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Synchronization of Fractional Reaction-Diffusion Neural Networks With Time-Varying Delays and Input Saturation

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ABSTRACT This study is concerned with a synchronization problem of two fractional reaction-diffusion neural networks with input saturation and time-varying delays by the Lyapunov direct method. We extend the traditional ellipsoid method by giving the novel definition of the ellipsoid and linear region of the saturated, which makes our method succinct and effective. First, we linearize the saturation terms by the properties of convex hulls. Then, by using a new Lyapunov-Krasovskii functional, we give the synchronization criteria and estimate the domain of attraction. All the results are presented in the form of linear matrix inequalities(LMIs). Finally, two numerical experiments verify the validity and reliability of our method.

INDEX TERMS Fractional reaction-diffusion, neural networks, Riemann-Liouville, input saturation.

I. INTRODUCTION

After the conception of "small world" [1] came up, the related research of complex networks has entered a rapid development stage. Complex networks is the network dynamically evolving in time whose structure is regular and complex [2]. It is an abstract description of the interaction between individuals in nature over time. Therefore, complex networks can describe not only the whole but also local behavior.

As one kind of complex networks, the neural networks has attracted many scholars' interest because it can simulate many practical problems. Under the existing theoretical framework, the neural networks are described in two parts: the topological structure and the dynamical model. From the point of view of the dynamical model, previous studies mainly focused on the ODEs model. Still, in practice, the reaction-diffusion phenomenon cannot be ignored due to the necessity of describing the behavior of substance in space. Thus reaction-diffusion neural networks have become a research hotspot in recent years [3]–[5]. On the other hand, as an extension of the integral order reaction-diffusion equation, the fractional-order reaction-diffusion equation can model more complex phenomena due to its non-local properties. It has achieved great success in such fields as anomalous diffusion [6], image enhancement [7], and porous media seepage [8], [9]. For neural networks, the existing researches mainly focus on the problems of ODEs with Caputo and Riemann-Liouville derivative [10], utilize the Laplace transform and properties of Mittag-Leffler function to obtain stability conditions [11]–[14]. On the other hand, the adaptive control law also attracts the attention of scholars. In [15], an adaptive sliding mode control method was presented for a class of fractional-order nonlinear time-delay systems with uncertainties to solve the target output tracking problem. By employing Hermitian form Lyapunov functionals and fractional skills, [16] present some sufficient criteria for fractional complex projective synchronization. In [17], sufficient conditions for the global asymptotical stabilization of a class of fractional-order nonautonomous systems had been obtained by constructing quadratic Lyapunov functions and utilizing a new property for Caputo fractional derivative. In [18], the sliding mode control problem for a normalized singular fractional-order system with matched uncertainties was investigated. The global stabilization criteria were given in [19] for fractional memristor-based neural networks with the aid of Lyapunov functions and the comparison principle. In general, the above research mainly focuses on the

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ODE system. Only the recent work [20] concern about the fractional reaction-diffusion neural networks(FRDNNs) problem with Riemann-Liouville derivative. Hence, the study of neural networks with fractional order reaction-diffusion model will further develop related fields.

Considering almost all practical applications, the time delay is unavoidable. Hence in this paper, we also take this factor into account. In many cases, researchers will construct a special Lyapunov functional to solve such problems: the Lyapunov-Krasovskii functional. The Lyapunov-Krasovskii stability theorem for fractional systems with delay had been investigated in literature [21]. And the Lyapunov-Krasovskii functional has many applications on the stability criterion and controller designing [22]–[25].

From the earlier discussion, synchronization between the nodes of neural networks is a widespread phenomenon. Usually, we need to introduce some controllers to synchronize the nodes in the neural networks. Fortunately, there many synchronization control strategies such as pinning control [26], [27], sliding mode control [28], adaptive control [29], and sampled-data control [30], which have been implemented can be applied on this topic.

On the other hand, when designing the controller for synchronization, input saturation cannot be neglected due to the maximum power or the propagation and reaction rate. Once the control signal reaches or exceeds the saturation state, the system will become hard to control or completely uncontrollable. At present, some methods such as ellipsoid method [31], anti-windup [32], [33], have been applied to solve such problems. The problem of adaptive neural control for a class of strict-feedback stochastic nonlinear systems with multiple time-varying delays subject to input saturation has been investigated in [34], neural network-based adaptive control for spacecraft under actuator failures and input saturations has been handled in [35], and [36] investigates reliable estimation problem for Markovian jump neural networks with sensor saturation. There exists extensive research on the control systems with saturation [31], [37]–[41]. The ellipsoid method [31] is simple and reliable, has been applied successfully in some discrete or ODEs model [42], [43], but no application is seen in PDEs models such as reaction-diffusion problems. In fact, some successful methods in the ODE model cannot be directly applied to the PDE model, and we must consider the evolution of the model in the whole space. Compared to other anti-windup methods, which usually introduce a dead zone, the main advantage of this method is that it is easy to linearize the saturation controller by introducing the auxiliary gain function. The estimation of the domain of attraction can be obtained by solving LMIs.

To the best of our knowledge, synchronization of FRDNNs with input saturation has not yet been fully investigated, which has theoretical and practical value to study. We hope that by putting forward such a Riemann-Liouville neural network, combined with some existing research basis, we can contribute to technology development in related fields and get some more universal results. Hence, motivated by the above reasons, the synchronization of FRDNNs with input saturation is investigated in this paper. We mainly intend to extend the ellipsoid method [44] combining with the Lyapunov-Krasovskii functional to the field of fractional partial differential model.

In this paper, we will focus on the synchronization of FRDNNs with time-varying delays and input saturation. Linearization of the saturated input is by using the properties of the convex hulls. The main contributions and innovations of this paper are as follows:

a) New definitions of the ellipsoid and linear region of the saturated are given for the FRDNNs input saturation problem.

b) A novel Lyapunov-Krasovskii functional is employed.

c) The saturation controller based on the convex hulls is extended to Riemann-Liouville FRDNNs. Meanwhile, the designed method can be easily extended to the system with Neumann boundary conditions.

d) The domain of attraction is also estimated to ensure that the initial value range does not exceed the saturation input's control capacity.

This paper is organized as follows. Section II gives some basic concepts, symbols, assumptions, and lemmas that are needed in the later proof process. In section III-IV we give the criterion of synchronization and the estimation of the domain of attraction. In Section V, we verify the theorem given in Section III by some numerical example. In Section VI, we summarize this paper and look forward to future research.

Notation: Throughout this paper, R^n denotes the n-dimensional Euclidean vector space, I_n denotes the $n \times n$ identity matrix, $\otimes$ denotes the Kronecker product.

II. PRELIMINARIES

Problem Formulation: In this paper, we set the response system as the following Riemann-Liouville FRDNNs

$$\begin{aligned} {}_{t_0}^{R}\partial_t^{\alpha} u_i &= d_i \Delta u_i - c_i u_i(x,t) + \sum_{j=1}^{n} a_{ij} f_j(u_j(x,t)) \\ &\quad + \sum_{j=1}^{n} g_{ij} f_j(u_j(x, t-\tau(t)) + b_i sat(J_i(x,t)), \\ &\quad t \geq 0, \quad i = 1, 2, \ldots, N, \end{aligned} \tag{1}$$

with the Dirichlet boundary conditions and initial conditions as

$$\begin{aligned} {}_{t_0}^{R}\partial_t^{\alpha-1} u_i(x,s) &= \phi_i(x,s), (x,s) \in \Omega \times [-\tau, 0], \\ u_i(x,t) &= 0, \quad (x,t) \in \partial\Omega \times [-\tau, +\infty], \end{aligned} \tag{2}$$

where $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ is the Laplace diffusion operator on Ω; $\phi_i(x,t)$ is a bounded continuous function; $x \in R^n$ is spatial independent variable; $u_i(x,t) \in R^n$ are the n-dimensional state of the i-th neuron at time t; c_i and d_i are $n \times n$ dimensional constant diagonal matrix where c_i represents the rate with which the ith neuron will reset its potential to the resting state when disconnected from the networks and external inputs in space x, and d_i represents the transmission diffusion coefficient along the ith neuron; a_{ij} and g_{ij} are $n \times n$ dimensional constant matrix where a_{ij} denote the connection strength, and g_{ij} are the coupling strength between the ith and the jth nodes; f_j are the excitation function of the jth node; $\tau(t)$ is the time-varying delay satisfying $0 \leq \tau(t) \leq \tau$ and $0 \leq \dot{\tau}(t) \leq \sigma \leq 1$; $J_i(x,t)$ are the control input and

$$sat(J_i(x,t)) = sign(J_i(x,t))min\{|J_i(x,t)|, \overline{J}_i\} \tag{3}$$

is the saturation function with the input saturation upbound $\overline{J}_i$. ${}_{t_0}^{R}\partial_t^{\alpha} u_i(t)$ denotes the α order Riemann-Liouville derivative which is defined as [45]

$${}_{t_0}^{R}\partial_t^{\alpha} u_i(x,t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dx^n} \displaystyle\int_{t_0}^{t} \dfrac{u_i(x,s)}{(t-s)^{\alpha-n+1}} ds \\ while\ n-1 < \alpha < n, \\ \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{u_i(x,s)}{(t-s)^{\alpha+1}} ds, \\ while\ \alpha < 0, \end{cases} \tag{4}$$

where $\Gamma(\cdot)$ denotes the Gamma function. Then, we set the drive system as

$$\begin{aligned} {}_{t_0}^{R}\partial_t^{\alpha} v_i &= d_i \Delta v_i - c_i v_i(x,t) + \sum_{j=1}^{N} a_{ij} f_j(v_j(x,t)) \\ &\quad + \sum_{j=1}^{N} g_{ij} f_j(v_j(x,t-\tau(t))), \quad t \geq 0,\ i=1,2,\ldots,N, \end{aligned} \tag{5}$$

with

$$\begin{aligned} {}_{t_0}^{R}\partial_s^{\alpha-1} v_i(x,s) &= \varphi_i(x,s), (x,s) \in \Omega \times [-\tau, 0], \\ v_i(x,t) &= 0, \quad (x,t) \in \partial\Omega \times [-\tau, +\infty]. \end{aligned} \tag{6}$$

Let $e_i(x,t) = u_i(x,t) - v_i(x,t)$ as the synchronization error function, then we have the error system as

$$\begin{aligned} &{}_{t_0}^{R}\partial_t^{\alpha} e_i(x,t) \\ &\quad = d_i \Delta e_i(x,t) - c_i e_i(x,t) \\ &\qquad + \sum_{j=1}^{N} a_{ij} \left(f_j(u_j(x,t)) - f_j(v_j(x,t))\right) \\ &\qquad + \sum_{j=1}^{N} g_{ij} \left(f_j(u_j(x,t-\tau(t))) - f_j(v_j(x,t-\tau(t)))\right) \\ &\qquad + b_i sat(J_i(x,t)), \quad t \geq 0,\ i = 1,2,\ldots,N, \end{aligned} \tag{7}$$

with

$$\begin{aligned} {}_{t_0}^{R}\partial_s^{\alpha-1} e_i(x,s) &= e_{0i}(x,s), (x,s) \in \Omega \times [-\tau, 0], \\ e_i(x,t) &= 0, \quad (x,t) \in \partial\Omega \times [-\tau, +\infty]. \end{aligned} \tag{8}$$

Next, some useful definitions are presented.

Definition 1: Define $\varepsilon(P,\rho) = \{\bar{e}^T(x,t) \in R^n : \bar{e}^T(t) P \bar{e}(t) \leq \frac{\rho}{V(\Omega)}, \bar{e}^T(t) = \max\{e(x,t), x \in \Omega\}\}$, where P is a positive definite matrix, $V(\Omega)$ denotes the volume of Ω.

Definition 2: The range of state values in which the control input remains linear with respect to $e_i(x,t)$ is defined as $L(K) = \{\bar{e}_i(t) \in R^n : k_l \bar{e}_i(t) \leq \bar{J}_i, \bar{e}_i(t) = \max\{e_i(x,t), x \in \Omega\}, l = 1,2,\ldots,n, i \in N\}$.

Remark 1: Definition 1 and 2 extends the conception of the ellipsoid and linear region of the saturated in [44]. By introducing the spatial variables, we use the maximum value of the function on the definition domain to represent the function's properties. We can find that this definition is very convenient to deal with the FRDNNs problem in later proof.

Definition 3 ([44]): The convex hulls of e_i is defined as

$$co\{e_i : i \in [1,N]\} := \left\{ \sum_{i=1}^{N} \theta_i e_i : \sum_{i=1}^{N} \theta_i = 1, \theta_i \geq 0 \right\}.$$

Definition 4 ([44]): For initial condition $\phi(t_0)$, the domain of attraction for u is defined as

$$S := \left\{ \phi(t_0) : \lim_{t \to \infty} u(t, \phi(t_0)) = 0 \right\}.$$

The assumptions given below are essential assets to achieve the main results of this paper.

Assumption 1 ([29]): For any $u(x,t), v(x,t) \in R^n$, there exist constants $\delta_i > 0 (j = 1,2,\ldots,N)$, such that:

$$|f_i(u) - f_i(v)| \leq \delta_i |u - v|, \tag{9}$$

and $\delta_{\max} = \max\{\delta_i\}$.

The following important lemmas will be employed during the proof process in the later section.

Lemma 1 ([20]): Let $u(x,t) \in C^n[\Omega \times [t_0, +\infty]]$ be a continuous function with the Riemann-Liouville fractional-order derivative existing, then the following inequality holds:

$$\frac{1}{2} {}_{t_0}^{R}\partial_t^{\alpha} u^T(x,t) P u(x,t) \leq u(x,t) P {}_{t_0}^{R}\partial_t^{\alpha} u^T(x,t), \quad \forall \alpha \in (0,1), \quad t > t_0.$$

where $P \in R^{n \times n}$ is a positive definite matrix.

Then, inspired by [43], we note that the saturation terms' expressions can be treated independently of spatial coordinates. Thus we can give the expressions of $sat(Kx(x,t))$ as the following lemma:

Lemma 2: Let Θ be the set of $n \times n$ diagonal matrices whose diagonal elements are either 1 or 0. Suppose each element of Θ is labeled as Θ_l and denote $\Theta_l^- = I_n - \Theta_l$. Clearly, if $\Theta_l \in \Theta$, then $\Theta_l^- \in \Theta$. Let $K, H \in R^{n \times n}$, then, for any $u(x,t) \in L(H)$, we have $sat(Ku(x,t)) = \sum_{l=1}^{2^n} \theta_l (\Theta_l K + \Theta_l^- H) u(x,t)$, where $0 \leq \theta_l \leq 1 (l = 1,2,\ldots,2^n)$ are some scalars satisfying . $\sum_{l=1}^{2^n} \theta_l = 1$.

Lemma 3 ([46]): For any vector $x, y \in R^n$, positive definite matrix $H \in R^{n\times n}$, the following inequality holds

$$\pm 2xy \le x^T Hx + y^T H^{-1} y.$$

Hence, according to Lemma 2 and Kronecker product properties, the synchronization errors (7) can be rewritten into a compact form as

$$\begin{aligned}
&{}^R_{t_0}\partial_t^\alpha e(x,t) \\
&\quad = D\Delta e(x,t) - Ce(x,t) \\
&\qquad + A\left(f(u(x,t)) - f(v(x,t))\right) \\
&\qquad + G\left(f(u(x,t-\tau(t))) - f(v(x,t-\tau(t)))\right) \\
&\qquad + B\sum_{l=1}^{2^n}\theta_l(I_N\otimes\Theta_l K + I_N\otimes\Theta_l^- H)e(x,t), \qquad (10)
\end{aligned}$$

where $D = diag\{d_i\}$, $C = diag\{c_i\}$, $G = \{g_{ij}\}$, $B = diag\{b_i\}$ with compatible dimension and $f(u(x,t)) = (f_1(u_1(x,t))\ldots f_N(u_N(x,t)))^T$.

III. MAIN RESULTS

In this section, we will derive sufficient conditions for synchronization of the systems with the Dirichlet boundary and control input saturation, that is:

Theorem: Suppose the assumption 1 holds, then system (1) and (5) will achieve synchronization if there exists a positive definite matrix Q and arbitrary matrix K, H such that

$$\widetilde{\Phi} = \begin{pmatrix} \widetilde{\Phi}_{1,1} & 0 & 0 \\ * & \widetilde{\Phi}_{2,2} & 0 \\ * & * & \widetilde{\Phi}_{3,3} \end{pmatrix} \le 0, \qquad (11)$$

and

$$\varepsilon(I,\rho) \subseteq L(H), \qquad (12)$$

where

$$\begin{cases}
\widetilde{\Phi}_{1,1} = \frac{1}{2}AA^T + \frac{1}{2}\delta^T\delta + \frac{1}{2}GG^T + Q - C \\
+B\sum_{l=1}^{2^n}\theta_l(I_N\otimes\Theta_l K + I_N\otimes\Theta_l^- H), \\
\widetilde{\Phi}_{2,2} = -D, \\
\widetilde{\Phi}_{3,3} = -(1-\sigma)Q + \frac{1}{2}\delta^T\delta.
\end{cases} \qquad (13)$$

with known matrix A, B, C, D, G.

Proof: Choose the following Lyapunov functional

$$V(t) = V_1(t) + V_2(t), \qquad (14)$$

where

$$V_1(t) = \frac{1}{2}\int_\Omega {}^R_{t_0}\partial_t^{\alpha-1}(e^T(x,t)e(x,t))dx, \qquad (15)$$

$$V_2(t) = \int_{t-\tau(t)}^t\int_\Omega e^T(x,s)Qe(x,s)dxds. \qquad (16)$$

Thus $V(t) \ge 0$ holds obviously. Then, according to Lemma 1, we get the derivative of $V_1(t)$ along the trajectories of system (10) as follows:

$$\begin{aligned}
&\dot{V}_1(t) \\
&\le \int_\Omega e^T(x,t)\,{}^R_{t_0}\partial_t^\alpha e(x,t)dx \\
&= \int_\Omega e^T(x,t)[D\Delta e(x,t) - Ce(x,t) \\
&\quad + A\left(f(u(x,t)) - f(v(x,t))\right) \\
&\quad + G\left(f(u(x,t-\tau(t))) - f(v(x,t-\tau(t)))\right) \\
&\quad + B\sum_{l=1}^{2^n}\theta_l(I_N\otimes\Theta_l K + I_N\otimes\Theta_l^- H)e(x,t)]dx \\
&= \int_\Omega e^T(x,t)D\Delta e(x,t)dx - \int_\Omega e^T(x,t)Ce(x,t)dx \\
&\quad + \int_\Omega e^T(x,t)A\left(f(u(x,t)) - f(v(x,t))\right)dx \\
&\quad + \int_\Omega e^T(x,t)G\left(f(u(x,t-\tau(t))) - f(v(x,t-\tau(t)))\right)dx \\
&\quad + \int_\Omega e^T(x,t)B\sum_{l=1}^{2^n}\theta_l(I_N\otimes\Theta_l K + I_N\otimes\Theta_l^- H)e(x,t)dx.
\end{aligned} \qquad (17)$$

Utilizing Green's formula and the boundary conditions, we have

$$\begin{aligned}
&\int_\Omega e^T(x,t)D\Delta e(x,t)dx \\
&\quad = e^T(x,t)De_x(x,t)\Big|_0^l - \int_0^l e_x{}^T(x,t)De_x(x,t)dx \\
&\quad = -\int_0^l e_x{}^T(x,t)De_x(x,t)dx. \qquad (18)
\end{aligned}$$

According to assumption (A1) and lemma 3, the third and fourth term satisfy the inequalities

$$\begin{aligned}
&\int_\Omega e^T(x,t)A\left(f(u(x,t)) - f(v(x,t))\right)dx \\
&\quad \le \frac{1}{2}\int_\Omega e^T(x,t)AA^Te(x,t)dx \\
&\qquad + \frac{1}{2}\int_\Omega e^T(x,t)\delta^T\delta e(x,t)dx \qquad (19)
\end{aligned}$$

and

$$\int_\Omega e^T(x,t)G\left(f(u(x,t-\tau(t)))-f(v(x,t-\tau(t)))\right)dx$$
$$\leq \frac{1}{2}\int_\Omega e^T(x,t)GG^Te(x,t)dx$$
$$+\frac{1}{2}\int_\Omega e^T(x,t-\tau(t))\delta^T\delta e(x,t-\tau(t))dx \tag{20}$$

Substituting (18)-(20) into (17), we have

$$\dot{V}_1(t)$$
$$\leq -\int_\Omega {e_x}^T(x,t)De_x(x,t)dx-\int_\Omega e^T(x,t)Ce(x,t)dx$$
$$+\frac{1}{2}\int_\Omega e^T(x,t)AA^Te(x,t)dx$$
$$+\frac{1}{2}\int_\Omega e^T(x,t)\delta^T\delta e(x,t)dx$$
$$+\frac{1}{2}\int_\Omega e^T(x,t)GG^Te(x,t)dx$$
$$+\frac{1}{2}\int_\Omega e^T(x,t-\tau(t))\delta^T\delta e(x,t-\tau(t))dx$$
$$+\int_\Omega e^T(x,t)B\sum_{l=1}^{2^n}\theta_l(I_N\otimes\Theta_lK+I_N\otimes\Theta_l^-H)e(x,t)dx. \tag{21}$$

Similarly, the derivative of $V_2(t)$ satisfies the following inequality

$$\dot{V}_2(t)$$
$$\leq \int_\Omega e^T(x,t)Qe(x,t)dx$$
$$-(1-\sigma)\times\int_\Omega e^T(x,t-\tau(t))Qe(x,t-\tau(t))dx. \tag{22}$$

Substituting (21) and (22) into (14), we have

$$\dot{V}(t)$$
$$=\dot{V}_1(t)+\dot{V}_2(t)$$
$$\leq -\int_\Omega {e_x}^T(x,t)De_x(x,t)-\int_\Omega e^T(x,t)Ce(x,t)dx$$
$$+\frac{1}{2}\int_\Omega e^T(x,t)AA^Te(x,t)dx$$
$$+\frac{1}{2}\int_\Omega e^T(x,t)\delta^T\delta e(x,t)dx$$
$$+\frac{1}{2}\int_\Omega e^T(x,t)GG^Te(x,t)dx$$
$$+\frac{1}{2}\int_\Omega e^T(x,t-\tau(t))\delta^T\delta e(x,t-\tau(t))dx$$
$$+\int_\Omega e^T(x,t)B\sum_{l=1}^{2^n}\theta_l(I_N\otimes\Theta_lK+I_N\otimes\Theta_l^-H)e(x,t)dx$$
$$+\int_\Omega e^T(x,t)Qe(x,t)dx$$
$$-(1-\sigma)\times\int_\Omega e^T(x,t-\tau(t))Qe(x,t-\tau(t))dx. \tag{23}$$

Thus, according to the condition (11), we have

$$\dot{V}(t)=\int_\Omega \widetilde{e}^T(x,t)\widetilde{\Phi}\widetilde{e}(x,t)dx\leq 0, \tag{24}$$

where $\widetilde{e}(x,t)=[e^T(x,t),{e_x}^T(x,t),e^T(x,t-\tau(t))]^T$ and

$$\begin{cases}\widetilde{\Phi}_{1,1}=\frac{1}{2}AA^T+\frac{1}{2}\delta^T\delta+\frac{1}{2}GG^T+Q-C\\ +B\sum_{l=1}^{2^n}\theta_l(I_N\otimes\Theta_lK+I_N\otimes\Theta_l^-H),\\ \widetilde{\Phi}_{2,2}=-D,\\ \widetilde{\Phi}_{3,3}=-(1-\sigma)Q+\frac{1}{2}\delta^T\delta.\end{cases} \tag{25}$$

Since (12) is equivalent to

$$\min\{\bar{e}^T(t)\bar{e}(t):h_i\bar{e}(t)=\bar{J}_i\}\geq\rho, \tag{26}$$

we can transform it as

$$\min\{\bar{e}^T(t)\bar{e}(t):h_i\bar{e}(t)=\bar{J}_i\}$$
$$=\int_\Omega \bar{J}_i^2(h_i{h_i}^T)^{-1}dx=M\bar{J}_i^2(h_i{h_i}^T)^{-1}, \tag{27}$$

through the Lagrange multiplier method, where $M=V(\Omega)$ denotes the volume of Ω. Thus we have

$$h_i{h_i}^T\leq\frac{M{\bar{J}_i}^2}{\rho}. \tag{28}$$

According to the Schur complement, (28) can be expressed as the following LMIs form

$$\begin{pmatrix} I & h_i\\ {h_i}^T & \frac{M{\bar{J}_i}^2}{\rho}\end{pmatrix}\geq 0. \tag{29}$$

Thus, system (1) and (5) can achieve synchronization under the saturation input control.

Meanwhile, according to (14) and we have

$$V(0)=\frac{1}{2}\int_\Omega {}^R_{t_0}\partial_0^{\alpha-1}(e^T(x,0)e(x,0))dx$$
$$+\int_{-\tau(0)}^{0}\int_\Omega e^T(x,s)Qe(x,s)dxds=\vartheta, \tag{30}$$

where ϑ is a constant. Accordingly, since $\dot{V}(t)\leq 0$, it concludes that $\int_\Omega {}^R_{t_0}\partial_t^{\alpha-1}(e^T(x,t)e(x,t))\leq V(t)\leq V(0)=\vartheta$.

In other words, for any initial value $e(x,0) \in \varepsilon(I,\rho)$, $e(x,t)$ will not leave $\varepsilon(I,\rho)$ indicating that for all $t > 0$, $e(x,t) \in \varepsilon(I,\rho) \subseteq L(H)$ holds.

The proof is completed.

Remark 1: In [47] when dealing with the Lyapunov functional V, the fractional derivation is directly carried out and get the Mittag-Leffler stability. For the time-delay problem, fractional derivation on the functional V cannot work, so to use Lyapunov-Krasovskii functional and derivate the functional V with respect to t is a more convenient way.

Remark 2: The Lyapunov-Krasovskii functional presented in our paper is a traditional and mature approach for related works of ODEs. Still, research seldom handles the Riemann-Liouville derivative with reaction-diffusion and saturation comprehensively, so we have made an original exploration of this issue.

Corollary: Assume that $\tau(t) \equiv 0$, then system (1) and (5) can reach synchronization with the feedback control

$$J_i(x,t) = k_i e_i(x,t), \tag{31}$$

if the following conditions

$$\widetilde{\Phi} = \begin{pmatrix} \widetilde{\Phi}_{1,1} & 0 \\ * & \widetilde{\Phi}_{2,2} \end{pmatrix} \leq 0, \tag{32}$$

and

$$\varepsilon(I,\rho) \subseteq L(H), \tag{33}$$

hold, where

$$\begin{cases} \widetilde{\Phi}_{1,1} = \frac{1}{2}AA^T + \delta^T\delta + \frac{1}{2}GG^T + Q - C - Q \\ +B\sum_{l=1}^{2^n}\theta_l(I_N \otimes \Theta_l K + I_N \otimes \Theta_l^- H), \\ \widetilde{\Phi}_{2,2} = -D. \end{cases} \tag{34}$$

with known matrix A, B, C, D, G, positive definite matrix Q and arbitrary matrix K, H.

IV. ESTIMATE THE DOMAIN OF ATTRACTION

Due to the nonlinear influence of saturation, the stability region is often local. In this section, we will give sufficient conditions for the initial conditions which can ensure the two system reach synchronization during a finite time.

It is difficult to deal with spatial variables, so we simplify the problem. Consider the set of the maximum values of the initial value of the system in its domain

$$\chi = \{\|e_1(x,t)\|_{\max}, \|e_2(x,t)\|_{\max}, \ldots, \|e_N(x,t)\|_{\max}\} \tag{35}$$

which conform to some certain shape reference set χ_R, then we hope that the shape reference set can fill the attraction region of the system as fully as possible. That is to solve the problem

$$\begin{aligned} &\sup_{\substack{Q>0,D,A,G,\\C,B,K,H}} \gamma \\ &s.t. \\ &\quad a)\ \gamma\chi \subset \varepsilon(I,\rho), \\ &\quad b)\ \widetilde{\Phi} \leq 0, \\ &\quad c)\ \varepsilon(I,\rho) \subseteq L(H). \end{aligned} \tag{36}$$

If χ_R is a polygon, i.e.

$$\chi_R = co\{e_1, e_2, \ldots, e_N\} \tag{37}$$

thus the constraint $a)$ is equivalent to $\gamma^2 M e_{i\max}^T e_{i\max} \leq \rho$, $i = 1, \ldots, N$. According to the Schur component, we have

$$\begin{aligned} &\frac{M}{\rho} e_{i\max}^T e_{i\max} \leq \frac{1}{\gamma^2} \Leftrightarrow \begin{pmatrix} \frac{1}{\gamma^2} & e_{i\max}^T \\ e_{i\max} & \frac{\rho}{M} \end{pmatrix} \geq 0, \\ &i = 1, \ldots, N. \end{aligned} \tag{38}$$

Hence, we get (38) as the sufficient condition for the domain of initial conditions that can ensure the two systems can achieve synchronization under the above theorem conditions.

Also, we can solve the following optimization problem to get the maximal volume of $\varepsilon(I,\rho)$,

$$\begin{aligned} &\min_{\substack{Q>0,D,A,G,\\C,B,K,H}} \zeta \\ &s.t. \\ &\quad a)\ \widetilde{\Phi} \leq 0, \\ &\quad b)\ \varepsilon(I,\zeta^{-1}) \subseteq L(H). \end{aligned} \tag{39}$$

where $\zeta = \frac{1}{\rho}$.

Remark 3: It should note that (38) is "sufficient" enough, which means that the estimation of the domain of attraction is often smaller than its theoretical one. In other words, the initial conditions obtained by the above methods are usually safe enough, but we still hope to find more conservative laws in our future work.

V. NUMERICAL EXAMPLES

In this section, we will give two examples. In Example 1, We take the parameters satisfying all the Theorem conditions to test the feedback control capability. Then, we will test the tolerance upbound of the initial errors in Example 2.

Example 1: Consider two four-nodes FDRNNs defined on $\Omega \times [-\tau, +\infty) = [-1,1] \times [-\tau, +\infty)$ with the following parameters:

$$\begin{aligned} \alpha &= 0.75, n = 1, N = 4, \\ \overline{J}_i &= 50, B = I_4, \\ D &= diag\{0.3, 0.2, 0.35, 0.4\}, \\ C &= diag\{-6, -3, -4, -3.6\}, \\ A = G &= \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0.4 & 1 \\ 1 & 0.4 & 0.7 & 0.2 \\ 1 & 1 & 0.2 & 2 \end{pmatrix}, \\ f(u(x,t)) &= \tanh(u(x,t)), \\ \tau(t) &= \frac{0.1e^t}{1+e^t}, \end{aligned}$$

and the initial value are given as $u_{0i}(x) = 0.41\sin(2\pi ix)$ and $v_{0i}(x) = 0$, $i = 1, 2, 3, 4$. Then according to the definition (4), it can be transformed as $u_i(x, \bar{t}) = 0.66\sin(2\pi ix)$ and $v_i(x, \bar{t}) = 0$ for $\bar{t} \in [-\tau, 0]$ approximately. Thus, we can get the maximal $\rho \approx 1.4625$ by solving (39).

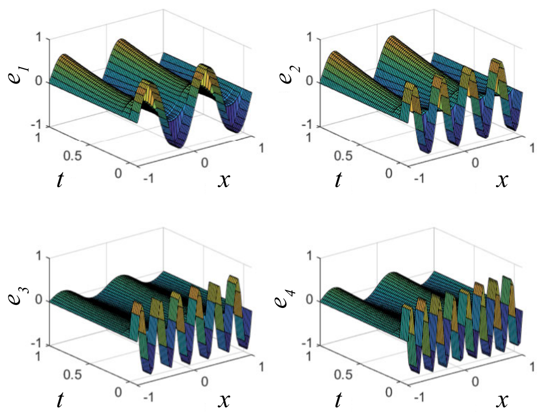

FIGURE 1. System errors without control of example 1.

FIGURE 1 shows the error system keeps oscillating in a large range without input control, which implies that the two neural networks cannot reach synchronization due to the influence of reaction terms and time-varying delays.

We use the MATLAB LMI control toolbox to solve the LMIs in the Theorem and get the feasible solution through the above parameters. Based on these solutions, we can choose the feedback gain matrix

$$K = diag\{-25.3473, -24.0324, -23.1751, -23.4426\}.$$

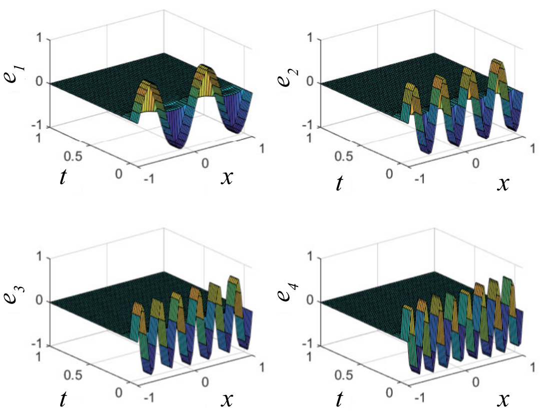

FIGURE 2. System errors with control of example 1.

With the above control gain, FIGURE 2 illustrates that the errors between the two neural networks achieve the neighborhood of 0 on the entire domain. Looking at it from another angle, as FIGURE 3 depicts, the errors between two systems decay very quickly under the proposed control input.

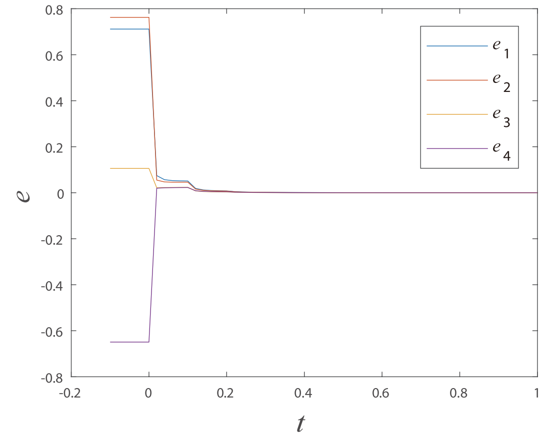

FIGURE 3. System errors with control at $x = 0.18$ of example 1.

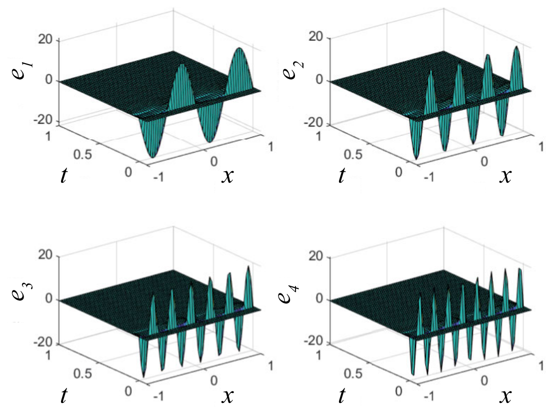

FIGURE 4. Control input of example 1.

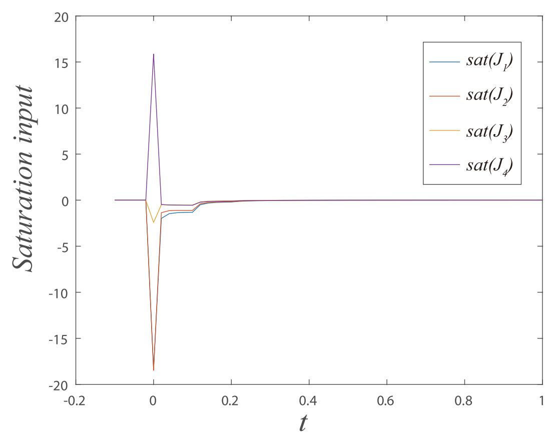

FIGURE 5. Control input at $x = 0.18$ of example 1.

Then FIGURE 4 and 5 shows the input control signal of each node. In this situation, they didn't trigger saturation. Next, Example 2 will test the robustness of the designed control law.

Example 2: Consider the parameters in Example 1, and we will replace them with some "sick" initial conditions to test the maximal tolerance of the initial errors. From (38),

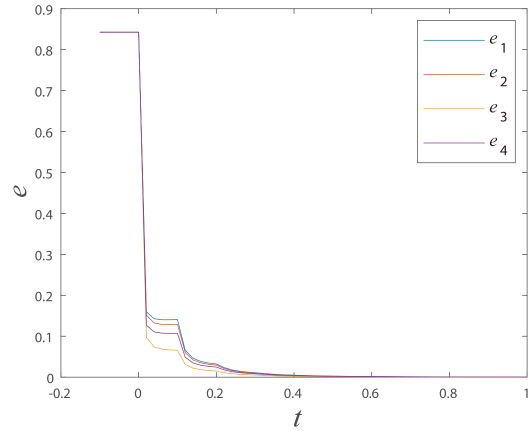

FIGURE 6. System errors of example 2 with $e_{imax} = 0.8426$ at $x = 0.18$.

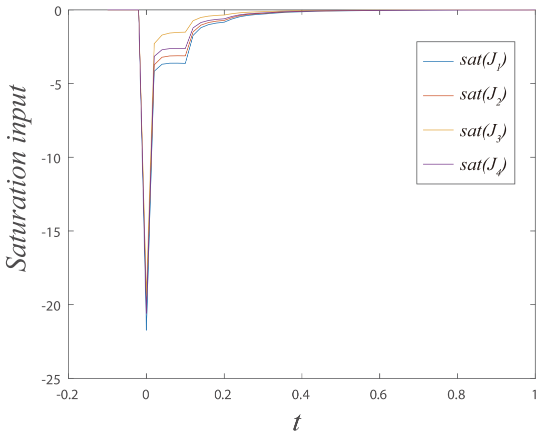

FIGURE 7. Control input of example 2 with $e_{imax} = 0.8426$ at $x = 0.18$.

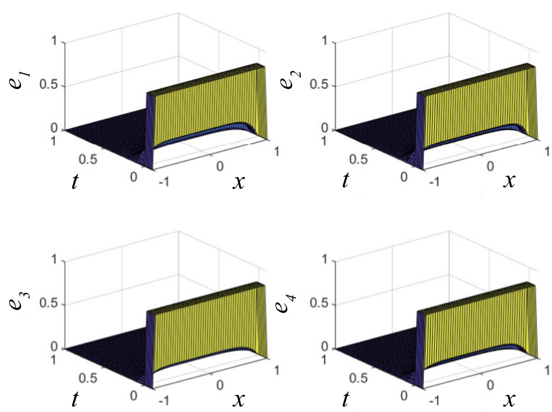

FIGURE 8. System errors of example 2 with $e_{imax} = 0.8426$.

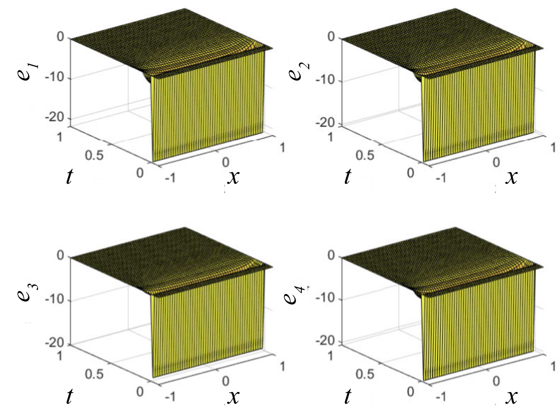

FIGURE 9. Saturation control input of example 2 with $e_{imax} = 0.8426$.

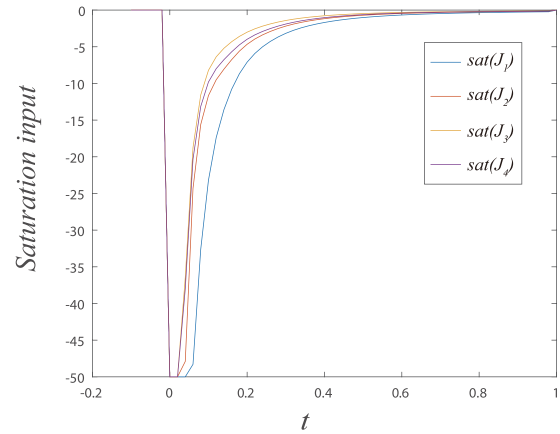

FIGURE 10. Saturation control input of example 2 with $e_{imax} = 20$ at $x = 0.18$.

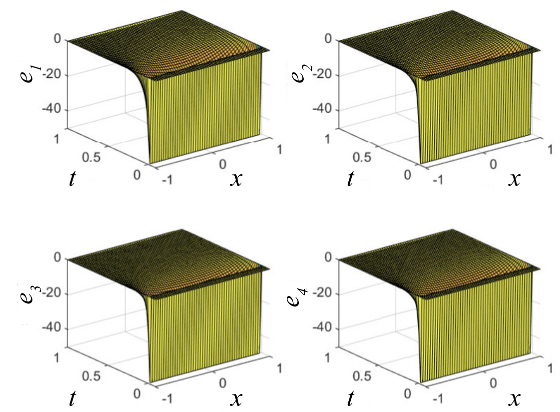

FIGURE 11. Saturation control input of example 2 with $e_{imax} = 20$.

naturally, except for the two boundaries, we can take the initial errors as the maximum value on the interval value on the whole domain, that is $u_{0i}(x) = e_{imax}$, $v_{0i}(x) = 0$. Let $\gamma = 1$, thus we have $e_{imax} \approx 0.8426$, the numerical experiment indicate that two system can reach synchronization as FIGURE 6 illustrate.

Increasing e_{imax} to 20, we found that although the control input has reached the saturation state, the error system can still approach the neighborhood of zero in finite time according to FIGURE 10-12.

Continue increase e_{imax} to 25, we found that the errors of node 1 increase rapidly as FIGURE 13 and 14 illustrate which

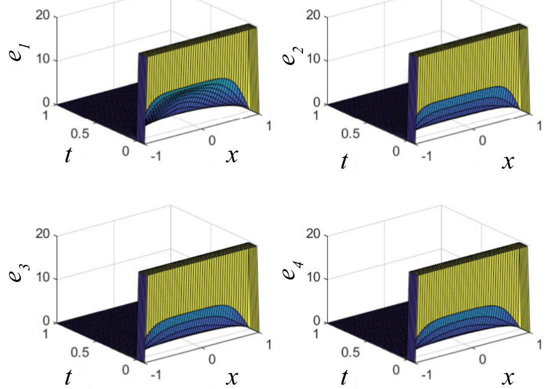

FIGURE 12. System errors of example 2 with $e_{imax} = 20$.

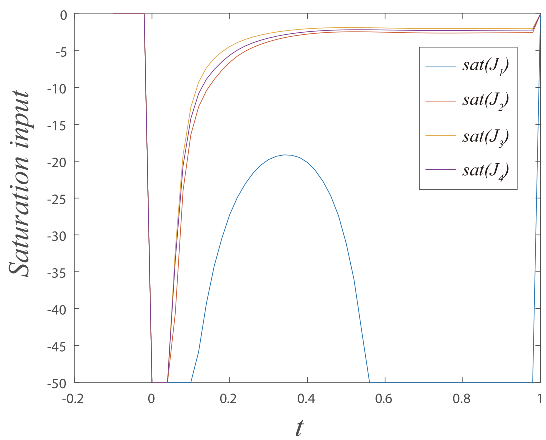

FIGURE 13. Saturation control input of example 2 with $e_{imax} = 25$ at $x = 0.18$.

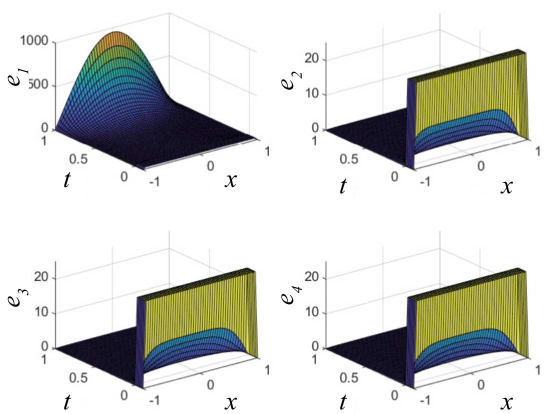

FIGURE 14. System errors of example 2 with $e_{imax} = 25$.

indicate that under the saturation bound $\overline{J}_i = 50$ the systems cannot synchronize.

VI. CONCLUSION

In this work, firstly, the definitions of the ellipsoid and the linear region of the saturated are extended to PDEs case. Under this framework, we construct a suitable Lyapunov-Krasovskii functional for synchronizing two fractional reaction-diffusion neural networks and obtain sufficient conditions under saturated control inputs by using convex hulls and some Riemann-Liouville fractional integral properties. Besides, we estimate the domain of attraction. All the conditions are presented in the form of LMIs thus can easily be solved by the MATLAB toolbox. At last, two numerical experiments show that the proposed control laws are reliable when trigger saturation state. Meanwhile, the designed control law is safe enough with our estimation of the domain of attraction. As we can see, our method is simple and sufficient, but the estimation of the domain of attraction is too small. In our future work, we can find some more suitable inequalities to achieve more conservative conditions and apply our approach on network consensus, fault-tolerant, adaptive fuzzy control, etc.

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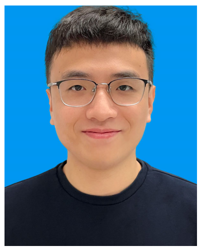

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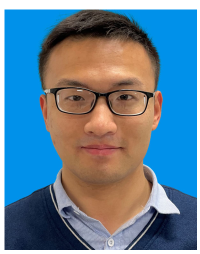

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