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Partial Anti-Synchronization in a Class of Chaotic and Hyper-Chaotic Systems

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ABSTRACT This paper investigates the partial anti-synchronization in a class of chaotic and hyper-chaotic systems. Firstly, the existence of the partial anti-synchronization for the chaotic system is proved. By a systematic method including two algorithms, all solutions of the partial anti-synchronization for a given chaotic system are then derived, and physical controllers are designed. Finally, some illustrative examples with numerical simulations are used to verify the validity and effectiveness of the theoretical results.

INDEX TERMS Partial, anti-synchronization, dynamic feedback control, existence, algorithm.

I. INTRODUCTION

It was well known that anti-synchronization in chaotic oscillators was found to be interesting. Anti-synchronization is essentially the same type of synchronism first investigated by Pecora and Carroll [1]. The difference is that anti-synchronization allows for the coexistence of chaotic attractors which are symmetric to each other. Anti-synchronization is a phenomenon whereby the state vectors of the slaved systems have the same amplitude but opposite signs as those of the master system. Therefore, the sum of two signals are expected to converge to zero when anti-synchronization appears. Anti-synchronization has been found distinctive applications in some fields. For example, in communication system, system's security and secrecy can be deeply strengthened by transforming from complete synchronization and anti-synchronization periodically in the process of digital signal transmission [2]. Hence, further research about anti-synchronization should be concerned in both theory and practice. It has been reported that anti-phase synchronization can theoretically occur in a subsystem of hyper-chaotic systems with symmetry, we firstly introduce the main results in [3] in next.

Consider an N -dimensional continuous flow

$$\dot{z} = F(z, p) \quad (1)$$

with the decomposition of the system into a driving subsystem x (dimension N_x) and a subsystem y to be synchronized

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(dimension N_y , where $N_x + N_y = N$), i.e.,

$$\dot{x} = f(x), \quad (2)$$

$$\dot{y} = h(x, p)G(y), \quad (3)$$

where $f(x)$ is a continuous vector function that generates a chaotic attractor, $h(x, p)$ is a scalar driving function, and $G(y)$ is a vector function that possesses symmetry. For simplicity, the reflecting symmetry in $G(y)$ is considered, i.e., $G(-y) = -G(y)$.

The replica of the subsystem to be synchronized is

$$\dot{y}' = h(x, p)G(y'), \quad (4)$$

where $y' \in N_y$ is the state. By choosing initial conditions of the System (3) and System (4), the result: $y(t) \rightarrow -y'(t)$, as $t \rightarrow \infty$ is ensued.

From then on, a variety of results have been obtained to realize the anti-synchronization in the chaotic and hyper-chaotic systems, such as, anti-synchronization between two identical systems, even anti-synchronization between two different systems, see Refs. [4]–[26] and the references therein. However, there is an important fundamental problem that has not been fully considered, i.e., the existence of the anti-synchronization for a given chaotic system was not put forward. In [22], we firstly gave a necessary and sufficient condition for the existence of the anti-synchronization for a given chaotic system. That is, for a given chaotic system: $\dot{q}_m = H(q_m)$, $q_m \in R^n$ is the state variable, its existence of the anti-synchronization occurs if and only if $H(-q_m) = -H(q_m)$. If this condition is satisfied, the corresponding slave

system is: $\dot{q}_s = H(q_s) + B_q u_q$, where $B_q \in R^r$, $u_q \in R^r$, $r \geq 1$, is the designed controller with $u_q = u_q(x, e)$, $u_q(x, 0) = 0$ and $e = q_m + q_s$. That is to say, $e = 0$ should be the equilibrium point of the uncontrolled sum system: $\dot{e} = H(q_s) + H(q_m)$, which ensures the feasibility of design physical controller $u_q = u_q(q_m, e)$. Without this condition, although the anti-synchronization between two identical systems or different systems has been realized, the obtained controllers were not both simple and physical, and the condition $u_q(q_m, 0) = 0$ was not satisfied inevitably.

Let's take the results in [8] for example. Consider the following Chen chaotic system

$$\begin{aligned} \dot{x}_1 &= a(y_1 - x_1), \\ \dot{y}_1 &= (c - a)x_1 - x_1z_1 + cy_1, \\ \dot{z}_1 &= x_1y_1 - bz_1, \end{aligned} \tag{5}$$

where $a = 35, b = 32, c = 28$.

The slave Chen system is given as

$$\begin{aligned} \dot{x}_2 &= a(y_2 - x_2) + U_1, \\ \dot{y}_2 &= (c - a)x_2 - x_2z_2 + cy_2 + U_2, \\ \dot{z}_2 &= x_2y_2 - bz_2 + U_3, \end{aligned} \tag{6}$$

and the controller $U = (U_1, U_2, U_3)^T$ is designed as

$$\begin{aligned} U_1 &= -e_1, \\ U_2 &= x_2z_2 + x_1z_1 - e_1, \\ U_3 &= -x_1y_1 - x_2y_2 - e_3, \end{aligned} \tag{7}$$

where $e = x + y$.

Since, $x_2 = e_1 - x_1, z_2 = e_3 - z_1$, and then

$$U_2 = x_2z_2 + x_1z_1 - e_1 = e_1e_3 - x_1e_3 - z_1e_1 + 2x_1z_1. \tag{8}$$

Obviously,

$$U_2(x_1, y_1, z_1, 0) = 2x_1z_1 \neq 0. \tag{9}$$

In addition, $U = (U_1, U_2, U_3)^T$ is a three input controller, which is not a physical controller since the single input controller is usually designed in applications. In other words, there is only one control channel for a given system whose dimension is not greater than three.

As a matter of fact, only few chaotic and hyper-chaotic systems satisfy the condition of existence of anti-synchronization. Such as, the modified Chua system, see Ref. [22]. But, the famous Lorenz system does not satisfy this condition. For a given chaotic or hyper-chaotic system, it is natural to find a subsystem which can satisfies the condition of existence of anti-synchronization for a given chaotic and hyper-chaotic system. This question is not only essential in theory but also significant in applications, which push us to investigate this question. Furthermore, if this condition is satisfied, how many solutions of this question can be derived. Therefore, for a given chaotic or hyper-chaotic system, it is critical to have a systematic method that can be used for proving the existence of the partial anti-phase synchronization, deriving all solutions for this type of synchronization

phenomenon and then designing the corresponding physical controller. To address this critical problem, this study is to develop such a systematic method that can be applied to a class of chaotic and hyper-chaotic systems. What's more, how to design the simple and physical controllers is also important. Note that the dynamic feedback control method has some advantages over other methods, it is also used preferentially in this paper. And notice that the subsystem of a given chaotic or hyper-chaotic system is usually simple, thus the linear feedback control method is also considered in some cases.

Motivated by the above conclusions, we study the existence of partial anti-synchronization problem in a class of chaotic and hyper-chaotic systems by the dynamic control method and linear feedback control method. The main contributions of this paper are listed as follows:

- 1) The existence of the partial anti-synchronization problem in such system is firstly proved.
- 2) Two algorithms are proposed to obtain the solutions of the partial anti-synchronization for this chaotic system.
- 3) Physical controllers are designed to realize the existence of anti-synchronization in such system.

The rest of this paper is organized as follows. Section 2 introduces the preliminary knowledge, and Section 3 presents the problem formation. Section 4 presents the main results of this paper, Section 5 provides the illustrative examples with numerical simulation, and Section 6 gives the conclusions.

II. PRELIMINARY

Consider the following chaotic system:

$$\dot{x} = f(x), \tag{10}$$

where $x \in R^n$ is the state, $f(x) \in R^n$ is a continuous vector function, i.e.,

$$x = \begin{pmatrix} X \\ Z \end{pmatrix}, f(x) = \begin{pmatrix} G_1(X, Z) \\ G_2(X, Z) \end{pmatrix}, \tag{11}$$

$X \in R^m, Z \in R^{n-m}, m \geq 1, G_1(X, Z) \in R^m, G_1(-X, Z) = -G_1(X, Z)$ and $G_2(X, Z) \in R^{n-m}$.

The system (10) is rewritten as follows

$$\dot{X} = G_1(X, Z), \tag{12}$$

$$\dot{Z} = G_2(X, Z). \tag{13}$$

Let the system (10) be the master system, then the slave system is described as

$$\dot{y} = f(y) + Bu, \tag{14}$$

where $y \in R^n$ is the state, $f(y) \in R^n$ is a continuous vector function, $B \in R^{n \times r}$ is a constant matrix, $r \geq 1, u \in R^r$ is the controller to be designed, i.e.,

$$y = \begin{pmatrix} Y \\ Z \end{pmatrix}, f(y) = \begin{pmatrix} G_1(Y, Z) \\ G_2(Y, Z) \end{pmatrix}, \tag{15}$$

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{16}$$

where $Y \in R^m$, $Z \in R^{n-m}$, $m \geq 1$, $G_1(Y, Z) \in R^m$, $G_1(-Y, Z) = -G_1(Y, Z)$ and $G_2(Y, Z) \in R^{n-m}$, $B_1 \in R^{m \times r}$, $B_2 \in R^{(n-m) \times r}$.

The system (14) is rewritten as follows

$$\dot{Y} = G_1(Y, Z) + B_1 u, \tag{17}$$

$$\dot{Z} = G_2(X, Z) + B_2 u. \tag{18}$$

Let $E = X + Y$, then the sum system is described as follows

$$\dot{E} = G_1(Y, Z) + G_1(X, Z) + B_1 u, \tag{19}$$

where $e \in R^m$ is the state, B_1 is given in (16), u is the designed controller.

Next, a definition is presented as follows.

Definition 1 [4]: Consider the sum system (19). If $\lim_{t \rightarrow \infty} \|E(t)\| = 0$, then the master system (12) and the slave system (17) are called to be achieved the anti-synchronization, which implies the partial anti-synchronization of the master system (10) and the slave system (14) is realized.

Remark 1: For the subsystem (12), if $G_1(X, Z) = M(Z)X$, where $M(Z) \in R^{m \times m}$ is a matrix about Z , then projective synchronization between the master system (10) and the slave system (14) can be realized. according to our previous results in [27]. Moreover, considering system (10), if the projective synchronization exists, then the partial anti-synchronization of such a system can be realized. However, the reverse is not necessarily true.

For the development of this paper, the dynamic feedback control method is introduced at first.

Lemma 1 [27]: Consider the following system:

$$\dot{x} = h(x) + bu, \tag{20}$$

where $x \in R^n$ is the state, $h(x) \in R^n$ is a vector function with $h(0) = 0$, $b \in R^{n \times r}$ is a constant matrix, $r \geq 1$, $u \in R^r$ is the controller to be designed. If $(h(x), b)$ can be stabilized, then a dynamic feedback controller u is designed as:

$$u = Kx, \tag{21}$$

where $K = k(t)b^T$, and the feedback gain $k(t)$ is updated by the following law:

$$\dot{k}(t) = -\gamma \|x(t)\|^2, \tag{22}$$

where $\gamma > 0$.

III. PROBLEM FORMULATION

Consider the following chaotic system:

$$\dot{z} = F(z), \tag{23}$$

where $z \in R^n$, $F(z) \in R^n$ is a vector function.

The main goal of this paper is to investigate the partial anti-synchronization for the given chaotic system (23) in the following three aspects:

- 1) The existence of the partial anti-synchronization.
- 2) The solutions of the partial anti-synchronization for this chaotic system.

- 3) The implementation of the partial anti-synchronization, i.e., a simple and physically implementable controller is designed for such a problem.

IV. MAIN RESULT

A. THE EXISTENCE OF THE PARTIAL ANTI-SYNCHRONIZATION

In this subsection, the existence of the partial anti-synchronization for a given chaotic system is presented.

Theorem 1: Consider the chaotic System (23). If there exists a non-singular transformation matrix T by which the System (23) can be transferred into the System (10), i.e.,

$$x = \begin{pmatrix} X \\ Z \end{pmatrix} = Tz, f(x) = \begin{pmatrix} G_1(X, Z) \\ G_2(X, Z) \end{pmatrix} = TF(z), \tag{24}$$

then, the System (23) achieves partial anti-synchronization by the controllers u in the following form:

$$u = u(X, Z, E), u(X, Z, 0) = 0. \tag{25}$$

Proof: If the condition of this theorem is satisfied, then $E = 0$ is the equilibrium point of the following system:

$$\dot{E} = G_1(Y, Z) + G_1(X, Z) = G_1(E - X, Z) + G_1(X, Z). \tag{26}$$

Thus, the controller u given in Equation (25) can stabilize the sum System (19). It means that the partial anti-synchronization in the System (23) is realized.

B. THE SOLUTIONS OF THE PARTIAL ANTI-SYNCHRONIZATION

How to find the non-singular matrix T to transfer the System (23) into the System (10) naturally arises. Then, if this matrix T exists, how many matrices like T can be obtained? There are two cases are discussed in this paper.

Case 1: the following whole Equation (29) has solutions.

For the System (23), if the following algebraic equation about α :

$$F(\alpha z) \equiv \alpha F(z), \tag{27}$$

where α is given as follows

$$\alpha = \begin{pmatrix} \alpha_1 & 0 & 0 & \dots & 0 \\ 0 & \alpha_2 & 0 & \dots & 0 \\ 0 & 0 & \alpha_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_n \end{pmatrix}, \tag{28}$$

$|\alpha_i| = 1, i \in \Lambda = \{1, 2, \dots, n\}$, i.e.,

$$\begin{cases} F_1(\alpha z) \equiv \alpha_1 F(z), \\ F_2(\alpha z) \equiv \alpha_2 F(z), \\ \vdots \\ F_n(\alpha z) \equiv \alpha_n F(z), \end{cases} \tag{29}$$

has a solution in the following form:

$$\beta^{(s)} = \begin{pmatrix} \alpha_{i_1} \\ \vdots \\ \alpha_{i_{s-1}} \\ \alpha_{i_s} \\ \alpha_{i_{s+1}} \\ \vdots \\ \alpha_{i_n} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \leftarrow s, \quad (30)$$

where $s \geq 1$ is the position of the last $\alpha_{i_j} = -1, i_j \in \Lambda, j = 1, 2, \dots, n$ then, the matrix T can be determined by the following algorithm:

Algorithm 1: Let $k = 1, s$ is the number of α_j , where $\alpha_j = -1, j \in \Lambda,$

$$\min \{j | \alpha_j = -1, j \in \Lambda\} := i_k, \quad (31)$$

where “:=” stands for definition.
while $k \leq s$ do

$$k = k + 1, \quad (32)$$

$$\min_{j \in \Lambda} \{\alpha_j = -1, j \neq i_1, i_2, \dots, i_{k-1}\} := i_k, \quad (33)$$

Thus, let

$$X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_s \end{pmatrix} = \begin{pmatrix} z_{i_1} \\ z_{i_2} \\ \vdots \\ z_{i_s} \end{pmatrix}. \quad (34)$$

Next, let $k = s + 1,$

$$\min \{j | \alpha_j = 1, j \in \Lambda\} := i_k, \quad (35)$$

while $k \leq n$ do

$$k = k + 1, \quad (36)$$

$$\min \{\alpha_j = 1, j \neq i_{s+1}, i_{s+2}, \dots, i_{k-1}, j \in \Lambda\} := i_k, \quad (37)$$

Thus, let

$$Z = \begin{pmatrix} Z_{s+1} \\ Z_{s+2} \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} z_{i_{s+1}} \\ z_{i_{s+2}} \\ \vdots \\ z_{i_n} \end{pmatrix}. \quad (38)$$

By Algorithm 1, the transform T is obtained as follows

$$T = \begin{pmatrix} \delta_n^{i_1} \\ \vdots \\ \delta_n^{i_s} \\ \vdots \\ \delta_n^{i_n} \end{pmatrix}, \quad (39)$$

$$\delta_n^{i_j} = (0 \quad \dots \quad 0 \quad 1 \quad 0 \quad \dots \quad 0) \in R^n, \quad (40)$$

\uparrow
 i_1

where $i_j \in \Lambda, j = 1, 2, \dots, n.$

For example, for the chaotic system: $\dot{z} = F(z), z \in R^3, F(z) \in R^3.$ If $\alpha_1 = -1, \alpha_2 = 1, \alpha_3 = -1,$ then $s = 2, i_1 = 1, i_2 = 3, i_3 = 2.$ By Algorithm 1, we obtain

$$T = \begin{pmatrix} \delta_3^{i_1} \\ \delta_3^{i_2} \\ \delta_3^{i_3} \end{pmatrix} = \begin{pmatrix} \delta_3^1 \\ \delta_3^3 \\ \delta_3^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (41)$$

By $T,$ the system $\dot{z} = F(z)$ is transferred into the following systems:

$$\dot{X} = G_1(X, Z), \quad (42)$$

$$\dot{Z} = G_2(X, Z), \quad (43)$$

and

$$\begin{pmatrix} X \\ Z \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ Z \end{pmatrix} = Tz = \begin{pmatrix} z_1 \\ z_3 \\ z_2 \end{pmatrix}, \quad (44)$$

$$f(x) = \begin{pmatrix} G_1(X, Z) \\ G_2(X, Z) \end{pmatrix} = TF(z) = \begin{pmatrix} F_1(z) \\ F_3(z) \\ F_2(z) \end{pmatrix}. \quad (45)$$

Case 2: the whole Equation (29) has no solution.

In this case, we should find the possible solutions of the part of the Equation (29) by the following algorithm:

Algorithm 2: Let $l = 1,$ where l is the number of $\alpha_j = 1, k$ is the position of the first $\alpha_j = 1, j \in \Lambda, 1 \leq k \leq n,$ we should check

$$\Gamma_k^l = \begin{pmatrix} \Gamma_{k,1}^l \\ \vdots \\ \Gamma_{k,k-1}^l \\ \Gamma_{k,k}^l \\ \Gamma_{k,k+1}^l \\ \vdots \\ \Gamma_{k,n}^l \end{pmatrix} = \begin{pmatrix} -1 \\ \vdots \\ -1 \\ 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} \leftarrow k, \quad (46)$$

is the solution of the following equation:

$$\begin{cases} F_1(\alpha z) \equiv \alpha_1 F(z), \\ \vdots \equiv \vdots \\ F_{k-1}(\alpha z) \equiv \alpha_{k-1} F(z), \\ F_{k+1}(\alpha z) \equiv \alpha_{k+1} F(z), \\ \vdots \equiv \vdots \\ F_n(\alpha z) \equiv \alpha_n F(z), \end{cases} \quad (47)$$

or not. If yes, the matrix T is obtained as follows

$$T = \begin{pmatrix} \delta_n^{i_1} \\ \vdots \\ \delta_n^{i_{k-1}} \\ \delta_n^{i_k} \\ \delta_n^{i_{k+1}} \\ \vdots \\ \delta_n^{i_n} \end{pmatrix} = \begin{pmatrix} \delta_n^1 \\ \vdots \\ \delta_n^{k-1} \\ \delta_n^k \\ \delta_n^{k+1} \\ \vdots \\ \delta_n^n \end{pmatrix}. \quad (48)$$

It is noted that there are $C_n^1 = n$ cases.

Next, $l = 2$, where l is also the number of $\alpha_j = 1$, k is also the position of the first $\alpha_j = 1$, $k + 1$ is the position of the second $\alpha_j = 1$, $j \in \Lambda$, $1 \leq k \leq n - 1$, we should check

$$\Gamma_k^l = \begin{pmatrix} \Gamma_{k,1}^l \\ \Gamma_{k,2}^l \\ \vdots \\ \Gamma_{k,k-1}^l \\ \Gamma_{k,k}^l \\ \Gamma_{k,k+1}^l \\ \Gamma_{k,k+2}^l \\ \vdots \\ \Gamma_{k,n}^l \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \\ 1 \\ 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} \leftarrow k, \quad (49)$$

is the solution of the following equation:

$$\begin{cases} F_1(\alpha z) \equiv \alpha_1 F(z), \\ \vdots \equiv \vdots \\ F_{k-1}(\alpha z) \equiv \alpha_{k-1} F(z), \\ F_{k+2}(\alpha z) \equiv \alpha_{k+2} F(z), \\ \vdots \equiv \vdots \\ F_n(\alpha z) \equiv \alpha_n F(z), \end{cases} \quad (50)$$

or not. If yes, the matrix T is obtained as follows

$$T = \begin{pmatrix} \delta_n^{i_1} \\ \vdots \\ \delta_n^{i_{k-1}} \\ \delta_n^{i_k} \\ \delta_n^{i_{k+1}} \\ \vdots \\ \delta_n^{i_{n-1}} \\ \delta_n^{i_n} \end{pmatrix} = \begin{pmatrix} \delta_n^1 \\ \vdots \\ \delta_n^{k-1} \\ \delta_n^{n-1} \\ \delta_n^n \\ \vdots \\ \delta_n^{k-1} \\ \delta_n^k \end{pmatrix}. \quad (51)$$

It is noted that there are $C_n^2 = \frac{n(n-1)}{2}$ cases. while $l \leq n - 1$ do

The similar procedure in Case: $l = 1$ and Case: $l = 2$ should be done to find the matrix T .

In summarizing, by Algorithm 1 and Algorithm 2, the matrix T is obtained if the whole Equation (29) or the parts of the Equation (29) has solutions.

C. IMPLEMENTATION OF THE PARTIAL ANTI-SYNCHRONIZATION

In this subsection, both simple and physically implementable controllers are designed for such problem.

According to the results in [27], [28], we propose the following conclusion.

Theorem 2: Consider the sum System (19). If $(G_1(Y, Z) + G_1(X, Z), B_1)$ can be stabilized, then the controller u is designed as follows

$$u = KE, \quad (52)$$

where $K = k(t)B_1^T$, and $k(t)$ is updated by the following update law:

$$\dot{k} = -\gamma \|E\|^2, \quad (53)$$

where $\gamma > 0$, it means that the master System (10) and the slave System (14) reach the partial anti-phase synchronization.

Proof: Since $(G_1(Y, Z) + G_1(X, Z), B_1)$ can be stabilized, according to Lemma 1, the controller u is designed in Equation (52).

Especially, if $G_1(X, Z) = M(Z)X$, the following results are obtained.

Theorem 3: Consider the sum System (19). If $(M(Z), B_1)$ can be stabilized, then the controller u is designed as follows

$$u = KE, \quad (54)$$

where $K = k(t)B_1^T$, and $k(t)$ is updated by the following update law

$$\dot{k} = -\gamma \|E\|^2, \quad (55)$$

where $\gamma > 0$, which implies that the master System (10) and the slave System (14) achieve the partial anti-synchronization.

Proof: Since $(M(Z), B_1)$ can be stabilized, according to Lemma 1, the controller u is designed in Equation (54).

Theorem 4: Consider the sum System (19). If $(M(Z), B_1)$ can be stabilized, then the controller u is designed as follows

$$u = -K(Z)E, \quad (56)$$

where $K(Z)$ such that $(M(Z) - B_1K(Z))$ is Hurwitz whatever Z is, i.e., the master System (10) and the slave System (14) achieve the partial anti-synchronization.

V. ILLUSTRATIVE EXAMPLES WITH NUMERICAL SIMULATIONS

In this section, three examples with numerical simulations are used to demonstrate the effectiveness and validity of the proposed results.

Example 1: The Lorenz system [28]

$$\dot{z} = F(z) = \begin{pmatrix} F_1(z) \\ F_2(z) \\ F_3(z) \end{pmatrix} = \begin{pmatrix} 10(z_2 - z_1) \\ 28z_1 - z_2 - z_1z_3 \\ -\frac{8}{3}z_3 + z_1z_2 \end{pmatrix}, \quad (57)$$

where $z \in R^3$ is the state.

According to the algebraic Equation (29), i.e.,

$$\begin{cases} F_1(\alpha z) - \alpha_1 F_1(z) = 10(\alpha_2 - \alpha_1)z_2 \equiv 0, \\ F_2(\alpha z) - \alpha_2 F_2(z) = 28(\alpha_1 - \alpha_2)z_1 \\ -(\alpha_1\alpha_3 - \alpha_2)z_1z_3 \equiv 0, \\ F_3(\alpha z) - \alpha_3 F_3(z) = (\alpha_1\alpha_2 - \alpha_3)z_1z_2 \equiv 0, \end{cases} \quad (58)$$

it results in

$$\begin{cases} \alpha_2 = \alpha_1, \\ \alpha_1\alpha_3 = \alpha_2, \\ \alpha_1\alpha_2 = \alpha_3, \end{cases} \quad (59)$$

Solving the Equation (59), we only obtain one solution:

$$\beta^{(2)} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_{i_1} \\ \alpha_{i_2} \\ \alpha_{i_3} \end{pmatrix}. \quad (60)$$

By Algorithm 1, we obtain

$$T = \begin{pmatrix} \delta_3^{i_1} \\ \delta_3^{i_2} \\ \delta_3^{i_3} \end{pmatrix} = \begin{pmatrix} \delta_3^1 \\ \delta_3^2 \\ \delta_3^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (61)$$

By T , the system Lorenz System (57) is divided into the following systems:

$$\dot{X} = M(Z)X, \quad (62)$$

$$\dot{Z} = G_2(X, Z), \quad (63)$$

where

$$M(Z) = \begin{pmatrix} -10 & 10 \\ 28 - Z & -1 \end{pmatrix}, \quad (64)$$

$$G_2(X, Z) = -\frac{8}{3}Z + X_1X_2, \quad (65)$$

then the slave Lorenz system is given as

$$\dot{Y} = M(Z)Y + B_1u, \quad (66)$$

$$\dot{Z} = G_2(X, Z) + B_2u, \quad (67)$$

where $M(Z)$ is given in Equation (64), $G_2(X, Z)$ is given in Equation (65),

$$B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad B_2 = 0. \quad (68)$$

Let $E = X + Y$, then the sum system is given as

$$\dot{E} = M(Z)E + B_1u, \quad (69)$$

where B_1 is given in Equation (68), u is the controller to be designed.

We can obtain that $(M(Z), B_1)$ is controllable, which implies $(M(Z), B_1)$ can be stabilized, and according to Theorem 3, the controller is designed as

$$u = KE = k(t)B_1^T E = k(t) \begin{pmatrix} 0 & 1 \end{pmatrix} E = k(t)E_2, \quad (70)$$

where $\dot{k}(t) = -\gamma \|E\|^2$ and $\gamma > 0$.

According to Theorem 4, the controller is designed as

$$u = K(Z)E = (Z - 28 \quad 0) E = (Z - 28)E_1. \quad (71)$$

Numerical simulation is done initial conditions: $X_1(0) = 1, X_2(0) = -2, Z_3(0) = 3, Y_1(0) = 5, Y_2(0) = -6$, and $k(0) = -1$. Figure 1 shows that the sum system is asymptotically stable, Figure 2 shows the states X_1, X_2 and the states Y_1, Y_2 , respectively. Figure 3 shows that the feedback gain $k(t)$ tends to constant.

Remark 2: In order to get a good performance of simulation, γ is often chosen a big positive number. In this paper, γ is set 1.

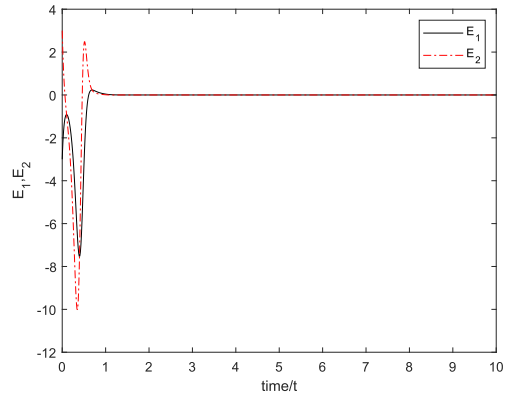


FIGURE 1. The sum system is asymptotically stable.

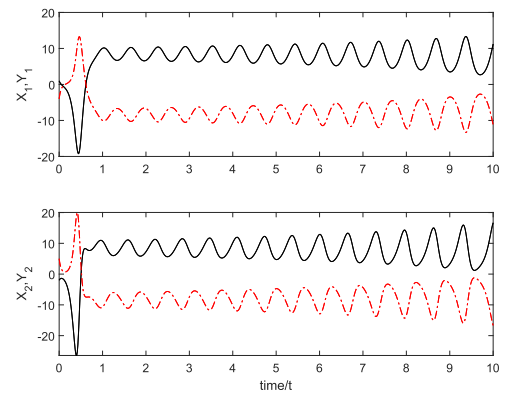


FIGURE 2. The states X_1, X_2 and the states Y_1, Y_2 .

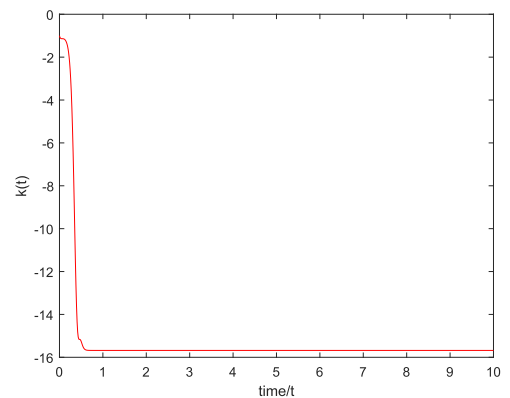


FIGURE 3. $k(t)$ tends to a constant.

Example 2: The Chen-Lee system [29]:

$$\dot{z} = F(z) = \begin{pmatrix} F_1(z) \\ F_2(z) \\ F_3(z) \end{pmatrix} = \begin{pmatrix} -z_2z_3 + 5z_1 \\ z_1z_3 - 10z_2 \\ \frac{1}{3}z_1z_2 - 3.8z_3 \end{pmatrix}, \quad (72)$$

where $z \in R^3$ is the state.

According to the algebraic Equation (29), it results in

$$\begin{cases} \alpha_2\alpha_3 = \alpha_1, \\ \alpha_1\alpha_3 = \alpha_2, \\ \alpha_1\alpha_2 = \alpha_3, \end{cases} \quad (73)$$

We obtain that three solutions of for the Equation (73), i.e.,

$$\beta_1^{(2)} = \begin{pmatrix} \alpha_{i_1} \\ \alpha_{i_2} \\ \alpha_{i_3} \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ \alpha_3 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad (74)$$

$$\beta_2^{(2)} = \begin{pmatrix} \alpha_{i_1} \\ \alpha_{i_3} \\ \alpha_{i_2} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_3 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad (75)$$

$$\beta_3^{(2)} = \begin{pmatrix} \alpha_{i_1} \\ \alpha_{i_2} \\ \alpha_{i_3} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}. \quad (76)$$

For the Case 1: $\alpha_1 = 1, \alpha_2 = \alpha_3 = -1$. By Algorithm 1, we obtain

$$T = \begin{pmatrix} \delta_3^{i_1} \\ \delta_3^{i_2} \\ \delta_3^{i_3} \end{pmatrix} = \begin{pmatrix} \delta_3^2 \\ \delta_3^3 \\ \delta_3^1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (77)$$

By T , the Chen-Lee System (72) is divided into the following systems:

$$\dot{X} = M(Z)X, \quad (78)$$

$$\dot{Z} = G_2(X, Z), \quad (79)$$

where

$$M(Z) = \begin{pmatrix} -10 & Z \\ \frac{1}{3}Z & -3.8 \end{pmatrix}, \quad (80)$$

$$G_2(X, Z) = 5Z - X_1X_2. \quad (81)$$

Next, the slave Chen-Lee system is given as

$$\dot{Y} = M(Z)Y + B_1u, \quad (82)$$

$$\dot{Z} = G_2(X, Z) + B_2u, \quad (83)$$

where $M(Z)$ is given in Equation (80) and $G_2(X, Z)$ is given in Equation (81), and

$$B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad B_2 = 0. \quad (84)$$

We can obtain that $(M(Z), B_1)$ is controllable, which implies $(M(Z), B_1)$ can be stabilized, and according to Theorem 3, the controller is designed as

$$u = KE = k(t)B_1^T E = k(t) \begin{pmatrix} 0 & 1 \end{pmatrix} E = k(t)E_2. \quad (85)$$

According to Theorem 4, the controller is designed as

$$u = K(Z)E = \begin{pmatrix} -\frac{1}{3} & 0 \end{pmatrix} E = -\frac{1}{3}E_1. \quad (86)$$

Numerical simulation is done initial conditions: $X_1(0) = 1, X_2(0) = -2, Z_3(0) = 3, Y_1(0) = -4, Y_2(0) = 5$, and $k(0) = -1$. Figure 4 shows that the sum system is asymptotically stable, Figure 5 shows the states X_1, X_2 and the states Y_1, Y_2 , respectively. Figure 6 shows the feedback gain $k(t)$ tends to a constant.

For the Case 2 and Case 3, we can obtain the corresponding results by similar procedure, thus, we omit them here.

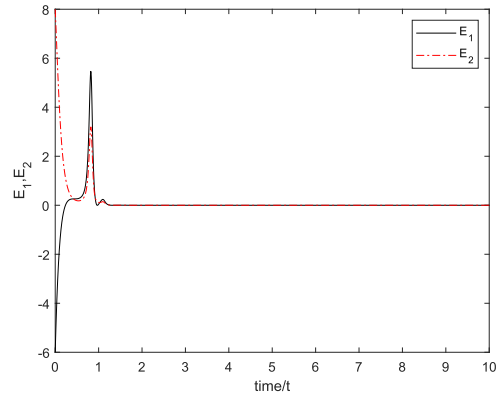


FIGURE 4. The sum system is asymptotically stable.

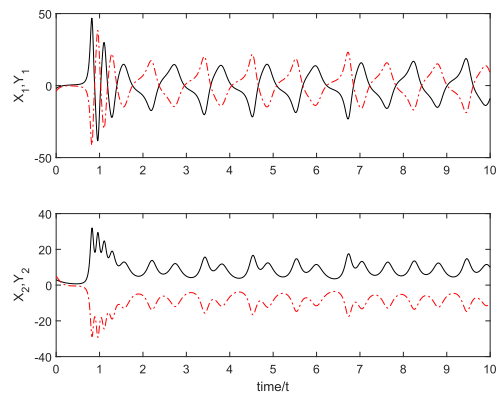


FIGURE 5. The states X_1, X_2 and the states Y_1, Y_2 .

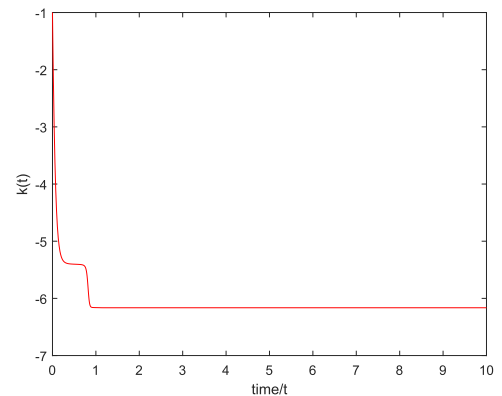


FIGURE 6. $k(t)$ tends to a constant.

Example 3: The hyper-Chen system [31]:

$$\dot{z} = F(z) = \begin{pmatrix} F_1(z) \\ F_2(z) \\ F_3(z) \\ F_4(z) \end{pmatrix} = \begin{pmatrix} -37z_1 + 37z_2 \\ -9z_1 - z_1z_3 + 26z_2 \\ -3z_3 + z_1z_2 + z_1z_3 - z_4 \\ -8z_4 + z_2z_3 - z_1z_3 \end{pmatrix}, \quad (87)$$

where $z \in R^4$ is the state.

According to the algebraic Equation (29), it results in

$$\begin{cases} \alpha_1 = \alpha_2, \\ \alpha_1\alpha_3 = \alpha_2, \\ \alpha_3 = \alpha_4, \\ \alpha_1\alpha_2 = \alpha_3, \\ \alpha_1 = 1, \\ \alpha_2\alpha_3 = \alpha_4, \\ \alpha_1\alpha_3 = \alpha_4. \end{cases} \quad (88)$$

We get

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad (89)$$

is the solo solution of the Equation (88). Thus, the Equation (88) has no solution given in Equation (30).

By Algorithm 2, we obtain

$$\Gamma_1^2 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \quad (90)$$

is the solo solution of the following equation:

$$\begin{cases} \alpha_1 = \alpha_2, \\ \alpha_1\alpha_3 = \alpha_2, \end{cases} \quad (91)$$

then $s = 2, i_1 = 1, i_2 = 2, i_3 = 3, i_4 = 4$. By Algorithm 2, we get

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (92)$$

By T , the hyper-Chen System (87) is divided into the following systems:

$$\dot{X} = M(Z)X, \quad (93)$$

$$\dot{Z} = G_2(X, Z), \quad (94)$$

where

$$M(Z) = \begin{pmatrix} -37 & 27 \\ -9 - Z_3 & 26 \end{pmatrix}, \quad (95)$$

$$G_2(X, Z) = \begin{pmatrix} -3Z_3 + X_1X_2 + X_1Z_3 - Z_4 \\ -38Z_4 + X_2Z_3 - X_1Z_3 \end{pmatrix}. \quad (96)$$

Next, the slave hyper-Chen system is given as

$$\dot{Y} = M(Z)Y + B_1u, \quad (97)$$

$$\dot{Z} = G_2(X, Z) + B_2u, \quad (98)$$

where $M(Z)$ is given in Equation (95) and $G_2(X, Z)$ is given in Equation (96), and

$$B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad B_2 = 0. \quad (99)$$

We can obtain that $(M(Z), B_1)$ is controllable, which implies $(M(Z), B_1)$ can be stabilized, and according to Theorem 3, the controller is designed as

$$u = KE = k(t)B_1^T E = k(t) \begin{pmatrix} 0 & 1 \end{pmatrix} E = k(t)E_2. \quad (100)$$

According to Theorem 4, the controller is designed as

$$u = K(Z)E = (Z_3 + 9 \quad -27) E = (Z_3 + 9)E_1 - E_2.$$

Numerical simulation is done initial conditions: $X_1(0) = 1, X_2(0) = -2, Z_3(0) = 3, Z_4(0) = 3, Y_1(0) = -4, Y_2(0) = 5$, and $k(0) = -1$. Figure 7 shows that the sum system is asymptotically stable, Figure 8 shows the states X_1, X_2 and the states Y_1, Y_2 , respectively. Figure 9 shows the feedback gain $k(t)$ tends to a constant.

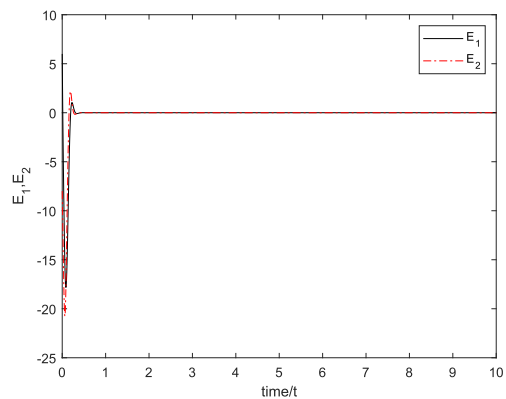


FIGURE 7. The sum system is asymptotically stable.

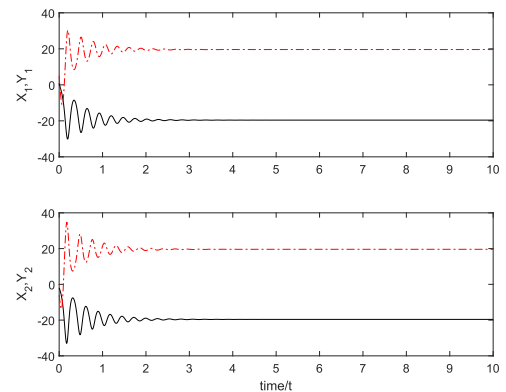


FIGURE 8. The states X_1, X_2 and the states Y_1, Y_2 .

Example 4: The parametrically excited pendulum [32]:

$$\dot{x} = f(x) = \begin{pmatrix} x_2 \\ -0.5x_2 - (1 + 2 \cos(2t)) \sin(x_1) \end{pmatrix}, \quad (101)$$

where $x \in R^2$ is the state.

It is easy to obtain that $f(-x) = -f(x)$, the anti-synchronization of the system (101) exists.

The slave system is given as

$$\dot{y} = \begin{pmatrix} y_2 \\ -0.5y_2 - (1 + 2 \cos(2t)) \sin(y_1) \end{pmatrix} + Bu, \quad (102)$$

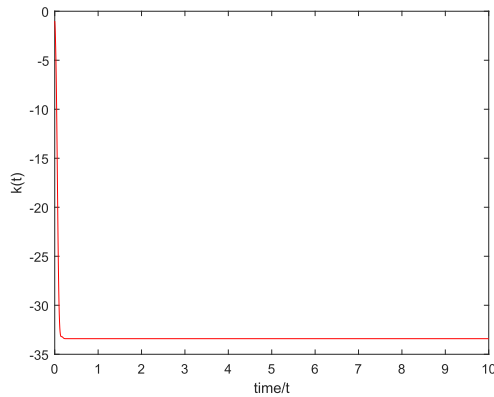


FIGURE 9. $k(t)$ tends to a constant.

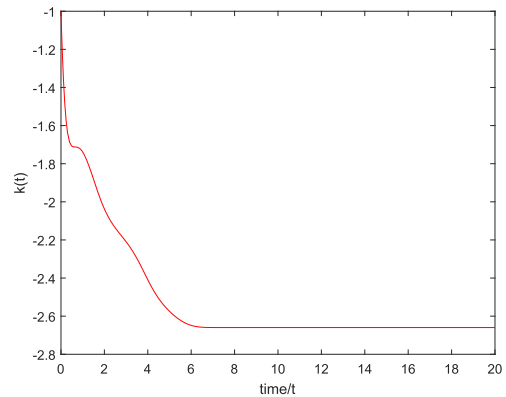


FIGURE 12. $k(t)$ converges to a negative constant.

where $u \in R$ is the controller to be designed, and

$$u = \begin{pmatrix} k(t)(x_1 + y_1) \\ 0 \end{pmatrix}, \quad (103)$$

and $k(t) = -(x_1 + y_1)^2$.

Numerical simulation is done initial conditions: $x_1(0) = 1$, $x_2(0) = -2$, $y_1(0) = -3$, $y_2(0) = 4$, and $k(0) = -1$. Figure 10 shows that the sum system is asymptotically stable, Figure 11 shows the states x_1, x_2 and the states y_1, y_2 , respectively. Figure 12 shows the feedback gain $k(t)$ tends to a constant.

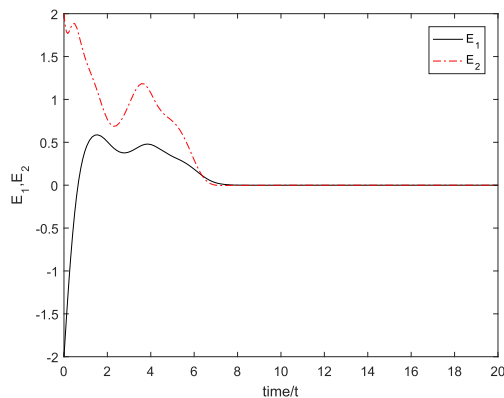


FIGURE 10. The sum system is asymptotically stable.

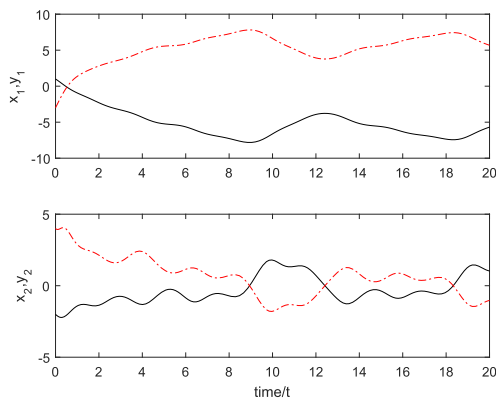


FIGURE 11. The states x_1, x_2 and the states y_1, y_2 .

Remark 3: From the four examples, it can be seen that the obtained controllers u are single input controllers, i.e., $u \in R$. Comparing with the existing results in [14], [17], these controllers are simple. In addition, the shortcoming for the proposed control strategy presented in Theorem 4 is that it has no universal equation, that is it is designed according to the matrices $M(Z)$ and B_1 .

Remark 4: The differences among the four examples are summaries as follows

- 1) For Example 1, Example 2 and Example 4, the investigated systems are chaotic, whereas the system in Example 3 is hyper-chaotic.
- 2) For Example 1, Example 2 and Example 3, the partial anti-synchronization problem of these studied systems exists, whereas anti-synchronization problem exists for the system given in Example 3.

VI. CONCLUSION

The partial anti-synchronization in a class of chaotic and hyper-chaotic systems has been investigated extensively in this paper. Firstly, the existence of partial anti-synchronization for the chaotic system has been proved. Then, by a systematic method including two algorithms, all solutions of the partial anti-synchronization for a given chaotic system have been derived, and physical controllers have been designed. In the end, some illustrative examples with numerical simulations have been used to verify the soundness and effectiveness of the theoretical results.

Inspired by the ideas in [33], in the future, more studies will be conducted on the application of the results of this study about anti-synchronization to nonlinear digital communication, and the design of chaotic and hyper-chaotic systems with system constraints that is referred to as fixed-time SOSM controller design with output constraints.

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