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Stabilization Domains for Second Order Delay Systems

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ABSTRACT In this paper, an analytic condition is given for determining delayed positive feedback controller for stabilizing an oscillatory system. The τ-decomposition and D-decomposition methods are employed in deriving this condition. The obtained results are then used to stabilize second order delay systems by a proportional controller. Under-damped and over-damped systems are treated separately, where the Smith-predictor structure is used in the over-damped case to obtain the stability conditions. Illustrative examples are given to show the effectiveness of the proposed approach.

INDEX TERMS Delayed positive feedback, τ-decomposition, D-composition, stability domains, time delay systems.

I. INTRODUCTION

It was shown in [1] that delayed positive feedback can stabilize oscillatory systems. It was also proven that a single integrator is stabilizable by a single delay and that a chain of integrators can be stabilized by multiple delays [2]. The continuing interest in stabilizing systems by delay can be explained and motivated by the form of the controller which is simple and easily implementable, as it consists usually of delayed versions of the output multiplied by constants. In [3], the authors presented a delay based control strategy, called the integral retarded controller, to control the velocity of a DC servomotor. The stabilization problem of a vibration system by delayed state difference feedback was investigated in [4]. In [5], stabilization of longitudinal vibrations along an elastic beam with delayed feedback is analyzed. Delayed feedback was also employed to stabilize nonlinear systems. In [6], stability of a class of generalized gyroscope systems under delayed feedback was investigated. In [7], the authors studied the stabilization problem of an unstable highly nonlinear hybrid stochastic differential delay equation by delay feedback control. In [8], delayed positive feedback was used in the study and analysis of coupled genetic oscillators.

Although stabilizing by delayed feedback has advantages such as ease of implementation and its simplicity, the introduction of delay comes with difficulties in analyzing stability of the closed loop system, which is challenging even in the case of linear time invariant systems. The difficulties are mainly due to the resulting infinite dimensional eigenvalue problem caused by the delay. In fact, studying stability of delay systems continue to be one of the active areas of research [9]–[10], as time delay appears in a natural way in many mathematical models of engineering and physical systems. It is well known that the delay term complicates the analysis of these systems, deteriorates the performance and may lead to instability. Stability results for time delay systems can be achieved using a time domain approach, where Lyapunov theorem and its extensions are exploited. Usually, this approach results in solving linear matrix inequalities LMI's. Aggregation methods were employed to get sufficient stability conditions, see [11]–[12]and the references therein.

Frequency domain methods are also used in stabilizing time delay systems. Many researchers used a parametric

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approach to study this problem by concentrating on concluding stability or instability of the system with respect to a single or multiple system parameters. The main approaches can be divided into two main streams. Those studying system's stability with respect to delay value, known as τ-decomposition methods [13], where it is focused on the partition of the delays parameter space into countably many regions having the same number of unstable poles [14]. The second approach, studies stability with respect to system parameters, not the delay, and it is known as the D-decomposition method. See [15]– [16] and the references therein for a detailed explanation and applications of the D-decomposition method. Both approaches rely on the continuity property of the roots of the quasi-polynomial with respect to delay and system parameters. The D-decomposition method was largely used in robust stability problems and design of low fixed order controllers for time delay systems, as it permits to decompose the controller's parameter space and determine stability boundaries using a set of parametric equations. These equations explicitly depend on the control parameters, which results in finding the stability regions and can help in obtaining analytical expressions for tuning the controllers. In this line of research, the D-decomposition method was used in [15] to determine stabilizing PID controllers for dead time systems; and in [17], the set of all stabilizing first order controllers were parameterized. Other methods were also used, such as the extensions of the Hermite-Biehler theorem, to find the set of all stabilizing PI and PID controllers for first-order systems with time delay in [18]–[19], respectively.

The main contribution of the paper is the joint use of the D-decomposition and τ-decomposition methods to find analytic conditions on the proportional and delay terms of a delayed positive controller to guarantee stability of the closed loop of an oscillatory system. Stabilization of oscillatory systems finds applications in many fields such as robotics and flexible structures [1]. A similar result was derived in [1] using the Nyquist criterion. Another advantage of the proposed method is its direct application in stabilizing second order delay systems by proportional controllers. It is also shown how to use this approach with a Smith predictor to stabilize second order time delay system. Moreover, the obtained results are applied to the consensus problem of multi-agent network system [20]–[22]. Finally, determining stabilizing regions for second order delay system can be considered as first step in obtaining optimal low order controllers by optimizing other performance indices using genetic algorithms or particle swarm optimization techniques.

The paper is organized as follows. In section II, we determine the stability domains in the space of the couple (k, τ) representing the parameters of a controller applied to an oscillating second order system. Next, we generalize the obtained results to a delayed second order system, where under-damped systems and over-damped systems are treated separately. Section IV is devoted to illustrative examples.

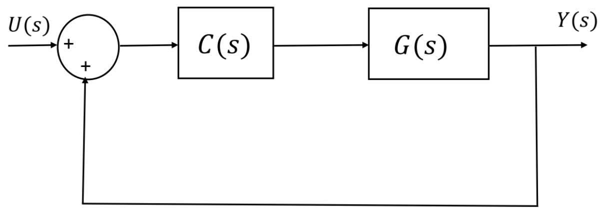

FIGURE 1. Block diagram of the system.

Finally, section V concludes the paper and draws some future alternatives.

II. SPECIAL SECOND ORDER SYSTEM

In this section, we consider determining the stabilizing regions in the (k, τ) space of a delayed positive feedback controller

$$u(t) = ky(t - \tau), \tag{1}$$

with a positive feedback k and a time delay represented by $\tau > 0$, applied to the system

$$\ddot{y}(t) + \alpha y(t) = u(t), \ \alpha > 0. \tag{2}$$

First, the admissible ranges of the positive feedback gain k are found by applying the D-decomposition method. Then, k is fixed within this range and the $\tau-$decomposition method is used to determine the stabilizing delays τ. By sweeping over the values of k, the complete set of stabilizing controllers can be obtained. The novelty in the developed method resides in the fact that the ranges of stabilizing (k, τ) values, for second order oscillatory system, are determined analytically. The closed-loop transfer function of Figure 1 is given by,

$$H(s) = \frac{C(s)G(s)}{1 - C(s)G(s)} = \frac{ke^{-\tau s}}{s^2 + \alpha - ke^{-\tau s}}. \tag{3}$$

Analytic determination of stability conditions for the closed loop system is a challenging task as a result of the obtained quasi-polynomial:

$$D(s, \tau) = s^2 + \alpha - ke^{-\tau s}. \tag{4}$$

That is why we will give a solution by dividing the original problem into two sub-problems.

A. DETERMINING ADMISSIBLE VALUES OF k

The D-decomposition technique relies on the continuity property of the roots of the quasi-polynomial (4) with respect to the coefficients. A necessary and sufficient condition for instability of a stable quasi-polynomial is the crossing of the imaginary axis by at least one of the roots. Suppose that τ is fixed, the previously mentioned result can be used to determine stabilizing intervals of k by partitioning the real line k as explained in the sequel. The determination of $D-$decomposition borders is obtained by setting $s = jw$ in (4), which yields

$$-w^2 + \alpha - ke^{-j\tau w} = 0. \tag{5}$$

the real part and imaginary part of (5) are equated to zero, to obtain:

$$\begin{cases} -w^2 + \alpha - kcos(\tau w) = 0, \\ sin(\tau w) = 0, \end{cases} \tag{6}$$

which gives the following equations

$$\begin{cases} -w^2 + \alpha - k(-1)^m = 0, \\ \tau w = m\pi \text{ where } \qquad m = 0, 1, 2, \ldots \end{cases} \tag{7}$$

Two cases are considered:

- Case 1: For even values of m, i.e. $m = 2l$, $l = 0, 1, 2, \ldots$, we have

$$\begin{cases} -w^2 + \alpha - k = 0, \\ w = \frac{2l\pi}{\tau}. \end{cases} \tag{8}$$

- Case 2: For odd values of m, i.e. $m = 2l + 1$, $l = 0, 1, 2, \ldots$, we have

$$\begin{cases} -w^2 + \alpha + k = 0, \\ w = \dfrac{(2l+1)\pi}{\tau}. \end{cases} \tag{9}$$

Equations (8) and (9) are written as follows:

$$\begin{cases} -(\frac{2l\pi}{\tau})^2 + \alpha - k = 0, \\ -(\frac{(2l+1)\pi}{\tau})^2 + \alpha + k = 0. \end{cases} \tag{10}$$

The resolution of this system of equations gives

$$\alpha = \frac{(8l^2 + 4l + 1)}{2}(\frac{\pi}{\tau})^2, \tag{11}$$

and

$$k = \frac{(4l+1)}{2}(\frac{\pi}{\tau})^2. \tag{12}$$

Using (11) and (12), the quasi-polynomial (4) has roots on the imaginary axis when

$$k = \frac{(4l+1)}{(8l^2+4l+1)}\alpha. \tag{13}$$

Note here that the above expression found for k is independent of τ. Taking account of the fact that $k > 0$, the admissible intervals as a function of l are obtained as follows:

$$0 < k < \frac{(4l+1)}{(8l^2+4l+1)}\alpha, \tag{14}$$

and

$$k > \frac{(4l+1)}{(8l^2+4l+1)}\alpha, \tag{15}$$

where $l = l = 0, 1, 2, \ldots$

B. COMPUTING STABILIZING VALUES OF τ

Once the admissible values of the gain k are determined, the conditions on the delay τ should be found. The τ-decomposition method is applied to obtain τ-values, which ensure stabilization of the system.

Given the characteristic function,

$$D(s, \tau) = n(s) + d(s)e^{-\tau s}, \tag{16}$$

where the polynomials $d(s)$ and $n(s)$ are coprime and degree$(n(s))$ $>$ degree$(d(s))$. The $\tau-$decomposition technique allows specifying the τ values so that $D(s, \tau)$ is stable.

We notice that if $s = jw$ is a root of $D(s, \tau)$, then its complex conjugate $\bar{s} = -jw$ is also a root, which gives the following system of equations:

$$\begin{cases} n(jw) + d(jw)e^{-j\tau w} = 0, \\ n(-jw) + d(-jw)e^{+j\tau w} = 0. \end{cases} \tag{17}$$

The elimination of the exponential term presented in (17) allows obtaining:

$$W(w^2) = n(jw)n(-jw) - d(jw)d(-jw) = 0. \tag{18}$$

However, when w_i verifies $W(w_i^2) = 0$ then $s = jw_i$ and $s = -jw_i$ denote the system characteristic roots for delays τ validating the equation below:

$$e^{-jw_i\tau} = -\frac{n(jw_i)}{d(jw_i)}. \tag{19}$$

The τ-decomposition technique is based on the algorithm written below:

1) Specify the delay free system roots (for $\tau = 0$), which provide hints about the delay τ impact on the evolution of the roots.
2) Compute the polynomial $W(w^2)$.
3) Calculate the real positive roots of $W(w^2)$ (represented by w_i^2 for $i \geq 1$). In case there is no positive real solution for W, then system stability is the same for all delay values. Moreover, it is important to determine the motion of the roots located near jw_i as τ increases. Thus, the sense of variation of $W(.)$ should be examined according to w^2. Obviously, the sense of variation of the functions $Re(s, .) : R^+ \rightarrow R, \tau \mapsto Re(s, \tau)$, where $Re(s, .)$ corresponds to the real part of s and $W(.) : R^+ \rightarrow R, w^2 \mapsto W(w^2)$ is the same [23].

$$signRe\frac{ds}{d\tau}|_{s=jw_i} = sign\frac{dW(w^2)}{dw^2}|_{w^2=w_i^2}. \tag{20}$$

Thus, the $\tau-$decomposition technique relies on the $Re(s, \tau)$ variation sense in the points jw_i neighborhood. In fact, when $\frac{dRe(s,\tau)}{d\tau} \leq 0$ or $\frac{dRe(s,\tau)}{d\tau} \geq 0$, then, for all $\tau_1 \geq \tau_2$ we get respectively $Re(s, \tau_1) \leq Re(s, \tau_2)$ or $Re(s, \tau_1) \geq Re(s, \tau_2)$. This result shows that roots move to the right-half or to the left-half complex plane according to the change of the delay value. This procedure allows determining the values of τ constituting the upper and lower-bounds of the system stability interval.

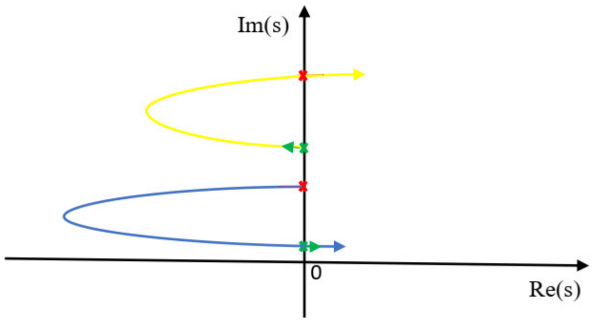

FIGURE 2. jw_l **producing the upper-bounds of stabilizing values of τ; jw_l producing the lower-bounds of stabilizing values of τ; Roots locus for decreasing value of τ and Roots locus for decreasing value of τ.**

We observe, from Figure 2, the effect of varying the time delay in the neighborhood of the crossing points. At these points, the delay bounds can be determined. In fact, for an increasing function $Re(s, \tau)$, as delay τ increases in the neighborhood of the points jw_l, $Re(s, \tau)$ will also augment as shown in Figure 2. As a consequence, the roots will migrate to the instability region. Hence, the upper bound of delay ensuring stability is obtained by increasing τ until the first pole of the system is located at the $j\omega$ axis. In a similar manner, if $Re(s, \tau)$ is a decreasing function of τ, the lower stability bound is calculated.

Applying this algorithm to the quasi-polynomial defined by (4), we get:

1) First compute zeros of the system without delay ($\tau = 0$). The system characteristic zeros provided by $s = \pm j\sqrt{\alpha - k}$, satisfying the following condition:

$$k < \alpha. \tag{21}$$

Obviously, the zeros are placed on the imaginary axis.

2) As

$$n(s) = s^2 + \alpha \quad \text{and} \quad d(s) = -k, \tag{22}$$

we obtain

$$\begin{aligned} W(w^2) &= n(jw)n(-jw) - d(jw)d(-jw), \\ &= (-w^2 + \alpha)^2 - k^2. \end{aligned}$$

3) Solving $W(w^2) = 0$ and considering the condition (21), the following roots are determined:

$$\begin{cases} w_1^2 = \alpha - k > 0, \\ w_2^2 = \alpha + k > 0. \end{cases}$$

By testing $Re(s, .)$ monotony in the points $\pm j\sqrt{w_1}$ neighborhood, we obtain

$$\begin{aligned} signRe\frac{ds}{d\tau}|_{s=j\sqrt{w_1}} &= sign\frac{dW(w^2)}{dw^2}|_{w^2=w_1^2}, \\ &= sign(-2k) < 0. \end{aligned}$$

In this case, $Re(s, .)$ is a decreasing function of τ. Therefore, increasing the value of τ will result in crossing the imaginary axis from right to left. Applying (19), we obtain:

$$cos(\tau\sqrt{\alpha - k}) = 1 \text{ and } sin(\tau\sqrt{\alpha - k}) = 0, \tag{23}$$

Thus,

$$\tau = \frac{2l\pi}{\sqrt{\alpha - k}} \text{ where } l = 0, 1, 2 \ldots \tag{24}$$

To maintain the roots in the left-half complex plane, the following relation should be satisfied

$$\tau > \frac{2l\pi}{\sqrt{\alpha - k}} \quad \text{where } l = 0, 1, 2 \ldots.$$

Similar arguments are applied in the case of the roots $\pm j\sqrt{w_2}$. The function $Re(s, .)$ monotony test is obtained applying the equation written below:

$$signRe\frac{ds}{d\tau}|_{s=j\sqrt{w_2}} = sign(2k) > 0. \tag{25}$$

Roots cross the imaginary axis from left to right for the following values of τ:

$$\tau = \frac{(2l+1)\pi}{\sqrt{\alpha + k}} \quad \text{where } l = 0, 1, 2 \ldots \tag{26}$$

Which gives the following condition

$$\tau < \frac{(2l+1)\pi}{\sqrt{\alpha + k}}.$$

Thus, the system stability is achieved for:

$$\frac{2l\pi}{\sqrt{\alpha - k}} < \tau < \frac{(2l+1)\pi}{\sqrt{\alpha + k}} \quad \text{where } l = 0, 1, 2 \ldots \tag{27}$$

Now, taking into account (14), (15), (21) and (27), we get the following conditions on the values of k and τ that guarantee stability of the closed loop system defined by (3):

$$\begin{cases} 0 < k < \dfrac{(4l+1)}{(8l^2 + 4l + 1)}\alpha, \\ \dfrac{2l\pi}{\sqrt{\alpha - k}} < \tau < \dfrac{(2l+1)\pi}{\sqrt{\alpha + k}}, \end{cases} \tag{28}$$

for $l = 0, 1, 2 \ldots$.

III. GENERAL SECOND ORDER DELAY SYSTEMS

In this section, taking advantage of the obtained result (28), we develop a method of stabilizing second order delay systems. Consider a prototype second order system with delay given by

$$G(s) = \frac{k_2 w_n^2 e^{-\tau s}}{s^2 + 2\zeta w_n s + w_n^2}, \tag{29}$$

where $\zeta > 0$ is the damping ratio, w_n is the natural frequency and k_2 a constant, to be stabilized by a proportional controller having the form

$$C(s) = k_1 e^{-\tau\zeta w_n}. \tag{30}$$

In this case, the transfer function of the closed-loop system of Figure 1 is given by

$$H(s) = \frac{k_1 k_2 w_n^2 e^{-\tau(s+\zeta w_n)}}{s^2 + 2\zeta w_n s + w_n^2 - w_n^2 k_1 k_2 e^{-\tau(s+\zeta w_n)}}, \tag{31}$$

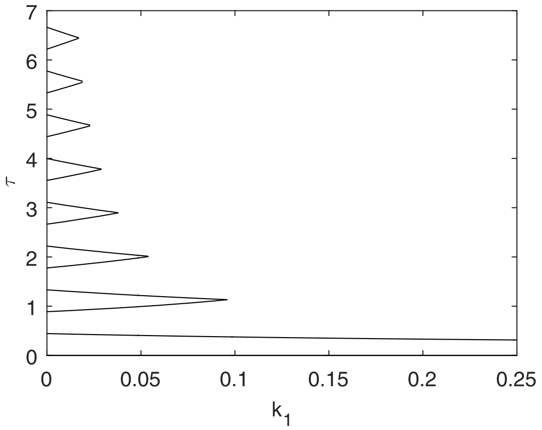

FIGURE 3. Stability regions (k_1, τ) for $\zeta = 0.707$ and $l = 0, 1, \ldots, 7$.

and the characteristic equation of (31) is as follows

$$D(s, \tau) = s^2 + 2\zeta w_n s + w_n^2 - w_n^2 k_1 k_2 e^{-\tau(s+\zeta w_n)}. \quad (32)$$

Now, we impose that the poles of the closed loop system are located to the left of the damping factor ζw_n by the change of variable $\mu = s + \zeta w_n$, we get

$$D(\mu, \tau) = \mu^2 + w_n^2(1 - \zeta^2) - k_1 k_2 w_n^2 e^{-\tau\mu}. \quad (33)$$

It is clear that equation (33) has the same form of (4) where $\alpha = w_n^2(1 - \zeta^2)$, $k = k_1 w_n^2$. The case $\zeta = 0$ corresponds to an oscillatory system and we can directly apply the results of the previous section. Two cases, namely the under-damped and over-damped case, will be studied.

A. UNDER-DAMPED CASE

In this case, we assume that the open loop system is under-damped ($\zeta < 1$), the stability regions of couple (τ, k_1) is provided applying the technique presented in Section II. Therefore, by using the τ-decomposition method, the stabilizing delay values are given by the following condition

$$\frac{\frac{2l\pi}{w_n}}{\sqrt{1 - \zeta^2 - k_1 k_2}} < \tau < \frac{\frac{(2l+1)\pi}{w_n}}{\sqrt{1 - \zeta^2 + k_1 k_2}}, \quad (34)$$

and the D-decomposition method permits the determination of k_1 stabilizing gain values given by

$$0 < k_1 \leq \frac{(4l + 1)}{(8l^2 + 4l + 1)} \frac{(1 - \zeta^2)}{k_2}, \quad l = 0, 1, 2, \ldots \quad (35)$$

For $w_n = 1$ and $\zeta = 0.707$, the stability regions are given in Figure 3. Figure 4 and Figure 5 show the stability region in the (k_1, τ, ζ) space for $l = 1$ and $l = 1, 2, \ldots, 6$, respectively.

B. OVER-DAMPED CASE

Consider the over-damped ($\zeta > 1$) case. We can just choose k_1 so that the polynomial $D(s, 0)$ is Hurwitz stable and the roots of $W(w^2)$ are all negative. However, the difficulty in satisfying these two constraints comes from the fact that we imposed the use of a proportional controller $C(s) = k_1 e^{-\zeta w_n}$. This may not always be feasible to achieve by the use of one

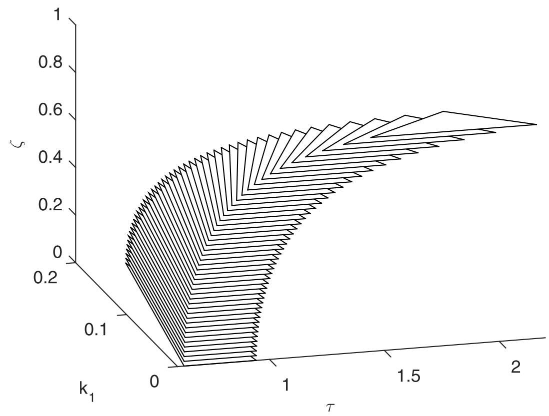

FIGURE 4. Stability regions (k_1, τ, ζ) for $l = 1$.

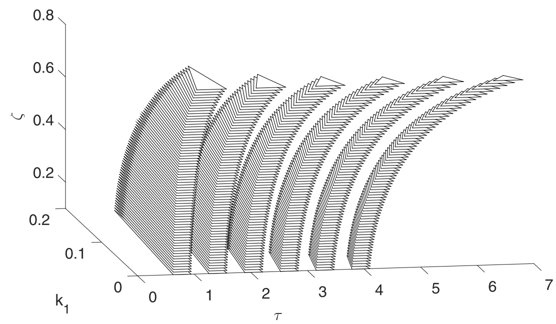

FIGURE 5. Stability regions (k_1, τ, ζ) for $l = 1, 2, \ldots, 6$.

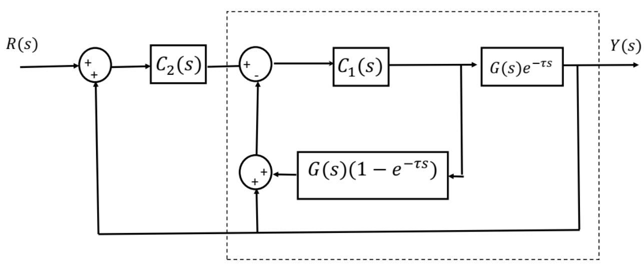

FIGURE 6. The Smith-predictor feedback system.

parameter k_1. To solve this problem, Smith's predictor is a key. Assuming that $C_1(s) = k_3$, the transfer function of the dashed block part in Figure 6 is given by

$$\begin{aligned} H_1(s) &= \frac{C_1(s)G(s)}{1 + C_1(s)G(s)} e^{-\tau s}, \quad (36) \\ &= \frac{k_2 k_3 w_n^2}{s^2 + 2\zeta w_n s + w_n^2 + k_2 k_3 w_n^2} e^{-\tau s}. \end{aligned}$$

By an appropriate choice of controller $C_1(s)$ the system can become an oscillatory type system. In fact, by choosing $C_1(s) = k_3 = \frac{\zeta^2}{k_2}$, and the same variable change $\mu = s + \zeta w_n$ we get

$$D(s) = s^2 + 2\zeta w_n s + w_n^2 + \zeta^2 w_n^2 = \mu^2 + w_n^2. \quad (37)$$

In the other words, the choice of

$$C_2(s) = k_4 e^{-(\tau+\tau_1)\zeta w_n} e^{-\tau_1 s}, \quad (38)$$

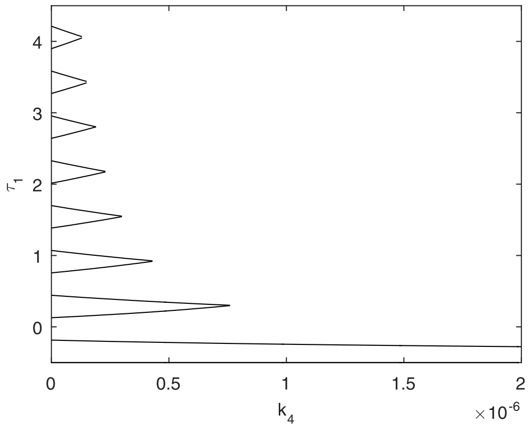

FIGURE 7. Stability regions (k_4, τ_1) for $\zeta = 50$, $w_n^2 = 10$ and $k_2 = 2$ and $\tau = 0.25$ seconds where $l = 0, 2, \ldots, 7$.

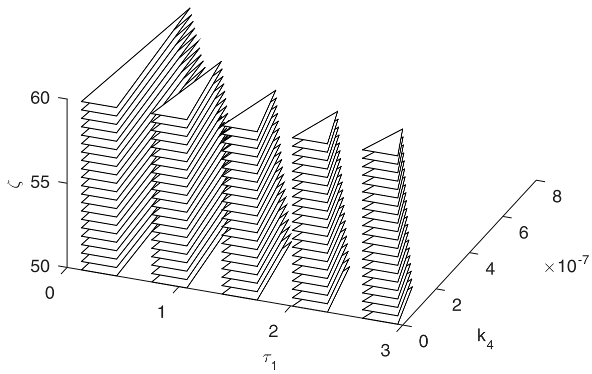

FIGURE 8. Stability regions (k_4, τ_1, ζ) for $w_n^2 = 10$ and $k_2 = 2$ and $\tau = 0.25$ seconds where $l = 1, 2, \ldots, 5$.

as a delayed positive feedback, the transfer function of the overall system presented in Figure 6 becomes

$$H(s) = \frac{k_4 w_n^2 \zeta^2 e^{-(\tau+\tau_1)\mu}}{\mu^2 + w_n^2 - \zeta^2 w_n^2 k_4 e^{-(\tau+\tau_1)\mu}}, \tag{39}$$

and the characteristic equation of (39) is as follows

$$D(\mu, \tau + \tau_1) = \mu^2 + w_n^2 - \zeta^2 w_n^2 k_4 e^{-(\tau+\tau_1)\mu}. \tag{40}$$

Therefore, we obtain in the same way, the relations defining the domain of (k_4, τ_1) stabilizing values, given by the following inequalities

$$\frac{\frac{2l\pi}{w_n}}{\sqrt{1 - \zeta^2 k_4}} - \tau < \tau_1 < \frac{\frac{(2l+1)\pi}{w_n}}{\sqrt{1 + \zeta^2 k_4}} - \tau, \tag{41}$$

and

$$0 < k_4 \leq \frac{(4l+1)}{(8l^2 + 4l + 1)} \frac{1}{\zeta^2}, l = 0, 1, 2, \ldots \tag{42}$$

Stability regions in the plane (k_4, τ_1), for $\zeta = 50$, $w_n^2 = 10$, $k_2 = 2$ and $\tau = 0.25$ seconds with $l = 0, 2, \ldots, 7$ are given in Figure 7. Finally, Figure 8 shows a 3D plot of the stability regions by varying the damping ratio.

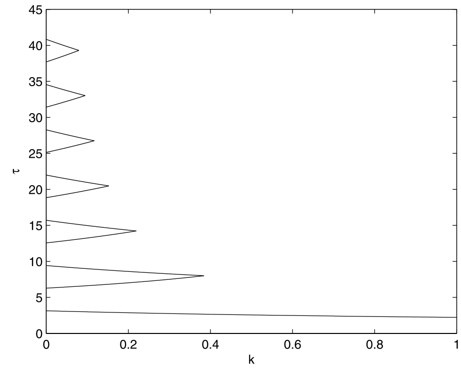

FIGURE 9. Stability regions for $\alpha = 1$ and $l = 0, 1, \ldots, 6$.

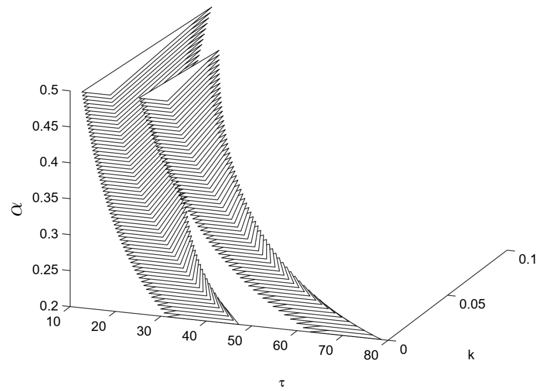

FIGURE 10. Stability regions in the (k, τ, α) plane for $l = 1, 2$.

IV. ILLUSTRATIVE EXAMPLES

In this section, we present four illustrative examples. In the first example, we consider the stabilization of an oscillatory system. In the second example, a DC motor modeled by a second order under-damped system, is treated. In the third example, the proposed method is applied to a TCP/AQM system, modeled by a second order over-damped delay system, in congestion control. A fourth example is given to show the application of the proposed approach to second order multi-agent system with delayed positive feedback controller.

Example 1: Consider stabilizing an oscillatory system with $\alpha = 1$. Figure 9 gives the stability regions for $l = 0, 1, \ldots, 6$ and Figure 10 illustrates the stabilizing regions in the (k, τ, α) plane using condition (28). The state responses are shown in Figures 11 and 12.

When supplied by voltage on the stator and on the rotor, a motor usually exerts a torque T that acts on the mechanical structure, which is characterized by the rotor inertia J and the viscous friction coefficient B. The equation that links the torque and the angular velocity w is given below

$$T(t) = K_t i(t) = J \frac{dw(t)}{dt} + Bw(t), \tag{43}$$

where K_t is the torque constant and i is the armature current. The velocity of the motor can be used to get the position θ as follows

$$w(t) = \frac{d\theta(t)}{dt}. \tag{44}$$

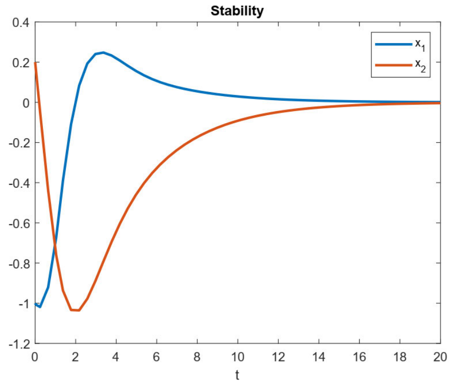

FIGURE 11. Response curves to initial conditions $x_1(0) = 0.1, x_2(0) = -1$, where $x_1 = y, x_2 = \dot{y}$, for $k = 0.8, \tau = 1$.

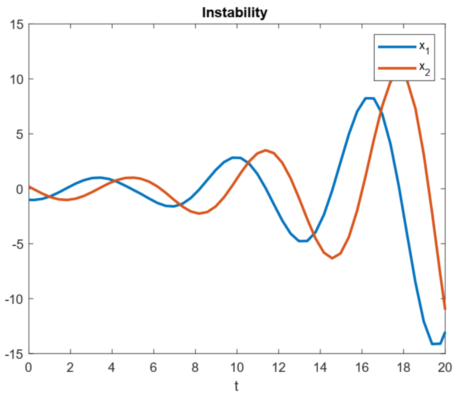

FIGURE 12. Response curves to initial conditions $x_1(0) = 0.1, x_2(0) = -1$, where $x_1 = y, x_2 = \dot{y}$, for $k = 0.8, \tau = 5$.

Substituting the (43) into (44), gives

$$K_t i(t) = J\frac{d^2\theta(t)}{dt^2} + B\frac{d\theta(t)}{dt}. \tag{45}$$

Therefore, the transfer function of the DC motor is given by

$$\frac{\theta(s)}{i(s)} = \frac{b}{s(s+a)}, \tag{46}$$

Example 2: The mechanical representation of a DC motor is shown in Figure 13. Now, we stabilize the DC motor by a positive delayed feedback controller as shown in Figure 14. The closed loop transfer where $G(s)$ is given by

$$G(s) = \frac{\theta(s)}{i(s)} = \frac{b}{s^2 + as + bk_1 - bk_2e^{-\tau s}}, \tag{47}$$

$$= \frac{b}{(s+\frac{a}{2})^2 + bk_1 - \frac{a^2}{4} - bk_2e^{-\tau s}},$$

$$= \frac{b}{\mu^2 + w_n^2 - ke^{-\tau s}}, \tag{48}$$

with $\mu = s + \frac{a}{2}$, $w_n^2 = bk_1 - \frac{a^2}{4}$ and $k = bk_2$. Choosing the same parameters given in [3], namely $a = 5$ and $b = 53$, we obtain an under-damped second order system.

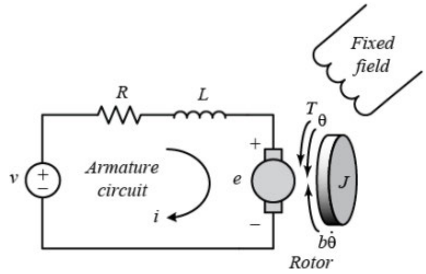

FIGURE 13. Mechanical representation of DC motor.

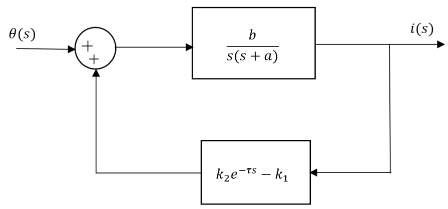

FIGURE 14. Block diagram of the DC servomotor in closed loop with the delayed positive feedback controller.

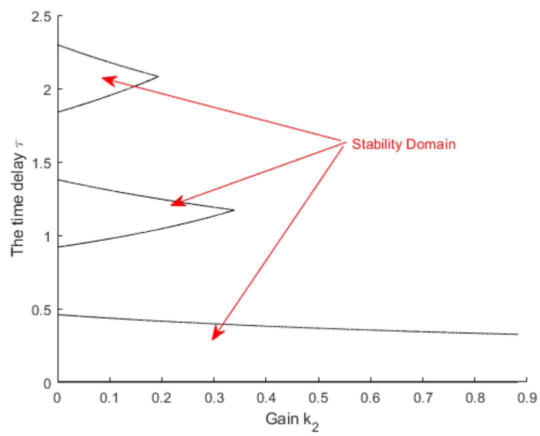

FIGURE 15. Stability regions (k_2, τ) for $l = 0, 1, 2..$

Therefore, by using the τ-decomposition method, the stabilizing delay values are given by the following condition

$$\frac{2l\pi}{\sqrt{bk_1 - \frac{a^2}{4} - bk_2}} < \tau < \frac{(2l+1)\pi}{\sqrt{bk_1 - \frac{a^2}{4} + bk_2}}, \tag{49}$$

and the D-decomposition method permits the determination of k_2 stabilizing gain values expressed as

$$0 < k_2 \leq \frac{(4l+1)}{(8l^2+4l+1)}(k_1 - \frac{a^2}{4b}), \quad l = 0, 1, 2, \ldots \tag{50}$$

Stability domains in the (k_2, τ_1) plane are given in Figures 15, 16 and 17 for different values of l and k_1. where $b = \frac{k_t}{J}$ and $a = \frac{B}{J}$.

Example 3: In [24], the dynamic model of transmission control protocol TCP flows is introduced by applying the fluid flow model without taking into accounts low beginning and timeout mechanism. Using this system together with delays in the network, an active queue management AQM is built.

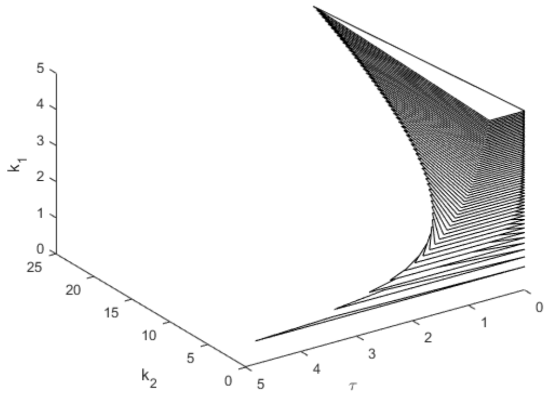

FIGURE 16. Stability regions (k_2, τ, k_1) for $l = 0$..

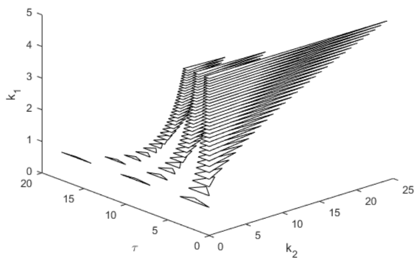

FIGURE 17. Stability regions (k_2, τ, k_1) for $l = 0, 1, 2$.

This model is depicted using the non-linear differential equation written below:

$$\begin{cases} \dot{W}(t) = \dfrac{1}{R(t)} - \dfrac{W(t)}{2} \times \dfrac{W(t-R(t))}{R(t-R(t))} p(t-R(t)), \\ \dot{q}(t) = \dfrac{NW(t)}{R(t)} - C, \\ R(t) = \dfrac{q(t)}{C(t)} + T_p. \end{cases} \tag{51}$$

where $\dot{W}(t)$ and $\dot{q}(t)$ are $W(t)$ and $q(t)$ time-derivatives, respectively. $W(t)$ represents the TCP window size, $q(t)$ denotes the queue length in the router, $p(t)$ denotes the probability packet marking/dropping ($p(t) \in [0\ 1]$), $R(t)$ denotes the round-trip time, $C(t)$ designates the capacity of the link, N the number of nodes and T_p refers to the delay of propagation. Equation (51) shows the progression of the TCP source congestion window $w(t)$. $p(t)$ signal represents the source loss rate, while delay τ corresponds to the RTT observed by the source. In system (51), (W, q) is the state of the system that should be controlled; p denotes the input. Thus, the equilibrium point (W^0, q^0, p^0) is given by resolving $\dot{W}(t) = 0$, assuming that $W(t - R_c) = W(t - R^0) = W_d^0$ and $q(t - R_c) = q(t - R^0) = q_d^0$.

After linearizing the nonlinear system [24], we obtain the following transfer function

$$G(s) = \frac{\frac{C^2}{2N} e^{-R_0 s}}{(s + \frac{2N}{R_0^2 C})(s + \frac{1}{R_0})} = \frac{k_2 w_n^2 e^{-R_0\, s}}{s^2 + 2\zeta w_n s + w_n^2}.$$

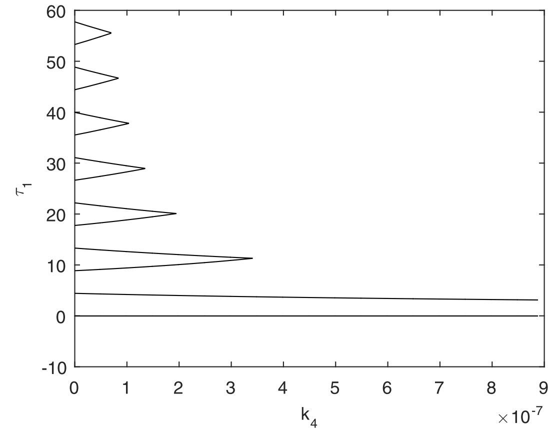

FIGURE 18. Stability domain $(k_4,\ \tau_1)$ for $N = 30$.

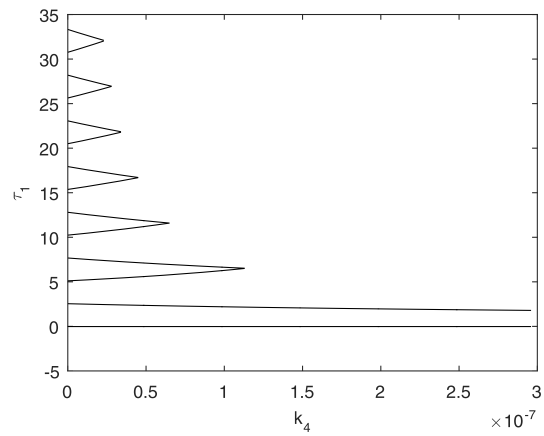

FIGURE 19. Stability domain $(k_4,\ \tau_1)$ for $N = 90$.

In this case, we have the following values: $w_n = \frac{1}{R_0}\sqrt{\frac{2N}{R_0 C}}$, $k_2 = \frac{R_0^3 C^3}{4N^2}$ and $\zeta = \frac{1}{2}\left(\sqrt{\frac{2N}{R_0 C}} + \sqrt{\frac{R_0 C}{2N}}\right)$ and $\tau = R_0$.
The transfer function of the applied proportional controller to the closed-loop AQM system is given by $C(s) = k_1 e^{-R_0 \zeta w_n}$. Since $\zeta > 1$, we find values of k_1 such that all roots of $D(s, 0)$ are situated in the left half of the complex plane. In addition, $W(w^2)$ must have no positive roots. These two conditions lead to the determination of the values of k_1 ensuring stability of the system

$$k_1 < \frac{4N^2}{R_0^3 C^3}. \tag{52}$$

Using the proposed method presented in the previous section, we can draw the domains of (k_1, τ_1) stabilizing values of the delayed positive feedback controller defined by (38). These domains are given in Figures 18, 19, 20, 21 and 22. Preliminary results for this example were given in [25].

Example 4: In this example we consider the consensus problem of a multi-agent network system. A second order linear dynamic model for each agent i is given below:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= a x_i + b v_i + u_i(t) \end{aligned} \tag{53}$$

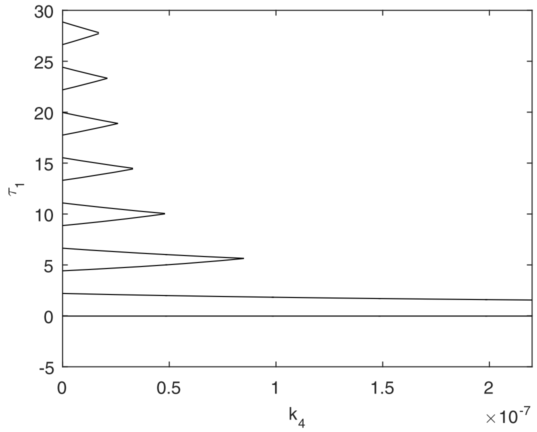

FIGURE 20. Stability domain (k_4, τ_1) for $N = 120$.

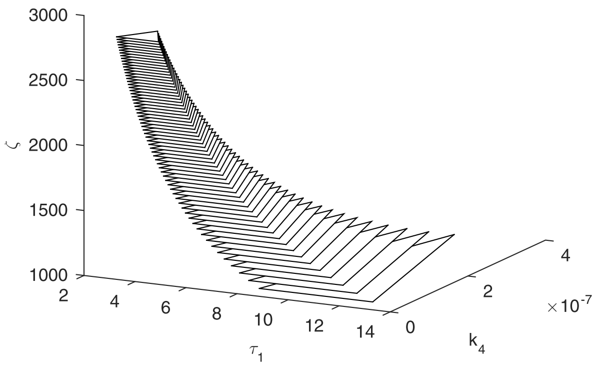

FIGURE 21. Stability domain (τ_1, k_4, ζ) for $l = 1$.

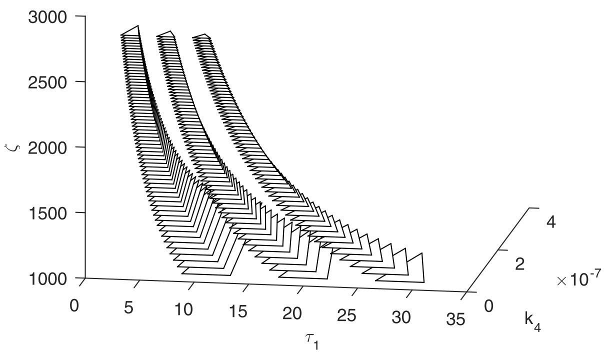

FIGURE 22. Stability domain (τ_1, k_4, ζ) for $l = 1, 2, 3$.

where $x_i \in R$, is the position state, $v_i \in R$ is the velocity state of i^{th} agent. The consensus protocol with delayed positive feedback can be written as

$$u_i(t) = -g_1 \sum_{\ell=1}^{n} a_{i,\ell}[x_\ell(t) - x_i(t)] + g_2 \sum_{\ell=1}^{n} a_{i,\ell}[x_\ell(t-\tau) - x_i(t-\tau)]. \quad (54)$$

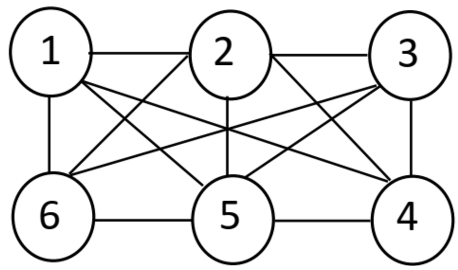

FIGURE 23. Network topology of 6 agents.

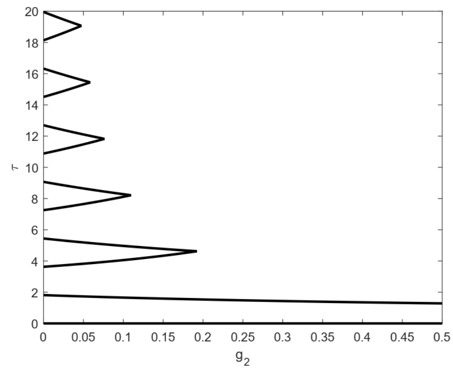

FIGURE 24. Stabilizing domains of (g_2, τ) for $a = 3, b = 0, g_1 = 0$, $l = 0, 1, \ldots, 5$.

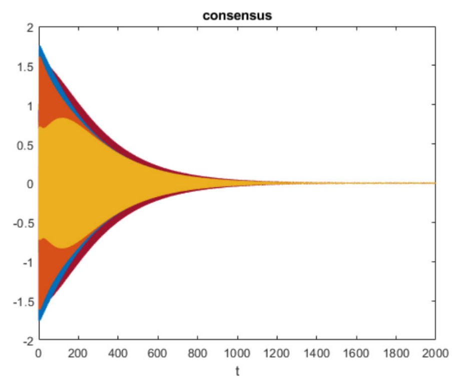

FIGURE 25. Consensus achieved for $a = 3$, $b = 0$ and $\tau = 8$ seconds, $g_1 = 0$ and $g_2 = 0.01$.

Similar to [22], we obtain its characteristic equation as follows:

$$D(s, \tau) \triangleq \prod_{i=1}^{n} f_i(s, \tau) = 0, \quad (55)$$

where $f_i(s, \tau)$, $i = 2, 3, \ldots, n$ are quasi-polynomials given by

$$f_i(s, \tau) = s^2 + bs + a + \lambda_i g_1 - \lambda_i g_2 e^{-\tau s}, \quad (56)$$

where λ_i, $i = 2, 3, \ldots, n$ are the non-zero eigenvalues of the Laplacian matrix of the graph. Then, using (28), we obtain the following result

$$0 < g_2 < \frac{4l+1}{8l^2+4l+1}\left(g_1 + \frac{a - \frac{b^2}{4}}{\lambda_{\max}}\right),$$
$$\max\{\tau_1, \tau_2\} < \tau < \min\{\tau_3, \tau_4\},$$

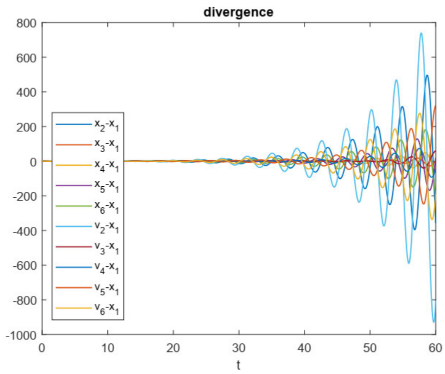

FIGURE 26. Consensus not achieved for a = 3, b = 0 and $\tau = 2$ seconds, $g_1 = 0$ and $g_2 = 0.2$.

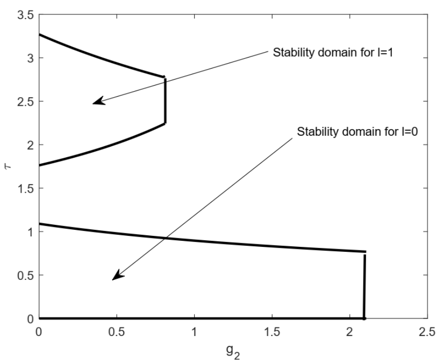

FIGURE 27. Stability domain of (g_2, τ) for $a = 0$, $b = 1.4$ and $g_1 = 2.2$.

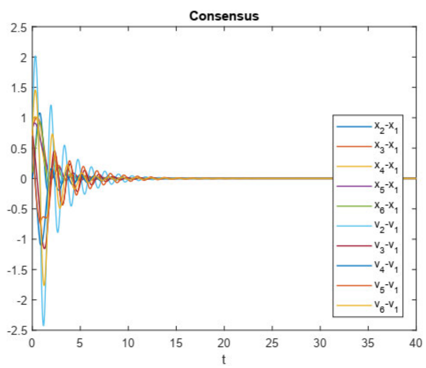

FIGURE 28. Consensus achieved for $a = 0$, $g_1 = 2.2$, $g_2 = 1$ and $\tau = 0.8$ seconds.

with

$$\tau_1 \triangleq \frac{2l\pi}{\sqrt{a - \frac{b^2}{4} + \lambda_{\max}(g_1 - g_2)}},$$

$$\tau_2 \triangleq \frac{(2l+1)\pi}{\sqrt{a - \frac{b^2}{4} + \lambda_{\max}(g_1 + g_2)}},$$

$$\tau_3 \triangleq \frac{2l\pi}{\sqrt{a - \frac{b^2}{4} + \lambda_{\min}(g_1 - g_2)}},$$

$$\tau_4 \triangleq \frac{(2l+1)\pi}{\sqrt{a - \frac{b^2}{4} + \lambda_{\min}(g_1 + g_2)}},$$

with $\lambda_{\max} = \max_{2 \le i \le n}\{\lambda_i\}$ and $\lambda_{\min} = \min_{2 \le i \le n}\{\lambda_i\}$.

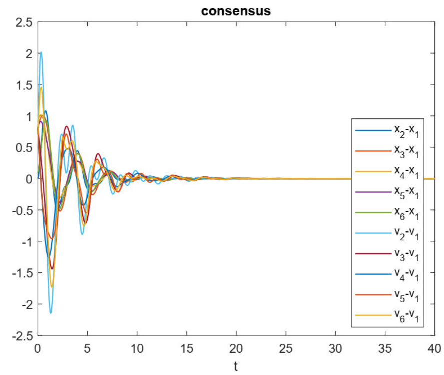

FIGURE 29. Consensus achieved for $a = 0$, $g_1 = 2.2$, $g_2 = 0.5$ and $\tau = 2.5$ seconds.

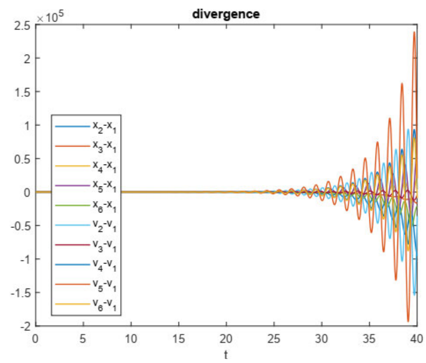

FIGURE 30. Consensus not achieved for $a = 0$; $b = 1.4$; $g_1 = 2.2$; $g_2 = 3$ and $\tau = 0.8$ seconds.

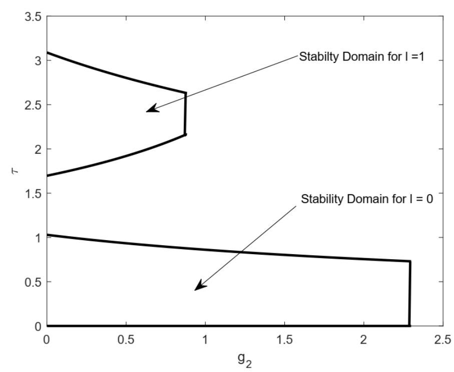

FIGURE 31. Stability domain of (g_2, τ) for $a = 1$, $b = 1.4$ and $g_1 = 2.2$.

For simulation, we consider a network with the following topology:
Its Laplacian matrix is given by

$$L = \begin{pmatrix} 4 & -1 & 0 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 & -1 & -1 \\ 0 & -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 & 0 \\ -1 & -1 & -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & 0 & -1 & 4 \end{pmatrix} \tag{57}$$

which leads to: $\lambda_{\min} = 4$, $\lambda_{\max} = 6$.
The stability domains for $a = 3$, $b = 0$, $g_1 = 0$, $l = 0, 1, \ldots, 5$ are given in Figure 24. The state responses are shown in Figures 25 and IV. For $a = 0$, $b = 1.4$, $g_1 = 2.2$,

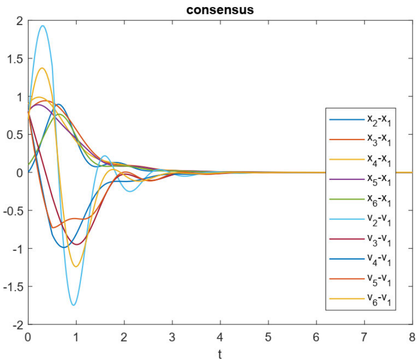

FIGURE 32. Consensus achieved for a=1, b=1.4 and $\tau = 0.5$ seconds, $g_1 = 2.2$ and $g_2 = 1.5$.

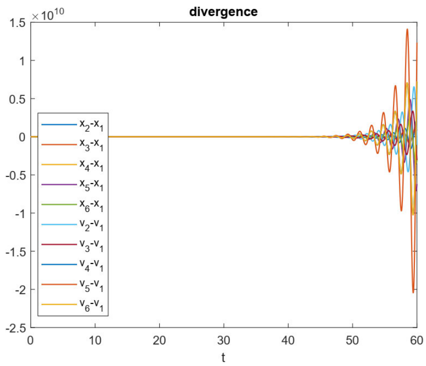

FIGURE 33. Consensus not achieved for a=1, b=1.4 and $\tau = 1.5$ seconds, $g_1 = 2.2$ and $g_2 = 2.5$.

$l = 0, 1$, the stability domains are given in Figure 27, its state responses are shown in Figures 28, 29 and 30. Finally, the stability domains for $a = 1$, $b = 1.4$, $g_1 = 2.2$, $l = 0, 1$ are given in Figure 31 and its state responses are shown in Figures 32 and 33.

V. CONCLUSION

Applying both τ-decomposition and D-decomposition techniques, stability of an oscillating second order systems with delayed positive feedback is examined. This method allows localizing stability domains of the parameters of a simple delayed positive feedback. Based on the obtained results, an extension was formulated to a delayed second order system. Four illustrative examples were given. In the first example, a delayed positive feedback for stabilizing an oscillatory system is given. In the second example, we stabilized an open loop under damped DC motor. In the third example, a controller was designed for AQM routers to address the congestion situations and minimize their impact. It consists of a simple positive feedback controller combined with Smith predictor. In the fourth example, the consensus problem of a second order multi-agent system with delayed positive feedback, was treated.

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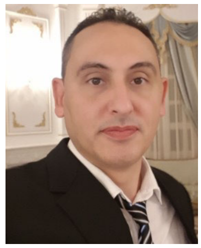

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