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Bounds on Covering Radius of Some Codes Over $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$

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ABSTRACT Let $R = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$ be a finite non-chain ring, where $u^2 = u$, $v^2 = v$, $uv = vu$. We give the lower and upper bounds on the covering radius of different types of repetition codes for Chinese Euclidean distance over R. Furthermore, we determine the upper bound on the covering radius of block repetition codes, simplex codes of types α and β, MacDonald codes of types α and β for Chinese Euclidean distance over R.

INDEX TERMS Covering radius, repetition codes, simplex codes, MacDonald codes.

I. INTRODUCTION

In 1994, Hammons *et al.* in [1] discovered that the best known nonlinear binary codes can be constructed by cyclic codes and Gray map over $\mathbb{Z}_4$. After this work, the coding theory over finite rings has attracted great attention from coding scholars. In order to obtain optimal codes over finite fields, many important research results have been determined by studying linear codes with special structures over finite rings [2]–[5]. Furthermore, the research on $\mathbb{Z}_p\mathbb{Z}_p[v]$-additive cyclic codes has also achieved some good results [6], [7].

The covering radius is an important geometric parameter that characterizes the maximum error-correcting capability of codes. Particularly, for codes applied in data compression, the covering radius is a measure of maximum distortion [8]. Therefore, the covering radius of codes has become a research hotspot in recent years. In 1978, Helleseth *et al.* in [9] studied the upper bounds on the covering radius of binary codes. In 1985, Cohen *et al.* in [10] and Graham *et al.* in [11] further studied the covering radius of binary linear codes and obtained some new results, respectively. Moreover, Levitin *et al.* in [12] discovered that the covering radius is used to upperbound the weight of zero neighbors in solving the minimum distance decoding problem. In 1999, the covering radius of codes over $\mathbb{Z}_4$ for Lee distance and Euclidean distance was studied in [13]. Later, Pandian *et al.* in [14] studied the covering radius of codes over $\mathbb{Z}_4$ for Chinese Euclidean distance. The covering radius of codes over $\mathbb{Z}_2 + u\mathbb{Z}_2$ with $u^2 = 0$ for Lee distance, Euclidean distance and Chinese Euclidean distance was studied in [15], [16].

Recently, the covering radius of codes over finite non-chain ring has been studied. In 2015, the covering radius of codes over $\mathbb{F}_2 + v\mathbb{F}_2$ for Lee distance was studied in [17]. Later, Gao *et al.* in [18] studied the covering radius of repetition codes, simplex codes and MacDonald codes over $\mathbb{F}_2 + v\mathbb{F}_2$ for Chinese Euclidean distance. Furthermore, Li *et al.* in [19] studied the covering radius of repetition codes, simplex codes and MacDonald codes over $\mathbb{F}_2R$ for Chinese Euclidean distance, where $R = \mathbb{F}_2 + v\mathbb{F}_2$.

Moreover, repetition codes are the simplest type of linear block codes with good error-correcting capability and have important applications in communication systems. Every nonzero codeword of the r-dimensional simplex code over finite field $\mathbb{F}_q$ has weight q^{r-1} and the simplex codes meet the Griesmer Bound. The MacDonald codes are punctured codes of the simplex codes and have many wide applications in authentication codes, association schemes and secret sharing schemes [20]–[22]. Therefore, it is very significant to study repetition codes, simplex codes and MacDonald codes. Until now, the research of different types of linear codes over $R = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$ has achieved many good results, where $u^2 = u$, $v^2 = v$, $uv = vu$. However, few coding scholars studied the covering radius of linear codes over R for Lee distance, Euclidean distance and Chinese Euclidean distance. Motivated by [18] and [19], we first consider the covering radius of repetition codes, simplex codes and MacDonald codes over R for Chinese Euclidean distance.

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The paper is organized as follows. In Section II, some basic results and the covering radius of codes for Chinese Euclidean distance over R are given. In Section III, we determine the lower and upper bounds on the covering radius of different types of repetition codes over R. In Section IV, we determine the upper bound on the covering radius of simplex codes of types α and β over R. In Section V, we determine the upper bound on the covering radius of MacDonald codes of types α and β over R. Section VI concludes the paper.

II. PRELIMINARIES

Let $R=\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2+uv\mathbb{F}_2$ be a finite commutative ring with characteristic 2, where $u^2=u$, $v^2=v$, $uv=vu$. Clearly, R is isomorphic to the quotient ring $\mathbb{F}_2[u,v]/\langle u^2-u, v^2-v, uv-vu\rangle$. Moreover, it is easy to observe that R is a Frobenius ring but not a local ring or a chain ring.

Definition 1: A linear code C of length n over R is an R-submodule of R^n.

For any $r\in R$, there exist $a,b,c,d\in\mathbb{F}_2$ such that r can be expressed as $r=a+ub+vc+uvd$. Define a Gray map ϕ from R to $\mathbb{F}_2$ as follows:

$$\theta: R\rightarrow \mathbb{F}_2$$
$$a+ub+vc+uvd\mapsto (a+b+c+d, a+c, a+b, a).$$

In [23], the authors described the notion of a Chinese Euclidean weight. For any $\boldsymbol{x}=(x_0,x_1,\cdots,x_{n-1})\in\mathbb{F}_2^n$, the Chinese Euclidean weight of $\boldsymbol{x}$ is defined as $w_{CE}(\boldsymbol{x})=\sum_{i=0}^{n-1}\{2-2cos(\frac{2\pi x_i}{4})\}$. Applying the conditions to the ring R and Gray map ϕ, for any $r\in R$, the Chinese Euclidean weight of r is defined as

$$w_{CE}(r)=\begin{cases}0, & r=0;\\ 2, & r=uv, v+uv, u+uv, 1+u+v+uv;\\ 4, & r=u, v, 1+u, 1+v, u+v, 1+u+v;\\ 6, & r=1+uv, 1+v+uv, 1+u+uv, u+v+uv;\\ 8, & r=1.\end{cases}$$

The Chinese Euclidean weight of $(r_0,r_1,\cdots,r_{n-1})\in R^n$ is defined as $\sum_{i=0}^{n-1}w_{CE}(r_i)$. For any two distinct codewords $\boldsymbol{c}_1,\boldsymbol{c}_2\in C$, the Chinese Euclidean distance is defined as $d_{CE}(\boldsymbol{c}_1,\boldsymbol{c}_2)=w_{CE}(\boldsymbol{c}_1-\boldsymbol{c}_2)$. The minimum Chinese Euclidean distance of C is defined as $d_{CE}(C)=min\{d_{CE}(\boldsymbol{c}_1,\boldsymbol{c}_2)|\boldsymbol{c}_1\neq\boldsymbol{c}_2,\boldsymbol{c}_1,\boldsymbol{c}_2\in C\}$. Clearly, for a linear code C, $d_{CE}(C)=min\{w_{CE}(\boldsymbol{c})|0\neq\boldsymbol{c}\in C\}$. If C is a linear code of length n over R with the number of codewords M and the minimum Chinese Euclidean distance d_{CE}, then we call it an (n,M,d_{CE}) code.

Definition 2: Let C be a linear code of length n over R. For any $\boldsymbol{y}\in R^n$, the Chinese Euclidean distance between $\boldsymbol{y}$ and C is defined as

$$d_{CE}(\boldsymbol{y},C)=\min\{d_{CE}(\boldsymbol{y},\boldsymbol{x})|\forall\boldsymbol{x}\in C\}.$$

The covering radius of C for Chinese Euclidean distance is defined as

$$r_{CE}(C)=\max\{d_{CE}(\boldsymbol{y},C)|\forall\boldsymbol{y}\in R^n\}.$$

TABLE 1. Repetition Codes of Length *n* over *R*.

Repetition Codes	Generator Matrix	Parameters of Repetition Codes
C_1	$(\overbrace{1\cdots1}^{n})$	$(n,16,2n)$
C_2	$(\overbrace{u\cdots u}^{n})$	$(n,4,2n)$
C_3	$(\overbrace{v\cdots v}^{n})$	$(n,4,2n)$
C_4	$(\overbrace{uv\cdots uv}^{n})$	$(n,2,2n)$
C_5	$(\overbrace{1+u\cdots1+u}^{n})$	$(n,4,2n)$
C_6	$(\overbrace{1+v\cdots1+v}^{n})$	$(n,4,2n)$
C_7	$(\overbrace{1+uv\cdots1+uv}^{n})$	$(n,8,2n)$
C_8	$(\overbrace{u+v\cdots u+v}^{n})$	$(n,8,2n)$
C_9	$(\overbrace{u+uv\cdots u+uv}^{n})$	$(n,2,2n)$
C_{10}	$(\overbrace{v+uv\cdots v+uv}^{n})$	$(n,2,2n)$
C_{11}	$(\overbrace{1+u+v\cdots1+u+v}^{n})$	$(n,4,2n)$
C_{12}	$(\overbrace{1+u+uv\cdots1+u+uv}^{n})$	$(n,8,2n)$
C_{13}	$(\overbrace{1+v+uv\cdots1+v+uv}^{n})$	$(n,8,2n)$
C_{14}	$(\overbrace{u+v+uv\cdots u+v+uv}^{n})$	$(n,8,2n)$
C_{15}	$(\overbrace{1+u+v+uv\cdots1+u+v+uv}^{n})$	$(n,2,2n)$

The following result of Mattson plays an important role in computing the covering radius of linear codes over R.

Proposition 1: [10] Let C_0 and C_1 be linear codes over R generated by matrices G_0 and G_1, respectively. If C is the linear code generated by

$$\left(\begin{array}{c|c}0 & G_1\\ \hline G_0 & A\end{array}\right)$$

then $r_d(C)\leq r_d(C_0)+r_d(C_1)$ and the covering radius of D, containing of C_0 and C_1, satisfies $r_d(D)\geq r_d(C_0)+r_d(C_1)$ for all distance d.

III. COVERING RADIUS OF REPETITION CODES

In this section, we consider repetition codes of length n over R. There are two types of repetition codes of length n over R. One type is the unit repetition code C_1, the other is zero divisor repetition code C_i for $2\leq i\leq 15$. We list them in TABLE 1.

The following result gives the lower and upper bounds on the covering radius of repetition codes $C_i (1\leq i\leq 15)$ of length n over R.

Theorem 1: $64\lfloor\frac{n}{16}\rfloor\leq r_{CE}(C_1)\leq 4n$, $2n+32\lfloor\frac{n}{14}\rfloor\leq r_{CE}(C_2)\leq 6n$, $2n+32\lfloor\frac{n}{14}\rfloor\leq r_{CE}(C_3)\leq 5n$, $2n+34\lfloor\frac{n}{14}\rfloor\leq r_{CE}(C_4)\leq 7n$, $60\lfloor\frac{n}{14}\rfloor\leq r_{CE}(C_5)\leq 6n$, $60\lfloor\frac{n}{14}\rfloor\leq r_{CE}(C_6)\leq 6n$, $58\lfloor\frac{n}{14}\rfloor\leq r_{CE}(C_7)\leq 5n$, $2n+32\lfloor\frac{n}{14}\rfloor\leq r_{CE}(C_8)\leq 6n$, $2n+34\lfloor\frac{n}{14}\rfloor\leq$

$r_{CE}(C_9) \le 7n$, $2n + 34\lfloor \frac{n}{14} \rfloor \le r_{CE}(C_{10}) \le 7n$, $60\lfloor \frac{n}{14} \rfloor \le r_{CE}(C_{11}) \le 6n$, $58\lfloor \frac{n}{14} \rfloor \le r_{CE}(C_{12}) \le 5n$, $58\lfloor \frac{n}{14} \rfloor \le r_{CE}(C_{13}) \le 5n$, $2n + 30\lfloor \frac{n}{14} \rfloor \le r_{CE}(C_{14}) \le 5n$, $2n + 34\lfloor \frac{n}{14} \rfloor \le r_{CE}(C_{15}) \le 7n$.

Proof See the appendix.

In the following, in order to get the upper bound on the covering radius of simplex codes of types α and β over R, we give the definition of block repetition codes.

Let $BRep^n$ be a block repetition code generated by

$$G = \Big(\overbrace{1\cdots 1}^{n_1} \overbrace{u\cdots u}^{n_2} \overbrace{v\cdots v}^{n_3} \overbrace{uv\cdots uv}^{n_4} \overbrace{1+u\cdots 1+u}^{n_5} \overbrace{1+v\cdots 1+v}^{n_6} \overbrace{1+uv\cdots 1+uv}^{n_7} \overbrace{u+v\cdots u+v}^{n_8} \overbrace{u+uv\cdots u+uv}^{n_9} \overbrace{v+uv\cdots v+uv}^{n_{10}} \overbrace{1+u+v\cdots 1+u+v}^{n_{11}} \overbrace{1+u+uv\cdots 1+u+uv}^{n_{12}} \overbrace{1+v+uv\cdots 1+v+uv}^{n_{13}} \overbrace{u+v+uv\cdots u+v+uv}^{n_{14}} \overbrace{1+u+v+uv\cdots 1+u+v+uv}^{n_{15}} \Big),$$

where $n = \sum_{i=1}^{15} n_i$. Then, we have that $BRep^n = \{\mu G | \mu \in R\}$.

Theorem 2: The upper bound on the covering radius of $BRep^n$ is given by

$$r_{CE}(BRep^n) \le 4n_1 + 5(n_7 + n_{13} + n_{14}) + 6(n_5 + n_6 + n_8 + n_9 + n_{11} + n_{12}) + 7(n_2 + n_3 + n_4 + n_9 + n_{10} + n_{15}),$$

where $n = \sum_{i=1}^{15} n_i$. Particularly, if $n_1 = n_2 = \cdots = n_{15}$, then $r_{CE}(BRep^n) \le 91n$.

Proof Let

$$\boldsymbol{x} = (\boldsymbol{x}_1|\boldsymbol{x}_2|\boldsymbol{x}_3|\boldsymbol{x}_4|\boldsymbol{x}_5|\boldsymbol{x}_6|\boldsymbol{x}_7|\boldsymbol{x}_8|\boldsymbol{x}_9|\boldsymbol{x}_{10}|\boldsymbol{x}_{11}|\boldsymbol{x}_{12}|\boldsymbol{x}_{13}|\boldsymbol{x}_{14}| \boldsymbol{x}_{15}) \in R^n,$$

where $\boldsymbol{x}_1$ has positions $a_i (0 \le i \le 15)$, $\boldsymbol{x}_2$ has positions $b_i (0 \le i \le 15)$, $\boldsymbol{x}_3$ has positions $d_i (0 \le i \le 15)$, $\boldsymbol{x}_4$ has positions $e_i (0 \le i \le 15)$, $\boldsymbol{x}_5$ has positions $f_i (0 \le i \le 15)$, $\boldsymbol{x}_6$ has positions $g_i (0 \le i \le 15)$, $\boldsymbol{x}_7$ has positions $h_i (0 \le i \le 15)$, $\boldsymbol{x}_8$ has positions $j_i (0 \le i \le 15)$, $\boldsymbol{x}_9$ has positions $k_i (0 \le i \le 15)$, $\boldsymbol{x}_{10}$ has positions $l_i (0 \le i \le 15)$, $\boldsymbol{x}_{11}$ has positions $m_i (0 \le i \le 15)$, $\boldsymbol{x}_{12}$ has positions $r_i (0 \le i \le 15)$, $\boldsymbol{x}_{13}$ has positions $t_i (0 \le i \le 15)$, $\boldsymbol{x}_{14}$ has positions $p_i (0 \le i \le 15)$, $\boldsymbol{x}_{15}$ has positions $q_i (0 \le i \le 15)$, satisfying $\sum_{i=0}^{15} a_i = n_1$, $\sum_{i=0}^{15} b_i = n_2$, $\sum_{i=0}^{15} d_i = n_3$, $\sum_{i=0}^{15} e_i = n_4$, $\sum_{i=0}^{15} f_i = n_5$, $\sum_{i=0}^{15} g_i = n_6$, $\sum_{i=0}^{15} h_i = n_7$, $\sum_{i=0}^{15} j_i = n_8$, $\sum_{i=0}^{15} k_i = n_9$, $\sum_{i=0}^{15} l_i = n_{10}$, $\sum_{i=0}^{15} m_i = n_{11}$, $\sum_{i=0}^{15} r_i = n_{12}$, $\sum_{i=0}^{15} t_i = n_{13}$, $\sum_{i=0}^{15} p_i = n_{14}$, $\sum_{i=0}^{15} q_i = n_{15}$.

According to the proof process of Theorem 1, we get the expressions of A_ω, B_ω, D_ω, E_ω, F_ω, G_ω, H_ω, I_ω, J_ω, K_ω, L_ω, M_ω, R_ω, T_ω, P_ω, Q_ω. In the following proof process, for convenience we need to use the above expressions, but the subscript ω should be replaced with other letters, for example $A_a = n_1 - a_0 + 7a_1 + 3a_2 + 3a_3 + a_4 + 3a_5 + 3a_6 + 5a_7 + 3a_8 + a_9 + a_{10} + 3a_{11} + 5a_{12} + 5a_{13} + 5a_{14} + a_{15}$ (replace ω with a in A_ω, $\sum_{i=0}^{15} a_i = n_1$).

Since $BRep^n = \{c_0 = 0 \cdot G, c_1 = 1 \cdot G, c_2 = u \cdot G, c_3 = v \cdot G, c_4 = uv \cdot G, c_5 = (1+u) \cdot G, c_6 = (1+v) \cdot G, c_7 = (1+uv) \cdot G, c_8 = (u+v) \cdot G, c_9 = (u+uv) \cdot G, c_{10} = (v+uv) \cdot G, c_{11} = (1+u+v) \cdot G, c_{12} = (1+u+uv) \cdot G, c_{13} = (1+v+uv) \cdot G, c_{14} = (u+v+uv) \cdot G, c_{15} = (1+u+v+uv) \cdot G\}$, then we have that

$$d_{CE}(\boldsymbol{x}, c_0) = A_a + A_b + A_d + A_e + A_f + A_g + A_h + A_j + A_k + A_l + A_m + A_r + A_t + A_p + A_q.$$

$$d_{CE}(\boldsymbol{x}, c_1) = B_a + D_b + E_d + F_e + G_f + H_g + I_h + J_j + K_k + L_l + M_m + R_r + T_t + P_p + Q_q.$$

$$d_{CE}(\boldsymbol{x}, c_2) = D_a + D_b + F_d + F_e + A_f + K_g + K_h + K_j + K_k + A_l + F_m + F_r + D_t + D_p + A_q.$$

$$d_{CE}(\boldsymbol{x}, c_3) = E_a + F_b + E_d + F_e + L_f + A_g + L_h + L_j + A_k + L_l + F_m + E_r + F_t + E_p + A_q.$$

$$d_{CE}(\boldsymbol{x}, c_4) = F_a + F_b + F_d + F_e + A_f + A_g + A_h + A_j + A_k + A_l + F_m + F_r + F_t + F_p + A_q.$$

$$d_{CE}(\boldsymbol{x}, c_5) = G_a + A_b + L_d + A_e + G_f + Q_g + G_h + L_j + A_k + L_l + Q_m + G_r + Q_t + L_p + Q_q.$$

$$d_{CE}(\boldsymbol{x}, c_6) = H_a + K_b + A_d + A_e + Q_f + H_g + H_h + K_j + K_k + A_l + Q_m + Q_r + H_t + K_p + Q_q.$$

$$d_{CE}(\boldsymbol{x}, c_7) = I_a + K_b + L_d + A_e + G_f + H_g + I_h + J_j + K_k + L_l + Q_m + G_r + H_t + J_p + Q_q.$$

$$d_{CE}(\boldsymbol{x}, c_8) = J_a + K_b + L_d + A_e + L_f + K_g + J_h + J_j + K_k + L_l + A_m + A_r + K_t + J_p + A_q.$$

$$d_{CE}(\boldsymbol{x}, c_9) = K_a + A_b + A_d + A_e + A_f + K_g + K_h + K_j + K_k + A_l + A_m + A_r + K_t + K_p + A_q.$$

$$d_{CE}(\boldsymbol{x}, c_{10}) = L_a + A_b + A_d + A_e + L_f + A_g + L_h + L_j + A_k + L_l + A_m + L_r + A_t + L_p + A_q.$$

$$d_{CE}(\boldsymbol{x}, c_{11}) = M_a + F_b + F_d + F_e + Q_f + Q_g + Q_h + A_j + A_k + A_l + M_m + M_r + M_t + F_p + Q_q.$$

$$d_{CE}(\boldsymbol{x}, c_{12}) = R_a + F_b + E_d + F_e + G_f + Q_g + G_h + L_j + A_k + L_l + M_m + R_r + M_t + E_p + Q_q.$$

$$d_{CE}(\boldsymbol{x}, c_{13}) = T_a + D_b + F_d + F_e + Q_f + H_g + H_h + K_j + K_k + A_l + M_m + M_r + T_t + D_p + Q_q.$$

$$d_{CE}(\boldsymbol{x}, c_{14}) = P_a + D_b + E_d + F_e + L_f + K_g + J_h + J_j + K_k + L_l + F_m + E_r + D_t + P_p + A_q.$$

$$d_{CE}(\boldsymbol{x}, c_{15}) = Q_a + A_b + A_d + A_e + Q_f + Q_g + Q_h + A_j + A_k + A_l + Q_m + Q_r + Q_t + A_p + Q_q.$$

Therefore,

$$\begin{aligned} &d_{CE}(\boldsymbol{x}, BRep^n) \\ &= \min\{d_{CE}(\boldsymbol{x}, c_0), d_{CE}(\boldsymbol{x}, c_1), d_{CE}(\boldsymbol{x}, c_2), \\ &\quad d_{CE}(\boldsymbol{x}, c_3), d_{CE}(\boldsymbol{x}, c_4), d_{CE}(\boldsymbol{x}, c_5), d_{CE}(\boldsymbol{x}, c_6), d_{CE}(\boldsymbol{x}, c_7), \\ &\quad d_{CE}(\boldsymbol{x}, c_8), d_{CE}(\boldsymbol{x}, c_9), d_{CE}(\boldsymbol{x}, c_{10}), d_{CE}(\boldsymbol{x}, c_{11}), d_{CE}(\boldsymbol{x}, \\ &\quad c_{12}), d_{CE}(\boldsymbol{x}, c_{13}), d_{CE}(\boldsymbol{x}, c_{14}), d_{CE}(\boldsymbol{x}, c_{15})\} \leq 4n_1+5(n_7 \\ &\quad +n_{13}+n_{14})+6(n_5+n_6+n_8+n_9+n_{11}+n_{12})+7(n_2 \\ &\quad +n_3+n_4+n_9+n_{10}+n_{15}). \end{aligned}$$

This means that

$$\begin{aligned} r_{CE}(BRep^n) \leq\ & 4n_1+5(n_7+n_{13}+n_{14})+6(n_5+n_6+n_8 \\ & +n_9+n_{11}+n_{12})+7(n_2+n_3+n_4+n_9 \\ & +n_{10}+n_{15}). \end{aligned}$$

Let $BRep^m$ be a block repetition code generated by

$$G = \left(\overbrace{1\cdots 1}^{m_1}\overbrace{v\cdots v}^{m_2}\overbrace{1+u+v\cdots 1+u+v}^{m_3}\right)$$

where $m = m_1+m_2+m_3$. Then, we have that $BRep^n = \{\gamma G|\gamma \in R\}$. Similar to the proof process of Theorem 2, we can directly obtain the following result.

Theorem 3: The upper bound on the covering radius of $BRep^m$ is given by

$$r_{CE}(BRep^m) \leq 4m_1+7m_2+6m_3,$$

where $m = m_1+m_2+m_3$.

IV. SIMPLEX CODES OF TYPES α AND β OVER R

A type α simplex code S_k^α is a linear code over R. The generator matrix G_k^α of S_k^α is constructed inductively. Let G_k^α be a $k\times 2^{4k}$ matrix over R. Let

$$G_1^\alpha = \left(A_1|A_2|A_3|A_4\right),$$

$A_1 = (0\ \ 1\ \ u\ \ 1+u)$, $A_2 = (v\ \ uv\ \ (1+u)v\ \ 1+v)$, $A_3 = (1+uv\ \ 1+(1+u)v\ \ u+v\ \ u+uv)$, $A_4 = (u+(1+u)v\ \ 1+u+v\ \ 1+u+uv\ \ 1+u+(1+u)v)$. Then G_k is constructed inductively as follows

$$G_k^\alpha = \left(\begin{array}{c|c|c|c} \mathbf{0} & \mathbf{1} & \cdots & \mathbf{1}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{u}\ \boldsymbol{\mathcal{C}}\ (\mathbf{1}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{u})\boldsymbol{v} \\ \hline G_{k-1}^\alpha & G_{k-1}^\alpha & \cdots & G_{k-1}^\alpha \end{array}\right), \qquad (1)$$

where $\mathbf{0}, \mathbf{1}, \mathbf{1}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{u}\ \boldsymbol{\mathcal{C}}\ (\mathbf{1}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{u})\boldsymbol{v}$ are denoted as $0\cdots 0, 1\cdots 1, 1+u+(1+u)v\cdots 1+u+(1+u)v$. The bold characters below have the same meaning.

A type β simplex code S_k^β is a linear codes over R constructed by omitting some columns from G_k^α. Let λ_k be a $k\times\frac{2^{4k}-2^{2k}}{3}$ matrix over R. Let

$$\lambda_1 = \left(1\ u\ 1+u\ v\right)$$

and

$$\lambda_2 = \left(B_1|B_2\right),$$

where

$$B_1 = \left(\begin{array}{c|c|c|c|c} \mathbf{0} & \mathbf{1} & \boldsymbol{u} & \mathbf{1}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{u} & \boldsymbol{v} \\ \hline \lambda_1 & G_1^\alpha & G_1^\alpha & G_1^\alpha & G_1^\alpha \end{array}\right),$$

$$B_2 = \left(\begin{array}{c|c|c} \mathbf{1}\ \boldsymbol{\mathcal{C}}\ (\mathbf{1}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{u})\boldsymbol{v} & \boldsymbol{u}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{uv} & \mathbf{1}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{u}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{v} \\ \hline \lambda_1 & \lambda_1 & \lambda_1 \end{array}\right).$$

Then λ_k is constructed inductively as follows

$$\lambda_k = \left(C_1|C_2\right),$$

where

$$C_1 = \left(\begin{array}{c|c|c|c|c} \mathbf{0} & \mathbf{1} & \boldsymbol{u} & \mathbf{1}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{u} & \boldsymbol{v} \\ \hline \lambda_{k-1} & G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha \end{array}\right),$$

$$C_2 = \left(\begin{array}{c|c|c} \mathbf{1}\ \boldsymbol{\mathcal{C}}\ (\mathbf{1}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{u})\boldsymbol{v} & \boldsymbol{u}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{uv} & \mathbf{1}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{u}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{v} \\ \hline \lambda_{k-1} & \lambda_{k-1} & \lambda_{k-1} \end{array}\right).$$

Let δ_k be a $k\times\frac{2^{4k}-2^{2k}}{3}$ matrix over R. Let

$$\delta_1 = \left(1\ u\ 1+u\ 1+u+v\right)$$

and

$$\delta_2 = \left(D_1|D_2\right),$$

where

$$D_1 = \left(\begin{array}{c|c|c|c} \mathbf{0} & \mathbf{1} & \boldsymbol{u} & \mathbf{1}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{u} \\ \hline \delta_1 & G_1^\alpha & G_1^\alpha & G_1^\alpha \end{array}\right),$$

$$D_2 = \left(\begin{array}{c|c|c|c} \mathbf{1}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{u}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{v} & \boldsymbol{v} & \boldsymbol{uv} & (\mathbf{1}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{u})\boldsymbol{v} \\ \hline G_1^\alpha & \delta_1 & \delta_1 & \delta_1 \end{array}\right).$$

Then δ_k is constructed inductively as follows

$$\delta_k = \left(E_1|E_2\right),$$

where

$$E_1 = \left(\begin{array}{c|c|c|c} \mathbf{0} & \mathbf{1} & \boldsymbol{u} & \mathbf{1}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{u} \\ \hline \delta_{k-1} & G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha \end{array}\right),$$

$$E_2 = \left(\begin{array}{c|c|c|c} \mathbf{1}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{u}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{v} & \boldsymbol{v} & \boldsymbol{uv} & (\mathbf{1}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{u})\boldsymbol{v} \\ \hline G_{k-1}^\alpha & \delta_{k-1} & \delta_{k-1} & \delta_{k-1} \end{array}\right).$$

Let G_k^β be a $k\times(\frac{2^{2k}-1}{3})^2$ generator matrix of S_k^β. Let $G_1^\beta = \left(1\right)$ and

$$G_2^\beta = \left(\begin{array}{c|c|c|c} \mathbf{1} & 0 & \boldsymbol{v} & \mathbf{1}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{u}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{v} \\ \hline G_1^\alpha & 1 & \delta_1 & \lambda_1 \end{array}\right).$$

Then G_k^β is constructed inductively as follows

$$G_k^\beta = \left(\begin{array}{c|c|c|c} \mathbf{1} & \mathbf{0} & \boldsymbol{v} & \mathbf{1}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{u}\ \boldsymbol{\mathcal{C}}\ \boldsymbol{v} \\ \hline G_{k-1}^\alpha & G_{k-1}^\beta & \delta_{k-1} & \lambda_{k-1} \end{array}\right) \qquad (2)$$

Theorem 4: $r_{CE}(S_k^\alpha) \leq 91(\frac{2^{4k}-2^4}{15})+96$.

Proof: By using the computational algebra system Magma [24], we get that $r_{CE}(S_1^\alpha) = 96$. Next, we prove this result by induction on k. Firstly, we have $r_{CE}(S_1^\alpha) = 96$ for $k = 1$, which is consistent with our calculation. Secondly, assume that the result holds for $k-1$, i.e., $r_{CE}(S_{k-1}^\alpha) \leq 91(\frac{2^{4(k-1)}-2^4}{15})+96$. Finally, we prove that the result holds for k, i.e., $r_{CE}(S_k^\alpha) \leq 91(\frac{2^{4k}-2^4}{15})+96$.

According to Eq. (1), Proposition 1 and Theorem 2, we have that

$$\begin{aligned} r_{CE}(S_k^\alpha) \leq r_{CE}(S_{k-1}^\alpha) + r_{CE}((&\overbrace{1\cdots 1}^{2^{4(k-1)}}\overbrace{u\cdots u}^{2^{4(k-1)}} \\ &\overbrace{1+u\cdots 1+u}^{2^{4(k-1)}}\overbrace{v\cdots v}^{2^{4(k-1)}}\overbrace{uv\cdots uv}^{2^{4(k-1)}} \end{aligned}$$

$$\overbrace{(1+u)v\cdots(1+u)v}^{2^{4(k-1)}}\overbrace{1+v\cdots1+v}^{2^{4(k-1)}}$$
$$\overbrace{1+uv\cdots1+uv}^{2^{4(k-1)}}\overbrace{1+(1+u)v\cdots1+(1+u)v}^{2^{4(k-1)}}$$
$$\overbrace{u+v\cdots u+v}^{2^{4(k-1)}}\overbrace{u+uv\cdots u+uv}^{2^{4(k-1)}}$$
$$\overbrace{u+(1+u)v\cdots u+(1+u)v}^{2^{4(k-1)}}$$
$$\overbrace{1+u+v\cdots1+u+v}^{2^{4(k-1)}}$$
$$\overbrace{1+u+uv\cdots1+u+uv}^{2^{4(k-1)}}$$
$$\overbrace{1+u+(1+u)v\cdots1+u+(1+u)v}^{2^{4(k-1)}}))$$
$$\leq 91(2^{4(k-1)}+2^{4(k-2)}+\cdots+2^4)+96$$
$$=91(\frac{2^{4k}-2^4}{15})+96.$$

Theorem 5: $r_{CE}(S_k^\beta)\leq 5(\frac{2^{4k}-2^8}{9})-13(\frac{2^{2k}-2^4}{9})+106.$

Proof By using the computational algebra system Magma [24], we get that $r_{CE}(S_2^\beta)=106$. Next, we prove this result by induction on k. Firstly, we have $r_{CE}(S_2^\beta)=106$ for $k=2$, which is consistent with our calculation. Secondly, assume that the result holds for $k-1$, i.e., $r_{CE}(S_{k-1}^\beta)\leq 5(\frac{2^{4(k-1)}-2^8}{9})-13(\frac{2^{2(k-1)}-2^4}{9})+106$. Finally, we prove that the result holds for k, i.e., $r_{CE}(S_k^\alpha)\leq 5(\frac{2^{4k}-2^8}{9})-13(\frac{2^{2k}-2^4}{9})+106$.

According to Eq. (2), Proposition 1 and Theorem 2, we have that

$$r_{CE}(S_k^\beta)\leq r_{CE}(S_{k-1}^\beta)+r_{CE}((\overbrace{1\cdots1}^{2^{4(k-1)}}\overbrace{v\cdots v}^{\frac{2^{4(k-1)}-2^{2(k-1)}}{3}}$$
$$\overbrace{1+u+v\cdots1+u+v}^{\frac{2^{4(k-1)}-2^{2(k-1)}}{3}})),$$
$$\leq r_{CE}(S_{k-1}^\beta)+4\cdot2^{4(k-1)}+7(\frac{2^{4(k-1)}-2^{2(k-1)}}{3})$$
$$+6(\frac{2^{4(k-1)}-2^{2(k-1)}}{3})$$
$$=r_{CE}(S_{k-1}^\beta)+\frac{25}{3}\cdot2^{4(k-1)}-\frac{13}{3}\cdot2^{2(k-1)}$$
$$\leq\frac{25}{3}(2^{4(k-1)}+\cdots+2^{4\times2})-\frac{13}{3}(2^{2(k-1)}+\cdots$$
$$+2^{2\times2})+106$$
$$=5(\frac{2^{4k}-2^8}{9})-13(\frac{2^{2k}-2^4}{9})+106.$$

V. MacDonald CODES OF TYPES α AND β OVER R

The MacDonald codes of type α over R can be constructed from the generator matrix G_k^α of simplex code S_k^α. For $2\leq u\leq k-1$, let $G_{k,u}^\alpha$ be the matrix obtained from G_k^α by deleting columns corresponding to the columns of G_u^α, i.e.,

$$G_{k,u}^\alpha=\left(G_k^\alpha\backslash\frac{\mathbf{0}}{G_u^\alpha},\right)$$

where $\left(A\backslash B\right)$ denotes the matrix obtained from the matrix A by deleting the matrix B and $\mathbf{0}$ is a $(k-u)\times2^{4u}$ zero matrix.

Definition 3: A linear code $C_{k,u}^\alpha$ generated by $G_{k,u}^\alpha$ is called a type α MacDonald code.

The MacDonald codes of type β over R can be constructed from the generator matrix G_k^β of simplex code S_k^β. For $2\leq u\leq k-1$, let $G_{k,u}^\beta$ be the matrix obtained from G_k^β by deleting columns corresponding to the columns of G_u^β, i.e.,

$$G_{k,u}^\beta=\left(G_k^\beta\backslash\frac{\mathbf{0}}{G_u^\beta},\right)$$

where $\left(A\backslash B\right)$ denotes the matrix obtained from the matrix A by deleting the matrix B and $\mathbf{0}$ is a $(k-u)\times(\frac{2^{2u}-1}{3})^2$ zero matrix.

Definition 4: A linear code $C_{k,u}^\beta$ generated by $G_{k,u}^\beta$ is called a type β MacDonald code.

Theorem 6: $r_{CE}(C_{k,u}^\alpha)\leq 91(\frac{2^{4k}-2^{4r}}{15})+r_{CE}(C_{r,u}^\alpha)$ for $u<r\leq k$.

Proof Since $2\leq u<r\leq k$, then the minimum value of k is 3. Next, we prove this result by induction on k. If $k=3$, then $k=r$ and $r_{CE}(C_{3,u}^\alpha)=r_{CE}(C_{3,u}^\alpha)$, which is consistent with the above result. Assuming that the result holds for $k-1$, i.e., $r_{CE}(C_{k-1,u}^\alpha)\leq 91(\frac{2^{4(k-1)}-2^{4r}}{15})+r_{CE}(C_{r,u}^\alpha)$. Finally, we prove that the result holds for k, i.e., $r_{CE}(C_{k,u}^\alpha)\leq 91(\frac{2^{4k}-2^{4r}}{15})+r_{CE}(C_{r,u}^\alpha)$.

According to Proposition 1 and Theorem 2, we have that

$$r_{CE}(C_{k,u}^\alpha)\leq r_{CE}(C_{k-1,u}^\alpha)+r_{CE}((\overbrace{1\cdots1}^{2^{4(k-1)}}\overbrace{u\cdots u}^{2^{4(k-1)}}$$
$$\overbrace{1+u\cdots1+u}^{2^{4(k-1)}}\overbrace{v\cdots v}^{2^{4(k-1)}}\overbrace{uv\cdots uv}^{2^{4(k-1)}}$$
$$\overbrace{(1+u)v\cdots(1+u)v}^{2^{4(k-1)}}\overbrace{1+v\cdots1+v}^{2^{4(k-1)}}$$
$$\overbrace{1+uv\cdots1+uv}^{2^{4(k-1)}}\overbrace{1+(1+u)v\cdots1+(1+u)v}^{2^{4(k-1)}}$$
$$\overbrace{u+v\cdots u+v}^{2^{4(k-1)}}\overbrace{u+uv\cdots u+uv}^{2^{4(k-1)}}$$
$$\overbrace{u+(1+u)v\cdots u+(1+u)v}^{2^{4(k-1)}}$$
$$\overbrace{1+u+v\cdots1+u+v}^{2^{4(k-1)}}$$
$$\overbrace{1+u+uv\cdots1+u+uv}^{2^{4(k-1)}}$$
$$\overbrace{1+u+(1+u)v\cdots1+u+(1+u)v}^{2^{4(k-1)}}))$$
$$\leq 91(2^{4(k-1)}+2^{4(k-2)}+\cdots+2^{4r})$$
$$+r_{CE}(C_{r,u}^\alpha)$$
$$=91(\frac{2^{4k}-2^{4r}}{15})+r_{CE}(C_{r,u}^\alpha).$$

Theorem 7: $r_{CE}(C^{\beta}_{k,u}) \leq 5(\frac{2^{4k}-2^{4r}}{9}) - 13(\frac{2^{2k}-2^{2r}}{9}) + r_{CE}(C^{\beta}_{r,u})$ for $u < r \leq k$.

Proof It is clear that the minimum value of k is 3. Next, we prove this result by induction on k. If $k = 3$, then $k = r$ and $r_{CE}(C^{\beta}_{3,u}) = r_{CE}(C^{\beta}_{3,u})$, which is consistent with the above result. Assume that the result holds for $k-1$, i.e., $r_{CE}(C^{\beta}_{k-1,u}) \leq 5(\frac{2^{4(k-1)}-2^{4r}}{9}) - 13(\frac{2^{2(k-1)}-2^{2r}}{9}) + r_{CE}(C^{\beta}_{r,u})$. Finally, we prove that the result holds for k, i.e., $r_{CE}(C^{\alpha}_{k,u}) \leq 5(\frac{2^{4k}-2^{4r}}{9}) - 13(\frac{2^{2k}-2^{2r}}{9}) + r_{CE}(C^{\beta}_{r,u})$.

According to Proposition 1 and Theorem 2, we have that

$$\begin{aligned} r_{CE}(C^{\beta}_{k,u}) &\leq r_{CE}(C^{\beta}_{k-1,u}) + r_{CE}((\overbrace{1\cdots 1}^{2^{4(k-1)}}\ \overbrace{v\cdots v}^{\frac{2^{4(k-1)}-2^{2(k-1)}}{3}} \\ &\quad \overbrace{1+u+v\cdots 1+u+v}^{\frac{2^{4(k-1)}-2^{2(k-1)}}{3}})), \\ &\leq r_{CE}(C^{\beta}_{k-1,u}) + 4\cdot 2^{4(k-1)} \\ &\quad + 7(\frac{2^{4(k-1)}-2^{2(k-1)}}{3}) + 6(\frac{2^{4(k-1)}-2^{2(k-1)}}{3}) \\ &= r_{CE}(C^{\beta}_{k-1,u}) + \frac{25}{3}\cdot 2^{4(k-1)} - \frac{13}{3}\cdot 2^{2(k-1)} \\ &\leq \frac{25}{3}(2^{4(k-1)}+\cdots+2^{4r}) - \frac{13}{3}(2^{2(k-1)}+\cdots \\ &\quad +2^{2r}) + r_{CE}(C^{\beta}_{r,u}) \\ &= 5(\frac{2^{4k}-2^{4r}}{9}) - 13(\frac{2^{2k}-2^{2r}}{9}) + r_{CE}(C^{\beta}_{r,u}). \end{aligned}$$

VI. CONCLUSION

In this paper, we study some upper bounds on the covering radius of repetition codes, simplex codes and MacDonald codes for Chinese Euclidean distance over $R = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$ with $u^2 = u$, $v^2 = v$, $uv = vu$. Unfortunately, the lower bound on the covering radius of these codes is not given. This will be our follow-up research direction. Furthermore, research on the covering radius of repetition codes, simplex codes and MacDonald codes for different distances over $\mathbb{F}_p + u\mathbb{F}_p + v\mathbb{F}_p + uv\mathbb{F}_p$ will be an open interesting problem in the future, where p is an odd prime.

APPENDIX

Proof of Theorem 1 Let

$$\begin{aligned} \boldsymbol{x}_u &= (\overbrace{1,\ldots,1}^{s}, \overbrace{v,\ldots,v}^{s}, \overbrace{uv,\ldots,uv}^{s}, \overbrace{1+u,\ldots,1+u}^{s}, \\ &\quad \overbrace{1+v,\ldots,1+v}^{s}, \overbrace{1+uv,\ldots,1+uv}^{s}, \overbrace{u+v,\ldots,u+v}^{s}, \\ &\quad \overbrace{u+uv,\ldots,v+uv}^{s}, \overbrace{v+uv,\ldots,v+uv}^{s}, \\ &\quad \overbrace{1+u+v,\ldots,1+u+v}^{s}, \overbrace{1+u+uv,\ldots,1+u+uv}^{s}, \\ &\quad \overbrace{1+v+uv,\ldots,1+v+uv}^{s}, \overbrace{u+v+uv,\ldots,u+v+uv}^{s}, \\ &\quad \overbrace{1+u+v+uv,\ldots,1+u+v+uv}^{n-13s}) \in R^n, s = \lfloor \frac{n}{14} \rfloor. \end{aligned}$$

The code $C_2 = \{(0,\ldots,0), (u,\ldots,u), (uv,\ldots,uv), (u+uv,\ldots,u+uv)\}$ generated by $(u\cdots u)$ is an $(n,4,2n)$ code. Then

$$\begin{aligned} d_{CE}(\boldsymbol{x}_u, (0,\ldots,0)) &= 2n + 32s, \\ d_{CE}(\boldsymbol{x}_u, (u,\ldots,u)) &= 6n - 24s, \\ d_{CE}(\boldsymbol{x}_u, (uv,\ldots,uv)) &= 4n + 4s, \\ d_{CE}(\boldsymbol{x}_u, (u+uv,\ldots,u+uv)) &= 4n + 4s. \end{aligned}$$

Therefore,

$$d_{CE}(\boldsymbol{x}_u, C_2) = \min\{2n+32s, 6n-24s, 4n+4s\}.$$

According to the definition of covering radius, it follows that

$$r_{CE}(C_2) \geq 2n + 32s.$$

Let $\boldsymbol{y}$ be any element of R^n with ω_0 coordinates as 0's, ω_1 coordinates as 1's, ω_2 coordinates as u's, ω_3 coordinates as v's, ω_4 coordinates as uv's, ω_5 coordinates as $(1+u)$'s, ω_6 coordinates as $(1+v)$'s, ω_7 coordinates as $(1+uv)$'s, ω_8 coordinates as $(u+v)$'s, ω_9 coordinates as $(u+uv)$'s, ω_{10} coordinates as $(v+uv)$'s, ω_{11} coordinates as $(1+u+vu)$'s, ω_{12} coordinates as $(1+u+uv)$'s, ω_{13} coordinates as $(1+v+uv)$'s, ω_{14} coordinates as $(u+v+uv)$'s, ω_{15} coordinates as $(1+u+v+uv)$'s. Then $\sum_{i=0}^{15}\omega_i = n$. Since $C_2 = \{(0,\ldots,0), (u,\ldots,u), (uv,\ldots,uv), (u+uv,\ldots,u+uv)\}$, then we get that

$$\begin{aligned} &d_{CE}(\boldsymbol{y}, (0,\ldots,0)) \\ &\ = n - \omega_0 + 7\omega_1 + 3\omega_2 + 3\omega_3 + \omega_4 \\ &\quad + 3\omega_5 + 3\omega_6 + 5\omega_7 + 3\omega_8 + \omega_9 + \omega_{10} + 3\omega_{11} + 5\omega_{12} \\ &\quad + 5\omega_{13} + 5\omega_{14} + \omega_{15}, \\ &d_{CE}(\boldsymbol{y}, (u,\ldots,u)) \\ &\ = n + 3\omega_0 + 3\omega_1 - \omega_2 + 3\omega_3 + \omega_4 \\ &\quad 7\omega_5 + 3\omega_6 + 5\omega_7 + 3\omega_8 + \omega_9 + 5\omega_{10} + 3\omega_{11} + 5\omega_{12} \\ &\quad + \omega_{13} + \omega_{14} + 5\omega_{15}, \\ &d_{CE}(\boldsymbol{y}, (uv,\ldots,uv)) \\ &\ = n + \omega_0 + 5\omega_1 + \omega_2 + \omega_3 - \omega_4 \\ &\quad 5\omega_5 + 5\omega_6 + 7\omega_7 + 5\omega_8 + 3\omega_9 + 3\omega_{10} + \omega_{11} + 3\omega_{12} \\ &\quad + 3\omega_{13} + 3\omega_{14} + 3\omega_{15}, \\ &d_{CE}(\boldsymbol{y}, (u+uv,\ldots,u+uv)) \\ &\ = n + \omega_0 + 5\omega_1 + \omega_2 \\ &\quad + 5\omega_3 + 3\omega_4 + 5\omega_5 + \omega_6 + 3\omega_7 + \omega_8 - \omega_9 + 3\omega_{10} \\ &\quad + 5\omega_{11} + 7\omega_{12} + 3\omega_{13} + 3\omega_{14} + 3\omega_{15}. \end{aligned}$$

Therefore,

$$\begin{aligned} &d_{CE}(\boldsymbol{y}, C_2) \\ &\ = \min\{d_{CE}(\boldsymbol{y}, (0,\ldots,0)), d_{CE}(\boldsymbol{y}, (u,\ldots,u)), \\ &\quad d_{CE}(\boldsymbol{y}, (uv,\ldots,uv)), d_{CE}(\boldsymbol{y}, (u+uv,\ldots,u+uv))\} \leq 6n. \end{aligned}$$

As a consequence,

$$2n + 32\lfloor \frac{n}{14} \rfloor \leq r_{CE}(C_2) \leq 6n.$$

Similar to the proof process of $r_{CE}(C_2)$, we can get $2n + 32\lfloor\frac{n}{14}\rfloor \leq r_{CE}(C_3) \leq 5n$, $2n + 34\lfloor\frac{n}{14}\rfloor \leq r_{CE}(C_4) \leq 7n$, $60\lfloor\frac{n}{14}\rfloor \leq r_{CE}(C_5) \leq 6n$, $60\lfloor\frac{n}{14}\rfloor \leq r_{CE}(C_6) \leq 6n$, $58\lfloor\frac{n}{14}\rfloor \leq r_{CE}(C_7) \leq 5n$, $2n + 32\lfloor\frac{n}{14}\rfloor \leq r_{CE}(C_8) \leq 6n$, $2n+34\lfloor\frac{n}{14}\rfloor \leq r_{CE}(C_9) \leq 7n$, $2n+34\lfloor\frac{n}{14}\rfloor \leq r_{CE}(C_{10}) \leq 7n$, $60\lfloor\frac{n}{14}\rfloor \leq r_{CE}(C_{11}) \leq 6n$, $58\lfloor\frac{n}{14}\rfloor \leq r_{CE}(C_{12}) \leq 5n$, $58\lfloor\frac{n}{14}\rfloor \leq r_{CE}(C_{13}) \leq 5n$, $2n + 30\lfloor\frac{n}{14}\rfloor \leq r_{CE}(C_{14}) \leq 5n$, $2n + 34\lfloor\frac{n}{14}\rfloor \leq r_{CE}(C_{15}) \leq 7n$, so we omit them.

Let z be any element of R^n. Then we have that

$$\begin{aligned}
&d_{CE}(z, (0, \ldots, 0)) \\
&= n - \omega_0 + 7\omega_1 + 3\omega_2 + 3\omega_3 + \omega_4 \\
&\quad + 3\omega_5 + 3\omega_6 + 5\omega_7 + 3\omega_8 + \omega_9 + \omega_{10} + 3\omega_{11} + 5\omega_{12} \\
&\quad + 5\omega_{13} + 5\omega_{14} + \omega_{15} = A_\omega,
\end{aligned}$$

$$\begin{aligned}
&d_{CE}(z, (1, \ldots, 1)) \\
&= n + 7\omega_0 - \omega_1 + 3\omega_2 + 3\omega_3 + 5\omega_4 \\
&\quad + 3\omega_5 + 3\omega_6 + \omega_7 + 3\omega_8 + 5\omega_9 + 5\omega_{10} + 3\omega_{11} + \omega_{12} \\
&\quad + \omega_{13} + \omega_{14} + 5\omega_{15} = B_\omega,
\end{aligned}$$

$$\begin{aligned}
&d_{CE}(z, (u, \ldots, u)) \\
&= n + 3\omega_0 + 3\omega_1 - \omega_2 + 3\omega_3 + \omega_4 \\
&\quad + 7\omega_5 + 3\omega_6 + 5\omega_7 + 3\omega_8 + \omega_9 + 5\omega_{10} + 3\omega_{11} + 5\omega_{12} \\
&\quad + \omega_{13} + \omega_{14} + 5\omega_{15} = D_\omega,
\end{aligned}$$

$$\begin{aligned}
&d_{CE}(z, (v, \ldots, v)) \\
&= n + 3\omega_0 + 3\omega_1 + 3\omega_2 - \omega_3 + \omega_4 \\
&\quad + 3\omega_5 + 7\omega_6 + 5\omega_7 + 3\omega_8 + 5\omega_9 + \omega_{10} + 3\omega_{11} + \omega_{12} \\
&\quad + 5\omega_{13} + \omega_{14} + 5\omega_{15} = E_\omega.
\end{aligned}$$

$$\begin{aligned}
&d_{CE}(z, (uv, \ldots, uv)) \\
&= n + \omega_0 + 5\omega_1 + \omega_2 + \omega_3 - \omega_4 \\
&\quad + 5\omega_5 + 5\omega_6 + 7\omega_7 + 5\omega_8 + 3\omega_9 + 3\omega_{10} + \omega_{11} + 3\omega_{12} \\
&\quad + 3\omega_{13} + 3\omega_{14} + 3\omega_{15} = F_\omega.
\end{aligned}$$

$$\begin{aligned}
&d_{CE}(z, (1 + u, \ldots, 1 + u)) \\
&= n + 3\omega_0 + 3\omega_1 + 7\omega_2 \\
&\quad + 3\omega_3 + 5\omega_4 - \omega_5 + 3\omega_6 + \omega_7 + 3\omega_8 + 5\omega_9 + \omega_{10} \\
&\quad + 3\omega_{11} + \omega_{12} + 5\omega_{13} + 5\omega_{14} + \omega_{15} = G_\omega.
\end{aligned}$$

$$\begin{aligned}
&d_{CE}(z, (1 + v, \ldots, 1 + v)) \\
&= n + 3\omega_0 + 3\omega_1 + 3\omega_2 \\
&\quad + 7\omega_3 + 5\omega_4 + 3\omega_5 - \omega_6 + \omega_7 + 3\omega_8 + \omega_9 + 5\omega_{10} \\
&\quad + 3\omega_{11} + 5\omega_{12} + \omega_{13} + 5\omega_{14} + \omega_{15} = H_\omega.
\end{aligned}$$

$$\begin{aligned}
&d_{CE}(z, (1 + uv, \ldots, 1 + uv)) \\
&= n + 5\omega_0 + \omega_1 + 5\omega_2 \\
&\quad + 5\omega_3 + 7\omega_4 + \omega_5 + \omega_6 - \omega_7 + \omega_8 + 3\omega_9 + 3\omega_{10} \\
&\quad + 5\omega_{11} + 3\omega_{12} + 3\omega_{13} + 3\omega_{14} + 3\omega_{15} = I_\omega.
\end{aligned}$$

$$\begin{aligned}
&d_{CE}(z, (u + v, \ldots, u + v)) \\
&= n + 3\omega_0 + 3\omega_1 + 3\omega_2 \\
&\quad + 3\omega_3 + 5\omega_4 + 3\omega_5 + 3\omega_6 + \omega_7 - \omega_8 + \omega_9 + \omega_{10} \\
&\quad + 7\omega_{11} + 5\omega_{12} + 5\omega_{13} + \omega_{14} + 5\omega_{15} = J_\omega.
\end{aligned}$$

$$\begin{aligned}
&d_{CE}(z, (u + uv, \ldots, u + uv)) \\
&= n + \omega_0 + 5\omega_1 + \omega_2 \\
&\quad + 5\omega_3 + 3\omega_4 + 5\omega_5 + \omega_6 + 3\omega_7 + \omega_8 - \omega_9 + 3\omega_{10} \\
&\quad + 5\omega_{11} + 7\omega_{12} + 3\omega_{13} + 3\omega_{14} + 3\omega_{15} = K_\omega.
\end{aligned}$$

$$\begin{aligned}
&d_{CE}(z, (v + uv, \ldots, v + uv)) \\
&= n + \omega_0 + 5\omega_1 + 5\omega_2 \\
&\quad + \omega_3 + 3\omega_4 + \omega_5 + 5\omega_6 + 3\omega_7 + \omega_8 + 3\omega_9 - \omega_{10} \\
&\quad + 5\omega_{11} + 3\omega_{12} + 7\omega_{13} + 3\omega_{14} + 3\omega_{15} = L_\omega.
\end{aligned}$$

$$\begin{aligned}
&d_{CE}(z, (1 + u + v, \ldots, 1 + u + v)) \\
&= n + 3\omega_0 + 3\omega_1 \\
&\quad + 3\omega_2 + 3\omega_3 + \omega_4 + 3\omega_5 + 3\omega_6 + 5\omega_7 + 7\omega_8 + 5\omega_9 \\
&\quad + 5\omega_{10} - \omega_{11} + \omega_{12} + \omega_{13} + 5\omega_{14} + \omega_{15} = M_\omega.
\end{aligned}$$

$$\begin{aligned}
&d_{CE}(z, (1 + u + uv, \ldots, 1 + u + uv)) \\
&= n + 5\omega_0 + \omega_1 \\
&\quad + 5\omega_2 + \omega_3 + 3\omega_4 + \omega_5 + 5\omega_6 + 3\omega_7 + 5\omega_8 + 7\omega_9 \\
&\quad + 3\omega_{10} + \omega_{11} - \omega_{12} + 3\omega_{13} + 3\omega_{14} + 3\omega_{15} = R_\omega.
\end{aligned}$$

$$\begin{aligned}
&d_{CE}(z, (1 + v + uv, \ldots, 1 + v + uv)) \\
&= n + 5\omega_0 + \omega_1 \\
&\quad + \omega_2 + 5\omega_3 + 3\omega_4 + 5\omega_5 + \omega_6 + 3\omega_7 + 5\omega_8 + 3\omega_9 \\
&\quad + 7\omega_{10} + \omega_{11} + 3\omega_{12} - \omega_{13} + 3\omega_{14} + 3\omega_{15} = T_\omega.
\end{aligned}$$

$$\begin{aligned}
&d_{CE}(z, (u + v + uv, \ldots, u + v + uv)) \\
&= n + 5\omega_0 + \omega_1 \\
&\quad + \omega_2 + \omega_3 + 3\omega_4 + 5\omega_5 + 5\omega_6 + 3\omega_7 + \omega_8 + 3\omega_9 \\
&\quad + 3\omega_{10} + 5\omega_{11} + 3\omega_{12} + 3\omega_{13} - \omega_{14} + 7\omega_{15} = P_\omega.
\end{aligned}$$

$$\begin{aligned}
&d_{CE}(z, (1 + u + v + uv, \ldots, 1 + u + v + uv)) \\
&= n + \omega_0 \\
&\quad + 5\omega_1 + 5\omega_2 + 5\omega_3 + 3\omega_4 + \omega_5 + \omega_6 + 3\omega_7 + 5\omega_8 + 3\omega_9 \\
&\quad + 3\omega_{10} + 3\omega_{11} + 3\omega_{12} + 3\omega_{13} + 7\omega_{14} - \omega_{15} = Q_\omega.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&d_{CE}(z, C_1) \\
&= \min\{A_\omega, B_\omega, D_\omega, E_\omega, F_\omega, G_\omega, H_\omega, I_\omega, J_\omega, \\
&\quad K_\omega, L_\omega, M_\omega, R_\omega, T_\omega, P_\omega, Q_\omega\} \leq 4n.
\end{aligned}$$

Let

$$\begin{aligned}
&\boldsymbol{x}_1 \\
&= (\overbrace{0, \ldots, 0}^{t}, \overbrace{1, \ldots, 1}^{t}, \overbrace{u, \ldots, u}^{t}, \overbrace{v, \ldots, v}^{t}, \overbrace{uv, \ldots, uv}^{t}, \\
&\quad \overbrace{1 + u, \ldots, 1 + u}^{t}, \overbrace{1{+}v, \ldots, 1{+}v}^{t}, \overbrace{1 + uv, \ldots, 1 + uv}^{t}, \\
&\quad \overbrace{u + v, \ldots, u + v}^{t}, \overbrace{u{+}uv, \ldots, v{+}uv}^{t}, \overbrace{v + uv, \ldots, v + uv}^{t}, \\
&\quad \overbrace{1 + u + v, \ldots, 1 + u + v}^{t}, \overbrace{1 + u + uv, \ldots, 1 + u + uv}^{t}, \\
&\quad \overbrace{1 + v + uv, \ldots, 1 + v + uv}^{t}, \overbrace{u + v + uv, \ldots, u + v + uv}^{t}, \\
&\quad \overbrace{1 + u + v + uv, \ldots, 1 + u + v + uv}^{n-13s}) \in R^n, \quad t = \lfloor\frac{n}{16}\rfloor.
\end{aligned}$$

Then we have that

$$\begin{aligned}
d_{CE}(\boldsymbol{x}_1,(0,\ldots,0)) &= 2n+32t,\\
d_{CE}(\boldsymbol{x}_1,(1,\ldots,1)) &= 6n-32t,\\
d_{CE}(\boldsymbol{x}_1,(u,\ldots,u)) &= 6n-32t,\\
d_{CE}(\boldsymbol{x}_1,(v,\ldots,v)) &= 6n-32t.\\
d_{CE}(\boldsymbol{x}_1,(uv,\ldots,uv)) &= 4n,\\
d_{CE}(\boldsymbol{x}_1,(1+u,\ldots,1+u)) &= 2n+32t,\\
d_{CE}(\boldsymbol{x}_1,(1+v,\ldots,1+v)) &= 2n+32t,\\
d_{CE}(\boldsymbol{x}_1,(1+uv,\ldots,1+uv)) &= 4n,\\
d_{CE}(\boldsymbol{x}_1,(u+v,\ldots,u+v)) &= 6n-32t,\\
d_{CE}(\boldsymbol{x}_1,(u+uv,\ldots,u+uv)) &= 4n,\\
d_{CE}(\boldsymbol{x}_1,(v+uv,\ldots,v+uv)) &= 4n,\\
d_{CE}(\boldsymbol{x}_1,(1+u+v,\ldots,1+u+v)) &= 2n+32t,\\
d_{CE}(\boldsymbol{x}_1,(1+u+uv,\ldots,1+u+uv)) &= 4n,\\
d_{CE}(\boldsymbol{x}_1,(1+v+uv,\ldots,1+v+uv)) &= 4n,\\
d_{CE}(\boldsymbol{x}_1,(u+v+uv,\ldots,u+v+uv)) &= 8n-64t,\\
d_{CE}(\boldsymbol{x}_1,(1+u+v+uv,\ldots,1+u+v+uv)) &= 64t.
\end{aligned}$$

Therefore,

$$\begin{aligned}
d_{CE}(\boldsymbol{x}_1,C_1) &= \min\{2n+32t,6n-32t,4n,8n-64t,64t\}\\
&\geq 64t.
\end{aligned}$$

This means that $64\lfloor\frac{n}{16}\rfloor \leq r_{CE}(C_1) \leq 4n$.

Next, we give the Magma calculation program for $r_{CE}(S_1^{\alpha})$. Similarly, we can get $r_{CE}(S_2^{\beta})$.

```
procedure inc_adic(~v,n,adic)
v[n]:=v[n]+1;
for i:=0~to n-2~do
if v[n-i] ne adic then
break;
end if;
v[n-i -1]:=v[n-i -1]+1;
v[n-i ]:=0;
end for;
end procedure;
F:=GF(2);
P<u,v>:=PolynomialRing(F,2);
R<u,v>:=quo<P|u^2-u,v^2-v,u*v-v*u>;
function weight(c,n)
wt:=0;
for i:=1~to n do
if c[i] in [u,1+u,v,1+v,u+v,1+u+v]
then wt+:=4;
elif c[i] in [u*v,(1+u)*v,u+u*v,
1+u+(1+u)*v] then wt+:=2;
elif c[i] in [1+u*v,1+(1+u)*v,u+(1+u)*v,
1+u+u*v] then wt+:=6;
elif c[i] eq $1$~then wt+:=8;
end if;
end for;
return wt;
end function;
function weight(c)
wt:=0;
if c in [u,1+u,v,1+v,u+v,1+u+v]
then wt:=4;
elif c in [u*v,(1+u)*v,u+u*v,1+u+(1+u)*v]
then wt:=2;
elif c in [1+u*v,1+(1+u)*v,u+(1+u)*v,
1+u+u*v] then wt:=6;
elif c eq $1$~then wt:=8;
end if;
return wt;
end function;
Rset:=[R|0,1,u,1+u,v,u*v,(1+u)*v,1+v,
1+u*v,1+(1+u)*v,u+v,u+u*v,u+(1+u)*v,
1+u+v,1+u+u*v,1+u+(1+u)*v];
n:=#R;
G:=Rset;
C:=[];
for r in Rset do
c:=[];
for i:=1~to n do
c[i]:=r*G[i];
end for;
Include(~C,c);
end for;
vec:=[];
for i:=1~to n-1~do
vec[i]:=0;
end for;
vec[n]:=1;
RC:=0;
while vec[1] ne n do
y:=[];
for i:=1~to n do
y[i]:=Rset[vec[i]+1];
end for;
if y in C then
inc_adic(~vec,n,n);
continue;
end if;
dyc:=8*n;
for c in C do
d:=0;
for i:=1~to n do
d+:=weight(y[i]-c[i]);
end for;
if d lt dyc then
dyc:=d;
end if;
end for;
if dyc gt RC then
RC:=dyc;
printf "R(C)=
for i:=1~to n do
```

```
printf "
end for;
printf "]\n";
end if;
inc_adic(~vec,n,n);
end while;
```

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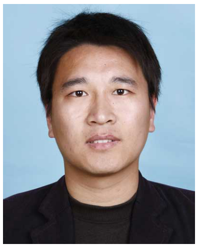

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