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A Unified Framework Design for Finite-Time Bipartite Consensus of Multi-Agent Systems

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ABSTRACT In this paper, the finite-time bipartite consensus (FTBC) problem is investigated for the multi-agent system (MAS) with detail balanced structure. To realize FTBC of MAS, a unified protocol framework is developed. Some criteria are established for realizing FTBC. It is worth noticing that estimations of settling time can be given in form of mathematical expression. The unified framework can bring in various protocols by choosing different parameters, which extends previous results. Finally, two numerical examples are provided to illustrate the effectiveness and superiority of corresponding theoretical results.

INDEX TERMS Finite-time bipartite consensus, Filippov solutions, detail balance digraph, structurally balanced signed graph, weighted signed average consensus, settling time.

I. INTRODUCTION

Coordination control of MAS has been a focus in many disciplines in past decades. The reason is that MAS has been applied into many practical applications. For example, the consensus of MAS has been applied in sensor networks [1]–[4], robot teams [5]–[8], distributed computation [9], [10], and so on. For coordination control of MAS, consensus problem is one of the most important topics. It aims to guarantee agent states converge to the same ideal values by some suitable protocols. Most existing works [1]–[4], [11], [12] on consensus of MAS are based on the fundamental assumption that the interactions among the agents are cooperative. However, there are cooperatives and antagonism simultaneously in practical applications, such as two-party political systems, trust networks, and so on. To deal with this kind of network consensus problems, lots of works [13]–[28] focused on consensus problem of MAS including antagonistic interactions, which are said to be bipartite consensus. Generally, such phenomena can be characterized by a signed graph whose edge weights can be both positive and negative, which denotes that cooperative and antagonism, respectively. Bipartite consensus (BC) implies that the final states are the same in modulus but not sign (direction).

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Since the concept of BC of MAS was firstly proposed in [13], asymptotic BC of MAS under undirected and digraph has been studied extensively in the existing literature.

However, asymptotic consensus protocol can't guarantee consensus is achieved in a finite time, which sometimes becomes a deficiency for practical systems, the reason is that sometimes consensus needs to be achieved in a finite time for some practical applications. To overcome the drawback, the finite-time bipartite consensus was proposed, and this kind of consensus has been a focus in control discipline recently. The reason is that FTBC owns many ideal performances such as higher convergence speed, better robustness, and disturbance rejection. Lots of results on FTBC have been reported in [29]–[35] and references therein. For example, [29] and [30] investigated FTBC problem of MAS under undirected topology. [31] investigated FTBC problem of MAS under directed topology structure. In virtue of homogeneity, [34] investigated FTBC problem of MAS with detail balanced structure. As a special finite-time bipartite consensus, fixed-time BC of MAS was discussed in [36]–[38]. Deng *et al.* proposed a continuous fixed-time bipartite consensus (FDTBC) protocol in [37]. An FDTBC protocol framework was proposed in [38] for undirected topology structure. Under this framework, both discontinuous FTBC protocols and discontinuous FDTBC protocol can be constructed by choosing suitable parameters. Moreover, settling

time estimations are presented in form of a mathematical expression. Similarly, synchronization of discontinuous complex networks was also discussed in [39] under a unified protocol framework.

Motivated by the above observations, a natural question will be asked. Under detail balanced structure, can discontinuous FTBC protocols and continuous FTBC protocols be constructed under a unified protocol framework? Besides, can corresponding settling time estimation be given in form of mathematical expression as well? This is an interesting and open problem, at present. To solve this problem, a unified FTBC protocol framework is designed for MAS with detail balanced structure in this article. The contributions of this article can be summed up as follows. Firstly, a unified framework is developed to construct FTBC protocol. Noticing that both continuous protocols and discontinuous protocols can be obtained under this framework. Secondly, by finite-time stability theorem, Lyapunov function, Filippov solution, and graph theory, a rigorous proof is carried out to obtain an estimation of settling time in form of mathematical expression. It is shown that the settling time can be estimated by protocol parameters, topology information, and initial values. Finally, some sufficient conditions are proposed to solve FTBC problem of MAS under the detail balanced structure.

The rest of the article is arranged as follows. Preliminaries are given in Section II. In Section III our main results are presented. Simulation examples are provided in Section IV. Section V concludes this article with some conclusions.

Notations: R^N , $R^{N \times N}$, R and R^+ denote N -dimensional Euclidean space, $N \times N$ dimensional real matrix space, real number set and positive real number set, respectively. Index set $\Pi_N = \{1, \dots, N\}$. A^T denotes transpose of matrix or vector A . Notation $A \geq 0$ ($A > 0$) means that A is a semi-positive definite (positive definite) matrix. Set $S = \{\text{diag}\{\sigma_1, \dots, \sigma_N\} | \sigma_i \in \{-1, 1\}\}$. $\text{sign}(\cdot)$ is signum function. $|\cdot|$ denotes absolute of a real number. $(\cdot)^{|\alpha|} = \text{sign}(\cdot) \cdot |\cdot|^\alpha$ for $\alpha \geq 0$. For a positive vector $\xi = [\xi_1, \dots, \xi_N]^T \in R^N$, $\xi_{\min} = \min_{i \in \Pi_N} \{\xi_i\}$, $\xi_{\max} = \max_{i \in \Pi_N} \{\xi_i\}$.

II. PRELIMINARIES

A. SET-VALUED DERIVATIVE AND FINITE-TIME STABILITY OF NONLINEAR SYSTEM

For every point x in the set $\Sigma \subset R^n$, if there is a nonempty set $\chi(x) \subset R^n$, then the map $x \mapsto \chi(x)$ is said to be a set-valued map from Σ to $\mathcal{B}(R^n)$, where $\mathcal{B}(R^n)$ denotes the set consisting of all the subsets of R^n .

Consider a dynamical system described by the following differential equation:

$$\dot{x}(t) = h(x(t)), \quad x(0) = x_0, \quad (1)$$

where $x(t) \in R^n$ stands for state of system (1), $h(\cdot) : R^n \rightarrow R^n$ is a nonlinear function or vector field. When $h(\cdot)$ is continuous, from the well-known Peano's theorem, the existence of a continuous differentiable solution can be guaranteed for (1). When it is discontinuous but locally measurable, the solution of (1) is understood in sense of Filippov in this article.

Definition 1: [42] A vector function $x(t)$ defined on the interval $[0, T^*)$ is called Filippov solution of system (1) if it is absolutely continuous on any compact subinterval of $[0, T^*)$ ($T^* > 0$), and it satisfies the differential inclusion $\dot{x}(t) \in \mathcal{K}[h](x(t))$ for almost all $t \in [0, t^*)$, the set-valued map $\mathcal{K}[h] : R^n \rightarrow R^n$ is defined as following

$$\mathcal{K}[h](x(t)) = \bigcap_{\delta > 0} \bigcup_{\mu(S)=0} \overline{\text{co}} \{h(B(x, \delta)) \setminus S\},$$

where $\overline{\text{co}}(\cdot)$ is convex closure, μ stands for Lebesgue measure, $B(x, \delta)$ denotes the open ball centered at x with radius $\delta > 0$.

In order to deal with the finite-time stability smoothly, here one assumes that the Filippov solutions of system (1) exist on interval $[0, \infty)$ and $h(0) = 0$.

Definition 2: [43]–[49] Assume $V : R^n \rightarrow R$ is a local Lipschitz function. The Clarke upper generalized derivative of V at x in the direction of $v \in R^n$ is defined by

$$V^0(x, v) = \limsup_{y \rightarrow x} \sup_{l \rightarrow 0^+} \frac{V(y + lv) - V(x)}{xl},$$

where ∂V denotes generalized gradient of V , which is defined as follows.

$$\partial V(x) = \overline{\text{co}} \left\{ \lim_{i \rightarrow +\infty} \nabla V(x_i) : x_i \rightarrow x, x_i \notin S \cup \Omega_V \right\},$$

where $\overline{\text{co}}$ denotes the convex hull, Ω_V denotes points set where V is not differentiable, $S \subset R^n$ denotes any set of measure zero.

The set-value Lie derivative of V with respect to $f(\cdot)$ at z is denoted as

$$L_f V(z) = \{a \in R | \exists v \in \mathbb{F}(f)(z) \text{ with } v^T \xi = a, \forall \xi \in \partial V(y)\},$$

moreover,

$$V'(x, v) = \lim_{l \rightarrow 0^+} \frac{V(y + lv) - V(y)}{l}$$

exists and $V'(x, v) = V^0(x, v)$.

Definition 3: [49] Suppose that regular function $V : R^n \rightarrow R_+$ satisfies local Lipschitz condition, and function $x(t) : [0, +\infty) \rightarrow R^n$ is absolutely continuous on any compact set of interval $[0, +\infty)$, then $x(t)$ and $V(x(t))$ are differentiable in almost everywhere in the interval $[0, +\infty)$, and one has

$$\frac{dV(x(t))}{dt} = \langle \zeta, \dot{x}(t) \rangle, \quad \forall \zeta \in \partial V(x(t)).$$

Definition 4: [50] The origin of system (1) is said to be globally finite-time stable if the following statements hold for any solution $x(t, x_0)$ of equation (1):

(a) Lyapunov stability: For any $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that $\|x(t, x_0)\| < \varepsilon$ for any $\|x_0\| < \delta$ and $t \geq 0$.

(b) Finite-time convergence: There exists a function $T : R^n \rightarrow (0, +\infty)$, called settling time function, such that

$$\lim_{t \rightarrow T(x_0)} x(t, x_0) = 0, \quad x(0, x_0) = x_0, \text{ for } t \geq T(x_0).$$

Lemma 1: [49] For real numbers $y_i \geq 0, i \in \Pi_N, 0 < \theta < 1$, one has the following inequality

$$\sum_{i=1}^N y_i^\theta \geq \left(\sum_{i=1}^N y_i \right)^\theta.$$

Proof: Applying the fundamental inequality $y^p \geq y$ for any $y \in (0, 1]$ and $p \in (0, 1]$, one has

$$\frac{\sum_{i=1}^N \xi_i^\theta}{\left(\sum_{i=1}^N \xi_i \right)^\theta} = \sum_{i=1}^N \left(\frac{\xi_i}{\sum_{i=1}^N \xi_i} \right)^\theta \geq \sum_{i=1}^N \frac{\xi_i}{\sum_{i=1}^N \xi_i} = 1,$$

which means that the conclusion in Lemma 2.5 holds. This is complete proof.

Lemma 2: [50] If function $U(t) : [0, +\infty) \rightarrow [0, +\infty)$ is differential in interval $[0, +\infty)$, and it satisfies inequality:

$$\frac{dU(t)}{dt} \leq K(U(t))^\beta,$$

where parameters $K > 0, 1 > \beta > 0$, then $U(t)$ converges to zero in a finite time $T > 0$ and $U(t) = 0$, for $t \geq T$ and associated settling time T satisfies

$$T \leq \frac{(U(0))^{1-\beta}}{K(1-\beta)}.$$

Lemma 3: [40], [41], [49], [50] Assume function $x(t) : [0, +\infty) \rightarrow R^n$ is absolutely continuous on any compact subset of interval $[0, +\infty)$, and function $U(t) : [0, +\infty) \rightarrow [0, +\infty)$ is C-regular. If there exist a function $\gamma(\tau) : [0, +\infty) \rightarrow [0, +\infty)$ with $\gamma(\tau) > 0$ for $\forall \tau \geq 0$ such that $\dot{U}(t) < -\gamma(U(t))$ and $t^* = \int_0^{U(0)} \frac{d\tau}{\gamma(\tau)} < +\infty$. Then one has $U(t) = 0$ for any $t \geq t^*$.

Remark 1: Lemma 2 is usually said to be the well-known finite-time Lyapunov stability theorem, where the Lyapunov function has to be differentiable. While the Lyapunov function U in Lemma 3 just satisfies C-regular conditions, which is weaker than the differentiability. And it is to be applied to deal with the finite-time stability problem of discontinuous system in this article.

B. SIGNED GRAPHS

A directed signed graph consists of vertex set $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$, edge set $\mathcal{E} = \{e_{ij} = (v_i, v_j)\} \subset \mathcal{V} \times \mathcal{V}$ and adjacency matrix A , and it is denoted by $\mathcal{G}(A)$. Here matrix $A = [a_{ij}] \in R^{N \times N}$ is defined as $a_{ij} \neq 0$ if and only if $(v_j, v_i) \in \mathcal{E}$, where $(v_j, v_i) \in \mathcal{E}$ implies that there is an edge between agent v_j and agent v_i , and agent v_j transmits information to agent $v_i, a_{ij} = 0$, otherwise. Here we assume graph doesn't contain self-loop, i.e., $a_{ii} \equiv 0$, and graph $\mathcal{G}(A)$ is digon sign-symmetric, i.e., $a_{ij}a_{ji} \geq 0$ for $\forall i \neq j, i, j \in \Pi_N$. The Laplacian matrix of graph $\mathcal{G}(A)$ is defined as $L(A) = D - A$, where D is a diagonal matrix, which is defined as $d_{ii} = \sum_{j=1}^N |a_{ij}|, d_{ij} = 0$ for $i \neq j$. A path (v_{i_1}, v_{i_m}) is a sequence of distinctive edges in \mathcal{E} , and these edges satisfy $(v_{i_k}, v_{i_{k+1}}) \in \mathcal{E}$, for $k \in \Pi_{m-1}$, where $v_{i_k}, k \in \Pi_m$, are distinctive vertices. If there is a path between any pair vertices, graph $\mathcal{G}(A)$ is said to be strongly

connected. $\mathcal{G}(A)$ contains a spanning tree if there is a vertex (called root vertex) that can reach all the other vertices by some directed path.

To state our main results, some definitions and lemmas, which are necessary and helpful to derivation of our main results, are listed as follows.

Definition 5: [34]: Directed graph $\mathcal{G}(A)$ is said to be detail balanced if there is a positive vector $\xi = [\xi_1, \dots, \xi_N]^T \in R^N$, where $\xi_i > 0$, such that $\xi_i a_{ij} = \xi_j a_{ji}$ for any $i, j \in \Pi_N$.

Definition 6: [39] Directed signed graph $\mathcal{G}(A)$ is called structurally balanced if there are two vertex sets V_1 and V_2 satisfying $V_1 \cup V_2 = V$, and $V_1 \cap V_2 = \emptyset$; Besides, one has $a_{ji} \geq 0$ if $i, j \in V_r, r \in \Pi_2$ and $a_{ji} \leq 0$, when index $i \in V_r, j \in V_{3-r}, r \in \Pi_2$.

Lemma 4: [39]: Directed signed graph $\mathcal{G}(A)$ is structurally balanced if and only if there exists a diagonal matrix $S = \text{diag}\{\sigma_1, \dots, \sigma_N\}$, where $\sigma_i \in \{-1, 1\}, i \in \Pi_N$, such that $SAS \geq 0$.

III. MAIN RESULTS

In this section, FTBC problem of MAS under detail balanced structure is to be discussed.

Consider the MAS consists of N agents, and the dynamics of i -th agent is described as follows.

$$\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_0 \quad i \in \Pi_N, \quad (2)$$

where $x_i(t) \in R$ stands for state of agent i , and $u_i(t) \in R^n$ stands for input, which is said to be protocol to be designed. To simplify expression, denote $x(t) = [x_1(t), \dots, x_N(t)]^T$.

Definition 7: [39] MAS (2) achieves FTBC if there is a settling time $T(x_0)$ such that

$$\lim_{t \rightarrow T(x_0)} |x_i(t)| = c, \quad |x_i(t)| = c, \quad t \geq T(x_0) \quad \forall i \in \Pi_N,$$

where c is a constant.

To solve FTBC problem of system (2), a unified FTBC protocol framework is designed as follows:

$$u_i = \sum_{k=1}^N a_{ik} (\tau_{ki})^{[p]}, \quad i \in \Pi_N, \quad (3)$$

where $\tau_{ki} = x_k - \text{sign}(a_{ik})x_i, 0 \leq p < 1$.

Remark 2: When power parameter $0 < p < 1$, protocol (3) degenerates into the one in [28], and which solved signed average consensus problem. Therefore, the protocol in [28] can be regarded as a special case of protocol (3), which solved FTBC problem of MAS under undirected and connected signed graph. In [32], protocol (3) with $p > 0$ was applied to solve FTBC problem of MAS (2) via homogeneity. However, the deficiency of [32] is that settling time cannot be estimated by mathematical expression. Besides, (3) is discontinuous protocol when $p = 0$, this case was not investigated in [32]. In addition, [39] only solved finite-time SAC problem of MAS (2) via a discontinuous protocol.

Remark 3: When power parameter $p = 0$ protocol (3) is discontinuous. When $0 < p < 1$ protocol (3) is continuous. Therefore, the two cases need to be discussed, respectively.

Based on above analysis, It can be seen that (3) is an unified framework, which unifies continuous and discontinuous protocols into a unified framework. In addition, we will prove that corresponding settling times can be estimated through mathematical expression in this article.

Remark 4: To improve the convergence rate, gain coefficient $\zeta > 0$ can be introduced into (3), then (3) can be rewritten as

$$u_i = \zeta \sum_{k=1}^N a_{ik} (\tau_{ki})^{[p]}. \quad (4)$$

To simplify expression, here we choose $\zeta = 1$.

Before moving on, a necessary lemma is introduced as follows.

Lemma 5: Assume that the signed graph $\mathcal{G}(A)$ is detail balanced with positive vector $\xi = [\xi_1, \dots, \xi_N]^T$ and structurally balanced. Then, under protocol (3), the weighted signed-average $\phi^*(t)$ satisfies

$$\dot{\phi}^*(t) = \omega \sum_{i=1}^N \xi_i \sigma_i x_i(t) = \phi^*(0),$$

where $\omega = \frac{1}{\sum_{i=1}^N \xi_i}$, and $\sigma_i, i \in \Pi_N$ are defined in Lemma 2.11.

Proof: Due to $\mathcal{G}(A)$ is structurally balanced, according to Lemma 4, one can find a matrix S such that $SAS \geq 0$ and $\sigma_i \sigma_j = \text{sign}(a_{ij})$, where $\sigma_i \in \{-1, 1\}$. Therefore, one can obtain $\sigma_i = \text{sign}(\sigma_i)$, and the derivative of $\phi^*(t)$ can be calculated as follows:

$$\begin{aligned} \dot{\phi}^*(t) &= \omega \sum_{i=1}^N \xi_i \sigma_i u_i(t) \\ &= \omega \sum_{i=1}^N \sum_{k=1}^N \xi_i |a_{ik}| (\sigma_k x_k - \sigma_i x_i)^{[p]} \\ &= \frac{\omega}{2} \sum_{i=1}^N \sum_{k=1}^N \xi_i |a_{ik}| (\sigma_k x_k - \sigma_i x_i)^{[p]} \\ &\quad + \frac{\omega}{2} \sum_{i=1}^N \sum_{k=1}^N \xi_i |a_{ki}| (\sigma_k x_k - \sigma_i x_i)^{[p]} \\ &= \frac{\omega}{2} \sum_{k=1}^N \sum_{i=1}^N \xi_i |a_{ik}| (\sigma_k x_k - \sigma_i x_i)^{[p]} \\ &\quad + \frac{\omega}{2} \sum_{i=1}^N \sum_{k=1}^N \xi_i |a_{ki}| (\sigma_k x_k - \sigma_i x_i)^{[p]} \\ &= \frac{\omega}{2} \sum_{i=1}^N \sum_{k=1}^N \xi_i |a_{ik}| (\sigma_k x_k - \sigma_i x_i)^{[p]} \\ &\quad - \frac{\omega}{2} \sum_{i=1}^N \sum_{k=1}^N \xi_i |a_{ik}| (\sigma_k x_k - \sigma_i x_i)^{[p]} = 0. \end{aligned}$$

Thus $\dot{\phi}^*(t) = \phi^*(0)$, for all $t \geq 0$. This is complete proof.

Obviously, Lemma 5 means that $\phi^*(t)$ is weighted signed average of initial states. Thus, when final consensus state is

$\phi^*(0)$, the corresponding consensus can be called weighted signed average consensus (WSAC), which can be regarded as the extension of signed average consensus in [30], [36], [37], [39]. The reason is that when $\xi_i \equiv 1, i \in \Pi_N$, WSAC equals to SAC.

Theorem 1: Assume the signed graph $\mathcal{G}(A)$ has detail-balanced structure with $\xi = [\xi_1, \dots, \xi_N]^T$. If the graph is structurally balanced, under the protocol (3) with $0 < p < 1$, MAS (2) can achieve FTBC and associated settling time is estimated by

$$T(x_0) \leq T_1 = \frac{V^{1-\alpha}(0)}{k_0(1-\alpha)},$$

where parameter $\alpha = \frac{1+p}{2} \in (\frac{1}{2}, 1)$, and parameter

$$k_0 = \frac{1}{2} (2\lambda_2(L(B)))^\alpha \left(\frac{2}{\xi_{\max}} \right)^\alpha$$

is to be determined later.

Proof: From Lemma 4, there is a matrix S such that $SAS \geq 0$ since graph $\mathcal{G}(A)$ is structurally balanced. Let $z_i(t) = \sigma_i x_i(t)$. Thus one has $z(t) = [z_1(t), \dots, z_N(t)]^T = Sx(t)$, $z(0) = Sx(0)$. Moreover, one has

$$\dot{z}_i(t) = \sigma_i \dot{x}_i(t) = \sigma_i u_i(t), i \in \Pi_N.$$

Substituting protocol (3) into above equation yields

$$\dot{z}_i(t) = \sum_{k=1}^N |a_{ik}| (z_k - z_i)^{[p]}.$$

Let $e_i(t) = z_i(t) - \phi^*(0)$, then one has

$$\begin{aligned} \dot{e}_i(t) &= \dot{z}_i(t) \\ &= \sum_{k=1}^N |a_{ik}| |\text{sign}(z_k - z_i)| |z_k - z_i|^p \\ &= \sum_{k=1}^N |a_{ik}| |\text{sign}(e_k - e_i)| |e_k - e_i|^p. \end{aligned} \quad (5)$$

Set $e(t) = [e_1(t), \dots, e_N(t)]^T$. Through simple mathematical operation, one can obtain $\xi^T e(t) = 0$.

Consider the following candidate Lyapunov function:

$$V = \frac{1}{2} \sum_{k=1}^N \xi_k e_k^2(t).$$

It is easy to verify that V is a continuous, differential, positive definite, and radically unbounded function. In addition, it satisfies inequalities:

$$\frac{1}{2} \xi_{\min} \sum_{k=1}^N e_k^2(t) \leq V \leq \frac{1}{2} \xi_{\max} \sum_{k=1}^N \xi_k e_k^2(t).$$

Furthermore, the following inequalities can be obtained:

$$\frac{2V}{\xi_{\max}} \leq \sum_{k=1}^N e_k^2(t) \leq \frac{2V}{\xi_{\min}}.$$

The derivative of V along (5) can be calculated as follows:

$$\begin{aligned}
 \dot{V} &= \sum_{i=1}^N \xi_i e_i(t) \dot{e}_i(t) \\
 &= \sum_{i=1}^N \sum_{k=1}^N |\xi_i a_{ik}| (e_k - e_i)^{[p]} \\
 &= \frac{1}{2} \sum_{i=1}^N \sum_{k=1}^N |\xi_i a_{ik}| (e_k - e_i)^{[p]} \\
 &\quad + \frac{1}{2} \sum_{k=1}^N \sum_{i=1}^N |\xi_i a_{ik}| (e_k - e_i)^{[p]} \\
 &= \frac{1}{2} \sum_{i=1}^N \sum_{k=1}^N |\xi_i a_{ik}| (e_k - e_i)^{[p]} \\
 &\quad - \frac{1}{2} \sum_{i=1}^N \sum_{k=1}^N |\xi_i a_{ik}| (e_k - e_i)^{[p]} \\
 &= -\frac{1}{2} \sum_{i=1}^N \sum_{k=1}^N |\xi_i a_{ik}| (e_k - e_i)(e_k - e_i)^{[p]} \\
 &= -\frac{1}{2} \sum_{i=1}^N \sum_{k=1}^N |\xi_i a_{ik}| |e_k - e_i|^{1+p} \\
 &\leq -\frac{1}{2} \left(\sum_{i=1}^N \sum_{k=1}^N |\xi_i a_{ik}|^{\frac{2}{1+p}} (e_k - e_i)^2 \right)^{\frac{1+p}{2}}, \quad (6)
 \end{aligned}$$

where Lemma 1 is inserted to prove above inequality due to $0 < p < 1$. According to detail balanced conditions, one has $|\xi_i a_{ik}|^{\frac{2}{1+p}} = |\xi_k a_{ki}|^{\frac{2}{1+p}}$, for $\forall i, k \in \Pi_N$. Set matrix $B = [b_{ik}] \in R^{N \times N}$, which is defined by $b_{ik} = |\xi_k a_{ki}|^{\frac{2}{1+p}}$, if $i \neq k$, and $b_{ik} = 0$, for $i = k$. Since $A = A^T$ the graph $\mathcal{G}(B) = \mathcal{G}(\mathcal{V}, \mathcal{E}, B)$ is an undirected and connected. According to matrix $B^T = B$, one has the following inequality:

$$\sum_{i=1}^N \sum_{k=1}^N |\xi_i a_{ik}|^{\frac{2}{1+p}} (e_k - e_i)^2 \geq 2\lambda_2(L(B)) e^T e, \quad (7)$$

where $L(B)$ denotes Laplacian matrix of the graph $\mathcal{G}(B)$.

Invoking (6) and (7), one can obtain

$$\begin{aligned}
 \dot{V} &\leq -\frac{1}{2} (2\lambda_2(L(B)))^{\frac{1+p}{2}} (e^T e)^{\frac{1+p}{2}} \\
 &\leq -\frac{1}{2} (2\lambda_2(L(B)))^{\frac{1+p}{2}} \left(\frac{2}{\xi_{\max}} \right)^{\frac{1+p}{2}} V^{\frac{1+p}{2}} \\
 &= -k_0 V^\alpha, \quad (8)
 \end{aligned}$$

where $\alpha = \frac{p+1}{2}$. According to Lemma 2 and inequality (8), V converges to zero in a finite time, and associated settling time is estimated by $T(x(0)) \leq T_1$. Thus we have $\lim_{t \rightarrow T_1} |x_i(t)| = |\phi^*(0)|$ and $|x_i(t)| = |\phi^*(0)|$, for $t \geq T_1$. According to Definition 7, FTBC problem of MAS (2) is solved. This is complete proof.

Next, the case that $p = 0$ is to be discussed. Obviously, power parameter $p = 0$ means that (3) is a discontinuous protocol $u_i = \sum_{k=1}^N a_{ik} \text{sign}(\tau_{ki})$ and $\dot{x}_i(t) = \sum_{k=1}^N a_{ik} \text{sign}(\tau_{ki})$

is a discontinuous system. By Filippov solution, one can obtain the following result.

Theorem 2: Suppose graph $\mathcal{G}(A)$ is structurally balanced and detail balanced structure with $\xi = [\xi_1, \dots, \xi_N]^T$. Then under the protocol (3) with $p = 0$, MAS (2) can achieve FTBC and associated settling time can be estimated by

$$T(x_0) \leq T_2 = \frac{2V^{\frac{1}{2}}(0)}{(2\lambda_2(L(A_0)))^{\frac{1}{2}} \left(\frac{2}{\xi_{\max}} \right)^{\frac{1}{2}}},$$

where parameter $\lambda_2(L(A_0))$ is to be determined later.

Proof: Similar to Theorem 1, one can obtain

$$\begin{aligned}
 \dot{z}_i(t) &= \sum_{k=1}^N |a_{ik}| \text{sign}(z_k - z_i), \\
 \dot{e}_i(t) &= \sum_{k=1}^N |a_{ik}| \text{sign}(e_k - e_i), \quad (9)
 \end{aligned}$$

which is a obviously discontinuous system. To guarantee the existence of Filippov solution for equation (9), denote the right side of (9) as $f(e)$. By the Filippov regularization, the Filippov solution of equation (9) can be defined as absolutely continuous function, which satisfies differential inclusion:

$$\dot{e}_i(t) \in \mathcal{K}[f](e(t)) = \sum_{k=1}^N |a_{ik}| \text{SIGN}(e_k - e_i), \quad i \in \Pi_N.$$

where the set-value function $\text{SIGN}(\cdot)$ satisfying $\text{SIGN}(\mu) = \text{sign}(\mu)$, if $\mu \neq 0$, $\text{SIGN}(\mu) \in [-1, 1]$ if $\mu = 0$. Detailed explanations about existence of Filippov solution of equation (9) can refer to references [51], [52]. Then, the candidate Lyapunov function is still taken as $V = \sum_{i=1}^N \xi_i e_i^2(t)$. The set-valued Lie derivative of V can be calculated as following:

$$\begin{aligned}
 \mathcal{L}V &= \left[\frac{\partial V}{\partial x} \right]^T \cdot \mathcal{K}[f](e(t)) \\
 &= 2 \sum_{i=1}^N \xi_i e_i \sum_{k=1}^N |a_{ik}| \text{SIGN}(e_k - e_i) \\
 &= - \sum_{i=1}^N \sum_{k=1}^N \left(|\xi_i a_{ik}|^2 (e_k - e_i)^2 \right)^{\frac{1}{2}} \\
 &\leq - \left(\sum_{i=1}^N \sum_{k=1}^N |\xi_i a_{ik}|^2 (e_k - e_i)^2 \right)^{\frac{1}{2}} \\
 &\leq - \left(2\lambda_2(L(A_0)) e^T e \right)^{\frac{1}{2}} \\
 &= - (2\lambda_2(L(A_0)))^{\frac{1}{2}} (e^T e), \quad (10)
 \end{aligned}$$

where $L(A_0)$ is Laplacian of the graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, A_0)$, matrix $A_0 = [a_{0ij}] \in R^{N \times N}$ is defined as $a_{0ij} = |\xi_i a_{ik}|^2$, $i \neq j$, $a_{0ii} = 0$ for $i, j \in \Pi_N$. According to detail balanced condition, one has $A_0^T = A_0$. Thus graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, A_0)$ can be regarded as an undirected connected graph, which leads

to $L(A_0)$ be a semi-positive definite matrix. Furthermore, one can obtain algebra connectivity $\lambda_2(L(A_0)) > 0$. Moreover, by Theorem 1, one has $\frac{2V}{\xi_{\max}} \leq \sum_{i=1}^N e_i^2(t) = e^T e$. Inserting this inequality into inequality (10), one can obtain $\mathcal{L}V \leq -(2\lambda_2(L(A_0)))^{\frac{1}{2}} \left(\frac{2}{\xi_{\max}}\right)^{\frac{1}{2}} V^{\frac{1}{2}}$. Invoking comparison principle and Lemma 2, one concludes that V converges to zero in a finite time, that is to say $\lim_{t \rightarrow T(x_0)} V = 0$ and $V = 0$ for $t > T(x_0)$. At the same time, associated settling time $T(x_0)$ satisfies

$$T(x_0) \leq T_2 = \frac{2V^{\frac{1}{2}}(0)}{(2\lambda_2(L(A_0)))^{\frac{1}{2}} \left(\frac{2}{\xi_{\max}}\right)^{\frac{1}{2}}}.$$

Noticing that definition of V and $e_i = \sigma_i x_i - \phi^*(0)$, one has $\lim_{t \rightarrow T_2} e_i = 0$ and $e_i = 0$ for $t > T_2$. Thus we have $\lim_{t \rightarrow T_2} |x_i| = |\phi^*(0)|$ and $|x_i(t)| = |\phi^*(0)|$, for $t > T_2$, and any $i \in \Pi_N$. Therefore the FTBC problem of MAS (2) is solved. This is complete proof.

Subsequently, we consider finite-time stability of MAS (2) under structurally unbalanced signed graph $\mathcal{G}(A)$.

Theorem 3: If the signed graph $\mathcal{G}(A)$ has detail balanced structure, and it is structurally unbalanced, under the protocol (3) with $0 < p < 1$, MAS (2) can realize finite time stability and associated settling time is estimated by

$$T(x_0) \leq T_3 = \frac{U^{1-\alpha}(0)}{k_1(1-\alpha)},$$

where parameters α, k_1 are to be determined later.

Proof: Consider MAS (2) with protocol (3) under structurally unbalanced signed graph $\mathcal{G}(A)$. Which is described as the following equations:

$$\dot{x}_i = \sum_{k=1}^N a_{ik} \text{sign}(\tau_{ki}) |\tau_{ki}|^p, \quad i \in \Pi_N, \quad (11)$$

where $\tau_{ki} = x_k - \text{sign}(a_{ki})x_i$. The candidate Lyapunov function is taken as $U = \sum_{i=1}^N \xi_i x_i^2$. Then the derivative of U along system (11) can be calculated as follows.

$$\begin{aligned} \dot{U} &= 2 \sum_{i=1}^N \xi_i x_i \dot{x}_i \\ &= 2 \sum_{i=1}^N \xi_i x_i \sum_{k=1}^N a_{ik} \text{sign}(\tau_{ki}) |\tau_{ki}|^p \\ &= \sum_{i=1}^N \xi_i x_i \sum_{k=1}^N a_{ik} \text{sign}(\tau_{ki}) |\tau_{ki}|^p \\ &\quad + \sum_{k=1}^N \xi_k x_k \sum_{i=1}^N a_{ki} \text{sign}(\tau_{ki}) |\tau_{ki}|^p \\ &= - \sum_{i=1}^N \sum_{k=1}^N |\xi_k a_{ki}| |x_k - \text{sign}(a_{ki})x_i|^{1+p} \\ &= - \sum_{i=1}^N \sum_{k=1}^N \left(|\xi_k a_{ki}|^{\frac{2}{p+1}} |x_k - \text{sign}(a_{ki})x_i|^2 \right)^{\frac{p+1}{2}} \end{aligned}$$

$$\begin{aligned} &\leq - \left(\sum_{i=1}^N \sum_{k=1}^N |\xi_k a_{ki}|^{\frac{2}{p+1}} (x_k - \text{sign}(a_{ki})x_i)^2 \right)^{\frac{p+1}{2}} \\ &= - \left(\sum_{k=1}^N \sum_{i=1}^N b_{ki} (x_k - \text{sign}(a_{ki})x_i)^2 \right)^{\frac{p+1}{2}}. \quad (12) \end{aligned}$$

Denote matrix $B_p = [b_{pki}] \in R^{N \times N}$, which is defined as $b_{pki} = \text{sign}(a_{ki}) |\xi_k a_{ki}|^{\frac{2}{p+1}}$, $L(B_p) = [l_{pki}] \in R^{N \times N}$ with $l_{pki} = \sum_{i=1}^N |\xi_k a_{ki}|^{\frac{2}{p+1}}$ if $i = k$ and $l_{pki} = -b_{pki}$ if $i \neq k$. Then $L(B_p)$ is Laplacian of the graph $\mathcal{G}(B_p)$. Due to graph $\mathcal{G}(B_p)$ is detail-balanced and structurally unbalanced, $L(B_p)$ is a positive definite matrix. Therefore, its eigenvalues can be arranged as $0 < \lambda_1(B_p) \leq \lambda_2(B_p) \leq \dots \leq \lambda_N(B_p)$. Which leads to $\lambda_1(B_p)x^T x \leq x^T L(B_p)x$ for any $x \in R^N$. Detailed explanation can refer to reference [13]. Thus, one has

$$\sum_{k=1}^N \sum_{i=1}^N b_{ki} (x_k - \text{sign}(a_{ki})x_i)^2 \geq 2\lambda_1(B_p)x^T x. \quad (13)$$

Based on inequalities (12) and (13), one can obtain the following inequality

$$\dot{U} \leq -(2\lambda_1(B_p))^{\frac{p+1}{2}} (x^T x)^{\frac{p+1}{2}}. \quad (14)$$

In addition, inequality $\frac{U}{\xi_{\max}} \leq \sum_{i=1}^N x_i^2 = x^T x \leq \frac{U}{\xi_{\min}}$ can be obtained from $U = \sum_{i=1}^N \xi_i x_i^2$. Inserting it into inequality into (14), one can obtain the following inequality

$$\dot{U} \leq -(2\lambda_1(B_p))^{\frac{p+1}{2}} \left(\frac{U}{\xi_{\max}} \right)^{\frac{p+1}{2}} = -k_1 U^\alpha, \quad (15)$$

where parameters $\alpha = \frac{p+1}{2}, k_1 = -(2\lambda_1(B_p))^\alpha \left(\frac{1}{\xi_{\max}}\right)^\alpha$.

According to Lemma 1 and inequality (15), U converges to zero in a finite time, and associated settling time can be estimated by T_3 . This is complete proof.

Remark 5: From Theorem 3, it can be seen that for the structurally unbalanced signed graphs, the finite time bipartite consensus problem is equal to the finite-time stability problem. That is to say, Theorem 3 solved finite time stability problem of continuous system (11) in virtue of positive definite matrix $L(B_p)$. If power parameter $p = 0$, the finite time stability of discontinuous system needs to be analyzed by Filippov solutions and set-valued Lie derivative.

Remark 6: It follows from Theorem 1, Theorem 2 and Theorem 3, the FTBC (or finite time stability) under the structurally balanced (unbalanced) signed graphs can be achieved under the protocol (3) with different power parameters. Specifically, if power parameter $p = 0$, corresponding finite-time stability has to be analyzed by Filippov solution.

Similar to Theorem 3, another result can be given as follows.

Theorem 4: If the signed graph $\mathcal{G}(A)$ is detail balanced and structurally unbalanced, MAS (2) under protocol (3) with $p = 0$ can achieve finite-time stability and associated settling

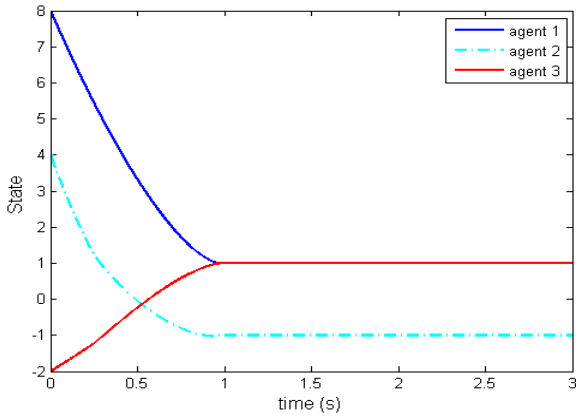


FIGURE 1. State trajectories of MAS (2) under protocol (3) with $p = \frac{1}{3}$ and topology $\mathcal{G}(A_1)$.

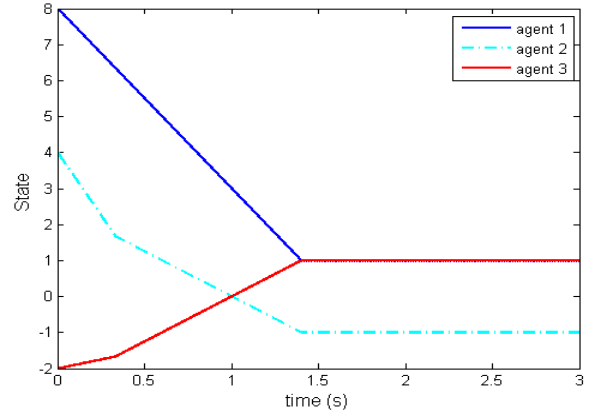


FIGURE 2. State trajectories of MAS (2) under protocol (3) with $p = 0$ and topology $\mathcal{G}(A_1)$.

time is estimated by $T(x_0) \leq T_4 = \frac{2U^{\frac{1}{2}}(0)}{k_3}$, where parameter $k_3 = (2\lambda_1(B_0))^{\frac{1}{2}} \left(\frac{1}{\xi_{\max}}\right)^{\frac{1}{2}}$, matrix $B_0 = [b_{0kj}] \in R^{N \times N}$ with $b_{0kj} = \text{sign}(a_{ki}) |\xi_k a_{ki}|^2$, $i, j \in \Pi_N$, $\lambda_1(B_0)$ is the smallest eigenvalue of matrix $L(B_0)$.

Obviously, matrix $L(B_0)$ can be regarded as Laplacian matrix of graph $\mathcal{G}(B_0)$, Due to the proof of Theorem 4 is similar to Theorem 3, to save place, the proof of Theorem 3.11 is omitted here. Interested readers can finish it according to Theorem 3.

In addition, due to undirected and connected signed graph is a special detail balanced signed graph, thus based on Theorem 1, Theorem 2, Theorem 3, and Theorem 4, one has the following corollary.

Corollary 1: Suppose signed graph $\mathcal{G}(A)$ is connected. If $\mathcal{G}(A)$ is structurally balanced, then MAS (2) under the protocol (3) can achieve FTBC; If $\mathcal{G}(A)$ is structurally unbalanced, then the MAS (2) can achieve finite-time stability under protocol (3).

The proof of Corollary 1 is easy to finish according to proofs of Theorem 1, Theorem 2, Theorem 3, and Theorem 4. To save space, corresponding proof is omitted here as well.

IV. NUMERICAL SIMULATIONS

In this section, two simulation examples are provided to illustrate the effectiveness of theoretical results in Section III. Our objective is to demonstrate the effectiveness of protocol (3) under conditions in Theorem 3.6, Theorem 3.7, Theorem 3.8, and Theorem 3.11, respectively.

Example: Consider a multi-agent systems under $\mathcal{G}(A_1)$ with $\xi = [2, 1, 3]^T$, and associated weighted adjacency matrix A_1 satisfies $A_1 = \begin{bmatrix} 0 & -2 & 3 \\ -4 & 0 & -3 \\ 2 & -1 & 0 \end{bmatrix}$. Obviously, it is

easy to prove that $\mathcal{G}(A_1)$ satisfies detail-balanced condition. To verify protocol (3) is effective under the conditions in Theorem 3.6, set $p = \frac{1}{3}$, the initial states of agents are chosen

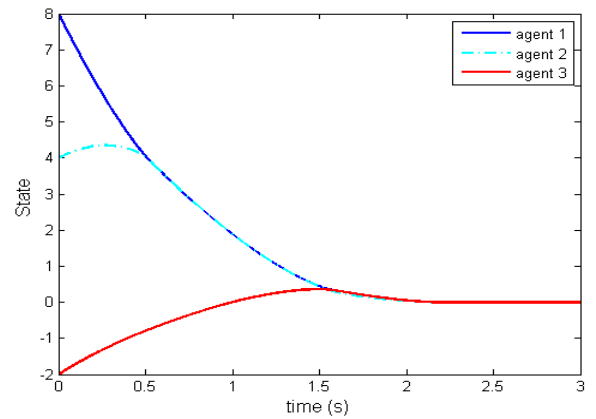


FIGURE 3. State trajectories of MAS (2) under protocol (3) with $p = \frac{1}{3}$ and topology $\mathcal{G}(A_2)$.

as $x(0) = [8, 4, -2]^T$. In addition, one can find a matrix

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ which ensures } SA_1S \geq 0. \text{ Then it is easy}$$

to get $\xi_{\max} = 3, \xi_{\min} = 1$. According to Lemma 3.5, one gets $\phi^*(x(0)) = 1$. According to Theorem 3.6, the corresponding theoretical value of settling time satisfies $T_1 \leq 2.8892s$. The evolutions of individuals' state $x_i(t)$ are plotted in Fig.1. One can see that the finite-time consensus is achieved, the final consensus state satisfying $x_i(t) = \phi^*(x(0)) = 1$ and the simulations match the theoretical results perfectly. Moreover, to verify the effectiveness of protocol (3) with $p = 0$ under the conditions in Theorem 3.7, corresponding evolutions of individuals' state $x_i(t)$ are plotted in Fig.2. By Theorem 3.7, associated theoretical value of settling time is $T_2 = 6.224s$. From Fig.2, one can see that simulation results accord with corresponding theoretical results. To demonstrate protocol (3) with $p = \frac{1}{3}$ is effective under the conditions in Theorem 3.8, one assumes that corresponding communication

topology is $\mathcal{G}(A_2)$, where $A_2 = \begin{bmatrix} 0 & 2 & 3 \\ 4 & 0 & -3 \\ 2 & -1 & 0 \end{bmatrix}$ other parame-

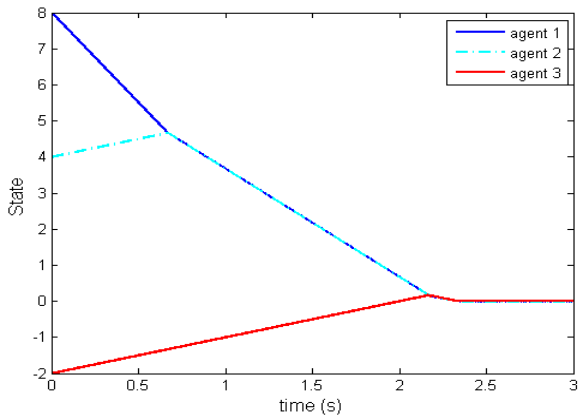


FIGURE 4. State trajectories of MAS (2) under protocol (3) with $p = 0$ and topology $\mathcal{G}(A_2)$.

ters and initial states are invariant. According to Theorem 3.8, associated estimation of settling time is $T_3 = 6.224s$. The corresponding simulation results are shown in Fig.3., which shows that finite-time stability is achieved. Moreover, from Fig.3., one can find that the real settling time is about 2.2s. This simulation supports our theoretical analysis. Set protocol parameter $p = 0$, other parameters are invariant as well. By Theorem 3.11, the corresponding theoretical value of estimation of settling time can be obtained $T_4 = 9.3699s$. Corresponding simulation results are shown in Fig.4., which accords well with theoretical results in Theorem 3.11.

V. CONCLUSION

In this article, we investigated FTBC problem of MAS under detail-balanced structure. The novelty of this article can be summed as follows. (1) Based on finite time stability theorem and structurally balanced graph theory, a weighted signed average consensus protocol framework is proposed to construct FTBC protocol; (2) Worth noticing that the final consensus state is weighted signed average of initial states, and associated settling time is a finite time, which can be estimated explicitly in form of mathematical expression by protocol parameters, initial conditions, and communication topology information; (3) By Filippov solution, a class of discontinuous protocol is proved to be effective as well under structurally balanced structure. For this case, existing literatures did not mention it. However, there are lots of remained works to be done, such as the case that random disturbance, communication delay are involved in the dynamics of MAS. These are also our future work.

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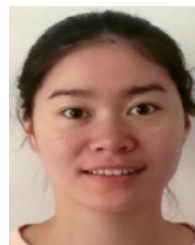
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