(\varphi; \boldsymbol{X}) = {} & n\log 2 + n\log\alpha + n\log\theta + n\log\lambda \\ & -\alpha\sum_{i=1}^n x_i + (\theta-1)\sum_{i=1}^n \log\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right) \\ & -2\sum_{i=1}^n \log(1+e^{-\alpha x_i}) + \lambda\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta} \\ & -n\log(e^{\lambda}-1) \end{aligned} \tag{23}$$

The maximum likelihood estimators say $\hat{\varphi} = (\hat{\alpha}, \hat{\theta}, \hat{\lambda})^T$ can be obtained by maximizing the log-likelihood function which can be achieve by solving the nonlinear likelihood equations obtained by differentiating (23) as

$$\frac{\partial \log \ell}{\partial \lambda} = \frac{n}{\lambda} - \frac{ne^{\lambda}}{e^{\lambda}-1} + \sum_{i=1}^{n}\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta}, \quad (24)$$

$$\frac{\partial \log \ell}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log(1-e^{-\alpha x_i}) - \sum_{i=1}^{n}\log(1+e^{-\alpha x_i}) + \lambda\sum_{i=1}^{n}\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta}\log\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right), \quad (25)$$

$$\frac{\partial \log \ell}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}x_i + (\theta-1)\sum_{i=1}^{n}\frac{\alpha e^{-\alpha x_i}}{1-e^{-\alpha x_i}} + (\theta+1)\sum_{i=1}^{n}\frac{\alpha e^{-\alpha x_i}}{1+e^{-\alpha x_i}} + 2\alpha\theta\lambda\sum_{i=1}^{n}\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta-1}\frac{e^{-\alpha x_i}}{(1+e^{-\alpha x_i})^2}. \quad (26)$$

The above nonlinear equations can be solved by mathematical software such as `R` and `Mathematica`. The existence of the MLEs under some possible conditions are discussed below. Similar studies can be found in [32], [42]–[47].

Theorem 15: Let $\partial_\alpha(\alpha;\theta,\lambda,\boldsymbol{x})$ be the right hand side of (24), let θ and λ are true values of the parameters, then, $\partial_\alpha(\alpha;\theta,\lambda,\boldsymbol{x}) = 0$ has at least one real root for $\theta \geq 1$.

Proof: Let ∂_α be the right hand of (24), then, for $\theta \geq 1$, $\lim_{\alpha\to 0}\partial_\alpha = \infty$ and $\lim_{\alpha\to\infty}\partial_\alpha = -\sum_{i=1}^{n}x_i < 0$, hence, for $\theta \geq 1$, ∂_α is a continuous function that runs from positive to negative, thus, $\partial_\alpha = 0$ has at least one real root. □

Theorem 16: Let $\partial_\theta(\theta;\alpha,\lambda,\boldsymbol{x})$ be the right hand side of (25), let α and λ are true values of the parameters, then, $\partial_\theta(\theta;\alpha,\lambda,\boldsymbol{x}) = 0$ has at least one real root in the interval $\left(\frac{-n}{(\lambda+1)\sum_{i=1}^{n}\log\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)}, \frac{-n}{\sum_{i=1}^{n}\log\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)}\right)$.

Proof: From (25), let $\omega_\theta = \lambda\sum_{i=1}^{n}\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta}\log\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)$, then, $\lim_{\theta\to 0}\omega_\theta = \lambda\sum_{i=1}^{n}\log\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)$, therefore, $\lim_{\theta\to 0}\partial_\theta = \frac{n}{\theta} + \sum_{i=1}^{n}\log\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right) + \lim_{\theta\to 0}\omega_\theta > \frac{n}{\theta} + (1+\lambda)\sum_{i=1}^{n}\log\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right) > 0$ only if $\theta < \frac{-n}{(\lambda+1)\sum_{i=1}^{n}\log\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)}$.

Other way, $\lim_{\theta\to\infty}\omega_\theta = 0$, this implies that, $\lim_{\theta\to\infty}\partial_\theta = \frac{n}{\theta} + \sum_{i=1}^{n}\log\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right) + \lim_{\theta\to\infty}\omega_\theta < \frac{n}{\theta} + \sum_{i=1}^{n}\log\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right) < 0$ only if $\theta > \frac{-n}{\sum_{i=1}^{n}\log\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)}$. Hence $\partial_\theta = 0$ has at least one real root in $\left(\frac{-n}{(\lambda+1)\sum_{i=1}^{n}\log\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)}, \frac{-n}{\sum_{i=1}^{n}\log\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)}\right)$. □

Theorem 17: Let $\partial_\lambda(\lambda;\alpha,\theta,\boldsymbol{x})$ be the right hand side of (26), let α and θ are true values of the parameters, then, $\partial_\lambda(\lambda;\alpha,\theta,\boldsymbol{x}) = 0$ has at least one real root for $\frac{n}{2} < \sum_{i=1}^{n}\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta} < n$.

Proof: From (26), let $\omega_\lambda = \frac{n}{\lambda} - \frac{ne^{\lambda}}{e^{\lambda}-1}$, then $\lim_{\lambda\to 0}\omega_\lambda = \lim_{\lambda\to 0}\frac{n(e^{\lambda}-\lambda e^{\lambda}-1)}{\lambda(e^{\lambda}-1)} = -\frac{n}{2}$, this implies $\lim_{\lambda\to 0}\partial_\lambda = \lim_{\lambda\to 0}\omega_\lambda + \sum_{i=1}^{n}\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta} = -\frac{n}{2} + \sum_{i=1}^{n}\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta} > 0$ only if $\sum_{i=1}^{n}\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta} > \frac{n}{2}$. On the other hand, $\lim_{\lambda\to\infty}\omega_\lambda = -n$, therefore, $\lim_{\lambda\to\infty}\partial_\lambda = \lim_{\lambda\to\infty}\omega_\lambda + \sum_{i=1}^{n}\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta} = -n + \sum_{i=1}^{n}\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta} < 0$ only if $\sum_{i=1}^{n}\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta} < n$. Thus, ∂_λ has at least one root if $\frac{n}{2} < \sum_{i=1}^{n}\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta} < n$. □

An approximate confidence interval for the parameters of the CPGHL can be establish by the regularity conditions that the parameters are in the interior of the parameter space but not on the boundary. We can established the asymptotic normality distribution for the α, θ and λ as $n \to \infty$, it is a three dimensional normal distribution with zero means and covariance matrix I^{-1}, where $I(\varphi) = -[\frac{\partial^2 \log \ell}{\partial\varphi\partial\varphi^T}]$, and the element of $I(\varphi)$ can be deduce from appendix V.

A $100(1-\xi)$ asymptotic confidence interval for each parameter φ_r is given by

$$ACI_r = \left(\hat{\varphi}_r - w_{\frac{\xi}{2}}\sqrt{\hat{I_{rr}}}, \hat{\varphi}_r + w_{\frac{\xi}{2}}\sqrt{\hat{I_{rr}}}\right),$$

where $\hat{I_{rr}}$ is the (r,r) diagonal element of $I_n(\hat{\varphi})^{-1}$ for $r = 1, 2, 3$, and $w_{\frac{\xi}{2}}$ is the quantile $1-\frac{\xi}{2}$ of the standard normal distribution.

2) LEAST SQUARE METHOD

The least squares estimation technique [48] for the CPGHL can be achieved as follow. Let $X_1, X_2, \ldots, X_n$ be an ordered random sample of size $n \geq 1$ from CPGHL, then the least squares estimators for the vector of parameters $\varphi = (\alpha,\theta,\lambda)^T$, say $\bar{\varphi} = (\bar{\alpha},\bar{\theta},\bar{\lambda})^T$ can be obtained by minimizing $\mathcal{L}(\varphi)$ with respect to φ,

$$\mathcal{L}(\varphi) = \sum_{i=1}^{n}\left(\frac{e^{\lambda\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta}}-1}{e^{\lambda}-1} - \frac{i}{n+1}\right)^2,$$

equivalently is to finding the solution of the following equations which can be done numerically using `R software`, `Mathematica` among others.

$$\frac{\partial\mathcal{L}(\varphi)}{\partial\alpha} = \frac{4\theta\lambda}{(e^{\lambda}-1)}\sum_{i=1}^{n}\left(\frac{e^{\lambda\kappa_i^{\theta}}-1}{e^{\lambda}-1} - \frac{i}{n+1}\right) \times \frac{x_i e^{-\alpha x_i}\kappa_i^{\theta-1}e^{\lambda\kappa_i^{\theta}}}{(1+e^{-\alpha x_i})^2},$$

$$\frac{\partial\mathcal{L}(\varphi)}{\partial\theta} = \frac{2\lambda}{(e^{\lambda}-1)}\sum_{i=1}^{n}\left(\frac{e^{\lambda\kappa_i^{\theta}}-1}{e^{\lambda}-1} - \frac{i}{n+1}\right)\kappa_i^{\theta}e^{\lambda\kappa_i^{\theta}}\log\kappa_i,$$

$$\frac{\partial \mathcal{L}(\varphi)}{\partial \lambda} = \frac{2}{(e^{\lambda}-1)} \sum_{i=1}^{n} \left(\frac{e^{\lambda \kappa_i^{\theta}}-1}{e^{\lambda}-1} - \frac{i}{n+1} \right) \times \left[\kappa_i^{\theta} - \frac{e^{\lambda}}{e^{\lambda}-1} \right] e^{\lambda \kappa_i^{\theta}},$$

where $\kappa_i = \frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}$.

3) PERCENTILE METHOD

The CPGHL has an explicit cdf, therefore, the unknown parameters $\varphi = (\alpha, \theta, \lambda)^T$ can be estimated by equating the sample percentile points with the population percentile points. Let u_i denotes an estimate of $F(x_{i:n})$ in (4), then the percentile estimators, $\tilde{\varphi} = (\tilde{\alpha}, \tilde{\theta}, \tilde{\lambda})^T$ can be obtained by minimizing

$$\mathcal{P}(\varphi) = \sum_{i=1}^{n} \left(x_{i:n} - \frac{1}{\alpha} A_i \right)^2,$$

where

$$A_i = \left[\log\left(1 - K_{\lambda}^{1/\theta}(u_i)\right) - \log\left(1 + K_{\lambda}^{1/\theta}(u_i)\right) \right],$$

$K_{\lambda}(u_i) = \frac{\log(1+u_i(e^{\lambda}-1))}{\lambda}$ and the percentile function is given by (8). This can be achieved by finding the solution of the following equations using `R software`, `Mathematica` etc.

$$\frac{\partial \mathcal{P}(\varphi)}{\partial \alpha} = \frac{2}{\alpha^2} \sum_{i=1}^{n} \left(x_{i:n} - \frac{1}{\alpha} A_i \right) A_i,$$

$$\frac{\partial \mathcal{P}(\varphi)}{\partial \theta} = -\frac{2}{\alpha \theta^2} \sum_{i=1}^{n} \left(x_{i:n} - \frac{1}{\alpha} A_i \right) \times \left(\frac{K_{\lambda}^{1/\theta}(u_i) \log K_{\lambda}(u_i)}{1 - K_{\lambda}^{1/\theta}(u_i)} + \frac{K_{\lambda}^{1/\theta}(u_i) \log K_{\lambda}(u_i)}{1 + K_{\lambda}^{1/\theta}(u_i)} \right),$$

$$\frac{\partial \mathcal{P}(\varphi)}{\partial \lambda} = \frac{2}{\alpha} \sum_{i=1}^{n} \left(x_{i:n} - \frac{1}{\alpha} A_i \right) \times \left(\frac{\tau_i}{1 - K_{\lambda}^{1/\theta}(u_i)} + \frac{\tau_i}{1 + K_{\lambda}^{1/\theta}(u_i)} \right),$$

where

$$\tau_i = \frac{\partial}{\partial \lambda} K_{\lambda}^{1/\theta}(u_i) = \frac{1}{\theta} K_{\lambda}^{\frac{1}{\theta}-1}(u_i) \left[\frac{u_i e^{\lambda}}{\lambda(1+u_i(e^{\lambda}-1))} - \frac{1}{\lambda} K_{\lambda}(u_i) \right].$$

4) SIMULATION STUDY

The performance of the MLE, LSE and P methods were assessed by simulation studies. We generate $N = 10,000$ sample form CPGHL each of sample sizes $n = (30, 40, 50, \cdots, 500)$ for some parameters $(\alpha, \theta, \lambda)$ as $(0.5, 0.5, 0.5)$, $(0.5, 0.5, -1.0)$, $(1.2, 1.5, -1.5)$, $(1.0, 1.0, 1.0)$, $(1.5, 0.5, 1.5)$ and $(1.2, 1.1, 1.1)$. In comparison between the three different methods, we examine the bias (Bias) and mean square error (MSE) of the estimators. The average bias and the average mean square error are presented in the figures 7 to 12. From the figures we can deduce that both the MLEs, SLEs and Ps perform consistently, the MSE in both cases decreases to zero as the sample size increases; α, θ has smaller MSE than λ when the sample size is small; the bias in the MLEs and LSEs converges to zero as the sample size increases while the the bias of Ps goes to zero in most cases as the sample size increases; the bias is negative in some cases especially for the SLEs and Ps; the MLEs has the smaller MSE in most cases as compared to LSEs and Ps thus, maximum likelihood perform better followed by the least square method, and percentile method.

B. MAXIMUM LIKELIHOOD ESTIMATION FOR RIGHT CENSORED DATA

In this subsection, we discuss the maximum likelihood estimation of right censored data for the CPGHL distribution and examine it performance numerically using some parameter values by simulation study.

In right censoring, for a specific individuals under study, we assume that there is a lifetime T_i and a censoring time, Cr_i, $i = 1, 2, 3, \cdots, n$. The $T's$ and $Cr's$ are assumed to be independent and identically distributed with density function $f(x)$ and survival function $s(x)$, where $X_i = \min(T_i, Cr_i)$. This can be represented by pairs of random variable (X_i, δ_i), where δ indicates whether the lifetime X_i is censored $(\delta_i = 0)$ or $(\delta_i = 1)$ if the lifetime is completely observed. The log-likelihood function for CPGHL can be expressed as

$$\log \ell_C(\varphi) = n_i \log \alpha + n\delta_i \log \theta + n\delta_i \log \lambda - \alpha \delta_i \sum_{i=1}^{n} x_i - n \log(e^{\lambda}-1) - \delta_i \sum_{i=1}^{n} \log(1-e^{-\alpha x_i}) + \delta_i(\theta-1) \sum_{i=1}^{n} \log\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right) + \lambda \delta_i \sum_{i=1}^{n} \log\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta} \quad (27)$$

$$+ (1-\delta_i) \sum_{i=1}^{n} \log\left(e^{\lambda} - e^{\lambda\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta}} \right). \quad (28)$$

The MLEs of $\varphi = (\alpha, \theta, \lambda)^T$ that is $\widehat{\varphi} = (\widehat{\alpha}, \widehat{\theta}, \widehat{\lambda})^T$ are obtained by maximizing (28), this can be achieved by solution of the nonlinear equation derived from (28). The numerical solution of the nonlinear equation can be done using software such as `R` among others. The partial derivatives are as follows.

$$\frac{\partial \log \ell_C}{\partial \alpha} = \frac{n\delta_i}{\alpha} - \lambda \sum_{i=1}^{n} x_i + \delta_i(\theta-1) \sum_{i=1}^{n} \frac{\alpha e^{-\alpha x_i}}{1-e^{-\alpha x_i}} + \delta_i(\theta+1) \sum_{i=1}^{n} \frac{\alpha e^{-\alpha x_i}}{1+e^{-\alpha x_i}}$$

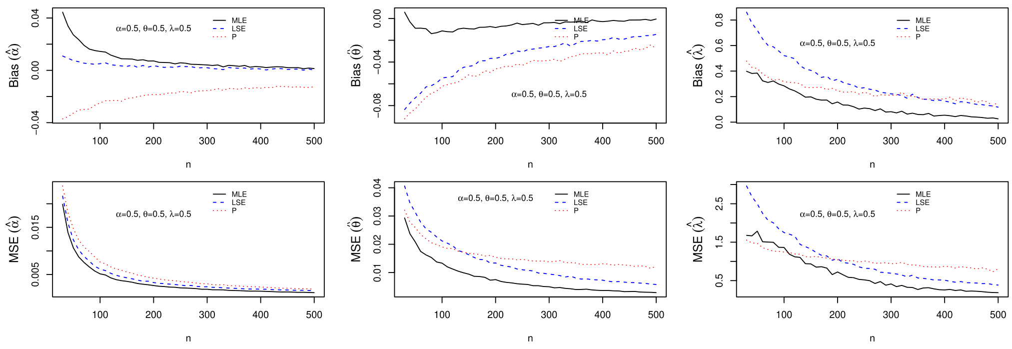

FIGURE 7. Plots of the bias and MSE of the estimated $\alpha = 0.5$, $\theta = 0.5$, $\lambda = 0.5$ for the MLE, LSE and P.

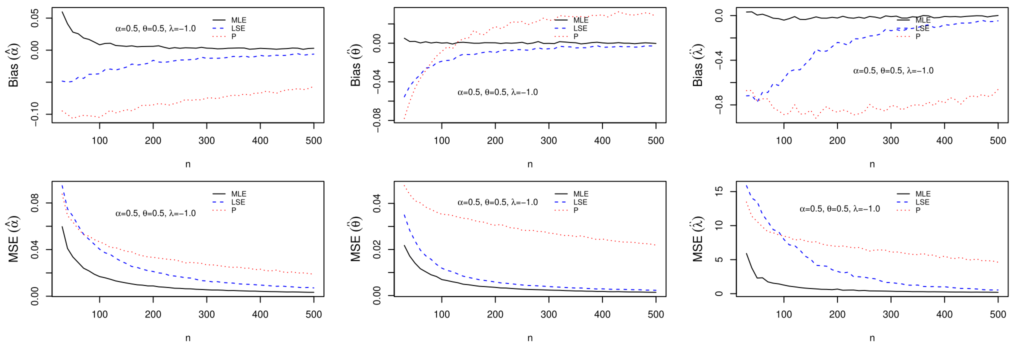

FIGURE 8. Plots of the bias and MSE of the estimated $\alpha = 0.5$, $\theta = 0.5$, $\lambda = -1.0$ for the MLE, LSE and P.

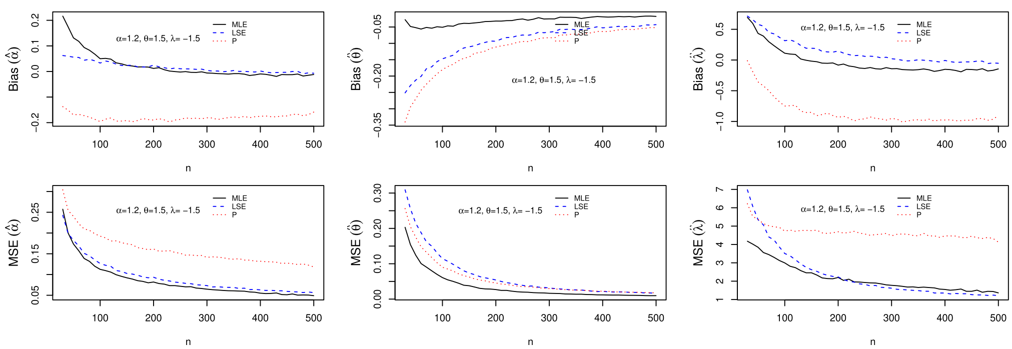

FIGURE 9. Plots of the bias and MSE of the estimated $\alpha = 1.2$, $\theta = 1.5$, $\lambda = -1.5$ for the MLE, LSE and P.

$$+\lambda\alpha\theta\delta_i\sum_{i=1}^{n}\frac{e^{-\alpha x_i}}{(1+e^{-\alpha x_i})^2}\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta-1}$$

$$-2\alpha\theta\lambda(1-\delta_i)\sum_{i=1}^{n}\frac{e^{-\alpha x_i}\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta-1}e^{\lambda\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)}}{\left[e^{\lambda}-e^{\lambda\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta}}\right](1+e^{-\alpha x_i})^2}$$

$$\frac{\partial \log \ell_C}{\partial\theta} = \frac{n\delta_i}{\theta}+\delta_i\sum_{i=1}^{n}\log\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right) + \lambda\delta_i\sum_{i=1}^{n}\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta}\log\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)$$

$$-(1-\delta_i)\lambda\sum_{i=1}^{n}\frac{\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta}\log\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)e^{\lambda\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta}}}{e^{\lambda}-e^{\lambda\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta}}}$$

$$\frac{\partial \log \ell_C}{\partial\lambda} = \frac{n\delta_i}{\lambda}-\frac{ne^{\lambda}}{e^{\lambda}-1}+\delta_i\sum_{i=1}^{n}\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta} + (1-\delta_i)\sum_{i=1}^{n}\frac{e^{\lambda}-\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta}e^{\lambda\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta}}}{e^{\lambda}-e^{\lambda\left(\frac{1-e^{-\alpha x_i}}{1+e^{-\alpha x_i}}\right)^{\theta}}}$$

An approximate confidence interval for the parameters of the CPGHL can be establish by following the regularity

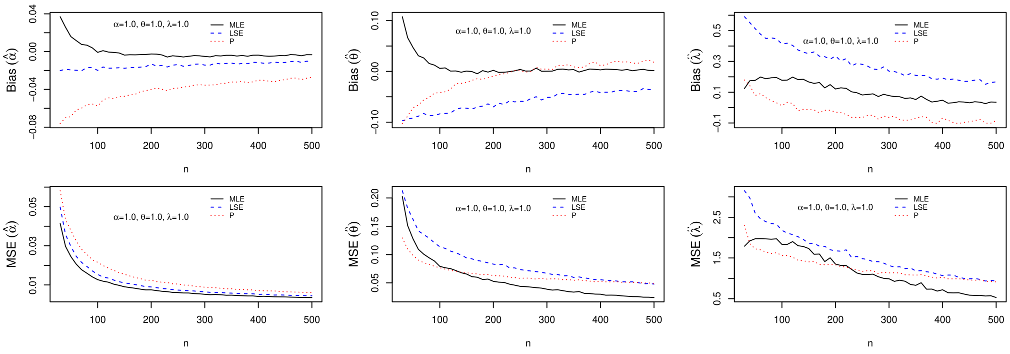

FIGURE 10. Plots of the bias and MSE of the estimated $\alpha = 1.0, \theta = 1.0, \lambda = 1.0$ for the MLE, LSE and P.

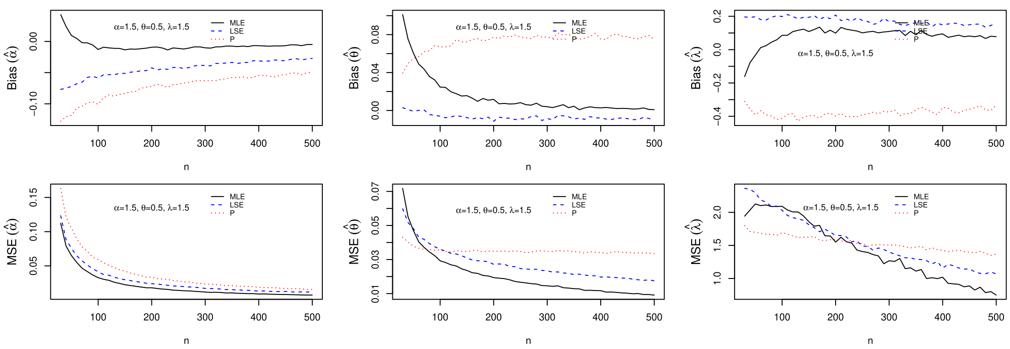

FIGURE 11. Plots of the bias and MSE of the estimated $\alpha = 1.5, \theta = 0.5, \lambda = 1.5$ for the MLE, LSE and P.

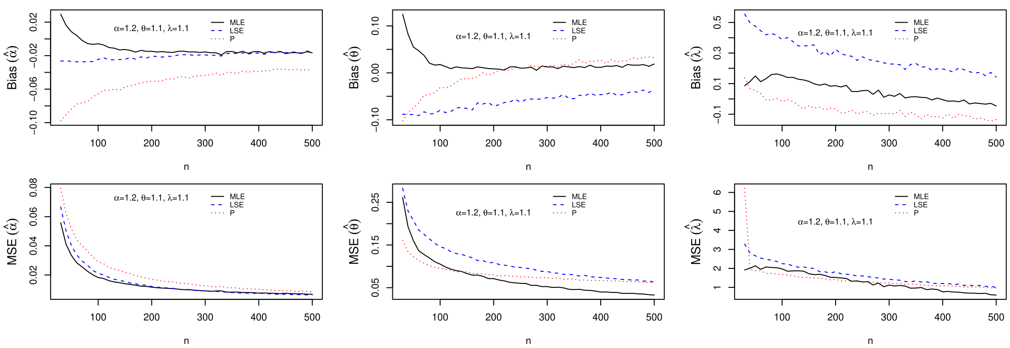

FIGURE 12. Plots of the bias and MSE of the estimated $\alpha = 1.2, \theta = 1.1, \lambda = 1.1$ for the MLE, LSE and P.

condition that the parameters are in the interior of the parameter space but not on the boundary. The asymptotic normality distribution for the α, θ and λ as $n \rightarrow \infty$ is a three dimensional normal distribution with zero means and covariance matrix I^{-1}, where $I(\varphi) = -[\frac{\partial^2 \log \ell_C}{\partial\varphi\partial\varphi^T}]$, and the element of $I(\varphi)$ are given in appendix V. A $100(1-\xi)$ asymptotic confidence interval for each parameter φ_r is given by $ACI_r = \left(\widehat{\varphi}_r - w_{\frac{\xi}{2}}\sqrt{\widehat{I_{rr}}}, \widehat{\varphi}_r + w_{\frac{\xi}{2}}\sqrt{\widehat{I_{rr}}}\right)$, where $\widehat{I_{rr}}$ is the (r,r) diagonal element of $I_n(\widehat{\varphi})^{-1}$ for $r = 1, 2, 3$, and $w_{\frac{\xi}{2}}$ is the quantile $1 - \frac{\xi}{2}$ of the standard normal distribution.

1) SIMULATION

The maximum likelihood estimation (MLE) for right censored data from the CPGHL is examined by simulation study. We generate a censored data of moderate size $N = 1000$ each of size $n = (30, 80, 130, \ldots, 330)$ which are randomly sampled from CPGHL for some different values of α, θ and λ. The percentage censoring is based on a sample size percentage $C\% = (10\%, 15\%, 20\%)$; the percentage is rounded to the nearest integer. The computations were perform using `R3.5.3`-software. The estimated values, bias, mean square error (MSE), average length of 95% confidence interval (ALCI) and coverage probability of the resulting assessment are presented in table 2. Observe from the table 2 that the estimated values converges to their true values as the sample size increases, thus the MLEs shows consistency. The MSE decreases as the sample size increases, this indicate that the MLEs behaved asymptotically unbiased estimators. Also the ALCI decreases as the sample size increases, but λ has the wider ALCI compared to α and θ. Notice that the bias is

TABLE 2. Estimated values (MLEs), MSE with bias in parenthesis, and average length of confidence interval with coverage probability (CP) in parenthesis of the simulated right censored data from CPGHL for some parameter values.

Sample size		Actual values			Estimated values			MSE(Bias)			ALCI(CP)		
$C\%$	n	α	λ	θ	$\widehat{\alpha}$	$\widehat{\lambda}$	$\widehat{\theta}$	$MSE(\widehat{\alpha})$	$MSE(\widehat{\lambda})$	$MSE(\widehat{\theta})$	$ALCI(\widehat{\alpha})$	$ALCI(\widehat{\lambda})$	$ALCI(\widehat{\theta})$
10%	30	0.8	-1.0	0.6	0.7889	-0.8468	0.5935	0.1068 (-0.111)	4.6927 (0.1532)	0.0312 (-0.0065)	1.4786 (0.94)	10.1367 (0.95)	0.7648 (0.98)
	80				0.7297	-1.0195	0.6024	0.0549 (-0.0704)	2.1433 (-0.0195)	0.0137 (0.0024)	0.9413 (0.95)	6.3337 (0.95)	0.4716 (0.98)
	130				0.7231	-1.0453	0.6028	0.00442 (-0.0759)	1.6775 (0.0404)	0.0087 (-0.0023)	0.7329 (0.92)	4.8353 (0.93)	0.3687 (0.95)
	180				0.7241	-0.9596	0.5977	0.0305 (-0.0759)	0.9306 (0.0404)	0.0069 (0.0023)	0.6147 (0.93)	3.7639 (0.93)	0.3143 (0.94)
	230				0.72506	-0.9676	0.6042	0.0258 (-0.0749)	0.7825 (0.0324)	0.0055 (0.0042)	0.5459 (0.92)	3.3151 (0.94)	0.2771 (0.95)
	280				0.7241	-0.9667	0.6028	0.0225 (-0.0759)	0.6302 (0.0332)	0.0042 (0.0028)	0.4973 (0.91)	2.9414 (0.94)	0.2532 (0.94)
	330				0.7299	-0.9154	0.5997	0.0188 (-0.0701)	0.4818 (0.0846)	0.0034 (-0.0003)	0.4479 (0.91)	2.6315 (0.94)	0.2335 (0.93)
10%	30	0.5	0.5	0.5	0.4571	0.3314	0.5574	0.0222 (-0.0430)	4.1798 (-0.1686)	0.0520 (0.0574)	0.6486 (0.96)	9.5285 (0.92)	0.9033 (0.98)
	80				0.4543	0.5712	0.5179	0.0107 (-0.0457)	2.2831 (0.0712)	0.0239 (0.0179)	0.3766 (0.95)	5.8355 (0.91)	0.5971 (0.98)
	130				0.4514	0.5891	0.5169	0.0073 (-0.0486)	1.7722 (0.0891)	0.0171 (0.0170)	0.2907 (0.92)	4.6956 (0.93)	0.4816 (0.98)
	180				0.4555	0.6152	0.5055	0.0056 (-0.0446)	1.1308 (0.1152)	0.0118 (0.0055)	0.2422 (0.90)	3.6216 (0.95)	0.4068 (0.97)
	230				0.4558	0.5898	0.5089	0.0047 (-0.0442)	0.7664 (0.0898)	0.0094 (0.0089)	0.2114 (0.89)	3.0441 (0.93)	0.3583 (0.95)
	280				0.4591	0.6638	0.4986	0.0041 (-0.0409)	0.7638 (0.1638)	0.0082 (-0.0014)	0.1873 (0.85)	2.7982 (0.94)	0.3268 (0.96)
	330				0.4587	0.6130	0.5033	0.0037 (-0.0413)	0.4574 (0.1130)	0.0061 (0.0033)	0.1759 (0.85)	2.5356 (0.96)	0.3043 (0.97)
15%	30	1.2	1.1	1.3	1.0419	0.8028	1.4749	0.1050 (-0.1581)	11.7650 (-0.2972)	0.3834 (0.1749)	1.2879 (0.94)	10.9617 (0.97)	2.6829 (0.98)
	80				1.0174	0.6870	1.4031	0.0754 (-0.1826)	4.4193 (-0.4130)	0.2057 (0.1031)	0.8418 (0.92)	7.9049 (0.87)	1.7362 (0.97)
	130				1.0256	0.8846	1.3485	0.0598 (-0.1744)	3.7296 (-0.2154)	0.1578 (0.0485)	0.6695 (0.87)	7.4117 (0.87)	1.5073 (0.96)
	180				1.0377	0.8918	1.3446	0.0522 (-0.1623)	3.0015 (-0.2082)	0.1239 (0.0446)	0.5524 (0.87)	5.3957 (0.88)	1.2470 (0.95)
	230				1.0401	0.9685	1.3363	0.0451 (-0.1608)	2.8714 (-0.1315)	0.1141 (0.0363)	0.4808 (0.82)	4.8724 (0.87)	1.1258 (0.95)
	280				1.0392	0.8950	1.3396	0.0430 (-0.1608)	2.1516 (-0.2050)	0.0880 (0.0396)	0.4436 (0.79)	4.2124 (0.89)	1.0194 (0.95)
	330				1.0452	0.9359	1.3412	0.0375 (-0.1548)	1.7339 (-0.1640)	0.0782 (0.0412)	0.4027 (0.75)	3.8925 (0.90)	0.9698 (0.95)
20%	30	1.5	-0.8	1.2	1.3527	-0.5290	1.1763	0.2552 (-0.1473)	4.3407 (0.2709)	0.1426 (-0.0237)	2.3134 (0.94)	11.6442 (0.96)	1.6596 (0.98)
	80				1.2281	-0.9242	1.1721	0.2035 (-0.2711)	3.5384 (-0.1242)	0.0644 (-0.0279)	1.6710 (0.92)	8.4763 (0.96)	0.9771 (0.97)
	130				1.1950	-1.1038	1.1839	0.2009 (-0.3049)	2.9572 (-0.3038)	0.0403 (-0.0161)	1.4161 (0.90)	7.1669 (0.96)	0.7564 (0.96)
	180				1.1862	-1.1503	1.1904	0.1885 (-0.3192)	2.3012 (-0.3530)	0.0245 (-0.0086)	1.2188 (0.91)	6.0431 (0.96)	0.6355 (0.96)
	230				1.1808	-1.1530	1.1914	0.1792 (-0.3192)	2.1060 (-0.3530)	0.0195 (-0.0086)	1.1378 (0.91)	5.6272 (0.95)	0.5694 (0.96)
	280				1.1732	-1.1989	1.1969	0.1776 (-0.3268)	1.8677 (-0.3989)	0.0143 (-0.0031)	1.0435 (0.92)	5.0867 (0.96)	0.5098 (0.96)
	330				1.1773	-1.1549	1.2002	0.1643 (-0.3227)	1.6318 (-0.3549)	0.0141 (0.0002)	1.0203 (0.92)	5.0379 (0.94)	0.4766 (0.96)

negative in some cases. The coverage probability approaches 0.95 in most cases.

IV. APPLICATIONS

Now, we illustrate the performance of the new distribution as compared with some other existing distributions using two real data applications in which one of them is a censored data. The models parameters are estimated by maximum likelihood procedure. In comparison, the competing distributions include the generalized half logistic (GHL) [16], Poisson half logistic (PHL) [37], power half logistic (PwHL) [49], Olapade half logistic (OHL) [50], half logistic Poisson (HLP) [41], exponentiated generalized standardized half logistic (EGSHL) [51], generalized half logistic Poisson (GHLP) [52], Beta half logistic (BHL) [53], type I half-logistic Burr X [54], and the Half logistic (HL).

A. FIRST DATA SET

The first data is the time between failures (1000's of hours) of a 180 ton of rear dump truck, provided by [55], also studied by [56], [57].

TABLE 3. MLEs, ℓ, AIC, BIC, CAIC, KS, AD, CvM and their p-values for the first data.

MLEs & Gof	Models									
	CPGHL	GHLP	GHL	PHL	EGSHL	OHL	HLP	HL	TIHL_{BX}	BHL
α	0.2092	0.2587	0.4726	0.7417	-	1.0001	0.2500	0.7417	0.0001	0.0073
β	-	0.6326	-	-	-	-	-	-	-	-
λ	-3.9983	3.5240	-	3.146×10^{-8}	-	-	5.3848	-	60.3530	-
θ	0.7119	-	0.4897	-	-	0.7159	-	-	0.2903	-
a	-	-	-	-	0.4156	-	-	-	-	0.5883
b	-	-	-	-	0.4787	-	-	-	-	92.4719
ℓ	-177.70	-179.54	-186.32	-214.08	-185.29	-214.02	-188.46	-214.08	-179.36	-180.69
AIC	361.39	365.09	376.65	432.17	374.57	432.04	380.92	430.17	364.71	367.37
BIC	369.90	373.60	382.32	437.84	380.25	437.71	386.59	433.00	373.22	375.88
CAIC	361.59	365.29	376.74	432.27	374.67	432.13	381.01	430.20	364.91	367.57
KS	0.0652	0.0886	0.1380	0.2788	0.1342	0.2766	0.1471	0.2788	0.0742	0.1014
p-value	0.6577	0.2766	0.0165	6.25×10^{-9}	0.0214	8.449×10^{-9}	0.0086	6.25×10^{-9}	0.4925	0.1500
AD	0.5316	0.8704	2.0502	1.9508	1.8867	1.8583	0.7221	1.9508	0.7956	1.0903
p-value	0.17113	0.0249	3.03×10^{-5}	5.328×10^{-5}	7.665×10^{-5}	9.008×10^{-5}	0.0582	5.28×10^{-5}	0.0382	0.0071
CvM	0.0796	0.1394	0.3351	0.3225	0.3120	0.3083	0.1149	0.3225	0.1257	0.1726
p-value	0.2077	0.0324	0.0013	0.00015	0.0002	0.0003	0.0693	0.0002	0.0495	0.0118

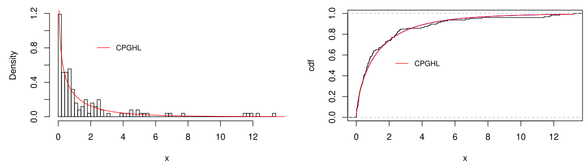

FIGURE 13. Plots of the histogram & estimated density of the CPGHL (left), and empirical cdf and estimated CPGHL cdf (right) for the first data.

We compare the fitted models by the model selection criteria known as the Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC). Further, the goodness of fit statistics so-called the Anderson-Darling (AD), Cramer-von Mises (CvM), and Kolmogorov Smirnov (KS) are considered. The model with the smallest value of these measures provide better fit to the data set. The numerical values of these measures of the CPGHL and the other competing models for the data are provided in table 3. One can see from table 3 the CPGHL has the smallest value of these measures, thus CPGHL represent the data better than the other competing distributions. The computations of AD and CvM was achieved using the `R` package `nortest` [58] and KS using `GLDEX` [59].

The computed information matrix of the CPGHL is given below and the 95% asymptotic confidence interval of the MLEs are: for α we get $(0, 0.4464)$, θ we have $(0.5993, 0.8246)$, and for λ it is $(-7.8766, -0.1205)$. Based on the MLEs, the asymptotic confidence interval appear good though λ has wider interval than α and θ. Figure 13 illustrates the plot of the histogram and estimated density of the CPGHL (left), while (right) is the empirical cdf and estimated CPGHL cdf for the first data set. Figure 14 shows the quantile-quantile plot and the plots of the profile log-likelihood function of CPGHL for the first data set.

$$I_n = \begin{pmatrix} 1364.41258 & - & - \\ -666.60277 & 649.46448 & - \\ -84.09984 & 43.485215 & 5.456968 \end{pmatrix},$$

and

$$I_n^{-1} = \begin{pmatrix} 0.01464785 & - & - \\ -0.00017251 & 0.00330300 & - \\ 0.22711940 & -0.02897941 & 3.91442281 \end{pmatrix}$$

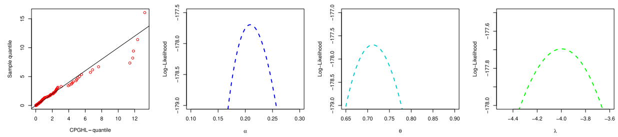

FIGURE 14. Plots of the quantile-quantile, and the plots of the profile log-likelihood function of CPGHL for the first data.

TABLE 4. MLEs, ℓ, AIC, BIC, CAIC, KS, AD, CvM and their p-values for the second data.

MLEs & Gof	Models								
	CPGHL	GHLP	GHL	PHL	EGSHL	OHL	HL	TIHL_{BX}	BHL
α	0.0119	0.0176	0.0275	0.0495	-	0.0496	0.0495	8.72×10^{-5}	7.358×10^{-4}
β	-	0.6561	-	-	-	-	-	-	-
λ	-5.0806	4.2981	-	4.037×10^{-8}	-	-	-	40.6600	-
θ	0.6948	-	0.4346	-	-	0.9992	-	0.2642	-
a	-	-	-	-	0.0238	-	-	-	0.5366
b	-	-	-	-	0.4876	-	-	-	0.5366
ℓ	-135.29	-137.03	-144.46	-151.80	-137.45	-151.80	-151.80	-136.66	-138.31
AIC	276.57	280.06	286.93	307.60	278.90	307.60	305.60	279.32	282.61
BIC	281.15	284.64	289.98	310.65	281.96	310.65	307.12	283.89	287.19
CAIC	277.37	280.86	287.31	307.99	279.29	307.99	305.72	280.12	283.41
KS	0.2064	0.2440	0.3073	0.4747	0.2701	0.4716	0.4740	0.2075	0.2712
p-value	0.0953	0.0287	0.0024	1.473×10^{-7}	0.0111	1.84×10^{-7}	1.55×10^{-7}	0.0923	0.0106
AD	1.4012	1.7197	2.4062	2.4034	1.7163	2.4034	2.4034	1.5921	1.9172
p-value	0.0011	0.0001	3.17×10^{-6}	3.23×10^{-6}	0.0002	3.23×10^{-6}	3.23×10^{-6}	0.0004	5.31×10^{-5}
CvM	0.2490	0.3109	0.4473	0.4471	0.3118	0.4471	0.4471	0.2866	0.3511
p-value	0.0012	0.0002	7.60×10^{-6}	7.63×10^{-6}	0.0002	7.63×10^{-6}	7.63×10^{-6}	0.0004	7.79×10^{-5}

B. SECOND DATA SET

The second data set is the intervals in days between successive failures of a piece of n software with values: 9, 12, 11, 4, 7, 2, 5, 8, 5, 7, 1, 6, 1, 9, 4, 1, 3, 3, 6, 1, 11, 33, 7, 91, 2, 1, 87, 47, 12, 9, 135, 258, 16, 35. Obtained from [60] also studied by [61]. The goodness of fit analysis includes the AIC, BIC, CAIC, AD, CvM, and KS. It is clear from table 4 CPGHL has the smallest value of these measures, thus CPGHL fitted the data better than the other distributions.

The computed information matrix of the CPGHL is given below and the 95% asymptotic confidence interval of the MLEs are: for α we get $(0, 0.0316)$, θ it is $(0.4951, 0.8937)$, and for λ we have $(-10.6641, 0.5028) - \{0\}$. Based on the MLEs, the asymptotic confidence interval is good but λ has wider interval than α and θ. Figure 15 show the plot of the histogram and estimated density of the CPGHL (left), while (right) is the empirical cdf and estimated CPGHL cdf for the second data. Figure 16 shows the quantile-quantile plot and the plots of the profile log-likelihood function of CPGHL for the second data.

$$I_n = \begin{pmatrix} 101966.8048 & - & - \\ -3619.3057 & 255.32731 & - \\ -312.3262 & 10.970780 & 1.080025 \end{pmatrix},$$

and

$$I_n^{-1} = \begin{pmatrix} 0.00010106 & - & - \\ 0.00039643 & 0.01033563 & - \\ 0.02519931 & 0.00965321 & 8.11509043 \end{pmatrix}$$

C. THIRD DATA SET

Below it is a censored data provided by [62] also studied by [63], it consists the ordered remission times (in months) of a random sample of 137 bladder cancer patients, the censored observations are 9 and represented by $'+'$. The data are: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 24.80+, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 0.87+, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90,2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 10.86+, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 4.33+, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 3.02+, 4.40, 5.85,8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 19.36+, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 4.65+, 6.76, 8.60+, 12.07, 21.73, 2.07, 3.36, 4.70+, 6.93, 8.65, 12.63, 22.69.

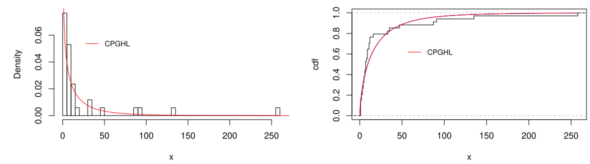

FIGURE 15. Plots of the histogram & estimated density of the CPGHL (left), and empirical cdf and estimated CPGHL cdf (right) for the second data.

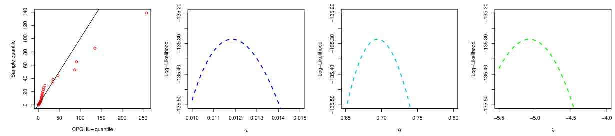

FIGURE 16. Plots of the quantile-quantile, and the plots of the profile log-likelihood function of CPGHL for the second data.

TABLE 5. MLEs, ℓ, AIC, CAIC for the third data.

MLEs& Gof	Models								
	CPGHL	GHLP	GHL	PHL	HL	HLP	PwHL	TIHL_{BX}	BHL
α	0.0693	0.1250	0.1361	0.0674	0.1397	0.0313	0.8933	9.692×10^{-4}	0.0041
β	-	1.3179	-	-	-	-	0.1877	-	-
λ	-3.7853	1.5150	-	-2.9249	-	6.6796	-	80.6320	-
θ	0.0693	-	0.9549	-	-	-	-	0.4407	-
a	-	-	-	-	-	-	-	-	1.1420
b	-	-	-	-	-	-	-	-	50.4250
ℓ_C	-419.48	-421.81	-424.99	-421.75	425.08	-421.92	-423.62	-423.95	-422.24
AIC	844.96	849.61	853.98	847.50	852.15	847.84	851.23	853.90	850.48
CAIC	845.14	849.79	854.07	847.59	852.18	847.93	851.32	854.08	850.66

The prepared model selection criterion are Akaike information criterion (AIC) and consistent Akaike information criterion (CAIC). The resulting values of these measures for each model are provided in table 5 which is in favor of our model.

The calculated information matrix I_n of the CPGHL is given below and the 95% asymptotic confidence interval of the MLEs are: for α we get $(0.0239, 0.1147)$, θ it is $(1.0008, 1.4106)$, and for λ we have $(-6.7301, -0.8406)$. Based on the MLEs, the asymptotic confidence interval of λ has wider interval than α and θ. Figure 17 give the plots of the Kaplan-Meier survival curve with an estimated CPGHL survival curve (left) and empirical and estimated CPGHL cumulative hazard rate curves (right) of the third data.

$$I_n = \begin{pmatrix} 29792.2611 & - & - \\ -2061.7108 & 234.76332 & - \\ -409.7842 & 27.853275 & 6.082246 \end{pmatrix},$$

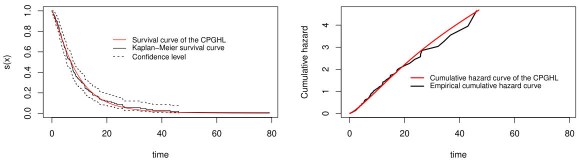

FIGURE 17. Plots of the Kaplan-Meier survival curve with an estimated CPGHL survival curve (left) and empirical and estimated CPGHL cumulative hazard rate curves (right) of the third data.

and

$$I_n^{-1} = \begin{pmatrix} 0.00053652 & - & - \\ 0.00092645 & 0.01092718 & - \\ 0.03190497 & 0.01237834 & 2.25728706 \end{pmatrix}$$

V. CONCLUSION

In this paper, a new model called complementary Poisson generalized half logistic (CPGHL) is proposed. The model possess an increasing, decreasing, unimodel and bathtub failure rate. Several mathematical and statistical properties of the model are explored and examine numerically such as the explicit expressions of the moments, quantile function, Moor's kurtosis and MacGillivray's skewness, mean deviations, Bonferroni and Lorenz curves, Shannon and Renyi entropy. We discuses the distribution of mixture of two CPGHL; the log-transform of the CPGHL and some related models; the asymptotic of the moment of residual life; order statistics and their moments; asymptotic distribution of the minimum and maximum order statistics, and the characterization of PHL by truncated moments of a certain function of a random variable. The estimation of the parameters was approached by maximum likelihood, least square, and percentile methods. Moreover, the estimation by maximum likelihood for censored data for the new model is discussed. The assessment of the proposed estimation techniques ware achieved by simulation studies. Three application of the model are provided for illustration in which one of them is a censored data; in both cases CPGHL fitted the three data better than some other existing distributions. Some of the advantages of this model is that it has three parameters, close form properties with their series representations converges with first few terms, and has flexibility to accommodate various failure rates; moreover, as we shown that the model parameter estimation can be achieved by different techniques, and we recommend further studies on the other estimation techniques such as Bayes estimation, maximum product spacing and minimum distance estimations, among others. We hoped that the model will attract wider applications in the field of physics, probability and statistics, engineering, computer science, stochastic, biomedical sciences, finance, information theory, reliability, and life-testing, etc.

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APPENDIX
ELEMENTS OF INFORMATION MATRIX

Let $v = 1 - e^{-\alpha x_i}$ and $u = 1 + e^{-\alpha x_i}$, then

$$\begin{aligned}\frac{\partial^2 \log \ell_C}{\partial \lambda^2} &= -\frac{n\delta_i}{\lambda^2} + \frac{ne^{-\lambda}}{(1-e^{-\lambda})^2} \\ &\quad + (1-\delta_i)\sum_{i=1}^{n} \frac{e^{\lambda} - (\frac{v}{u})^{2\theta} e^{\lambda(\frac{v}{u})^{\theta}}}{e^{\lambda} - e^{\lambda(\frac{v}{u})^{\theta}}} \\ &\quad - (1-\delta_i)\sum_{i=1}^{n} \frac{\left(e^{\lambda} - (\frac{v}{u})^{\theta} e^{\lambda(\frac{v}{u})^{\theta}}\right)^2}{(e^{\lambda} - e^{\lambda(\frac{v}{u})^{\theta}})^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial \log \ell_C}{\partial \theta^2} &= -\frac{n\delta_i}{\theta^2} + \lambda\delta_i \sum_{i=1}^{n} (\frac{v}{u})^{\theta} \log(\frac{v}{u}) \\ &\quad - \lambda(1-\delta_i)\sum_{i=1}^{n} \frac{(\frac{v}{u})^{2\theta} \log^2(\frac{v}{u}) e^{\lambda(\frac{v}{u})^{\theta}}}{e^{\lambda} - e^{\lambda(\frac{v}{u})^{\theta}}} \\ &\quad - \lambda\theta(1-\delta_i)\sum_{i=1}^{n} \frac{(\frac{v}{u})^{\theta} \log^2(\frac{v}{u}) e^{\lambda(\frac{v}{u})^{\theta}}}{e^{\lambda} - e^{\lambda(\frac{v}{u})^{\theta}}} \\ &\quad - \lambda^2(1-\delta_i)\sum_{i=1}^{n} \frac{(\frac{v}{u})^{2\theta} \log^2(\frac{v}{u}) e^{2\lambda(\frac{v}{u})^{\theta}}}{[e^{\lambda} - e^{\lambda(\frac{v}{u})^{\theta}}]^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial \log \ell_C}{\partial \theta \partial \lambda} &= \delta_i \sum_{i=1}^{n} (\frac{v}{u})^{\theta} \log(\frac{v}{u}) \\ &\quad - \lambda(1-\delta_i)\sum_{i=1}^{n} \frac{(\frac{v}{u})^{2\theta} \log^2(\frac{v}{u}) e^{\lambda(\frac{v}{u})^{\theta}}}{e^{\lambda} - e^{\lambda(\frac{v}{u})^{\theta}}} \\ &\quad - (1-\delta_i)\lambda\sum_{i=1}^{n} \frac{(\frac{v}{u})^{\theta} \log^2(\frac{v}{u}) e^{\lambda(\frac{v}{u})^{\theta}}}{e^{\lambda} - e^{\lambda(\frac{v}{u})^{\theta}}} \\ &\quad + \lambda(1-\delta_i)\sum_{i=1}^{n} \frac{\left(e^{\lambda} - \lambda(\frac{v}{u})^{\theta} \log(\frac{v}{u}) e^{\lambda(\frac{v}{u})^{\theta}}\right)}{(e^{\lambda} - e^{\lambda(\frac{v}{u})^{\theta}})^2} \\ &\quad \times (\frac{v}{u})^{\theta} \log(\frac{v}{u}) e^{\lambda(\frac{v}{u})^{\theta}}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \log \ell_C}{\partial \theta \partial \alpha} &= 2\delta_i \sum_{i=1}^{n} \frac{x_i e^{-\alpha x_i}}{u^2(\frac{v}{u})} \\ &\quad + 2\lambda\delta_i\theta \sum_{i=1}^{n} \frac{x_i e^{-\alpha x_i}}{u^2} (\frac{v}{u})^{\theta-1} \log(\frac{v}{u}) \\ &\quad + 2\lambda\delta_i \sum_{i=1}^{n} \frac{x_i e^{-\alpha x_i}}{u^2} (\frac{v}{u})^{\theta-1}\end{aligned}$$

$$- 2\theta\lambda^2(1-\delta)\sum_{i=1}^{n}\frac{\frac{x_i e^{-\alpha x_i}}{u^2}(\frac{v}{u})^{2\theta-1}\log(\frac{v}{u})e^{\lambda(\frac{v}{u})^\theta}}{e^\lambda - e^{\lambda(\frac{v}{u})^\theta}}$$
$$- 2\lambda(1-\delta)\sum_{i=1}^{n}\frac{\frac{x_i e^{-\alpha x_i}}{u^2}(\frac{v}{u})^{\theta-1}\log(\frac{v}{u})e^{\lambda(\frac{v}{u})^\theta}}{e^\lambda - e^{\lambda(\frac{v}{u})^\theta}}$$
$$- 2\lambda(1-\delta)\sum_{i=1}^{n}\frac{\frac{x_i e^{-\alpha x_i}}{u^2}(\frac{v}{u})^{\theta-1}e^{\lambda(\frac{v}{u})^\theta}}{e^\lambda - e^{\lambda(\frac{v}{u})^\theta}}$$
$$- 2\theta\lambda^2(1-\delta)\sum_{i=1}^{n}\frac{\frac{x_i e^{-\alpha x_i}}{u^2}(\frac{v}{u})^{2\theta-1}\log(\frac{v}{u})e^{2\lambda(\frac{v}{u})^\theta}}{(e^\lambda - e^{\lambda(\frac{v}{u})^\theta})^2}$$

$$\frac{\partial^2 \log \ell_C}{\partial\lambda\partial\alpha} = \alpha\theta\delta_i\sum_{i=1}^{n}\frac{e^{-\alpha x_i}}{u^2}(\frac{v}{u})^{\theta-1}$$
$$- 2(1-\delta_i)\alpha\theta\sum_{i=1}^{n}\frac{e^{-\alpha x_i}(\frac{v}{u})^{\theta-1}e^{\lambda(\frac{v}{u})^\theta}}{(e^\lambda - e^{\lambda(\frac{v}{u})^\theta})u^2}$$
$$- 2(1-\delta_i)\alpha\theta\lambda\sum_{i=1}^{n}\frac{e^{-\alpha x_i}(\frac{v}{u})^{\theta}e^{\lambda(\frac{v}{u})^\theta}(e^\lambda - (\frac{v}{u})^\theta e^{\lambda(\frac{v}{u})^\theta})}{(e^\lambda - e^{\lambda(\frac{v}{u})^\theta})^2\, u^2}$$

$$\frac{\partial^2 \log \ell_C}{\partial\alpha^2} = -\frac{n\delta_i}{\alpha^2} + \delta_i(\theta-1)\sum_{i=1}^{n}\frac{e^{-\alpha x_i}}{v}$$
$$- \delta_i\alpha(\theta-1)\sum_{i=1}^{n}\frac{x_i e^{-\alpha x_i}}{v} - \alpha\delta_i(\theta-1)\sum_{i=1}^{n}\frac{x_i e^{-2\alpha x_i}}{v^2}$$
$$+ \delta_i(\theta+1)\sum_{i=1}^{n}\frac{e^{-\alpha x_i}}{u} - \delta_i(\theta+1)\sum_{i=1}^{n}\frac{x_i e^{-\alpha x_i}}{u}$$
$$- \delta_i(\theta+1)\sum_{i=1}^{n}\frac{x_i e^{-2\alpha x_i}}{u^2} - \delta_i\theta\lambda\sum_{i=1}^{n}\frac{x_i e^{-\alpha x_i}v^{\theta-1}}{u^{\theta+1}}$$
$$+ \delta_i\theta\lambda(\theta-1)\sum_{i=1}^{n}\frac{x_i e^{-2\alpha x_i}v^{\theta-2}}{u^{\theta+2}} + \delta_i\theta\lambda\sum_{i=1}^{n}\frac{e^{-\alpha x_i}v^{\theta-1}}{u^{\theta+1}}$$
$$+ \delta_i\alpha\theta\lambda(\theta+1)\sum_{i=1}^{n}\frac{x_i e^{-2\alpha x_i}v^{\theta-1}}{u^{\theta+2}}$$
$$- 2(1-\delta_i)\theta\lambda\sum_{i=1}^{n}\frac{e^{-\alpha x_i}v^{\theta-1}e^{\lambda(\frac{v}{u})^\theta}}{(e^\lambda - e^{\lambda(\frac{v}{u})^\theta})u^{\theta+1}}$$
$$+ 2(1-\delta_i)\alpha\theta\lambda\sum_{i=1}^{n}\frac{x_i e^{-\alpha x_i}v^{\theta-1}e^{\lambda(\frac{v}{u})^\theta}}{(e^\lambda - e^{\lambda(\frac{v}{u})^\theta})u^{\theta+1}}$$
$$- 2(1-\delta_i)\alpha\theta\lambda(\theta-1)\sum_{i=1}^{n}\frac{x_i e^{-2\alpha x_i}v^{\theta-2}e^{\lambda(\frac{v}{u})^\theta}}{(e^\lambda - e^{\lambda(\frac{v}{u})^\theta})u^{\theta+1}}$$
$$- 4(1-\delta_i)\alpha\theta^2\lambda^2\sum_{i=1}^{n}\frac{x_i e^{-2\alpha x_i}v^{2(\theta-1)}e^{\lambda(\frac{v}{u})^\theta}}{(e^\lambda - e^{\lambda(\frac{v}{u})^\theta})u^{2\theta+2}}$$
$$- 2(1-\delta_i)\alpha\theta(\theta+1)\lambda\sum_{i=1}^{n}\frac{x_i e^{-2\alpha x_i}v^{\theta-1}e^{\lambda(\frac{v}{u})^\theta}}{(e^\lambda - e^{\lambda(\frac{v}{u})^\theta})u^{\theta+2}}$$
$$- 4(1-\delta_i)\alpha\theta^2\lambda^2\sum_{i=1}^{n}\frac{x_i e^{-2\alpha x_i}v^{2(\theta-1)}e^{2\lambda(\frac{v}{u})^\theta}}{(e^\lambda - e^{\lambda(\frac{v}{u})^\theta})^2\, u^{2\theta+2}}$$

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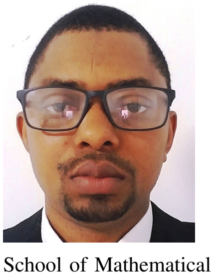

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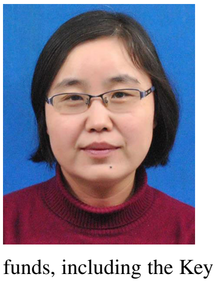

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