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Design of Rate-Compatible Polar Codes Based on Non-Uniform Channel Polarization

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ABSTRACT In this article we propose a technique for polar codes (PC) construction for any code length. By default, PC construction is limited to code length proportional to the power of two. To construction the code length arbitrary, puncturing, shortening and extension techniques must be applied. However, performance is degraded with the use of these techniques. Other ways to design polar codes with arbitrary code length but which have encoding and decoding with higher complexity such as multi-kernel, concatenated codes and specific constructions for belief propagation (BP) or successive cancellation list (SCL) decoding. The polarization theory is generalized for non-uniform channels (NUC) and with this approach we can construction rate-compatible PC and variable code length. We developed an implementation algorithm based on the of PC construction by Gaussian approximation (NUPGA). In a scenario where the transmission is over an additive white Gaussian noise (AWGN) channel and under successive cancellation (SC) decoding, the PC construction of arbitrary code length can be implemented with NUPGA. With NUPGA we re-polarize the projected synthetic channels by choosing more efficiently the positions of the information bits. In addition, we present a generalization of the Gaussian approximation (GA) for the polarization and re-polarization processes and an extension technique for PC. The PC construction based on NUPGA present better performance than the existing techniques as shown in the simulations of this work.

INDEX TERMS Polar codes, arbitrary-length, rate-compatible, non-uniform polarization, channel polarization, re-polarization.

I. INTRODUTION

Polar codes and channel polarization theory were introduced by Arikan [1] in 2009. Such codes constitute a powerful channel coding scheme, with a low complexity encoder and decoder. In a scenario where the decoder is SC and the channel is AWGN, for a long code length, the capacity can be achieved of binary symmetric discrete memoryless channels (B-DMCs). Choosing the most reliable channels is the basic premise for construction of a polar code. This channel reliability depends on the code length and the signal-to-noise ratio, being defined by the channel polarization theorem. The theorem proves that in channel polarization, for a code length long enough, the bit channels tend to two conditions: either they become noiseless or noisy. The 3GPP Group selected polar codes for the 5th generation (5G) [2], where they will be used for uplink/downlink channel control.

In [1] it is observed that the PC construction, in its standard form, is the limitation of the project in code length as a power multiple of two, i.e., $N = 2^n$, however, the code length flexibility is required for practical applications. There are several construction techniques applied in the standard polar code model proposed by Arikan [1] considering the SC decoder. Among the main PC construction techniques are: parameter Bhattacharyya [1], density evolution (DE) [3], [4] and [5], Gaussian approximation (GA) of density evolution [6] and the polarization weight (PW) [7], [8].

Both in [9] and in [10] they show a comparative study of the PC construction, their algorithm complexity and their performance. The scenario used is the AWGN channel and SC and SCL decoders for various rates and codewords. A performance study on the AWGN channel of the GA method in PC construction can be found at [11] and [12]. In [9], good

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design of PC was verified with several construction methods, with SC decoding and for various scenarios varying both the code length and the code rate. Polar codes can also be constructed and adapted to a specific decoder, for example, construction of polar codes for SCL decoding [13] and belief propagation (BP) decoding [14], [15]. In [16] the authors propose a genetic algorithm framework that jointly optimizes the PC construction and rate with a specific decoder.

We can group the PC construction of variable length in several techniques, the main ones being: arbitrary kernels techniques, multi-kernel techniques (MK), puncturing techniques, shortening techniques and extension techniques. Polarization matrices of various sizes, for example 3×3, 5×5 and 7×7, have been used to construction of polar codes of any length. BCH kernel matrices proposed in [17] and the code decompositions proposed in [18] both have restrictions on the size of the kernel matrices. Square polarizing kernels larger than two have been proposed in [19], [20] and [21], while a PC construction with mixed kernel sizes has been proposed in [22], [23]. By considering different polarizing kernels of alternate dimensions, MK improves block length flexibility. Although the general coding and decoding structure follows the same structure of standard polar codes, there is an increased complexity with the generalization. The PC construction using the Reed-Muller (RM) rule [24]–[26] can improve the performance of the error rate.

The main shortening and puncturing techniques can be found in [27]–[41]. Generally, puncturing or shortening causes a loss of performance because when the number of bits punctured or shortened increases the code length decreases, degrading the performance. In [27] a study on the main puncturing and shortening techniques is carried out, including the column weights (CW) and the reversal quasi-uniform puncturing scheme (RQUP). Recalling that one of the main limitations of the PC construction is regarding the length of the code, given by $N = 2^n$, that is, multiples of two. In the flexible length construction, a shortening or a puncturing design technique is chosen to obtain a length of $2^{n-1} < M < 2^n$. Puncturing techniques are applied in scenarios where the decoding is belief propagation (BP). We find in [28]–[30] the main studies on punctured PC. Among the techniques used we can mention the reduced generator matrix, exponent connection, minimum distance, stop tree drilling and schemes applied to hybrid automatic repeat request incremental redundancy (HARQ-IR). An important puncturing method, known as parallel concatenated polar (PCP), has been proposed in [28] and consists of the parallel concatenation of multiple polar codes which enables incremental retransmissions for HARQ-IR systems. It is rate compatible and together with the puncturing scheme allows a flexible length code construction.

Several of these can be found at [31]–[34]. In [35] and [36] we have a performance analysis of puncturing codes based on the DE construction technique. The shortening techniques, in turn, are applied to the construction of PC with SC and SCL decoding. As in the scheme we have adopted, the decoder is SC or SCL, it is a fact that the shortened bits are known. So, in these shortened bit positions, a likelihood ratio (LLR) is defined as infinite. An efficient shortening method is reported in [37], where the shortened bits set is optimized simultaneously with the frozen bits set. In [38] a shortening method is presented that produces good results and its technique is based on using column weight (CW) to reduce the size of the generating matrix. The main technique for reducing the generating matrix used in the shortened PC was proposed by [39] and is known as reversal quasi-uniform puncturing (RQUP). The polarization-driven (PD) shortening technique has been presented in [40] based on the reduction of the generating matrix along with a strategy for the choice of bits shortened according to the channel polarization indices associated with the line index of the generator matrix. Recently, the PW algorithm has been used in a puncturing and shortening technique as reported in [41].

In [29], [42] and [43] we find proposals for the PC extension. For HARQ schemes it was proposed that an arbitrary number of incremental coded bits can be generated by extending the polarization matrix such that multiple retransmissions are aggregated to produce a longer polar code with extra coding gain. In terms of complexity, it is similar to the standard polar code, both in the encoder and in the decoder. Nevertheless, there is a significant increase in complexity when designing flexible-length polar codes by concatenated codes [44], [48] and by asymmetric kernel construction [46]. In both cases the PC construction is specific to each kernel dimension without generalization gains. In [47] is presented a chained polar subcode technique for effective PC construction.

The main objective of this work is to demonstrate the use of non-uniform channel (NUC) polarization theory in the PC construction of arbitrary length [52]. We have expanded the conference article mentioned above by including further details in the description of the shortening algorithm proposed, another algorithm that performs extension, an analysis of the approach and extra simulation results of several application scenarios. In the works of [48] and [49] methods are proposed to generalize the channel polarization in scenarios of parallel transmission or when the channel parameter is unknown. In [48], a technique for PC construction for multichannel polar codes has been reported, including a scheme for modulation and bit interleaving, resulting in rate-matching compatibility. The PC construction scheme proposed in [49] deals with scenarios with parallel channels and random channel parameters.

In uniform polarization, the same value of the Bhattacharyya parameter is given for all channels according to the PC construction proposed by Arikan [1]. In terms of construction, GA is equivalent to the use of the same LLR defined for all channels. In the proposed Non-Uniform Polarization based on Gaussian Approximation (NUPGA) technique, we assign different LLR values for the channels with the guarantee of the validity of the channel polarization principle. Therefore, it is possible to construct polar codes of arbitrary length while maintaining their rate-compatibility.

Then, we present an algorithm for PC shortening and also an algorithm for PC extension, both based on the NUPGA technique. The NUPGA-based algorithms jointly implement the technique for shortened or extended channels with the re-polarization of the channels. We also present a generalization of the GA algorithm, which is used for both polarization of the initial channel and re-polarization of the shortened PC, and a simplified construction technique for extended polar codes. The existing techniques are compared with the proposed NUPGA technique in various simulations exploring different combinations of code length and code rate. A key feature of the proposed designs is that the encoder and decoder structures are the same as that of the original polar codes [1] and require the same complexity.

Very briefly, we report below the main contributions in this article:

- a new technique for constructing polar codes of arbitrary length based on NUPGA along with a proof that it achieves capacity;
- application of NUPGA as a technique for shortening and extension of polar codes;
- an extensive simulation study that compares the NUPGA-based and existing design techniques.

In the following we have structured the content of this work into several sections. In Section II, we have the basic definitions about PC, notation used and its method of encoding and decoding. In Section III, the channel polarization theory is revisited [1]. In Section IV, we present the non-uniform polarization of channels, and using the induction method to compare with the uniform polarization of channels, we show that it also achieves channel capacity. In Section V, we present the algorithms for implementing shortening and extension techniques based on NUPGA. In Section VI we show the comparative simulations with the NUPGA and other techniques. In Section VII we have the conclusions of this work.

II. POLAR CODES

This article uses the same notation that [1] and in this section we present a brief PC description.

Given a B-DMC $W : \mathcal{X} \rightarrow \mathcal{Y}$, where $\mathcal{X} \in \{0, 1\}$ and $\mathcal{Y} \in \mathbb{R}$ and $W(y|x), x \in \mathcal{X}, y \in \mathcal{Y}$. We have that $W(y|x)$ is the channel transition probability, $x \in \mathcal{X}$, $y \in \mathcal{Y}$. For a scheme with $N = 2^n$ channels W independent, after a process of combining and splitting [1], N synthetic channels are generated and its defined as $W_N^{(i)}$ with $i = \{1, 2, \ldots, N\}$. To transmission the information bits, the most reliable sub-channels are chosen, represented by K. $\mathcal{A}$ is the set of K indices. In turn, $\mathcal{A}^c$ is its complementary set, containing the indices of the least reliable channels.

The PC is defined by three parameters: $N = 2^n$, $R = K/N$ and $\mathcal{A} \in [N]$ with K cardinality; that is, code length, code rate and information set, respectively. The encoding is given by $x_1^N = u_1^N \mathbf{G}_N$. $\mathbf{G}_N$ is the transformation matrix. $u_1^N \in \{0.1\}^N$ is the input block. $x_1^N \in \{0.1\}^N$ is the codeword, where $u_1^N = [u_{\mathcal{A}}, u_{\mathcal{A}^c}]$, with $u_{\mathcal{A}}$ are bits of information and $u_{\mathcal{A}^c}$ are frozen bits. We defined $\mathbf{G}_N = \mathbf{B}_N \mathbf{F}_2^{\otimes n}$, where $\otimes$ denotes the Kronecker product, $\mathbf{F}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{B}_N$ is the bit-reversal permutation matrix. A simplification without loss of generalization is the omission of $\mathbf{B}_N$.

The estimate $\hat{u}_i$ for SC over AWGN is given by

$$\hat{u}_i = \arg \max_{u_i \in \{0,1\}} W_N^{(i)}(y_1^N, u_1^{i-1}|u_i), \quad i \in A \tag{1}$$

The $\hat{u}_1^N = (\hat{u}_1, \ldots, \hat{u}_N)$ and $y_1^N = (y_1, \ldots, y_N)$, the likelihood ratio (LR) message of u_i, $LR(u_i) = \frac{W_N^{(i)}(y_1^N, \hat{u}_1^{i-1}|0)}{W_N^{(i)}(y_1^N, \hat{u}_1^{i-1}|1)}$, recursively using SC decoding [1]. Then, value of the $\hat{u}_i$ is:

$$\hat{u}_i = \begin{cases} h_i(y_1^N, \hat{u}_1^{i-1}), & \text{if } i \in \mathcal{A}. \\ u_i, & \text{if } i \in \mathcal{A}^c. \end{cases} \tag{2}$$

where $h_i : \mathcal{Y}^{\mathrm{N}} \times \mathcal{X}^{\,i-1} \rightarrow \mathcal{X}$, $i \in \mathcal{A}$, are decision functions defined as:

$$h_i(y_1^N, \hat{u}_1^{i-1}) = \begin{cases} 0, & \text{if } \dfrac{W_N^{(i)}(y_1^N, \hat{u}_1^{i-1}|0)}{W_N^{(i)}(y_1^N, \hat{u}_1^{i-1}|1)} \geq 1 \\ 1, & \text{otherwise.} \end{cases} \tag{3}$$

for $y_1^N \in \mathcal{Y}^{\mathrm{N}}$, $\hat{u}_1^{\,i-1} \in \mathcal{X}^{\,i-1}$.

We denote $L(i, j)$ as LR node, i being the line and j being the stage, following mapping of the decoding tree [1]. The values assumed by $L(i, j)$ can be obtained recursively [1] using the equations:

$$L(i, j+1) = \begin{cases} f(L(i, j), L(i + n/2^{j+1}, j)), & (f \text{ nodes}) \\ g(L(i - n/2^{j+1}, j), L(i, j), \hat{u}_{sum}), & (g \text{ nodes}) \end{cases} \tag{4}$$

where f and g functions were defined in [1] as:

$$f(a, b) = \frac{1 + ab}{a + b} \tag{5}$$

$$g(a, b, \hat{u}_{sum}) = a^{1-2\hat{u}_{sum}} b \tag{6}$$

where $\hat{u}_{sum}$ is the previous decoded bits. The $\hat{u}_{sum}$ estimated value is given by (2). So, the decision g nodes depends on the estimate of f nodes given by (2), that is, of previously decoded bits. In order to simplify the design of the decode, [50] proposes a LLR SC/SCL decoder. In this way, the previous equations (5) - (6), in the natural domain, are transformed into the logarithmic domain:

$$F(a, b) = 2\tanh^{-1}(\tanh(a/2)\tanh(b/2)) \tag{7}$$

$$G(a, b, \hat{u}_{sum}) = a(-1)^{\hat{u}_{sum}} + b. \tag{8}$$

Initially proposed for LDPC decoding and in order to further reduce the complexity of the decoder in (7), we can use a minimal sum approximation [50]:

$$F(a, b) \approx \operatorname{sign}(a)\operatorname{sign}(b) \min(|a|, |b|), \tag{9}$$

where (8)-(9) describe the LLR version of the SC algorithm.

In the SCL decoder [51], the candidate sequences set is given by $S^{(i)}$ for ith SC decoding step. And $|S^{(i)}|$ is the size of $S^{(i)}$. The maximum allowed list size is L and T a limit for $T \leq 1$ pruning. About the SCL decoding steps, we have:

- for each $S^{(i)}$, all i bits are estimated,
- generate two alternatives i, estimate $\hat{u}_i = 0$ or $\hat{u}_i = 1$ by SC;
- no action if the number of candidates $|S^{(i)}| < L$;
- otherwise, select $S^{(i)}$ with the highest probabilities up to the limit given by $|S^{(i)}|$;
- checks candidate to candidate $\hat{u}_i \in S^{(i)}$, if $P(\hat{u}_1^i) < T \max_{\hat{u}_1^i \in S^{(i)}} P(\hat{u}_1^i)$, discarding $\hat{u}_1^i$ from $S^{(i)}$.

Each candidate on the list is examined and the likelihood is calculated. The estimate with the maximum probability is selected:

$$\hat{u}_1^N = \arg \max_{\hat{u}_1^N \in S^{(i)}} \prod_{i=1}^{N} W(y_i|(u_1^N)_i). \tag{10}$$

III. IMPORTANT ASPECTS OF PC CONSTRUCTION

In this section, we deal with the theory for the PC construction. We also approach the generalization of the polarization theory for non-uniform channels (NUC). In this scenario, the main aspects of the channel polarization theory are maintained, namely, the conservation of the associated channel capacity and the induction to the polarized channel [1].

A. PC CONSTRUCTION

The main aspect in the PC construction is to find the best W for the information bits, $\mathcal{A}$, with the standard PCs are construction with $N = 2^n$. If $N \to \infty$, these bit channels are divided into either noise free or completely noisy channels. The $Z(W)$ [1] parameter and defined as

$$Z(W) = \sum_{y \in \mathcal{Y}} \sqrt{W(y|x=0)W(y|x=1)}, \tag{11}$$

where $W(Y|X)$ is the probability, $\mathcal{X} = \{0, 1\}$ and $\mathcal{Y} \in \mathbb{R}$, $x \in \mathcal{X}, y \in \mathcal{Y}$. For any B-DMC, the reliability of bit-channels can be recursively determined [1], and with the exception of the BEC channel, for all other channels its method of determination is approximate [1], and as seen earlier, several algorithms have been proposed [9]. The BEC channels are well studied in this regard. In it when $Z(W)$ is close to zero the channels are almost noiseless, while $Z(W)$ is close to one the channels are noisy. The essential idea is to choose the most reliable bit-channels (noise free channels) to transmit information bits ($\mathcal{A}$), while noisy bit channels known to both encoder and decoder are frozen ($\mathcal{A}^c$).

For the construction of arbitrary-length polar codes, a generalization of channel polarization is necessary for the definition of non-uniform polarization, maintaining the primary results of the channel polarization theory. First, we will generalize channel types to see if full capacity is maintained. Then, we will verify if the channel polarization theory is valid for non-uniform channels.

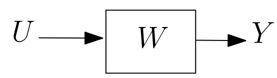

FIGURE 1. The channel W.

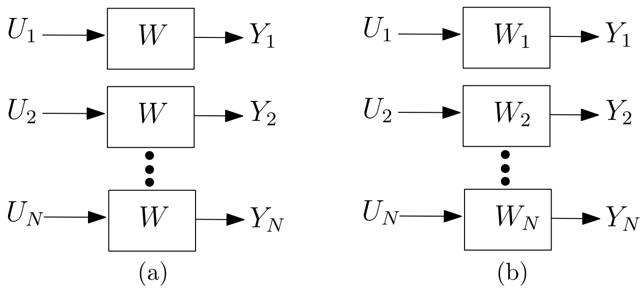

FIGURE 2. (a) Uniform DMC and (b) non-uniform DMC.

B. CHANNEL CAPACITY

In Fig.1 we show a B-DMC channel, designated by the symbol W, with input U and output Y.

The input symbol on B-DMC channel is considered a discrete random variable generated by U. Similarly, the symbol at the output of the channel is modeled by another discrete random variable Y. Then, a set of DMC channels from Fig.1 can be shown as in Fig.2a. The Bhattacharyya parameter is $Z(W)$ for all DMC channels. Consider the transmission of N different symbols $[u_1\ u_2 \cdots u_N]$ through the channel in a serial manner. These symbols that are transmitted serially, in our modeling are considered independent and identically distributed (i.i.d.) random variables. Without loss of generality, we consider that the transmission of each symbol is through each channel separately, as in Fig.2a, where $\mathbf{U} = [U_1\ U_2\ \cdots\ U_N]$ and $\mathbf{Y} = [Y_1\ Y_2\ \cdots\ Y_N]$. Therefore, the deduction of the system's capacity will be the same as if we use other channels, that is, non-uniform channels, as suggested in Fig.2b.

Now, the Bhattacharyya parameter is different for all B-DMC channels. The mutual information for Fig.2a and Fig.2b are shown below.

The mutual information is given by

$$\begin{aligned} I(\mathbf{U}; \mathbf{Y}) = I(U_1; \mathbf{Y}) + I(U_2; \mathbf{Y}|U_1) + I(U_3; \mathbf{Y}|U_1, U_2) \\ + \cdots + I(U_N; \mathbf{Y}|U_1, U_2, \cdots, U_{N-1}). \end{aligned} \tag{12}$$

We consider U_1 and $Y_1, Y_2, \cdots, Y_N$ independent from each other, so we have:

$$\begin{aligned} I(U_1; \mathbf{Y}) &= I(U_1; Y_1) \\ I(U_2; \mathbf{Y}|U_1) &= I(U_2; Y_2) \\ I(U_3; \mathbf{Y}|U_1, U_2) &= I(U_3; Y_3) \\ I(U_N; \mathbf{Y}|U_1, U_2, \cdots, U_{N-1}) &= I(U_N; Y_N) \end{aligned}$$

Then, (12) can be written as

$$\begin{aligned} I(\mathbf{U}; \mathbf{Y}) = I(U_1; Y_1) + I(U_2; Y_2) + I(U_3; Y_3) \\ + \cdots + I(U_N; Y_N) \end{aligned} \tag{13}$$

and let the capacity be $C = \max I(\mathbf{U}; \mathbf{Y})$, then we have

$$\begin{aligned} \max I(\mathbf{U}; \mathbf{Y}) &= \max I(U_1; Y_1) + \max I(U_2; Y_2) \\ &\quad + \cdots + \max I(U_N; Y_N) \\ \max I(\mathbf{U}; \mathbf{Y}) &= NC \end{aligned} \tag{14}$$

However, a key concept of channel polarization is that it consists of a method where the channel outputs depend on the other inputs as well. This implies the following inequalities:

$$\begin{aligned} I(U_1; \mathbf{Y}) &\neq I(U_1; Y_1) \\ I(U_2; \mathbf{Y}|U_1) &\neq I(U_2; Y_2) \\ I(U_3; \mathbf{Y}|U_1, U_2) &\neq I(U_3; Y_3) \\ I(U_N; \mathbf{Y}|U_1, U_2, \cdots, U_{N-1}) &\neq I(U_N; Y_N) \end{aligned}$$

That is, (12) cannot be simplified as in (13). In addition, it is necessary to ensure that

$$\begin{aligned} I(U_1; \mathbf{Y}) < I(U_3; \mathbf{Y}|U_1, U_2) \leq I(U_2; \mathbf{Y}|U_1) \\ < \cdots < I(U_N; \mathbf{Y}|U_1, U_2, \cdots, U_{N-1}), \end{aligned} \tag{15}$$

which means the individual capacities increase in an orderly manner but the total capacity remains constant, i.e., the total capacity of the channels is maintained, regardless of whether the channels are equal or not, that is, uniform or non-uniform, with different Bhattacharyya parameters. Then, for uniform channels, according to (14), and for non-uniform channels, with the inequality of (15), the capacity of the channels is conserved, and we show that non-uniform polarization schemes achieve symmetric capacity:

$$\max I(\mathbf{U}; \mathbf{Y}) = \sum_{i=1}^{N} C_i = NC \tag{16}$$

Therefore, we can devise methods to construct polar codes that take into account different Bhattacharyya parameters. Then we verify the convergence of the polarization theory [1] for NUC. The channel capacity, channel polarization and polarization convergence will be better studied in the next sections. Then, we show that all results remain valid for the case of generalized channels.

C. UNIFORM CONSTRUCTION

In the channel polarization process [1], before recursion, to we consider the channels W independent and with the identical parameter $Z(W)$. Let $W : \mathcal{X} \rightarrow \mathcal{Y}$ denote a symmetric B-DMC, with $\mathcal{X} = \{0, 1\}$ and $\mathcal{Y} \in \mathbb{R}$ and the channel transition probability $W(y|x)$, where $x \in \mathcal{X}$, $y \in \mathcal{Y}$. Denote $W^N : \mathcal{X}^N \rightarrow \mathcal{Y}^N$ with

$$W^N(y_1^N|x_1^N) = \prod_{i=1}^{N} W(y_i|x_i) \tag{17}$$

The mutual information is defined by [1]

$$I(W) = \sum_{y\in Y}\sum_{x\in X} \frac{1}{2} W(y|x) \log \frac{W(y|x)}{\frac{1}{2}W(y|0) + \frac{1}{2}W(y|1)}, \tag{18}$$

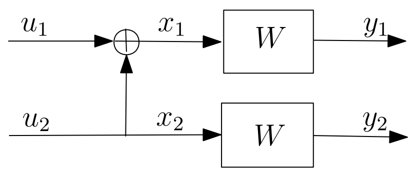

FIGURE 3. **The Channel W_2.**

where the base-2 logarithm $0 \leq I(W) \leq 1$ is employed. And $Z(W)$ parameter is given by [1]

$$Z(W) = \sum_{y\in Y} \sqrt{W(y|0)W(y|1)}, \tag{19}$$

where $0 \leq Z(W) \leq 1$. For any B-DMC W, we have

$$log \frac{2}{1 + Z(W)} \leq I(W) \leq \sqrt{1 + Z(W)^2}. \tag{20}$$

On the N independent channels of W we apply the polarization process. After the combination of channels and division operation, as described in [1], we obtain a set of polarized channels $W_N^{(i)} : \mathcal{X} \rightarrow \mathcal{Y} \times \mathcal{X}^{\mathrm{i}-1}$, $i = 1, 2, \ldots, N$. As defined in [1], this channel transition probability is given by

$$W_N^{(i)}(y_1^N, u_1^{(i-1)}|u_i) = \sum_{u_{i+1}^N \in X^{N-1}} \frac{1}{2^{N-1}} W_N(y_1^N|u_1^N), \tag{21}$$

where N is the code length.

According to [1], $N\ to\infty$, $I(W_N^{(i)})$ tends to 0 or 1.

In Fig.3 we show the process of creating the channel W_2: recursion step of combining two copies of W independent, that is, $\mathcal{X}^2 \rightarrow \mathcal{Y}^2$ which has the transition probabilities given by

$$W_2^{(1)}(y_1^2|u_1) = \sum_{u_2} \frac{1}{2} W(y_1|u_1 \oplus u_2)W(y_2|u_2). \tag{22}$$

$$W_2^{(2)}(y_1^2|u_1|u_2) = \frac{1}{2} W(y_1|u_1 \oplus u_2)W(y_2|u_2). \tag{23}$$

The channel polarization can be represented by a graphical representation called the construction tree [1]. We can see that a unique value of W is used in the construction, it is actually a simplification for the parameter $Z(W)$.

On the BEC channel, for example, being $(W_N^{(i)}, W_N^{(i)}) \rightarrow (W_{2N}^{(2i-1)}, W_{2N}^{(2i)})$ produces two B-DMC:

$$\begin{aligned} (W_N^{(i)}, W_N^{(i)}) &\rightarrow (W_{2N}^{(2i-1)}, W_{2N}^{(2i)}) \\ Z(W_{2N}^{(2i)}) &\leq 2Z(W_N^{(i)}) - Z(W_N^{(i)})^2 \\ Z(W_{2N}^{(2i-1)}) &\leq Z(W_N^{(i)})^2 \\ Z(W_{2N}^{(2i-1)}) &\leq Z(W_{2N}^{(2i)}) \end{aligned} \tag{24}$$

according to (18), we have

$$\begin{aligned} I(W_{2N}^{(2i)}) &= I(U_1; Y_1, Y_2), \\ I(W_{2N}^{(2i-1)}) &= I(U_2; Y_1, Y_2, U_1), \end{aligned} \tag{25}$$

where U_1 and U_2 are iid. From the chain rule, it follows that

$$I(W_{2N}^{(2i)}) + I(W_{2N}^{(2i-1)}) = I(U_1; Y_1, Y_2) + I(U_2; Y_1, Y_2, U_1) \\ = 2I(W_N^{(i)}) \tag{26}$$

and

$$I(W_{2N}^{(2i)}) = I(U_2; Y_1, Y_2, U_1) \\ \geq I(W_N^{(i)}), \tag{27}$$

which results in

$$I(W_{2N}^{(2i)}) \geq I(W_{2N}^{(2i-1)}) \tag{28}$$

For the case of the BEC channel [1]

$$Z(W_{2N}^{(2i-1)}) \leq 2Z(W_N^{(i)}) - Z(W_N^{(i)})^2 \tag{29}$$

$$Z(W_{2N}^{(2i)}) = Z(W_N^{(i)})^2. \tag{30}$$

where reliability and cumulative rate must satisfy [1]

$$\sum_{i=1}^{N} I(W_N^{(i)}) = NI(W). \tag{31}$$

$$\sum_{i=1}^{N} Z(W_N^{(i)}) \leq NZ(W). \tag{32}$$

In uniform channel polarization there is always a set of B-DMCs that reaches capacity when $N \to \infty$, is arbitrary small $\delta \leq 0$

$$\frac{|\{I(W_N^{(i)}) \in (1-\delta, 1]\}|}{N} \to \frac{I_s}{N}, \\ \frac{|\{I(W_N^{(i)}) \in [0, \delta)\}|}{N} \to 1 - \frac{I_s}{N}, \tag{33}$$

where the values of $I_N^{(i)}$ converge to $\{0, 1\}$. The term I_s is the sum of the mutual information, given by $I(\mathbf{U}; \mathbf{Y})$ in (13). For uniform construction it is equivalent to (31).

IV. PROPOSED NON-UNIFORM CONSTRUCTION

In this section, we show a generalization of the equations presented in Section III.B, we will see that channel polarization can also be applied to non-uniform channels (NUC).

We now have two channels W, which are independent and we will consider them to be non-uniform, so $W_{(i)} : \mathcal{X}_{(\mathrm{i})} \to \mathcal{Y}_{(\mathrm{i})}$, as shown in Fig. 5, where we rewrite (17) as

$$W(y_1^N | x_1^N) = \prod_{i=1}^{N'} W_{(i)}(y_i | x_i). \tag{34}$$

So N and N', where $|N| = |N'|$, and $W_{(i)}(y|x) = W_{(j)}(y|x)$ if $i = j$. The symmetric capacity (18) and the $Z(W)$ parameter (19) [1] for any $W_{(i)}$, are rewritten as

$$I(W_{(i)}) = \sum_{y \in Y} \sum_{x \in X} \frac{1}{2} W_{(i)}(y|x) \log \frac{W_{(i)}(y|x)}{\frac{1}{2}W_{(i)}(y|0) + \frac{1}{2}W_{(i)}(y|1)}, \tag{35}$$

$$Z(W_{(i)}) = \sum_{y_i \in Y} \sqrt{W_{(i)}(y_i|0) W_{(i)}(y_i|1)}. \tag{36}$$

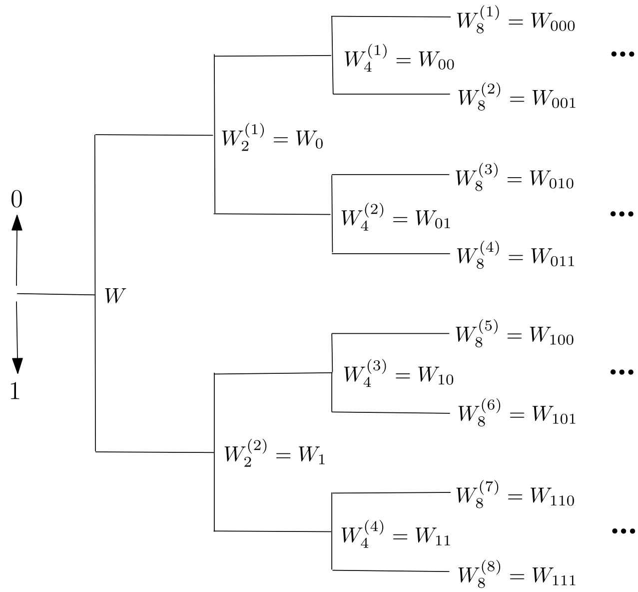

FIGURE 4. The channel construction.

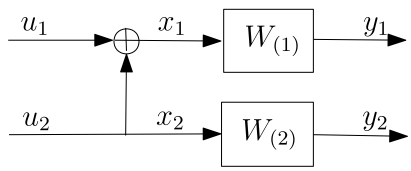

FIGURE 5. The NUC channel $W_{2'}$.

Note the (20) is equivalent as

$$\log \frac{2}{1 + Z(W_{(i)})} \leq I(W_{(i)}) \leq \sqrt{1 + Z(W_{(i)})^2}, \tag{37}$$

and (21) remains valid.

For W_2 we rewrite (22) and (23) as

$$W_2^{(1)}(y_1^2 | u_1) = \sum_{u_2} \frac{1}{2} W_{(1)}(y_1 | u_1 \oplus u_2) W_{(2)}(y_2 | u_2), \tag{38}$$

$$W_2^{(2)}(y_1^2, u_1 | u_2) = \frac{1}{2} W_{(1)}(y_1 | u_1 \oplus u_2) W_{(2)}(y_2 | u_2). \tag{39}$$

Using the BEC channel again as an example, for a comparison with Section III.B, for the case of the parameter Z we have:

$$\begin{aligned} Z(W_2^{(2)}) &= \sum_{y_1^2, u_1} \sqrt{W_2^{(2)}(y_1^2, u_1 | u_2 = 0) W_2^{(2)}(y_1^2, u_1 | u_2 = 1)} \\ &= \sum_{y_1^2, u_1} \frac{1}{2} \sqrt{W_{(1)}(y_1|u_1) W_{(2)}(y_2|0)} \\ &\quad \cdot \sqrt{W_{(1)}(y_1|u_1) W_{(2)}(y_2|1)} \\ &= \sum_{y_2, u_1} \sqrt{W_{(2)}(y_2|0) W_{(2)}(y_2|1)} \\ &\quad \cdot \sum_{y_1, u_1} \frac{1}{2} \sqrt{W_{(1)}(y_1|u_1) W_{(1)}(y_1|u_1)} \\ &= Z(W_{(2)}) Z(W_{(1)}). \end{aligned} \tag{40}$$

And (29) and (30):

$$Z(W_2^{(1)}) \leq Z(W_{(1)}) + Z(W_{(2)}) - Z(W_{(1)})Z(W_{(2)}), \quad (41)$$
$$Z(W_2^{(2)}) \leq Z(W_{(1)})Z(W_{(2)}), \quad (42)$$
$$Z(W_2^{(2)}) \leq Z(W_2^{(1)}). \quad (43)$$

So (31) and (32) are equivalent to

$$\sum_{i=1}^{N'} I(W_{N'}^{(i)}) = \sum_{i=1}^{N'} I(W_{(i)}), \quad (44)$$
$$\sum_{i=1}^{N'} Z(W_{N'}^{(i)}) \leq \sum_{i=1}^{N'} Z(W_{(i)}). \quad (45)$$

We can show that (26) can be obtained by performing the following operations:

$$\begin{aligned} I(W_2^{(1)}) &= I(Y_1, Y_2, U_1), \\ I(W_2^{(2)}) &= I(Y_1, Y_2, U_1; U_2), \\ I(W_2^{(1)}) + I(W_2^{(2)}) &= I(Y_1, Y_2, U_1) + I(Y_1, Y_2, U_1; U_2) \\ &= I(W_1) + I(W_2). \end{aligned}$$

So, for any set of B-DMC, we can rewrite (33) for the non-uniform polarization channel:

$$\begin{aligned} \lim_{N\to\infty} \frac{|\{\sum_1^N I(W_N^{(i)}) \in (1-\delta, 1]\}|}{N} &\to \frac{I_s}{N}, \\ \lim_{N\to\infty} \frac{|\{\sum_1^N I(W_N^{(i)})) \in [0, \delta)\}|}{N} &\to 1 - \frac{I_s}{N}. \end{aligned} \quad (46)$$

where the values of $I_N^{(i)}$ converge to $\{0, 1\}$, which is a novel result related to the established result for uniform channel polarization in (33). The term I_s is the sum of the mutual information, given by $I(\mathbf{U}; \mathbf{Y})$ in (13). For non-uniform construction it is equivalent to (44).

V. PROPOSED NUPGA DESIGN ALGORITHMS

In this section, we detail the NUPGA method and the implementation of the non-uniform construction algorithms. In Fig.6 we have a generalization of the recursive polarization process. The nodes implement the functions described in (29) and (30).

A. PROPOSED NUPGA-BASED SHORTENING

The PD [40] is a starting point to define the channels that will initially be shortened. Shortening techniques reduce the length of the codeword from N to M, that is, $2^{n-1} < M < 2^n$. The number of bits of information is represented by K. The symbols N and M represent, respectively, the code lengths of the standard PC and the shortened PC. Note that $K < M < N$. The indexes set of the shortened bits is represented by the symbol P, also called the shortening pattern. The cardinality of the shortened bits is represented by $|P| = N - M$. Thus, for shortened PC, the code rate is represented by $R = K/M$. Note that the decoder knows the shortened bits P. When decoding, the corresponding LLRs are set to infinity.

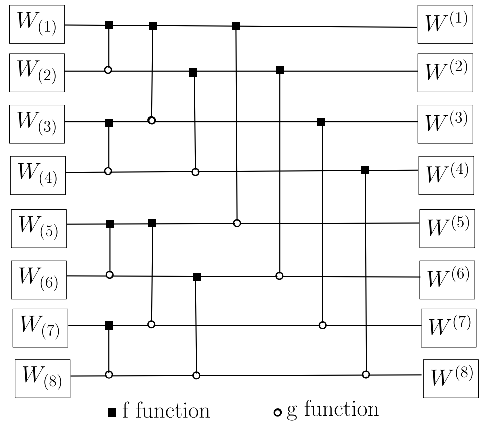

FIGURE 6. Alternative polarization tree.

Consider that the vector P contains the channels obtained by the PD [40]. In the first step, the codeword is generated by setting the set P set to zero. In the next step, the message length of the codeword is reduced by P. We remark that non-universality [1] is one of the main characteristics of PCs. For all simulations, this work adopts $SNR = 0dB$. And with code shortening, we have changes in the bit channels reliability, which deteriorates performance when compared to the original code. In this regard, the study in [40] indicates that the order of channel polarization does not change after shortening. For the shortened channels, we consider the parameter $Z(W)$ penalized a frozen bit, and will be used as input in the NUPGA method.

In AWGN channels, the LLRs of each subchannel, namely $L_N^{(i)}$, the channel polarization can be estimated with the recursive GA algorithm proposed by [6]

$$\begin{cases} E(L_N^{(2i-1)}) = \phi^{-1}(1 - (1 - \phi(E(L_{N/2}^{(i)})))^2) \\ E(L_N^{(2i)}) = 2E(L_{N/2}^{(i)}), \end{cases} \quad (47)$$

with $E[\cdot]$ being the expected value.

$$\phi(x) = \begin{cases} \exp(-0.4527x^{(0.86)} + 0.0218) & \text{if } 0 < x \leq 10 \\ \sqrt{\frac{\pi}{x}}(1 - \frac{10}{7x}) \exp(-\frac{x}{4}) & \text{if } x > 10 \end{cases} \quad (48)$$

In NUPGA, the GA (47) equation is generalized, making it possible to treat arbitrary lengths of code, with $f = E(L_{N'}^{(2i-1)})$ and $g = E(L_{N'}^{(2i)})$ according to Fig.6. This results in the following proposed recursions:

$$\begin{cases} E(L_{N'}^{(2i-1)}) = \phi^{-1}(1 - (1 - \phi(E(L_1^{(i)})))(1 - \phi(E(L_2^{(i)})))) \\ E(L_{N'}^{(2i)}) = E(L_1^{(i)})E(L_2^{(i)}). \end{cases} \quad (49)$$

In Algorithm 1 we have the description of the proposed NUPGA shortening algorithm. Suppose for example a scheme with $N = 4$ and $K = 2$. We have $P = \{1, 1, 1, 1\}$, vector not shortened with the result $F = \{0, 1, 0, 1\}$, as expected. The shortened vector $P = \{1, 1, 1, 0\}$ is applied

Algorithm 1 Proposed NUPGA Shortening Algorithm

1: **I**$NPUT$: N, code length
2: **I**$NPUT$: K, information bits
3: **I**$NPUT$: P, shortened bits
4: **I**$NPUT$: design-SNR $E_{dB} = (RE_b/N_o)$ in dB
5: **O**$UTPUT$: $F \in \{0, 1, \ldots, N-1\}$ with $|F| = N$
6: $S = 10^{EdB/10}$ and $n = log_2 N$
7: $L \in R^N$, Initialize $[E(L_1^{(i)})]_1^N = 4S$ [6]
8: Upgrade with shortening vector $[E(L_1^{(i)})]_1^N$ with P
9: **for** i = 1 to $n+1$ **do**
10: $d = 2^{(i-2)}$
11: **for** b = 1 to N step $2^{(i-1)}$ **do**
12: **for** k = 0 to $d-1$ **do**
13: **if** $E(L_{k+b}^{(i-1)}) = 0$ or $E(L_{k+b+d}^{(i-1)}) = 0$ **then**
14: $E(L_{k+b}^{(i)}) = E(L_{k+b}^{(i-1)})$
15: $E(L_{k+b+d}^{(i)}) = E(L_{k+b+d}^{(i-1)})$
e$ndif$
16: $E(L_{k+b}^{(i)}) = \phi^{-1}(1-(1-\phi(E(L_{k+b}^{(i-1)})))(1-$
17: $\phi(E(L_{k+b+d}^{(i-1)}))))$
18: $E(L_{k+b+d}^{(i)}) = E(L_{k+b}^{(i-1)})E(L_{k+b+d}^{(i-1)})$
e$ndfor$
e$ndfor$
e$ndfor$
19: $F = Find\ indices\ of\ smallest\ elements\ (E[L], K)$
20: return F

to the operation indicated in step 5 of Algorithm 1. Finally, $F = \{0, 1, 1, 0\}$ is the new set of information obtained.

B. PROPOSED NUPGA-BASED EXTENSION

A simple polar codes extension scheme can be implemented as suggested in Fig.7 [43]. An additional level of polarization is performed in P_1^M and the information bits u_1^k. The connection between the additional channels P_1^M and channels u_{k-M}^k is carried out by the linear polarization sequence of u_1^k. In the extension scheme the complexity is $NlogN$, less than the puncturing/shortening scheme which is $(N+1)log(N+1)$. This scheme is efficient for kernels with low dimension ($N < 512$) and for extension of $P < 50\%$ of N. Using the proposed NUPGA technique, we can consider all additional bit channels as $P_1^M = 0$ and as bits output from polarized channels u_{k-M}^k, uniform and limited to the length of the extension. For the encoder we have the same definition, that is, $P_1^M = 0$. In the decoder, we have

$$\hat{u}_1^k = f(LLR((u_1^k) + LLR(u_{k-M}^k)), \tag{50}$$

in the same order.

Note that according to (44), we have

$$\sum_{i=1}^{N} I(W_N^{(i)}) + \sum_{i=1}^{M} I(W_M^{(i)}) = \sum_{i=1}^{N} I(W_{(i)}) + \sum_{i=1}^{M} I(W_{(i)})$$

this only happens in the following two cases: either $W_N^{(i)}$ and $W_M^{(i)}$ are noise channel and both of the $I(\cdot)$ are equal to 1, or $W_N^{(i)}$ and $W_M^{(i)}$ are both perfect channel such that the two

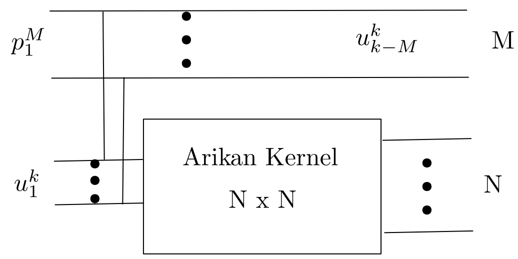

FIGURE 7. PC extension.

$I(\cdot)$ are equal to 0. Using NUPGA, when $W_N^{(i)}$ is perfect and $W_M^{(i)}$ is useless, the two $I(\cdot)$ are 1 and 0, respectively. In other words, if the extended bit channel $W_M^{(i)}$ is noise channel and excluding the case that both $W_N^{(i)}$ and $W_M^{(i)}$ are perfect channel, the re-polarization improves the reliability of the shortened channels. With regards to (45), we have

$$\sum_{i=1}^{N} Z(W_N^{(i)}) + \sum_{i=1}^{M} Z(W_M^{(i)}) \le \sum_{i=1}^{N} Z(W_{(i)}) + \sum_{i=1}^{M} Z(W_{(i)})$$

and with the use of NUPGA, we have

$$\sum_{i=1}^{M} Z(W_M^{(i)}) \le \sum_{i=1}^{N} Z(W_N^{(i)})$$

and then

$$\sum_{i=1}^{M} Z(W_{(i)}) \le \sum_{i=1}^{N} Z(W_{(i)})$$

which ensures that the extended channels will all be noisy.

This method is similar to the extension of the polarization matrix proposed in [42] and [43]. Note that the construction method for polar codes extension allows us to maintain the same encoder and decoder for codeword N. In the proposed NUPGA extension algorithm, the original codeword of the channels initially designed is first extended by adding new bits. Thus, it is possible to increase the code length by gradually adding new bits, making it possible to build codewords of any length. The information bit channels are polarized according to the reliability of the bit channel calculated from the new extended channels. The main idea of the proposed NUPGA extension algorithm is to generate the new bit channels as frozen bits and make the associated information bits more reliable than before. The extension length is ΔM and the new rate is $M = (N + \Delta M)$ which can still be decoded efficiently. Therefore, the extension channels are obtained with the proposed NUPGA extension algorithm. The details of the proposed NUPGA extension algorithm are shown in Algorithm 2.

VI. SIMULATION RESULTS

The performance of the NUPGA-based shortening and extension algorithms is assessed in this section against competing approaches such as CW [38], RQUP [39] and PD [40].For performance analysis we compared the BER and FER of

Algorithm 2 Proposed NUPGA Extension Algorithm

I$NPUT$: N, code length
I$NPUT$: K, information bits
I$NPUT$: P, shortened bits
I$NPUT$: design-SNR $E_{dB} = (RE_b/N_o)$ in dB
I$NPUT$: ΔM, extension length
O$UTPUT$: $F \in \{0, 1, \ldots, N + \Delta M - 1\}$
$S = 10^{EdB/10}$, $n = log_2 N$
$L \in R^N$, Initialize $[E(L_1^{(i)})]_1^N = 4S$ [6] and $[E(L_1^{(i)})]_N^{\Delta M} = 0$
Upgrade with vector $[E(L_1^{(i)})]_1^N$ with P
for i = 1 to $n + 1$ **do**
  $d = 2^{(i-2)}$
  for b = 1 to N step $2^{(i-1)}$ **do**
    for k = 0 to $d - 1$ **do**
      if $E(L_{k+b}^{(i-1)}) = 0$ or $E(L_{k+b+d}^{(i-1)}) = 0$ **then**
        $E(L_{k+b}^{(i)}) = E(L_{k+b}^{(i-1)})$
        $E(L_{k+b+d}^{(i)}) = E(L_{k+b+d}^{(i-1)})$
      e$ndif$
      $E(L_{k+b}^{(i)}) = \phi^{-1}(1 - (1 - \phi(E(L_{k+b}^{(i-1)})))(1-$
        $\phi(E(L_{k+b+d}^{(i-1)}))))$
      $E(L_{k+b+d}^{(i)}) = E(L_{k+b}^{(i-1)})E(L_{k+b+d}^{(i-1)})$
    e$ndfor$
  e$ndfor$
e$ndfor$
$F = Find\ indices\ of\ smallest\ elements\ (E[L], K)$
return F

each curve, Bit Error Rate and Frame Error Rate respectively. We adopt BPSK signaling over the AWGN channel for the evaluation. In the simulations, SC and SCL decoders were considered with randomly generated codewords and different code rates. In Fig.8 we compare NUPGA for shortening with [9] using an SC decoder, where we show that the performance of the shortened code is inferior to that of the standard code. A similar pattern can be seen in Fig.9 with the SCL decoder [51] and list size $L = 16$. Fig.10 shows the performance over AWGN of NUPGA for shortening with SCL decoding aided by Cyclic Redundancy Check (CRC) with size 24 (CA-SLC) [51] and $L = 16$. In Fig.11 we compare the performance of the proposed NUPGA extension algorithm with the PD and NUPGA shortening algorithms with SCL and $L = 16$.

In the first example, in Fig.8, we show the standard PC [9] called the mother code (MC). In the simulation we use MC with $N = 512$ and rate $R = 1/2$; $M = 320$ with $R = 1/2$. We compare them with CW [38], RQUP [39] and PD [40] in addition to the proposed NUPGA shortening techniques. In Fig.9, we show the performance for $M = 400$ with $R = 1/2$, with CA-SCL decoding with $L = 16$. And in Fig.10, the performance for: $M = 400$, $R = 1/4$, AWGN, CA-SCL, $L = 16$ and CRC with length 24. In Fig.11, we compare the performance of the shortened polar codes $M = 280$ and $k = 128$, the proposed NUPGA extension technique with three other curves: with MC [9] of length

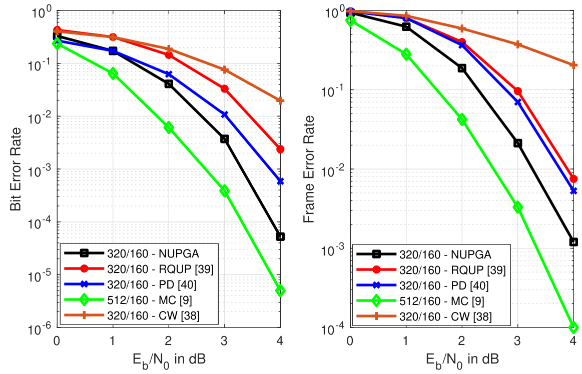

FIGURE 8. Performance of PC with $N = 512$ with $R = 1/2$ and $M = 320$ with $R = 1/2$.

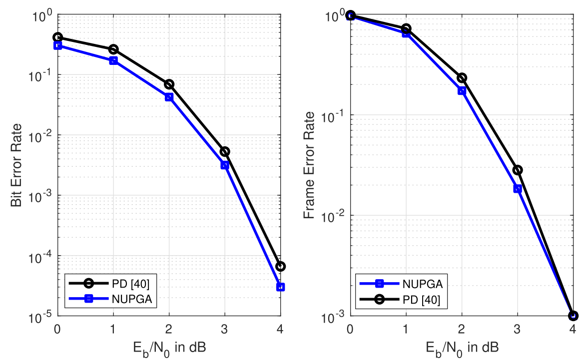

FIGURE 9. PD versus NUPGA: $M = 400$, $K = 200$ and CA-SCL with $L = 16$.

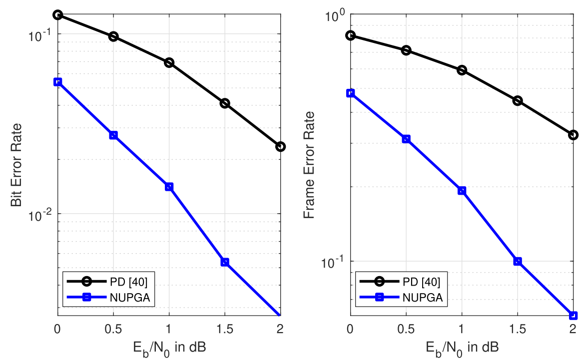

FIGURE 10. Performance of PD and NUPGA shortening algorithms both with $M = 400$ and $K = 50$ using CA-SCL with $L = 16$ and $CRC = 24$.

$N = 256$, the PD [40] algorithm and the NUPGA shortening under CA-SCL with $L = 16$ and CRC with length 24. The results show that the NUPAG extension technique outperforms the NUPGA shortening and the PD algorithms using list decoding with CRC. The shortening methods proposed by [38], [39] and [40] have a computational complexity of $\mathcal{O}$(P+K+NlogN) [27].Therefore, we can estimate the computational complexity of NUPGA at $\mathcal{O}$(P+K+2NlogN).

We notice in all simulations that the proposed NUPGA technique has gains in performance in the scenarios studied. As shown in Fig.10, we can see that for low rates under list decoding, the gain tends to be greater. In Fig.8 we observe that

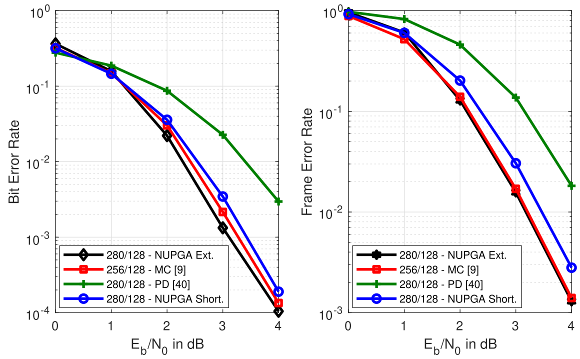

FIGURE 11. Performance of NUPGA extension with PD and NUPGA shortening, decoder CA-SCL with $L = 16$ and CRC with length 24.

the performance gain of NUPGA is of the order of 0.5 dB as compared to the approach of CW [38]. In Fig.9, we verify that the gain obtained by NUPGA is less than 0.1dB. In Fig.10 we notice that the gain is up to 1.2dB, which indicates that the NUPGA is advantageous for low rates. In Fig.11, we observe that the NUPGA extension algorithm achieves a performance improvement over that obtained by the NUPGA shortening algorithm, which is around 0.1 dB. We can see that according to the curves a modest incremental extension has good BER performance, maintaining the same FER performance as the original code [9].

VII. CONCLUSION

In this work, we propose an efficient technique for PC construction of arbitrary length through the theory of non-uniform channel polarization. For its implementation, we developed the NUPGA based on the generalization of the GA technique. Rate compatibility is maintained and the proposed approach achieves the channel's capacity. With the proposed NUPGA-based shortening and extension algorithms we can design PC for any codelength by re-polarizing the conventional PC. Since the performance of rate-compatible polar codes (including extended, punctured or shortened) deteriorates with the gap in length to the mother code, the proposed schemes considerably outperform existing schemes. Simulations show that the performance of NUPGA-based designs is better than competing techniques.

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