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Linearly Monotonic Convergence and Robustness of P-Type Gain-Optimized Iterative Learning Control for Discrete-Time Singular Systems

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ABSTRACT In this article, the repetitive finite-length linear discrete-time singular system is formulated as an input-output equation by virtue of the lifted-vector method and a gain-optimized P-type iterative learning control profile is architected by sequentially arguing the learning-gain vector in minimizing the addition of the quadratic norm of the tracking-error vector and the weighed quadratic norm of the compensation vector. By virtue of the elementary permutation matrix and the property of the quadratic function, the optimized-gain vector is solved and explicitly expressed by the system Markov matrix and the iteration-wise tracking error. Then the linearly monotonic convergence of the tracking error is derived under the assumption that the initial state of the dynamic subsystem is resettable. Furthermore, for the circumstance that the system parameters uncertainties exist, the quasi scheme is established by replacing the exact system Markov matrix with the approximated one in the optimized gain. Rigorous analysis conveys that the proposed gain-optimized scheme is robust to the system internal disturbance within a suitable range. The validity and effectiveness are demonstrated numerically.

INDEX TERMS Discrete-time singular systems, iterative learning control, linearly monotonic convergence, robustness, the optimized-gain vector, tuning factor.

I. INTRODUCTION

In mathematical, the hybrid differential/difference-algebraic equations constitute of a singular system, where the differential/difference equation describes the faster dynamic subsystem whilst the algebraic equation models the slower static subsystem, respectively [1]. Occasionally, a singular system is considered as a descriptor system, a semi-state system or a generalized system, etc. For the reason that the idea recommended, the dynamical behavior of a singular system has been recognized as a more reasonable feature in diverse areas such as the electricity grid [2], economy [3], power systems [4], robots science [5], chemical engineering [6] and biology [7], etc. Basing on the hybrid dynamic-static character of the singular systems, the essential and conceptual controllability and the observability [2], Lyapunov stability [8], the robust stability and the stabilization [9], and the optimal control [10], etc., have been adapted. Because of the complexity, it encounters quite difficult challenges for a singular system to examine the scientific, artificial, and mathematical deeds in pursuing efficacious control schemes.

For the desired trajectory tracking issue of a repetitively operational system, the iterative learning control (ILC) has been acknowledged as a fairly possible and effective strategy in the control perspective. The notion of ILC is inferred from the human being's learning experience while the same job is done repetitively [11] by investigating the consignment implementation mechanism in former repetitions. The primary concern of a proficient ILC is the compensation technique that can adjust the reference signal for the improvement of the output tracking [12]. The traditional compensation ideas for ordinary systems are the tracking-error feed-forward or feedback modes, including the P-type, PD-type, or D-type profiles [13], [16]. The ILC has attracted significant attention due to the straightforward narrative and less prior to the system knowledge [14]–[18].

Following up the ILC investigations for ordinary systems purely described by difference or differential equations,

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the ILC studies for singular systems have earned many efforts covering an open and a closed-loop ILC technique using the standard dynamic decomposition, the asymptotic convergence in the sense of lambda-norm measurement [19], [20] a closed-loop PD-type ILC technique via an initial state rectification [21], a D-type ILC handling random initial states [22], the P-type ILCs for singular impulsive system and distributed parameter systems [23], [24] and an iterative learning controller with packet loss [25] and so on.

By looking at the above-mentioned ILCs for singular systems, it is evident that the key ILC schemes are the traditional P-type or PD-type profiles with the learning gain being constant. Since the issues do not give a handy way for determining the learning gain, the learning program is usually made through experience. Furthermore, the only asymptotic convergence does not meet the practical requirement for the tracking performance assessment as it will likely trigger an overshoot of the tracking error in the beginning iterations and subsequently causes the system to hang. It can interfere with the practical implementation of the established ILCs. In addition, for improvement of the tracking accuracy and the convergence speed, a gain-regulative P-type ILC scheme for a singular system with measurement noise is proposed [26]. However, the gain regulation is implemented by pre-multiplying a diagonal matrix formed by a non-linear function with a constant learning gain matrix, which is hardly acceptable due to the acquisition of the fixed learning gain.

Indeed, some experimental system knowledge containing the design, selection of the parameters or the impulse response could be accessible through trials before the control method is implemented. In context, the known system information is workable to optimize the traditional ILC to overcome the deficiency of the fixed learning gain ILC mode. The current study about the optimal ILC has focused on ordinary systems. The discrete-time optimal ILC without analytical convergence has stemmed from the classical mathematical optimization methods [27]. Later, Owens *et al.* have developed the idea to norm-optimal ILC (NOILC) scheme for linear continuous-time systems, where the sequential objective function is formulated as the addition of the quadratic norm of the tracking-error vector and the weighed quadratic norm of the incremental-input vector of two consecutive iterations. The argued control input originally manifests a feedback mode that needs the complex calculation to solve the co-state system based on the matrix Riccati differential equation [28]. Importantly, it is definite that the finite-length feature of the linear discrete-time ILC system allows the dynamics to be transformed via the lifted-vector-matrix technique as an algebraic input-output transmission. This makes the NOILC advancement and modification spectacular and distinctive. The diversities are including the acceleration technique [29], the non-minimum phase effect [30], the auxiliary optimization [31], the application extension [32], [33] as well as the data-driven efficiency [34]. The existing Owens-type NOILCs are splendid indeed. However, the reluctance is no profound conversation about the rate of convergence. Simultaneously, a parameter-optimal ILC (POILC) with iteration-wise scalar learning gain has been designed in [35], where the performance index is the combination of the quadratic norm of the tracking-error vector and quadratic absolution of the scalar learning gain and the argument is the learning gain of the traditional feed-forward D-type ILC. Though the mechanism is originated from the ILC profile, the declining of the tracking error is inefficient due to the single-dimensional gain optimization. Progressively, a full-dimensional gain optimization has been raised, where the performance index is assessed as the additive quadratic norms of the tracking-error vector and the weighed learning-gain vector [12]. The idea is no doubt innovative. However, the convergence is incomplete for the case when some components of the tracking-error vector are zeros. Fortunately, an inspiring breakout is the rigorous strictly monotonic convergence of the full-dimensional parameter-optimal ILC with the iteration-wise tuning factor via an algebraic approach [36]. As a matter of fact is that the NOILC and POILC are coupled as linear quadratic multi-parameter optimization on behalf of the general concepts of NOILC and POILC [37].

Although the above NOILCs and POILCs have originally focused on ordinary systems, it is referential for the gain-optimized ILC to be utilizable for singular systems. For the regard, by reformulating the discrete-time difference-algebraic singular systems as an algebraic input-output transmission, a gain-adaptive ILC (GAILC) has been explored where the minimization problem is consisting of the quadratic norm of the tracking-error vector and the argument is the P-type iteration-time-wise learning-gain vector [38]. The optimal learning gain is solved in an explicit form and the monotonic convergence has been derived for the circumstance that the initial state is resettable and the exact parameters of the system are available. Though the convergence result seems perfect, the practical implementation is hardly realizable as it requires ultimate compensation as no compensation cost is considered. For usual implementation ease, the possible way is to evaluate the reduction of the tracking error and the compensation cost in a tradeoff manner. Motivated by the issue, this paper explores a P-type gain-optimized iterative learning control (GOILC) strategy for a linear discrete-time singular system. The scheme argues the sequential learning-gain vector while minimizing the sequential performance index composed by the additive quadratic tracking-error vector and the weighed quadratic compensation term by an iteration-wise tuning factor. The target is to explicate the optimal learning-gain vector and the linearly monotonic convergence. The subsequent work is to analyze the robustness of the GOILC to the system parameters uncertainties, respectively. By taking advantage of the permutation transformation and the matrix theory, the paper clarifies the existence and uniqueness of the iteration-time-variable learning-gain vector and deduces the convergence factor for linearly monotonic convergence in a rigorous manner, under the assumptions that the initial state of the dynamic subsystem is resettable and the system Markov parameters are precisely available. The paper also derives the robustness of the proposed GOILC to the system parameters uncertainties in a direct time-domain manner.

The remaining paper is arranged as follows. Section II formulates the discrete-time dynamic-static singular system as an input-output form and presents a gain-optimized iterative learning control scheme. Section III focuses on the solution of the optimized learning-gain vector and the linearly monotonic convergence of the GOILC under the assumption that the initial states are resettable. Section IV discusses the robustness of the GOILC to the system parameters uncertainties. The effectiveness and the validity are numerically verified in section V and the conclusion of the work is addressed in Section VI.

II. SYSTEM REFORMULATION AND P-TYPE GAIN-OPTIMIZED ITERATIVE LEARNING CONTROL STRATEGY

Consider a class of repetitive linear discrete-time single-input-single-output singular systems taking the form of

$$\begin{cases} \begin{bmatrix} \mathbf{I}_p & \mathbf{0}_{12} \\ \mathbf{0}_{21} & \mathbf{0}_{22} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}}_k(n+1) \\ \hat{\boldsymbol{x}}_k(n+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}}_k(n) \\ \hat{\boldsymbol{x}}_k(n) \end{bmatrix} \\ \qquad + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} \boldsymbol{u}_k(n), \\ y_k(n) = [\mathbf{C}_1 \mathbf{C}_2] \begin{bmatrix} \tilde{\boldsymbol{x}}_k(n) \\ \hat{\boldsymbol{x}}_k(n) \end{bmatrix}, n \in D; \end{cases} \tag{1}$$

where $\mathbf{I}_p$ denotes a p-th-order identity matrix, the subscript k presents the repetition index, $\tilde{\boldsymbol{x}}_k(n) \in R^p$ stands for the k-th-repetition p-dimensional state of the dynamic subsystems while $\hat{\boldsymbol{x}}_k(n) \in R^q$ is the k-th-repetition q-dimensional state of the static subsystems, respectively. In addition, $\mathbf{0}_{12}$, $\mathbf{0}_{21}$ and $\mathbf{0}_{22}$ represent zero matrices together with $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{21}$, $\mathbf{A}_{22}$, $\mathbf{B}_1$ and $\mathbf{B}_2$ are matrices with appropriate dimensions, respectively. $D = \{0, 1, \cdots, N-1\}$ is the set of the discrete-time variable n. In this paper, we propose the learning strategies under the assumption that the matrix $\mathbf{A}_{22}$ is non-singular satisfying $\mathbf{C}_2\mathbf{A}_{22}^{-1}\mathbf{B}_2 \neq 0$ and the initial state of the system (1) with respect to the dynamic subsystems being $\tilde{x}_k(0) = 0$.

Let

$$\boldsymbol{x}_k(n) = \begin{bmatrix} \tilde{\boldsymbol{x}}_k(n) \\ \hat{\boldsymbol{x}}_k(n) \end{bmatrix}, \bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\mathbf{A}_{11} & -\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\mathbf{A}_{12} \end{bmatrix},$$

$$\bar{\mathbf{B}}_1 = \begin{bmatrix} \mathbf{B}_1 \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\mathbf{B}_1 \end{bmatrix}, \bar{\mathbf{B}}_0 = \begin{bmatrix} 0 \\ -\mathbf{A}_{22}^{-1}\mathbf{B}_2 \end{bmatrix},$$

$$\tilde{\mathbf{B}}_1 = \left(\bar{\mathbf{A}}\bar{\mathbf{B}}_0 + \bar{\mathbf{B}}_1\right).$$

The state equation of the system (1) is inferred as

$$\boldsymbol{x}_k(n+1) = \bar{\mathbf{A}}\boldsymbol{x}_k(n) + \bar{\mathbf{B}}_1\boldsymbol{u}_k(n) + \bar{\mathbf{B}}_0\boldsymbol{u}_k(n+1). \tag{2}$$

Recursively, the output becomes

$$y_k(n) = \mathbf{C}\boldsymbol{x}_k(n) = \mathbf{C}\bar{\mathbf{A}}^{n-1}\tilde{\mathbf{B}}_1 u_k(0) + \mathbf{C}\bar{\mathbf{A}}^{n-2}\tilde{\mathbf{B}}_1 u_k(1) + \cdots + \mathbf{C}\tilde{\mathbf{B}}_1 u_k(n-1) + \mathbf{C}\bar{\mathbf{B}}_0 u_k(n), \text{ for all } n \geq 0. \tag{3}$$

Denote

$$\boldsymbol{y}_k = [y_k(0), y_k(1), y_k(2), \cdots, y_k(N)]^{\mathrm{T}},$$
$$\boldsymbol{u}_k = [u_k(0), u_k(1), u_k(2), \cdots, u_k(N)]^{\mathrm{T}},$$
$$m_0 = \mathbf{C}\bar{\mathbf{B}}_0, m_j = \mathbf{C}\bar{\mathbf{A}}^{j-1}\tilde{\mathbf{B}}_1, \quad j = 1, 2, \cdots, N,$$
$$\mathbf{M} = \begin{bmatrix} m_0 & 0 & 0 & \cdots & 0 \\ m_1 & m_0 & 0 & \cdots & 0 \\ m_2 & m_1 & m_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_N & m_{N-1} & m_{N-2} & \cdots & m_0 \end{bmatrix}.$$

Here $\mathbf{M}$ is a Toeplitz matrix composed of the impulse response sequence $\{m_0, m_1, \cdots, m_N\}$ of the system (3) and is usually named the Markov matrix of the singular system (1), assuming that $m_0 = \mathbf{C}\bar{\mathbf{B}}_0$ is nonzero.

Then, the input-output of the system (1) is lifted as

$$\boldsymbol{y}_k = \mathbf{M}\boldsymbol{u}_k. \tag{4}$$

Remark 1: It is worthy to notice that the singular system (1) is composed of the dynamic subsystem and the static subsystem in hybrid form. As the static subsystem is memoryless, the sampling instant n for the state $\hat{\boldsymbol{x}}_k(n)$ of the static subsystems lacks the end instant N if the sampling set is $D = \{0, 1, \cdots, N-1\}$. In order to present the hybrid dynamic-static equation (1) in a lifted-vector-matrix form (4), the paper acquiesces the sampling range for the sub-state $\hat{\boldsymbol{x}}_k(n)$ is $n \in D^+ = D \cup \{N\}$. Throughout the paper, the consideration of the sampling variable is $n \in D$ for the dynamic subsystem whilst $n \in D^+ = D \cup \{N\}$ for the static subsystem, respectively.

Besides, due to the static subsystem, the output equation (3) conveys that the sampling instant of the output $y_k(n)$ is synchronous with that of the control input $u_k(n)$ for the case when $\mathbf{C}_2\mathbf{A}_{22}^{-1}\mathbf{B}_2 \neq 0$, namely, $\mathbf{C}\bar{\mathbf{B}}_0 \neq 0$. This implies there is no leading of the sampling number of the output $y_k(n)$ in (3) to the input $u_k(n)$ for all $n \in D^+$. It is no other but the well-known concept that the relative degree of the singular systems (4) is zero while $\mathbf{C}\bar{\mathbf{B}}_0$ is nonzero. This essential and distinguishing attribute of the singular systems makes the learning compensation mode to differ from the derivative-type method for the ordinary systems.

In this paper, we will construct a P-type gain-optimized iterative learning control for a class of singular systems (4) under the assumption that $m_0 = \mathbf{C}\bar{\mathbf{B}}_0$ is nonzero.

Given that $\boldsymbol{y}_d = [y_d(0), y_d(1), \cdots, y_d(N)]^{\mathrm{T}}$ is a predetermined desired trajectory. While the systems (4) attempt to track $\boldsymbol{y}_d$ to enable the tracking error $\boldsymbol{e}_k = y_d - y_k$ to vanish as the repetition goes on, an iterative learning control strategy has established as

$$\boldsymbol{u}_1 : \text{arbitrarily given};$$
$$\boldsymbol{u}_{k+1} = \boldsymbol{u}_k + \mathbf{E}_k\boldsymbol{\Lambda}_k, \ k = 1, 2, \cdots. \tag{5}$$

Here

$$\boldsymbol{e}_k = \boldsymbol{y}_d - \boldsymbol{y}_k = [e_k(0), e_k(1), \ldots, e_k(N)]^{\mathrm{T}},$$
$$\mathbf{E}_k = \mathrm{diag}(\boldsymbol{e}_k) = \begin{bmatrix} e_k(0) & 0 & \cdots & 0 \\ 0 & e_k(1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_k(N) \end{bmatrix},$$
$$\boldsymbol{\Lambda}_k = [\boldsymbol{\Lambda}_k(0), \boldsymbol{\Lambda}_k(1), \cdots, \boldsymbol{\Lambda}_k(N)]^{\mathrm{T}}.$$

In particular, $\boldsymbol{\Lambda}_k$ is termed as the learning-gain vector that will solve by the minimization problems composed of

$$\arg\min_{\boldsymbol{\Lambda}_k} J(\boldsymbol{\Lambda}_k) = \|\boldsymbol{e}_{k+1}\|^2 + \eta_k \cdot \|\boldsymbol{u}_{k+1} - \boldsymbol{u}_k\|^2. \tag{6}$$

In the minimization problem (6), η_k is a sequential tuning factor that shows the concerning ratio of the compensation cost $\|\boldsymbol{u}_{k+1} - \boldsymbol{u}_k\|^2 = \|\mathbf{E}_k\boldsymbol{\Lambda}_k\|^2$ to the tracking-error energy $\|\boldsymbol{e}_{k+1}\|^2$ satisfying $0 \le \eta_k \le \eta_0$, for all $k = 1, 2, \cdots$. Here, η_0 is a constant upper bound of the tuning factor, which means that the concerning compensation cost is confined in an allowable range.

Remark 2: In the ILC (5), it is found that for each sampling instant n, $n \in D^+ = D \cup \{N\}$ we arrive $u_{k+1}(n) = u_k(n) + \boldsymbol{\Lambda}_k(n)e_k(n)$. It implies that the sampling discrete-time instant n of the compensation term $\boldsymbol{\Lambda}_k(n)e_k(n)$ is synchronous with the compensated input $u_k(n)$. Additionally, as the minimization problem (6) determines the learning-gain vector $\boldsymbol{\Lambda}_k$, it is reasonable to term the ILC (5) as a P-type gain-optimized iterative learning control (P-type GOILC) scheme.

For deriving the optimized-gain vector $\boldsymbol{\Lambda}_k$ of the minimization problem (6), what follows are some necessary propositions.

Denote two functions as

$$f(z) = a_0 + a_1 z + \cdots + a_m z^m, z \in [0, +\infty),$$
$$g(z) = a_{-l}z^{-l} + \cdots + a_{-1}z^{-1} + a_0 + a_1 z + \cdots + a_m z^m,$$
$$z \in (0, +\infty),$$

where m and l are positive integers and $a_{-l}, \cdots, a_{-1}, a_0, a_1, \cdots, a_m$ are constant coefficients, respectively.

Proposition 1: For a square matrix $\mathbf{P}$, if there exists a number $\lambda(\mathbf{P})$ and a vector ξ so that $\mathbf{P}\xi = \lambda(\mathbf{P})\xi$, then $f(\mathbf{P})\xi = f(\lambda(\mathbf{P}))\xi$ holds. In particular, if the matrix $\mathbf{P}$ is invertible, then $\lambda(\mathbf{P}) \neq 0$ and $g(\mathbf{P})\xi = g(\lambda(\mathbf{P}))\xi$. Here $\lambda(\mathbf{P})$ is noted as an eigenvalue of the matrix $\mathbf{P}$.

Throughout the context, denote $\lambda_{\min}(\mathbf{P}) = \min\{\lambda(\mathbf{P})\}$ and $\lambda_{\max}(\mathbf{P}) = \max\{\lambda(\mathbf{P})\}$, respectively.

Proposition 2: For any two invertible matrices $\mathbf{P}$ and $\mathbf{Q}$ with identical dimensions, the equation $\mathbf{P} - \mathbf{Q} = \mathbf{P}\left(\mathbf{Q}^{-1} - \mathbf{P}^{-1}\right)\mathbf{Q}$ identically holds.

Proposition 3: For any two $n \times n -$ dimensional real symmetric semi-positive definite matrices $\mathbf{P}$ and $\mathbf{Q}$ with identical dimensions, the inequality $\lambda_{\max}(\mathbf{P} + \mathbf{Q}) \le \lambda_{\max}(\mathbf{P}) + \lambda_{\max}(\mathbf{Q})$ holds.

Proposition 4: For a continuous function $h(z) = \frac{z}{1+z}$, $z \in [0, +\infty)$, $h(z)$ is monotonously increasing.

Definition 1: A nonnegative sequence $\{e_k \ge 0, k = 1, 2, \cdots\}$ is said to be linearly monotonically convergent, means that $e_{k+1} < e_k$, $\lim_{k\to+\infty} e_k = 0$ and $\lim_{k\to+\infty}\sup \frac{e_{k+1}}{e_k} = \rho < 1$.

III. SOLUTION OF THE OPTIMIZED-GAIN VECTOR AND THE LINEARLY MONOTONE CONVERGENCE

Theorem 1: Assume that the input $\boldsymbol{u}_k$ in (4) is updated by the input $\boldsymbol{u}_{k+1}$ generated by (5), the optimized-gain vector $\boldsymbol{\Lambda}_k$ of the minimization problem (6) is existent and unique.

Proof: From the GOILC (5), we have

$$\begin{aligned} \boldsymbol{e}_{k+1} &= \boldsymbol{y}_d - \boldsymbol{y}_{k+1} = \boldsymbol{y}_d - \boldsymbol{y}_k - \left(\boldsymbol{y}_{k+1} - \boldsymbol{y}_k\right) \\ &= \boldsymbol{e}_k - \mathbf{M}\mathbf{E}_k\boldsymbol{\Lambda}_k. \end{aligned} \tag{7}$$

Substituting the inner product of the equality (7) into the performance index (6), one yield

$$\begin{aligned} J(\boldsymbol{\Lambda}_k) = {} & \boldsymbol{\Lambda}_k^{\mathrm{T}}\mathbf{E}_k\left(\eta_k\mathbf{I} + \mathbf{M}^{\mathrm{T}}\mathbf{M}\right)\mathbf{E}_k\boldsymbol{\Lambda}_k \\ & -2\boldsymbol{e}_k^{\mathrm{T}}\mathbf{M}\mathbf{E}_k\boldsymbol{\Lambda}_k + \boldsymbol{e}_k^{\mathrm{T}}\boldsymbol{e}_k. \end{aligned} \tag{8}$$

It is seen that the objective function $J(\boldsymbol{\Lambda}_k)$ is a quadratic function concerning the argument vector $\boldsymbol{\Lambda}_k$ and its Hesse matrix is $\mathbf{E}_k\left(\eta_k\mathbf{I} + \mathbf{M}^{\mathrm{T}}\mathbf{M}\right)\mathbf{E}_k$, which is symmetric semi-positive definite. Then, if the objective function $J(\boldsymbol{\Lambda}_k)$ exists a unique stationary point, it must be the optimal solution.

Letting the gradient $\nabla J(\boldsymbol{\Lambda}_k) = \frac{\partial J(\boldsymbol{\Lambda}_k)}{\partial \boldsymbol{\Lambda}_k} = 0$ makes

$$\mathbf{E}_k\left(\eta_k\mathbf{I} + \mathbf{M}^{\mathrm{T}}\mathbf{M}\right)\mathbf{E}_k\boldsymbol{\Lambda}_k = \mathbf{E}_k\mathbf{M}^{\mathrm{T}}\boldsymbol{e}_k. \tag{9}$$

Observe that in the equation(9), the matrix $\mathbf{E}_k$ is diagonal and the diagonal elements match the components of the tracking-errors vector. For the reason that the GOILC (5) is to pursue the tracking error vanishing, it much possible that a few components of the vector $\boldsymbol{e}_k$ are null as the iteration is going on. There is no difference, however, the matrix is in all likelihood singular. Within the condition, the equation (9) is not absolutely determinative. The solution wishes to derive for the instances when the matrix $\mathbf{E}_k$ is non-singular and singular, respectively.

Case 1: The matrix $\mathbf{E}_k$ is non-singular.

Equation (9) obtains

$$\boldsymbol{\Lambda}_k = \mathbf{E}_k^{-1}\left(\eta_k\mathbf{I} + \mathbf{M}^{\mathrm{T}}\mathbf{M}\right)^{-1}\mathbf{M}^{\mathrm{T}}\boldsymbol{e}_k.$$

Keep in mind that the value $m_0 = \mathbf{C}\bar{\mathbf{B}}_0$ being nonzero means that the Toeplitz matrix $\mathbf{M}$ is invertible. Accordingly, the above equation is comparable to

$$\boldsymbol{\Lambda}_k = \mathbf{E}_k^{-1}\mathbf{M}^{-1}\left(\mathbf{I} + \eta_k\mathbf{M}^{-\mathrm{T}}\mathbf{M}^{-1}\right)^{-1}\boldsymbol{e}_k. \tag{10}$$

Case 2: If the matrix $\mathbf{E}_k$ is singular.

Without loss of generality, suppose that rank $(\mathbf{E}_k) = r_k < N + 1$ with $e_k(h_p) \neq 0$ for $p = 1, 2, \cdots, r_k$ and $e_k(s_q) = 0$ for $q = 1, 2, \cdots, N + 1 - r_k$ satisfying $h_1 < h_2 < \cdots < h_{r_k}$ and $s_1 < s_2 < \cdots < s_{N+1-r_k}$.

Denote

$$\tilde{e}_k = \left[e_k(h_1), e_k(h_2), \cdots, e_k\left(h_{r_k}\right)\right]^{\mathrm{T}}, \quad (11)$$
$$\bar{e}_k = \left[e_k(s_1), e_k(s_2), \cdots, e_k\left(s_{N+1-r_k}\right)\right]^{\mathrm{T}}, \quad (12)$$
$$\tilde{\mathbf{E}}_k = \mathrm{diag}(\tilde{e}_k), \quad (13)$$
$$\bar{\mathbf{E}}_k = \mathrm{diag}(\bar{e}_k), \quad (14)$$
$$\tilde{\mathbf{\Lambda}}_k = \left[\Lambda_k(h_1), \Lambda_k(h_2), \cdots, \Lambda_k\left(h_{r_k}\right)\right]^{\mathrm{T}}, \quad (15)$$
$$\bar{\mathbf{\Lambda}}_k = \left[\Lambda_k(s_1), \Lambda_k(s_2), \cdots, \Lambda_k\left(s_{N+1-r_k}\right)\right]^{\mathrm{T}}, \quad (16)$$
$$\mathbf{\Phi}_k = [\boldsymbol{\delta}_{h_1}, \boldsymbol{\delta}_{h_2}, \cdots, \boldsymbol{\delta}_{h_{r_k}}, \boldsymbol{\delta}_{s_1}, \boldsymbol{\delta}_{s_2}, \cdots, \boldsymbol{\delta}_{s_{N+1-r_k}}]. \quad (17)$$

It is understandable that the matrix $\tilde{\mathbf{E}}_k$ is invertible and $\bar{\boldsymbol{e}}_k = 0$ together with $\bar{\mathbf{E}}_k = \mathbf{0}$. Specifically, in the representation of $\mathbf{\Phi}_k$, $\boldsymbol{\delta}_i$ gives the $(N+1)$-dimensional unit vector whose the i-th element is unit and the others are zeros, for $i = h_1, h_2, \cdots, h_{r_k}, s_1, s_2, \cdots s_{N+1-r_k}$. It indicates that $\mathbf{\Phi}_k$ is a permutation matrix and orthogonal fulfilling $\mathbf{\Phi}_k^{-1} = \mathbf{\Phi}_k^{\mathrm{T}}$. For eases, $\mathbf{\Phi}_k$ is noted as a column-permutation matrix while $\mathbf{\Phi}_k^{\mathrm{T}}$ is a row-permutation matrix, respectively.

It is no hard to get through calculation

$$\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{E}_k\mathbf{\Phi}_k = \begin{bmatrix} \tilde{\mathbf{E}}_k & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{E}}_k \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{E}}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (18)$$
$$\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{\Lambda}_k = \begin{bmatrix} \tilde{\mathbf{\Lambda}}_k \\ \bar{\mathbf{\Lambda}}_k \end{bmatrix}, \quad (19)$$
$$\mathbf{\Phi}_k^{\mathrm{T}}\boldsymbol{e}_k = \begin{bmatrix} \tilde{\boldsymbol{e}}_k \\ \bar{\boldsymbol{e}}_k \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{e}}_k \\ 0 \end{bmatrix}, \quad (20)$$
$$\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{M}\mathbf{\Phi}_k = \begin{bmatrix} \tilde{\mathbf{M}}_k & \widehat{\mathbf{M}}_k \\ \bar{\mathbf{M}}_k & \check{\mathbf{M}}_k \end{bmatrix}. \quad (21)$$

Here

$$\tilde{\mathbf{M}}_k = \begin{bmatrix} m_0 & 0 & \cdots & 0 \\ m_{h_2-h_1} & m_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ m_{h_{r_k}-h_1} & m_{h_{r_k}-h_2} & \cdots & m_0 \end{bmatrix} \quad (22)$$

is an $r_k \times r_k$-dimensional square matrix while $\bar{\mathbf{M}}_k$, $\widehat{\mathbf{M}}_k$ and $\check{\mathbf{M}}_k$ are matrices with appropriate dimensions respectively. Visually, the matrix $\tilde{\mathbf{M}}_k$ is invertible by the assumption $m_0 = \mathbf{C}\bar{\mathbf{B}}_0 \neq 0$.

Pre-multiplying equation (9) by $\mathbf{\Phi}_k^{\mathrm{T}}$ and in view of $\mathbf{\Phi}_k^{-1} = \mathbf{\Phi}_k^{\mathrm{T}}$ provides

$$\left(\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{E}_k\mathbf{\Phi}_k\right)\left[\eta_k\mathbf{I} + \left(\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{M}\mathbf{\Phi}_k\right)^{\mathrm{T}}\left(\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{M}\mathbf{\Phi}_k\right)\right]$$
$$\left(\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{E}_k\mathbf{\Phi}_k\right)\left(\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{\Lambda}_k\right) = \left(\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{E}_k\mathbf{\Phi}_k\right)\left(\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{M}\mathbf{\Phi}_k\right)^{\mathrm{T}}\mathbf{\Phi}_k^{\mathrm{T}}e_k. \quad (23)$$

By means of denotations (18)-(21), the equation (23) converts into

$$\begin{bmatrix} \tilde{\mathbf{E}}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\left(\eta_k\mathbf{I} + \begin{bmatrix} \tilde{\mathbf{M}}_k^{\mathrm{T}} & \bar{\mathbf{M}}_k^{\mathrm{T}} \\ \widehat{\mathbf{M}}_k^{\mathrm{T}} & \check{\mathbf{M}}_k^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \tilde{\mathbf{M}}_k & \widehat{\mathbf{M}}_k \\ \bar{\mathbf{M}}_k & \check{\mathbf{M}}_k \end{bmatrix}\right)\begin{bmatrix} \tilde{\mathbf{E}}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{\mathbf{\Lambda}}_k \\ \bar{\mathbf{\Lambda}}_k \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{E}}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \tilde{\mathbf{M}}_k^{\mathrm{T}} & \bar{\mathbf{M}}_k^{\mathrm{T}} \\ \widehat{\mathbf{M}}_k^{\mathrm{T}} & \check{\mathbf{M}}_k^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \tilde{e}_k \\ 0 \end{bmatrix}. \quad (24)$$

Equivalently, the equation (24) becomes

$$\tilde{\mathbf{E}}_k\left(\eta_k\mathbf{I}_{r_k} + \left(\tilde{\mathbf{M}}_k^{\mathrm{T}}\tilde{\mathbf{M}}_k + \bar{\mathbf{M}}_k^{\mathrm{T}}\bar{\mathbf{M}}_k\right)\right)\tilde{\mathbf{E}}_k\tilde{\mathbf{\Lambda}}_k = \tilde{\mathbf{E}}_k\tilde{\mathbf{M}}_k^{\mathrm{T}}\tilde{\boldsymbol{e}}_k. \quad (25)$$
$$\mathbf{0}\bar{\mathbf{\Lambda}}_k = 0. \quad (26)$$

As the matrix $\left(\eta_k\mathbf{I}_{r_k} + \tilde{\mathbf{M}}_k^{\mathrm{T}}\tilde{\mathbf{M}}_k + \bar{\mathbf{M}}_k^{\mathrm{T}}\bar{\mathbf{M}}_k\right)$ is positive definite and thus invertible, for simplification, denote

$$\mathbf{\Psi}_k = \left(\eta_k\mathbf{I}_{r_k} + \left(\tilde{\mathbf{M}}_k^{\mathrm{T}}\tilde{\mathbf{M}}_k + \bar{\mathbf{M}}_k^{\mathrm{T}}\bar{\mathbf{M}}_k\right)\right)^{-1}. \quad (27)$$

It is evident that $\mathbf{\Psi}_k$ is symmetric. Then the solution to the equation (25) gives rise to

$$\tilde{\mathbf{\Lambda}}_k = \tilde{\mathbf{E}}_k^{-1}\mathbf{\Psi}_k\tilde{\mathbf{M}}_k^{\mathrm{T}}\tilde{\boldsymbol{e}}_k. \quad (28)$$

In the view of denotations (12), (14) and (16), it can be seen that $\bar{\mathbf{\Lambda}}_k$ is coupled with $\bar{\mathbf{E}}_k = \mathbf{0}$. It is thus congruent to let

$$\bar{\mathbf{\Lambda}}_k = 0. \quad (29)$$

Considering equations (28), (29) and denotation (19) together with the fact $\mathbf{\Phi}_k^{\mathrm{T}} = \mathbf{\Phi}_k^{-1}$, the solution to the equation (9) is encouraged as

$$\mathbf{\Lambda}_k = \mathbf{\Phi}_k\begin{bmatrix} \tilde{\mathbf{E}}_k^{-1}\mathbf{\Psi}_k\tilde{\mathbf{M}}_k^{\mathrm{T}}\tilde{\boldsymbol{e}}_k \\ 0 \end{bmatrix}. \quad (30)$$

When the formulation (10) is paired with (30), one gets

$$\mathbf{\Lambda}_k = \begin{cases} \mathbf{E}_k^{-1}\mathbf{M}^{-1}\left(\mathbf{I} + \eta_k\mathbf{M}^{-\mathrm{T}}\mathbf{M}^{-1}\right)^{-1}\boldsymbol{e}_k, \\ \qquad \text{if } \mathbf{E}_k \text{is nonsingular;} \\ \mathbf{\Phi}_k\begin{bmatrix} \tilde{\mathbf{E}}_k^{-1}\mathbf{\Psi}_k\tilde{\mathbf{M}}_k^{\mathrm{T}}\tilde{e}_k \\ 0 \end{bmatrix}, \\ \qquad \text{if } \mathbf{E}_k \text{is singular..} \end{cases} \quad (31)$$

This completes the proof.

Theorem 2: Assuming that the singular system (4) repetitively operates over the sampling set D^+ with the value $m_0 = \mathbf{C}\bar{\mathbf{B}}_0$ being nonzero while the initial state $\tilde{x}_k(0)$ of the dynamic subsystem is reset at zero, that is, $\tilde{x}_{k+1}(0) = 0$, for all $k = 1, 2, \cdots$. Under the condition of $0 < \eta_k \leq \eta_0$, for all $k = 1, 2, \cdots$, the P-type GOILC (5) with ITVLGV $\mathbf{\Lambda}_k$ solved by (31) is linearly monotonically convergent, namely, there exists such a constant ρ $(0 \leq \rho < 1)$ that, $\|\boldsymbol{e}_{k+1}\|^2 < \|\boldsymbol{e}_k\|^2$, $\lim_{k\to+\infty}\|\boldsymbol{e}_{k+1}\|^2 = 0$ and $\lim_{k\to+\infty}\sup\frac{\|\boldsymbol{e}_{k+1}\|^2}{\|\boldsymbol{e}_k\|^2} < \rho < 1$.

Proof: Following the derivation of Theorem 1, the proof is split into two cases when the matrix $\mathbf{E}_k$ is non-singular and singular, respectively.

Case 1: The matrix $\mathbf{E}_k$ is non-singular.

From the formulations (4), (5), (9) and (10), we have

$$\boldsymbol{e}_{k+1} = \boldsymbol{y}_d - \boldsymbol{y}_{k+1} = \boldsymbol{y}_d - \boldsymbol{y}_k - \left(\boldsymbol{y}_{k+1} - \boldsymbol{y}_k\right)$$
$$= \boldsymbol{e}_k - \mathbf{M}\mathbf{E}_k\mathbf{\Lambda}_k = \left(\mathbf{I} - \left(\mathbf{I} + \eta_k\mathbf{M}^{-\mathrm{T}}\mathbf{M}^{-1}\right)^{-1}\right)\boldsymbol{e}_k. \quad (32)$$

Here $\mathbf{M}^{-\mathrm{T}} = \left(\mathbf{M}^{\mathrm{T}}\right)^{-1}$.

Denote

$$\mathbf{S}_k = \mathbf{M}^{-\mathrm{T}}\mathbf{M}^{-1}, \tag{33}$$

$$\mathbf{H}_k = \left(\mathbf{I} + \eta_k \mathbf{M}^{-\mathrm{T}}\mathbf{M}^{-1}\right)^{-1} = \left(\mathbf{I} + \eta_k \mathbf{S}_k\right)^{-1}. \tag{34}$$

It is evident that the matrices $\mathbf{S}_k$ and $\mathbf{H}_k$ are real and symmetric.

Then, equality (32) becomes

$$\boldsymbol{e}_{k+1} = \left(\mathbf{I} - \mathbf{H}_k\right)\boldsymbol{e}_k. \tag{35}$$

Calculating inner product for both sides of the equation (35) arrives

$$\|\boldsymbol{e}_{k+1}\|^2 = \boldsymbol{e}_k^{\mathrm{T}}\left(\mathbf{I} - \mathbf{H}_k\right)^2 \boldsymbol{e}_k. \tag{36}$$

From the property of the multi-variable quadratic function, the equation (36) is evaluated as

$$\|\boldsymbol{e}_{k+1}\|^2 \le \lambda_{\max}\left(\left(\mathbf{I} - \mathbf{H}_k\right)^2\right)\|\boldsymbol{e}_k\|^2. \tag{37}$$

From Proposition 2, we have

$$\mathbf{I} - \mathbf{H}_k = \mathbf{I} - \left(\mathbf{I} + \eta_k \mathbf{S}_k\right)^{-1} = \eta_k \mathbf{S}_k \left(\mathbf{I} + \eta_k \mathbf{S}_k\right)^{-1}. \tag{38}$$

From Propositions 1 and 4, the equation (38) makes

$$\lambda_{\max}\left(\left(\mathbf{I} - \mathbf{H}_k\right)^2\right) = \lambda_{\max}\left(\eta_k^2 \mathbf{S}_k^2 \left(\mathbf{I} + \eta_k \mathbf{S}_k\right)^{-2}\right) = \left(\frac{\eta_k \lambda_{\max}\left(\mathbf{S}_k\right)}{1 + \eta_k \lambda_{\max}\left(\mathbf{S}_k\right)}\right)^2. \tag{39}$$

Recall that the matrix $\mathbf{M}$ is invertible. This implies that $\lambda_{\max}\left(\mathbf{S}_k\right) = \lambda_{\max}\left(\mathbf{M}^{-\mathrm{T}}\mathbf{M}^{-1}\right)$ is constantly positive.

Denote

$$\lambda_{\max}\left(\mathbf{S}_k\right) = \lambda_{\max}\left(\mathbf{M}^{-\mathrm{T}}\mathbf{M}^{-1}\right) = \lambda_0.$$

From Proposition 4, the condition of $\eta_k \le \eta_0$ yields

$$\left(\frac{\eta_k \lambda_{\max}\left(\mathbf{S}_k\right)}{1 + \eta_k \lambda_{\max}\left(\mathbf{S}_k\right)}\right)^2 \le \left(\frac{\eta_0 \lambda_0}{1 + \eta_0 \lambda_0}\right)^2. \tag{40}$$

Substituting the formulations (39) and (40) into (37) gives rise to

$$\|\boldsymbol{e}_{k+1}\|^2 \le \sigma_0 \|\boldsymbol{e}_k\|^2, \tag{41}$$

where $\sigma_0 = \left(\frac{\eta_0\lambda_0}{1+\eta_0\lambda_0}\right)^2 < 1$.

Case 2: If the matrix $\mathbf{E}_k$ is singular.

For this case, the inversion $\mathbf{E}_k$ does not exist. Thus we may not directly solve the equation. It needs to adopt a permutation transformation for reforming the equation (7).

Left-multiplying equation (7) by $\mathbf{\Phi}_k^{\mathrm{T}}$ and considering formula (27) and denotations (18)-(21) give rise to

$$\begin{aligned}\mathbf{\Phi}_k^{\mathrm{T}}\boldsymbol{e}_{k+1} &= \begin{bmatrix}\tilde{\boldsymbol{e}}_{k+1}\\ \bar{\boldsymbol{e}}_{k+1}\end{bmatrix} = \mathbf{\Phi}_k^{\mathrm{T}}\boldsymbol{e}_k - \left(\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{M}\mathbf{\Phi}_k\right)\left(\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{E}_k\mathbf{\Phi}_k\right)\left(\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{\Lambda}_k\right)\\ &= \begin{bmatrix}\tilde{\boldsymbol{e}}_k\\ 0\end{bmatrix} - \begin{bmatrix}\tilde{\mathbf{M}}_k & \widehat{\mathbf{M}}_k\\ \bar{\mathbf{M}}_k & \check{\mathbf{M}}_k\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{E}}_k & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{\Lambda}}_k\\ \bar{\mathbf{\Lambda}}_k\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{I}_{r_k} - \tilde{\mathbf{M}}_k\mathbf{\Psi}_k\tilde{\mathbf{M}}_k^{\mathrm{T}}\\ -\bar{\mathbf{M}}_k\mathbf{\Psi}_k\tilde{\mathbf{M}}_k^{\mathrm{T}}\end{bmatrix}\tilde{\boldsymbol{e}}_k = \begin{bmatrix}\mathbf{I}_{r_k} - \tilde{\mathbf{H}}_k\\ -\bar{\mathbf{M}}_k\tilde{\mathbf{M}}_k^{-1}\tilde{\mathbf{H}}_k\end{bmatrix}\tilde{\boldsymbol{e}}_k,\end{aligned} \tag{42}$$

where $\tilde{\mathbf{H}}_k = \tilde{\mathbf{M}}_k\mathbf{\Psi}_k\tilde{\mathbf{M}}_k^{\mathrm{T}}$.

From denotation (27), we have

$$\begin{aligned}\tilde{\mathbf{H}}_k &= \tilde{\mathbf{M}}_k\mathbf{\Psi}_k\tilde{\mathbf{M}}_k^{\mathrm{T}} = \tilde{\mathbf{M}}_k\left(\eta_k\boldsymbol{I}_{r_k} + \left(\tilde{\mathbf{M}}_k^{\mathrm{T}}\tilde{\mathbf{M}}_k + \bar{\mathbf{M}}_k^{\mathrm{T}}\bar{\mathbf{M}}_k\right)\right)^{-1}\tilde{\mathbf{M}}_k^{\mathrm{T}}\\ &= \left(\mathbf{I}_{r_k} + \eta_k\tilde{\mathbf{M}}_k^{-\mathrm{T}}\tilde{\mathbf{M}}_k^{-1} + \tilde{\mathbf{M}}_k^{-\mathrm{T}}\bar{\mathbf{M}}_k^{\mathrm{T}}\bar{\mathbf{M}}_k\tilde{\mathbf{M}}_k^{-1}\right)^{-1}\\ &= \left(\mathbf{I}_{r_k} + \eta_k\tilde{\mathbf{S}}_k + \bar{\mathbf{S}}_k\right)^{-1},\end{aligned} \tag{43}$$

where

$$\tilde{\mathbf{S}}_k = \tilde{\mathbf{M}}_k^{-\mathrm{T}}\tilde{\mathbf{M}}_k^{-1}, \tag{44}$$

$$\bar{\mathbf{S}}_k = \tilde{\mathbf{M}}_k^{-\mathrm{T}}\bar{\mathbf{M}}_k^{\mathrm{T}}\bar{\mathbf{M}}_k\tilde{\mathbf{M}}_k^{-1}. \tag{45}$$

Then $\tilde{\mathbf{S}}_k$ is symmetric positive definite and $\bar{\mathbf{S}}_k$ is symmetric semi-positive definite, respectively.

Then from proposition 2, we get

$$\begin{aligned}\mathbf{I}_{r_k} - \tilde{\mathbf{H}}_k &= \mathbf{I}_{r_k} - \left(\boldsymbol{I}_{r_k} + \eta_k\tilde{\mathbf{S}}_k + \bar{\mathbf{S}}_k\right)^{-1}\\ &= \left(\eta_k\tilde{\mathbf{S}}_k + \bar{\mathbf{S}}_k\right)\left(\boldsymbol{I}_{r_k} + \eta_k\tilde{\mathbf{S}}_k + \bar{\mathbf{S}}_k\right)^{-1}\\ &= \left(\eta_k\tilde{\mathbf{S}}_k + \bar{\mathbf{S}}_k\right)\tilde{\mathbf{H}}_k = \eta_k\tilde{\mathbf{S}}_k\tilde{\mathbf{H}}_k + \bar{\mathbf{S}}_k\tilde{\mathbf{H}}_k.\end{aligned} \tag{46}$$

Then

$$\begin{aligned}&\left(\mathbf{I}_{r_k} - \tilde{\mathbf{H}}_k\right)\left(\mathbf{I}_{r_k} - \tilde{\mathbf{H}}_k\right)\\ &= \left(\mathbf{I}_{r_k} - \tilde{\mathbf{H}}_k\right) - \tilde{\mathbf{H}}_k\left(\mathbf{I}_{r_k} - \tilde{\mathbf{H}}_k\right)\\ &= \left(\mathbf{I}_{r_k} - \tilde{\mathbf{H}}_k\right) - \eta_k\tilde{\mathbf{H}}_k\tilde{\mathbf{S}}_k\tilde{\mathbf{H}}_k - \tilde{\mathbf{H}}_k\bar{\mathbf{S}}_k\tilde{\mathbf{H}}_k.\end{aligned} \tag{47}$$

In addition,

$$\begin{aligned}&\left(\bar{\mathbf{M}}_k\tilde{\mathbf{M}}_k^{-1}\tilde{\mathbf{H}}_k\right)^{\mathbf{T}}\left(\bar{\mathbf{M}}_k\tilde{\mathbf{M}}_k^{-1}\tilde{\mathbf{H}}_k\right)\\ &= \tilde{\mathbf{H}}_k\left(\tilde{\mathbf{M}}_k^{-\mathrm{T}}\bar{\mathbf{M}}_k^{\mathrm{T}}\bar{\mathbf{M}}_k\tilde{\mathbf{M}}_k^{-1}\right)\tilde{\mathbf{H}}_k = \tilde{\mathbf{H}}_k\bar{\mathbf{S}}_k\tilde{\mathbf{H}}_k.\end{aligned} \tag{48}$$

Calculating inner product to both sides of the equation (42) and then substituting the equalities (47) and (48), we obtain

$$\begin{aligned}\left\|\mathbf{\Phi}_k^{\mathrm{T}}\boldsymbol{e}_{k+1}\right\|^2 &= \|\boldsymbol{e}_{k+1}\|^2\\ &= \tilde{\boldsymbol{e}}_k^{\mathrm{T}}\left(\mathbf{I}_{r_k} - \tilde{\mathbf{H}}_k\right)\tilde{\boldsymbol{e}}_k - \eta_k\tilde{\boldsymbol{e}}_k^{\mathrm{T}}\tilde{\mathbf{H}}_k\tilde{\mathbf{S}}_k\tilde{\mathbf{H}}_k\tilde{\boldsymbol{e}}_k\\ &\le \tilde{\boldsymbol{e}}_k^{\mathrm{T}}\left(\mathbf{I}_{r_k} - \tilde{\mathbf{H}}_k\right)\tilde{\boldsymbol{e}}_k\\ &\le \lambda_{\max}\left(\mathbf{I}_{r_k} - \tilde{\mathbf{H}}_k\right)\|\tilde{\boldsymbol{e}}_k\|^2 = \tilde{\sigma}\left(\eta_k\right)\|\boldsymbol{e}_k\|^2,\end{aligned} \tag{49}$$

where

$$\begin{aligned}\tilde{\sigma}\left(\eta_k\right) &= \lambda_{\max}\left(\mathbf{I}_{r_k} - \tilde{\mathbf{H}}_k\right)\\ &= \lambda_{\max}\left(\eta_k\tilde{\mathbf{S}}_k + \bar{\mathbf{S}}_k\right)\left(\mathbf{I}_{r_k} + \eta_k\tilde{\mathbf{S}}_k + \bar{\mathbf{S}}_k\right)^{-1}.\end{aligned}$$

From Propositions 1 and 3, we confirm

$$\tilde{\sigma}\left(\eta_k\right) = \frac{\lambda_{\max}\left(\eta_k\tilde{\mathbf{S}}_k + \bar{\mathbf{S}}_k\right)}{1 + \lambda_{\max}\left(\eta_k\tilde{\mathbf{S}}_k + \bar{\mathbf{S}}_k\right)}. \tag{50}$$

From Propositions 1 and 3, it is no difficult to obtain

$$\lambda_{\max}\left(\eta_k\tilde{\mathbf{S}}_k+\bar{\mathbf{S}}_k\right)\leq\eta_k\lambda_{\max}\left(\tilde{\mathbf{S}}_k\right)+\lambda_{\max}\left(\bar{\mathbf{S}}_k\right). \tag{51}$$

From the denotations (44) and (45) it is noticed that the matrices $\tilde{\mathbf{S}}_k$ and $\bar{\mathbf{S}}_k$ are totally dependent on the permutation matrix $\mathbf{\Phi}_k^{\mathrm{T}}$ and Markov matrix $\mathbf{M}$. Because the permutation $h_1, h_2, \cdots, h_{r_k}$ has a one-to-one correspondence to the matrix $\mathbf{\Phi}_k^{\mathrm{T}}$ and the possible permutations $h_1, h_2, \cdots, h_{r_k}$, for all indices $k = 1, 2, \cdots$ and $r_k = 1, 2, \cdots, N+1$, are finite, the possible forms of the matrices $\tilde{\mathbf{S}}_k$ and $\bar{\mathbf{S}}_k$ are limited. Therefore, the eigenvalues $\lambda_{\max}\left(\tilde{\mathbf{S}}_k\right)$ and $\lambda_{\max}\left(\bar{\mathbf{S}}_k\right)$ are uniformly upper-bounded.

Denote

$$\lambda_{\max}\left(\tilde{\mathbf{S}}_k\right)\leq\tilde{\lambda}_0, \tag{52}$$

$$\lambda_{\max}\left(\bar{\mathbf{S}}_k\right)\leq\bar{\lambda}_0. \tag{53}$$

Combining inequality (52) with (53) makes

$$\lambda_{\max}\left(\eta_k\tilde{\mathbf{S}}_k+\bar{\mathbf{S}}_k\right)\leq\eta_k\tilde{\lambda}_0+\bar{\lambda}_0. \tag{54}$$

From Proposition 4 and the condition of $\eta_k \leq \eta_0$, inequality (54) results in

$$\tilde{\sigma}\left(\eta_k\right)\leq\frac{\eta_0\tilde{\lambda}_0+\bar{\lambda}_0}{1+\eta_0\tilde{\lambda}_0+\bar{\lambda}_0}. \tag{55}$$

Denote

$$\tilde{\sigma}_0=\frac{\eta_0\tilde{\lambda}_0+\bar{\lambda}_0}{1+\eta_0\tilde{\lambda}_0+\bar{\lambda}_0}. \tag{56}$$

Inequalities (49) and (55) along with denotation, (56) arrives

$$\|\boldsymbol{e}_{k+1}\|^2\leq\tilde{\sigma}_0\|\boldsymbol{e}_k\|^2. \tag{57}$$

Let

$$\rho=\max\{\sigma_0,\tilde{\sigma}_0\}.$$

Hence, inequalities (41) and (57) achieve

$$\|\boldsymbol{e}_{k+1}\|^2\leq\rho\|\boldsymbol{e}_k\|^2.(0\leq\rho<1). \tag{58}$$

It is immediate to attain that $\|\boldsymbol{e}_{k+1}\|^2 < \|\boldsymbol{e}_k\|^2$, $\lim_{k\to+\infty}\|\boldsymbol{e}_{k+1}\|^2=0$ and $\lim_{k\to+\infty}\sup\frac{\|\boldsymbol{e}_{k+1}\|^2}{\|\boldsymbol{e}_k\|^2}<\rho<1$.

This completes the proof.

Remark 3: In Theorem 2, the linearly monotonic convergence is perfect under the assumption that the initial state of the dynamic subsystem is resettable to zero, that is, $\tilde{x}_{k+1}(0) = 0$, for all $k = 1, 2, \cdots$. This is ideally one of the basic assumptions for ILC investigations. In addition the formula (39) and (50) conveys that the smaller tuning factor will lead to a smaller convergence factor, namely, a faster convergence. Besides, the optimized ITVLGV $\mathbf{\Lambda}_k$ of the ILC scheme (5) is dependent upon the parameters exactness of the system (1). It is hardly realizable in engineering. However, because of unavoidable disturbance, the system parameters may have some uncertainties.

The next section is to analyze the robustness of the P-type GOILC (5) to the system parameters uncertainties, respectively.

IV. ROBUSTNESS OF THE P-TYPE GOILC

This part is going to the robustness analysis of the P-type GOILC (5) to the uncertainties of the parameters of the system (1), namely, system (4) in an additive form. For the regard, we acquiesce that the system dynamics is subject to the input-output equation (4), but the parameters uncertainties occur in the optimized-gain vector expression.

Assume that the parameters of the system (1) are identified in approximate ones formulated as $\mathbf{A}^* = \mathbf{A} + \Delta\mathbf{A}$, $\mathbf{B}^* = \mathbf{B} + \Delta\mathbf{B}$ and $\mathbf{C}^* = \mathbf{C} + \Delta\mathbf{C}$, where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are the precise parameters of the system (1) while $\Delta\mathbf{A}$, $\Delta\mathbf{B}$ and $\Delta\mathbf{C}$ represent uncertainties, respectively. In specific, by denoting $\omega_j = \mathbf{C}^*\left(\bar{\mathbf{A}}^*\right)^{j-1}\tilde{\mathbf{B}}_1^*$ as the approximation of $m_j = \mathbf{C}\bar{\mathbf{A}}^{j-1}\tilde{\mathbf{B}}_1$, for $j = 1, 2, \cdot\cdot\cdot, N$, the corresponding Markov matrix of $\mathbf{M}$ is approximated as

$$\mathbf{\Omega}=\begin{bmatrix}\omega_0&0&0&\cdots&0\\\omega_1&\omega_0&0&\cdots&0\\\omega_2&\omega_1&\omega_0&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\\omega_N&\omega_{N-1}&\omega_{N-2}&\cdots&\omega_0\end{bmatrix}.$$

For validity, assume that $\omega_0 = \mathbf{C}^*\bar{\mathbf{B}}_0^* = \bar{\mathbf{C}}_2^*\left(\bar{\mathbf{A}}_{22}^*\right)^{-1}\bar{\mathbf{B}}_2^* \neq 0$. This means that the parameter uncertainties do not change the nullity relative degree of the system (1). The detailed inference and notions can refer to the lifted-vector-matrix formulation (4) in Section II.

Denote $\Delta\mathbf{M} = \mathbf{\Omega} - \mathbf{M}$ as the disturbance matrix of the precise Markov matrix $\mathbf{M}$. Then, we have

$$\mathbf{\Omega}=\mathbf{M}+\Delta\mathbf{M}. \tag{59}$$

It is known that, in terms of the ILC scheme (5), Theorem 1 conveys that the optimized learning-gain vector $\mathbf{\Lambda}_k$, which is formulated in equations (10) and (30), is relied on the Markov matrix $\mathbf{M}$ and the tracking-error diagonal-spanned matrix $\mathbf{E}_k$. Because only the approximate parameters are available, we may substitute the Markov matrix $\mathbf{M}$ in the learning-gain vector formulations (10), and (30) by the approximate Markov matrix $\mathbf{\Omega}$ and reserve $\mathbf{E}_k$ as is so that the learning scheme still performs well under some appropriate uncertainties. As such, a quasi P-type GOILC strategy for the system (4) is established as

$$\begin{aligned}&\boldsymbol{u}_1:\text{arbitrarily given;}\\&\boldsymbol{u}_{k+1}=\boldsymbol{u}_k+\mathbf{E}_k\mathbf{\Theta}_k,\end{aligned} \tag{60}$$

where the iteration-time varying learning-gain vector $\mathbf{\Theta}_k$ is formulated as

If $\mathbf{E}_k$ is non-singular,

$$\mathbf{\Theta}_k=\mathbf{E}_k^{-1}\left(\eta_k\mathbf{I}+\mathbf{\Omega}^{\mathrm{T}}\mathbf{\Omega}\right)^{-1}\mathbf{\Omega}^{\mathrm{T}}\boldsymbol{e}_k. \tag{61}$$

If $\mathbf{E}_k$ is singular,

$$\mathbf{\Theta}_k=\mathbf{\Phi}_k\begin{bmatrix}\tilde{\mathbf{E}}_k^{-1}\left(\eta_k\mathbf{I}_{r_k}+\left(\tilde{\mathbf{\Omega}}_k^{\mathrm{T}}\tilde{\mathbf{\Omega}}_k+\bar{\mathbf{\Omega}}_k^{\mathrm{T}}\bar{\mathbf{\Omega}}_k\right)\right)^{-1}\tilde{\mathbf{\Omega}}_k^{\mathrm{T}}\tilde{e}_k\\\mathbf{0}\end{bmatrix}. \tag{62}$$

Here, $\tilde{\mathbf{\Omega}}_k$ and $\bar{\mathbf{\Omega}}_k$ are consisting of the expression as

$$\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{\Omega}\mathbf{\Phi}_k = \begin{bmatrix} \tilde{\mathbf{\Omega}}_k & \widehat{\mathbf{\Omega}}_k \\ \bar{\mathbf{\Omega}}_k & \breve{\mathbf{\Omega}}_k \end{bmatrix}. \tag{63}$$

In the expression (63), the dimensions of matrices $\tilde{\mathbf{\Omega}}_k$, $\bar{\mathbf{\Omega}}_k$, $\widehat{\mathbf{\Omega}}_k$ and $\breve{\mathbf{\Omega}}_k$ are identical to that of $\tilde{\mathbf{M}}_k$, $\bar{\mathbf{M}}_k$, $\widehat{\mathbf{M}}_k$ and $\breve{\mathbf{M}}_k$ in the formula (21), respectively.

Then

$$\mathbf{\Phi}_k^{\mathrm{T}}\mathbf{\Omega}\mathbf{\Phi}_k = \mathbf{\Phi}_k^{\mathrm{T}}\mathbf{M}\mathbf{\Phi}_k + \mathbf{\Phi}_k^{\mathrm{T}}\Delta\mathbf{M}\mathbf{\Phi}_k. \tag{64}$$

Here

$$\mathbf{\Phi}_k^{\mathrm{T}}\Delta\mathbf{M}\mathbf{\Phi}_k = \begin{bmatrix} \Delta\tilde{\mathbf{M}}_k \, \Delta\widehat{\mathbf{M}}_k \\ \Delta\bar{\mathbf{M}}_k \, \Delta\breve{\mathbf{M}}_k \end{bmatrix}. \tag{65}$$

Indeed, in the expression (65), the dimensions of matrices $\Delta\tilde{\mathbf{M}}_k$, $\Delta\bar{\mathbf{M}}_k$, $\Delta\widehat{\mathbf{M}}_k$ and $\Delta\breve{\mathbf{M}}_k$ are accordant to the ones of the matrices $\tilde{\mathbf{M}}_k$, $\bar{\mathbf{M}}_k$, $\widehat{\mathbf{M}}_k$ and $\breve{\mathbf{M}}_k$ in the formula (21), respectively.

Consequently, formulations (21), (63) and (65) attain

$$\tilde{\mathbf{\Omega}}_k = \tilde{\mathbf{M}}_k + \Delta\tilde{\mathbf{M}}_k, \tag{66}$$
$$\bar{\mathbf{\Omega}}_k = \bar{\mathbf{M}}_k + \Delta\bar{\mathbf{M}}_k. \tag{67}$$

Before going to the robustness, two groups of denotations are listed as follows.

Denote

$$\delta\mathbf{M} = \Delta\mathbf{M}\cdot\mathbf{M}^{-1}, \tag{68}$$
$$\mathbf{\Xi}_k = \mathbf{M}\left(\eta_k\mathbf{I} + \mathbf{\Omega}^{\mathrm{T}}\mathbf{\Omega}\right)^{-1}\mathbf{M}^{\mathrm{T}}, \tag{69}$$
$$\delta\Pi_k = \delta\mathbf{M}\mathbf{\Xi}_k^2\delta\mathbf{M}^{\mathrm{T}} - (\mathbf{I}-\mathbf{\Xi}_k)\,\mathbf{\Xi}_k\delta\mathbf{M}^{\mathrm{T}} - \delta\mathbf{M}\mathbf{\Xi}_k(\mathbf{I}-\mathbf{\Xi}_k), \tag{70}$$
$$\delta\mathbf{S}_k = \mathbf{M}^{-\mathrm{T}}\left(\Delta\mathbf{M}^{\mathrm{T}}\mathbf{M} + \mathbf{M}^{\mathrm{T}}\Delta\mathbf{M} + \Delta\mathbf{M}^{\mathrm{T}}\Delta\mathbf{M}\right)\mathbf{M}^{-1} = \delta\mathbf{M}^{\mathrm{T}} + \delta\mathbf{M} + \delta\mathbf{M}^{\mathrm{T}}\delta\mathbf{M}, \tag{71}$$
$$\delta\Sigma_k = \mathbf{H}_k\delta\mathbf{S}_k\mathbf{\Xi}_k^2\delta\mathbf{S}_k\mathbf{H}_k + (\mathbf{I}-\mathbf{H}_k)\,\mathbf{\Xi}_k\delta\mathbf{S}_k\mathbf{H}_k + \mathbf{H}_k\delta\mathbf{S}_k\mathbf{\Xi}_k(\mathbf{I}-\mathbf{H}_k), \tag{72}$$
$$\varsigma(\eta_k) = \lambda_{\max}(\delta\Pi_k + \delta\Sigma_k). \tag{73}$$

The group of denotations (68)-(73) will be adopted for robustness when the matrix $\mathbf{E}_k$ is non-singular.

From the expressions (70) and (72), it is confirmative that $\delta\Pi_k = \mathbf{0}$ and $\delta\Sigma_k = \mathbf{0}$, which implies that $\varsigma(\eta_k) = \lambda_{\max}(\delta\Pi_k + \delta\Sigma_k) = 0$ holds, if the relative uncertainty matrix $\delta\mathbf{M} = \Delta\mathbf{M}\cdot\mathbf{M}^{-1} \equiv \mathbf{0}$. Owing to the continuity of the eigenvalues with respect to the elements, we may believe that the value $\varsigma(\eta_k) = \lambda_{\max}(\delta\Pi_k + \delta\Sigma_k)$ may be significantly smaller in the circumstance while the uncertainty matrix $\Delta\mathbf{M}$ is constrained within an adequate range.

In addition, denote

$$\delta\tilde{\mathbf{M}}_k = \Delta\tilde{\mathbf{M}}_k\cdot\tilde{\mathbf{M}}_k^{-1}, \tag{74}$$
$$\delta\bar{\mathbf{M}}_k = \Delta\bar{\mathbf{M}}_k\cdot\tilde{\mathbf{M}}_k^{-1}, \tag{75}$$
$$\tilde{\mathbf{\Xi}}_k = \tilde{\mathbf{M}}_k\left(\eta_k\mathbf{I}_{r_k} + \left(\tilde{\mathbf{\Omega}}_k^{\mathrm{T}}\tilde{\mathbf{\Omega}}_k + \bar{\mathbf{\Omega}}_k^{\mathrm{T}}\bar{\mathbf{\Omega}}_k\right)\right)^{-1}\tilde{\mathbf{M}}_k^{\mathrm{T}}, \tag{76}$$
$$\delta\tilde{\Pi}_k = \delta\tilde{\mathbf{M}}_k\tilde{\mathbf{\Xi}}_k^2\delta\tilde{\mathbf{M}}_k^{\mathrm{T}} - \delta\tilde{\mathbf{M}}_k\tilde{\mathbf{\Xi}}_k\left(\mathbf{I}_{r_k} - \tilde{\mathbf{\Xi}}_k\right) - \left(\mathbf{I}_{r_k} - \tilde{\mathbf{\Xi}}_k\right)\tilde{\mathbf{\Xi}}_k\delta\tilde{\mathbf{M}}_k^{\mathrm{T}}, \tag{77}$$
$$\delta\bar{\Pi}_k = \delta\tilde{\mathbf{M}}_k\tilde{\mathbf{\Xi}}_k\bar{\mathbf{S}}_k\tilde{\mathbf{\Xi}}_k\delta\tilde{\mathbf{M}}_k^{\mathrm{T}} + \delta\tilde{\mathbf{M}}_k\tilde{\mathbf{\Xi}}_k\bar{\mathbf{S}}_k\tilde{\mathbf{\Xi}}_k + \tilde{\mathbf{\Xi}}_k\bar{\mathbf{S}}_k\tilde{\mathbf{\Xi}}_k\delta\tilde{\mathbf{M}}_k^{\mathrm{T}}, \tag{78}$$
$$\Delta\tilde{\mathbf{W}}_k = \Delta\tilde{\mathbf{M}}_k^{\mathrm{T}}\tilde{\mathbf{M}}_k + \tilde{\mathbf{M}}_k^{\mathrm{T}}\Delta\tilde{\mathbf{M}}_k + \Delta\tilde{\mathbf{M}}_k^{\mathrm{T}}\Delta\tilde{\mathbf{M}}_k, \tag{79}$$
$$\Delta\bar{\mathbf{W}}_k = \Delta\bar{\mathbf{M}}_k^{\mathrm{T}}\bar{\mathbf{M}}_k + \bar{\mathbf{M}}_k^{\mathrm{T}}\Delta\bar{\mathbf{M}}_k + \Delta\bar{\mathbf{M}}_k^{\mathrm{T}}\Delta\bar{\mathbf{M}}_k, \tag{80}$$
$$\delta\tilde{\mathbf{S}}_k = \tilde{\mathbf{M}}_k^{-\mathrm{T}}\Delta\tilde{\mathbf{W}}_k\tilde{\mathbf{M}}_k^{-1} = \delta\tilde{\mathbf{M}}_k^{\mathrm{T}} + \delta\tilde{\mathbf{M}}_k + \delta\tilde{\mathbf{M}}_k^{\mathrm{T}}\delta\tilde{\mathbf{M}}_k, \tag{81}$$
$$\delta\bar{\mathbf{S}}_k = \tilde{\mathbf{M}}_k^{-\mathrm{T}}\Delta\bar{\mathbf{W}}_k\tilde{\mathbf{M}}_k^{-1} = \delta\bar{\mathbf{M}}_k^{\mathrm{T}} + \delta\bar{\mathbf{M}}_k + \delta\bar{\mathbf{M}}_k^{\mathrm{T}}\delta\bar{\mathbf{M}}_k, \tag{82}$$
$$\delta\tilde{\mathbf{H}}_k = \tilde{\mathbf{\Xi}}_k\left(\delta\tilde{\mathbf{S}}_k + \delta\bar{\mathbf{S}}_k\right)\tilde{\mathbf{H}}_k, \tag{83}$$
$$\delta\tilde{\Sigma}_k = \delta\tilde{\mathbf{H}}_k\delta\tilde{\mathbf{H}}_k + \left(\mathbf{I}_{r_k} - \tilde{\mathbf{H}}_k\right)\delta\tilde{\mathbf{H}}_k + \delta\tilde{\mathbf{H}}_k\left(\mathbf{I}_{r_k} - \tilde{\mathbf{H}}_k\right), \tag{84}$$
$$\delta\bar{\Sigma}_k = \delta\tilde{\mathbf{H}}_k\bar{\mathbf{S}}_k\delta\tilde{\mathbf{H}}_k - \tilde{\mathbf{H}}_k\bar{\mathbf{S}}_k\delta\tilde{\mathbf{H}}_k - \delta\tilde{\mathbf{H}}_k\bar{\mathbf{S}}_k\tilde{\mathbf{H}}_k. \tag{85}$$
$$\tilde{\varsigma}(\eta_k) = \lambda_{\max}\left(\delta\tilde{\Pi}_k + \delta\bar{\Pi}_k + \delta\tilde{\Sigma}_k + \delta\bar{\Sigma}_k\right). \tag{86}$$

The notions (74)-(86) will be utilized for robustness when the matrix $\mathbf{E}_k$ is singular.

From the expressions (77), (78), (84) and (85), it is of course that $\tilde{\varsigma}(\eta_k) = 0$ if the relative uncertainty matrix $\delta\mathbf{M} = \Delta\mathbf{M}\cdot\mathbf{M}^{-1} \equiv \mathbf{0}$ which contains $\Delta\tilde{\mathbf{M}}_k = \mathbf{0}$ and $\Delta\bar{\mathbf{M}}_k = \mathbf{0}$.

In virtue of the continuity of the eigenvalues with respect to the elements, we may realize that the value $\tilde{\varsigma}(\eta_k)$ may be very smaller in the circumstance while the uncertainty matrix $\Delta\mathbf{M}$ is constrained within an adequate range.

Theorem 3: Assume that the system (4) repetitively operates over the sampling set D^+ with the value $m_0 = \mathbf{C}\bar{\mathbf{B}}_0$ being nonzero and the initial state $\tilde{x}_k(0)$ of the dynamic subsystem is resettable, that is, $\tilde{x}_{k+1}(0) = 0$, for all $k = 1, 2, \cdots$. Under the condition of $\eta_k \le \eta_0$, for all $k = 1, 2, \cdots$, the Quasi P-type GOILC (60) with ITVLGV $\mathbf{\Theta}_k$ formulated by (61) and (62) is linearly monotonically convergent if the relative uncertainties $\delta\mathbf{M} = \Delta\mathbf{M}\cdot\mathbf{M}^{-1}$, $\delta\tilde{\mathbf{M}}_k = \Delta\tilde{\mathbf{M}}_k\cdot\tilde{\mathbf{M}}_k^{-1}$ and $\delta\bar{\mathbf{M}}_k = \Delta\bar{\mathbf{M}}_k\cdot\tilde{\mathbf{M}}_k^{-1}$ are appropriately constrained so that the inequalities $\varsigma(\eta_k) \le \varsigma_0$ and $\sigma_0 + \varsigma_0 < 1$ holds for the case when the matrix $\mathbf{E}_k$ is non-singular. In contrast, the inequalities $\tilde{\varsigma}(\eta_k) \le \tilde{\varsigma}_0$ and $\tilde{\sigma}_0 + \tilde{\varsigma}_0 < 1$ are guaranteed for the case when the matrix $\mathbf{E}_k$ is singular, respectively.

In other words, there exists such a constant τ $(0 \le \tau < 1)$ that, $\|\boldsymbol{e}_{k+1}\|^2 < \|\boldsymbol{e}_k\|^2$, $\lim_{k\to+\infty}\|\boldsymbol{e}_{k+1}\|^2 = 0$ and $\lim_{k\to+\infty}\sup\frac{\|\boldsymbol{e}_{k+1}\|^2}{\|\boldsymbol{e}_k\|^2} < \tau < 1$.

Proof:

Case 1: The matrix $\mathbf{E}_k$ is non-singular.

Substituting the algorithm (60) with the learning-gain vector (61) and denotation (59) into the system (4) yields

$$\boldsymbol{e}_{k+1} = \boldsymbol{e}_k - \mathbf{M}\mathbf{E}_k\mathbf{\Theta}_k = \boldsymbol{e}_k - \mathbf{M}\left(\eta_k\mathbf{I} + \mathbf{\Omega}^{\mathrm{T}}\mathbf{\Omega}\right)^{-1}\mathbf{\Omega}^{\mathrm{T}}\boldsymbol{e}_k = \boldsymbol{e}_k - \mathbf{M}\left(\eta_k\mathbf{I} + \mathbf{\Omega}^{\mathrm{T}}\mathbf{\Omega}\right)^{-1}\mathbf{M}^{\mathrm{T}}\left(\mathbf{I} + \mathbf{M}^{-\mathrm{T}}\Delta\mathbf{M}^{\mathrm{T}}\right)\boldsymbol{e}_k. \tag{87}$$

Substituting denotations (69) and (59) into (87) gets

$$e_{k+1} = (\mathbf{I} - \Xi_k) e_k - \Xi_k \delta \mathbf{M}^{\mathrm{T}} e_k. \tag{88}$$

Calculating inner product to both sides of the equation (88) conducts

$$\begin{aligned} \|e_{k+1}\|^2 &= e_{k+1}^{\mathrm{T}} e_{k+1} \\ &= e_k^{\mathrm{T}} (\mathbf{I} - \Xi_k)^{\mathrm{T}} (\mathbf{I} - \Xi_k) e_k + e_k^{\mathrm{T}} \delta \mathbf{M} \Xi_k^2 \delta \mathbf{M}^{\mathrm{T}} e_k \\ &\quad - e_k^{\mathrm{T}} \left((\mathbf{I} - \Xi_k) \Xi_k \delta \mathbf{M}^{\mathrm{T}} + \delta \mathbf{M} \Xi_k (\mathbf{I} - \Xi_k) \right) e_k \\ &= e_k^{\mathrm{T}} (\mathbf{I} - \Xi_k)^{\mathrm{T}} (\mathbf{I} - \Xi_k) e_k + e_k^{\mathrm{T}} \delta \Pi_k e_k. \end{aligned} \tag{89}$$

Substituting notion (59) into (69) and considering formula (33) receive

$$\begin{aligned} \Xi_k &= \mathbf{M} \left(\eta_k \mathbf{I} + \left(\mathbf{M}^{\mathrm{T}} + \Delta \mathbf{M}^{\mathrm{T}} \right) (\mathbf{M} + \Delta \mathbf{M}) \right)^{-1} \mathbf{M}^{\mathrm{T}} \\ &= (\mathbf{I} + \eta_k \mathbf{S}_k + \delta \mathbf{S}_k)^{-1}. \end{aligned} \tag{90}$$

Recall that expression (34) delivers $\mathbf{H}_k = (\mathbf{I} + \eta_k \mathbf{S}_k)^{-1}$. From Proposition 2, the expressions (34) and (90), one achieves

$$\begin{aligned} \Xi_k &= \mathbf{H}_k + (\Xi_k - \mathbf{H}_k) = \mathbf{H}_k + \Xi_k \left(\mathbf{H}_k^{-1} - \Xi_k^{-1} \right) \mathbf{H}_k \\ &= \mathbf{H}_k - \Xi_k \delta \mathbf{S}_k \mathbf{H}_k. \end{aligned} \tag{91}$$

Therefore

$$\begin{aligned} &(\mathbf{I} - \Xi_k)^{\mathrm{T}} (\mathbf{I} - \Xi_k) \\ &= (\mathbf{I} - \mathbf{H}_k + \Xi_k \delta \mathbf{S}_k \mathbf{H}_k)^{\mathrm{T}} (\mathbf{I} - \mathbf{H}_k + \Xi_k \delta \mathbf{S}_k \mathbf{H}_k) \\ &= (\mathbf{I} - \mathbf{H}_k)^2 + \mathbf{H}_k \delta \mathbf{S}_k \Xi_k^2 \delta \mathbf{S}_k \mathbf{H}_k \\ &\quad + (\mathbf{I} - \mathbf{H}_k) \Xi_k \delta \mathbf{S}_k \mathbf{H}_k + \mathbf{H}_k \delta \mathbf{S}_k \Xi_k (\mathbf{I} - \mathbf{H}_k) \\ &= (\mathbf{I} - \mathbf{H}_k)^2 + \delta \Sigma_k. \end{aligned} \tag{92}$$

Substituting formulation (92) into (89) and considering inequalities (39) and (40) along with the assumption of $\varsigma(\eta_k) = \lambda_{\max}(\delta \Pi_k + \delta \Sigma_k) \le \varsigma_0$, we have

$$\begin{aligned} &\|e_{k+1}\|^2 \\ &= e_k^{\mathrm{T}} (\mathbf{I} - \mathbf{H}_k)^2 e_k + e_k^{\mathrm{T}} (\delta \Pi_k + \delta \Sigma_k) e_k \\ &\le \lambda_{\max} (\mathbf{I} - \mathbf{H}_k)^2 \|e_k\|^2 + \lambda_{\max} (\delta \Pi_k + \delta \Sigma_k) \|e_k\|^2 \\ &\le (\sigma_0 + \varsigma_0) \|e_k\|^2. \end{aligned} \tag{93}$$

Case 2: If the matrix $\mathbf{E}_k$ is singular.

Equation (76) is equivalent to

$$\left(\eta_k \mathbf{I}_{r_k} + \left(\tilde{\mathbf{\Omega}}_k^{\mathrm{T}} \tilde{\mathbf{\Omega}}_k + \bar{\mathbf{\Omega}}_k^{\mathrm{T}} \bar{\mathbf{\Omega}}_k \right) \right)^{-1} \tilde{\mathbf{M}}_k^{\mathrm{T}} = \tilde{\mathbf{M}}_k^{-1} \tilde{\Xi}_k. \tag{94}$$

Substituting expressions (66) and (94) into a learning-gain vector (62) makes

$$\mathbf{\Theta}_k = \mathbf{\Phi}_k \begin{bmatrix} \tilde{\mathbf{E}}_k^{-1} \tilde{\mathbf{M}}_k^{-1} \tilde{\Xi}_k \left(I_{r_k} + \delta \tilde{\mathbf{M}}_k^{\mathrm{T}} \right) \tilde{e}_k \\ 0 \end{bmatrix}. \tag{95}$$

Substituting the quasi P-type GOILC (60) into the system (4) yields

$$e_{k+1} = e_k - \mathbf{M} \mathbf{E}_k \mathbf{\Theta}_k. \tag{96}$$

Pre-multiplying the equation (96) by $\mathbf{\Phi}_k^{\mathrm{T}}$ and substituting the learning-gain vector (95), we get

$$\begin{aligned} &\mathbf{\Phi}_k^{\mathrm{T}} e_{k+1} \\ &= \mathbf{\Phi}_k^{\mathrm{T}} e_k - \left(\mathbf{\Phi}_k^{\mathrm{T}} \mathbf{M} \Phi_k \right) \left(\mathbf{\Phi}_k^{\mathrm{T}} \mathbf{E}_k \mathbf{\Phi}_k \right) \left(\mathbf{\Phi}_k^{\mathrm{T}} \mathbf{\Theta}_k \right) \\ &= \begin{bmatrix} \tilde{e}_k \\ 0 \end{bmatrix} - \begin{bmatrix} \tilde{\mathbf{M}}_k & \widehat{\mathbf{M}}_k \\ \bar{\mathbf{M}}_k & \check{\mathbf{M}}_k \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{E}}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \left(\mathbf{\Phi}_k^{\mathrm{T}} \mathbf{\Theta}_k \right) \\ &= \begin{bmatrix} \tilde{e}_k \\ 0 \end{bmatrix} - \begin{bmatrix} \tilde{\mathbf{M}}_k \tilde{\mathbf{E}}_k & \mathbf{0} \\ \bar{\mathbf{M}}_k \tilde{\mathbf{E}}_k & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{E}}_k^{-1} \tilde{\mathbf{M}}_k^{-1} \tilde{\Xi}_k \left(\mathbf{I}_{r_k} + \delta \tilde{\mathbf{M}}_k^{\mathrm{T}} \right) \tilde{e}_k \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I}_{r_k} - \tilde{\Xi}_k - \tilde{\Xi}_k \delta \tilde{\mathbf{M}}_k^{\mathrm{T}} \\ -\bar{\mathbf{M}}_k \tilde{\mathbf{M}}_k^{-1} \tilde{\Xi}_k - \bar{\mathbf{M}}_k \tilde{\mathbf{M}}_k^{-1} \tilde{\Xi}_k \delta \tilde{\mathbf{M}}_k^{\mathrm{T}} \end{bmatrix} \tilde{e}_k. \end{aligned} \tag{97}$$

Computing 2-norm for both sides of the equation (97) and considering denotation (48), we obtain

$$\begin{aligned} \|e_{k+1}\|^2 &= \tilde{e}_k^{\mathrm{T}} \left(\mathbf{I}_{r_k} - \tilde{\Xi}_k \right)^{\mathrm{T}} \left(\mathbf{I}_{r_k} - \tilde{\Xi}_k \right) \tilde{e}_k + \tilde{e}_k^{\mathrm{T}} \delta \tilde{\Pi}_k \tilde{e}_k \\ &\quad + \tilde{e}_k^{\mathrm{T}} \tilde{\Xi}_k \bar{\mathbf{S}}_k \tilde{\Xi}_k \tilde{e}_k + \tilde{e}_k^{\mathrm{T}} \delta \bar{\Pi}_k \tilde{e}_k. \end{aligned} \tag{98}$$

From equalities (66) and (67), we obtain

$$\begin{aligned} \tilde{\mathbf{\Omega}}_k^{\mathrm{T}} \tilde{\mathbf{\Omega}}_k &= \left(\tilde{\mathbf{M}}_k^{\mathrm{T}} + \Delta \tilde{\mathbf{M}}_k^{\mathrm{T}} \right) \left(\tilde{\mathbf{M}}_k + \Delta \tilde{\mathbf{M}}_k \right) \\ &= \tilde{\mathbf{M}}_k^{\mathrm{T}} \tilde{\mathbf{M}}_k + \Delta \tilde{\mathbf{M}}_k^{\mathrm{T}} \tilde{\mathbf{M}}_k + \tilde{\mathbf{M}}_k^{\mathrm{T}} \Delta \tilde{\mathbf{M}}_k + \Delta \tilde{\mathbf{M}}_k^{\mathrm{T}} \Delta \tilde{\mathbf{M}}_k \\ &= \tilde{\mathbf{M}}_k^{\mathrm{T}} \tilde{\mathbf{M}}_k + \Delta \tilde{\mathbf{W}}_k. \end{aligned} \tag{99}$$

Analogously, we reach

$$\bar{\mathbf{\Omega}}_k^{\mathrm{T}} \bar{\mathbf{\Omega}}_k = \bar{\mathbf{M}}_k^{\mathrm{T}} \bar{\mathbf{M}}_k + \Delta \bar{\mathbf{W}}_k. \tag{100}$$

Substituting (99) and (100) into (76) and considering denotations (47) and (48), we arrive

$$\begin{aligned} \tilde{\Xi}_k &= \tilde{\mathbf{M}}_k \left(\eta_k \mathbf{I}_{r_k} + \left(\tilde{\mathbf{\Omega}}_k^{\mathrm{T}} \tilde{\mathbf{\Omega}}_k + \bar{\mathbf{\Omega}}_k^{\mathrm{T}} \bar{\mathbf{\Omega}}_k \right) \right)^{-1} \tilde{\mathbf{M}}_k^{\mathrm{T}} \\ &= \tilde{\mathbf{M}}_k \left(\eta_k \mathbf{I}_{r_k} + \tilde{\mathbf{M}}_k^{\mathrm{T}} \tilde{\mathbf{M}}_k + \bar{\mathbf{M}}_k^{\mathrm{T}} \bar{\mathbf{M}}_k + \Delta \tilde{\mathbf{W}}_k + \Delta \bar{\mathbf{W}}_k \right)^{-1} \tilde{\mathbf{M}}_k^{\mathrm{T}} \\ &= \left(\mathbf{I}_{r_k} + \eta_k \tilde{\mathbf{S}}_k + \bar{\mathbf{S}}_k + \delta \tilde{\mathbf{S}}_k + \delta \bar{\mathbf{S}}_k \right)^{-1}. \end{aligned} \tag{101}$$

By virtue of Proposition 2, we achieve

$$\begin{aligned} \tilde{\Xi}_k &= \tilde{\mathbf{H}}_k + \tilde{\Xi}_k \left(\tilde{\mathbf{H}}_k^{-1} - \tilde{\Xi}_k^{-1} \right) \tilde{\mathbf{H}}_k \\ &= \tilde{\mathbf{H}}_k - \tilde{\Xi}_k \left(\delta \tilde{\mathbf{S}}_k + \delta \bar{\mathbf{S}}_k \right) \tilde{\mathbf{H}}_k = \tilde{\mathbf{H}}_k - \delta \tilde{\mathbf{H}}_k. \end{aligned} \tag{102}$$

Then equality (102) produces

$$\mathbf{I}_{r_k} - \tilde{\Xi}_k = \mathbf{I}_{r_k} - \tilde{\mathbf{H}}_k + \delta \tilde{\mathbf{H}}_k. \tag{103}$$

Therefore

$$\begin{aligned} &\left(\mathbf{I}_{r_k} - \tilde{\Xi}_k \right)^{\mathrm{T}} \left(\mathbf{I}_{r_k} - \tilde{\Xi}_k \right) \\ &= \left(\mathbf{I}_{r_k} - \tilde{\mathbf{H}}_k + \delta \tilde{\mathbf{H}}_k \right)^{\mathrm{T}} \left(\mathbf{I}_{r_k} - \tilde{\mathbf{H}}_k + \delta \tilde{\mathbf{H}}_k \right) \\ &= \left(\mathbf{I}_{r_k} - \tilde{\mathbf{H}}_k \right)^2 + \delta \tilde{\Sigma}_k. \end{aligned} \tag{104}$$

Besides, the formula (102) delivers

$$\begin{aligned} \tilde{\Xi}_k \bar{\mathbf{S}}_k \tilde{\Xi}_k &= \left(\tilde{\mathbf{H}}_k - \delta \tilde{\mathbf{H}}_k \right) \bar{\mathbf{S}}_k \left(\tilde{\mathbf{H}}_k - \delta \tilde{\mathbf{H}}_k \right) \\ &= \tilde{\mathbf{H}}_k \bar{\mathbf{S}}_k \tilde{\mathbf{H}}_k + \delta \bar{\Sigma}_k. \end{aligned} \tag{105}$$

Substituting expressions (104) and (105) into formula (98) achieves

$$\|\boldsymbol{e}_{k+1}\|^2 = \tilde{\boldsymbol{e}}_k^{\mathrm{T}}\left(\mathbf{I}_{r_k} - \tilde{\mathbf{H}}_k\right)^2 \tilde{\boldsymbol{e}}_k + \tilde{\boldsymbol{e}}_k^{\mathrm{T}}\tilde{\mathbf{H}}_k\bar{\mathbf{S}}_k\tilde{\mathbf{H}}_k\tilde{\boldsymbol{e}}_k + \tilde{\boldsymbol{e}}_k^{\mathrm{T}}\left(\delta\tilde{\Pi}_k + \delta\bar{\Pi}_k + \delta\tilde{\Sigma}_k + \delta\bar{\Sigma}_k\right)\tilde{\boldsymbol{e}}_k. \quad (106)$$

Similar to the derivation of expression (49), the first two terms in the right hand of the equation (106) are rewritten as

$$\tilde{\boldsymbol{e}}_k^{\mathrm{T}}\left(\mathbf{I}_{r_k} - \tilde{\mathbf{H}}_k\right)^2 \tilde{\boldsymbol{e}}_k + \tilde{\boldsymbol{e}}_k^{\mathrm{T}}\tilde{\mathbf{H}}_k\bar{\mathbf{S}}_k\tilde{\mathbf{H}}_k\tilde{\boldsymbol{e}}_k = \tilde{\boldsymbol{e}}_k^{\mathrm{T}}\left(\mathbf{I}_{r_k} - \tilde{\mathbf{H}}_k\right)\tilde{\boldsymbol{e}}_k - \eta_k\tilde{\boldsymbol{e}}_k^{\mathrm{T}}\tilde{\mathbf{H}}_k\tilde{\mathbf{S}}_k\tilde{\mathbf{H}}_k\tilde{\boldsymbol{e}}_k. \quad (107)$$

Substituting (107) into (106) and under the property of the matrix-weighing quadratic function, the equation (106) is evaluated as

$$\|\boldsymbol{e}_{k+1}\|^2 \le \left(\tilde{\sigma}(\eta_k) - \eta_k\bar{\sigma}(\eta_k) + \tilde{\varsigma}(\eta_k)\right)\|\boldsymbol{e}_k\|^2 \le \left(\tilde{\sigma}_0 + \tilde{\varsigma}_0\right)\|\boldsymbol{e}_k\|^2. \quad (108)$$

Here $\tilde{\sigma}_0$ is as shown in (56) and

$$\tilde{\varsigma}(\eta_k) \le \tilde{\varsigma}_0. \quad (109)$$

Let

$$\tau = \max\{\sigma_0 + \varsigma_0, \tilde{\sigma}_0 + \tilde{\varsigma}_0\}. \quad (110)$$

Then inequalities (93) and (108) makes

$$\|\boldsymbol{e}_{k+1}\|^2 \le \tau\|\boldsymbol{e}_k\|^2, (0 \le \tau < 1). \quad (111)$$

Therefore, inequality (111) arrives $\|\boldsymbol{e}_{k+1}\|^2 < \|\boldsymbol{e}_k\|^2$, $\lim_{k\to+\infty}\|\boldsymbol{e}_{k+1}\|^2 = 0$ and $\lim_{k\to+\infty}\sup\frac{\|\boldsymbol{e}_{k+1}\|^2}{\|\boldsymbol{e}_k\|^2} < \tau < 1$.

This completes the proof.

Remark 4: From the expressions of the uncertainty values $\varsigma(\eta_k)$ *and* $\tilde{\varsigma}(\eta_k)$ *respectively shown in* (73) and (86), *it is confirmed that those two values are the relative uncertainties. As discussed in remark 3 that the smaller tuning factor leads to a smaller convergence factor. As such a smaller convergence factor may tolerate the wider range of relative uncertainties* $\varsigma(\eta_k)$ *and* $\tilde{\varsigma}(\eta_k)$. *In virtue of the continuity of the eigenvalues with respect to the elements, we may realize that the value* $\varsigma(\eta_k)$ *and* $\tilde{\varsigma}(\eta_k)$ *may be very smaller in the circumstance while the uncertainty matrix* $\Delta\mathbf{M}$ *is constrained within an adequate range.*

As the relative uncertainties matrices $\delta\tilde{\mathbf{M}}_k = \Delta\tilde{\mathbf{M}}_k \cdot \tilde{\mathbf{M}}_k^{-1}$ *and* $\delta\bar{\mathbf{M}}_k = \Delta\bar{\mathbf{M}}_k \cdot \tilde{\mathbf{M}}_k^{-1}$ *are strongly relevant to the matrix* $\delta\mathbf{M} = \Delta\mathbf{M}\cdot\mathbf{M}^{-1}$, *the relative uncertainties degree is usually a positive correlation function of the value* $\Delta m = \left\|\Delta\mathbf{M}\cdot\mathbf{M}^{-1}\right\|_2$. *For the sake of simplicity,* $\Delta m = \left\|\Delta\mathbf{M}\cdot\mathbf{M}^{-1}\right\|_2$ *is used to measure the relative uncertainty. Here, it is worthy to mention that a larger relative uncertainty* Δm *does not means a worse convergence as* Δm *is a function of multiple uncertainties* $\Delta\mathbf{A}$, $\Delta\mathbf{B}$ *and* $\Delta\mathbf{C}$.

Besides, Theorem 3 conveys that the linearly monotone convergence is derived in a direct time domain manner and is guaranteed in a wider uncertainty scope.

V. NUMERICAL SIMULATIONS

In this simulation section, we shall consider the sampling set as $D = \{0, 1, 2, \cdots, 79\}$. The desired trajectory is preset as $y_d(n+1) = -5\sin(0.4(n+1)) + 10e^{-(n+1)}$, $n \in D^+ = D \cup \{80\}$. The tracking error in 2-norm is calculated as

$$\|\boldsymbol{e}_k\|_2 = \sqrt{\sum_{n=0}^{80}|e_k(n)|^2}.$$

Consider the following LDTI singular system

$$\begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_k(n+1) = \begin{bmatrix} -0.6 & -0.5 & 0.5 \\ 1 & -0.5 & -0.1 \\ a_1 & a_2 & a_3 \end{bmatrix} x_k(n) + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u_k(n), \\ y_k(n) = [c_1 \;\; c_2 \;\; c_3]\, x_k(n), n \in S. \end{cases} \quad (112)$$

For the system (112), the state pertaining to the dynamic subsystem is $\tilde{x}_k(n) = \left[x_k^{(1)}(n), x_k^{(2)}(n)\right]^{\mathrm{T}}$ while the state about the static subsystem is $\hat{x}_k(n) = x_k^{(3)}(n)$ respectively. The initial state relating to the dynamic subsystem is set as $\tilde{x}_k(0) = \tilde{x}_0 = [0 \;\; 0]^{\mathrm{T}}$.

Pick the parameters as $a_1 = 0.1, a_2 = 0, a_3 = 1, b_1 = 0.2$, $b_2 = 0.2$, $b_3 = -0.6$, $c_1 = 0.5$, $c_2 = 0.2$ and $c_3 = 2.5$. It is computed that $\rho(\mathbf{E}, \mathbf{A}) = (-0.58 + 0.71i, -0.58 - 0.71i)$ which signifies that the finite eigenvalue of the system (112) is within the unit circle, that is to say, the system (112) is stable. It is calculated that $m_0 = \mathbf{C}\bar{\mathbf{B}}_0 = 1.5 \neq 0$. This implies that the relative degree of the system (112) is zero.

Set the tuning-factor sequences as

$$\eta_k^{(1)} = 0.8, \quad \eta_k^{(2)} = 0.6\cdot 0.9^k, \quad \eta_k^{(3)} = 0.6\left(1 - \sin\left(k\pi/8\right)\right), \quad \eta_k^{(4)} \equiv 0.$$

Part 1: Validation and efficacy of GOILC (5)

This part is going for simulation when the initial state $\tilde{x}_{k+1}(0)$ of the dynamic subsystem is resettable to zero, that is, $\tilde{x}_{k+1}(0) = 0$, for all $k = 1, 2, \cdots$. As discussed in Theorem 2, simulation is also conducted for the cases when $\mathbf{E}_1$ is non-singular and singular, respectively.

Case 1-1: Matrix $\mathbf{E}_1$ is non-singular.

Choose the first iteration output as

$y_1(n+1) = y_d(n+1) + 0.6\sin(0.2(n+1)\pi) - 2.8$, $n \in D^+$.

Then it is assure that all components of the tracking error $\boldsymbol{e}_1 = \boldsymbol{y}_d - \boldsymbol{y}_1$ are nonzero. This implies that the matrix $\mathbf{E}_1$ is non-singular.

Whereas the GOILC (5) is forced on the system (112), the sequential tracking error tendencies have come into view. Fig.1 displays the tracking behavior for the tuning factor being fixed at $\eta_k = 0.8$, where the solid curve shows the desired trajectory while the dash-dotted, the dashed, and the

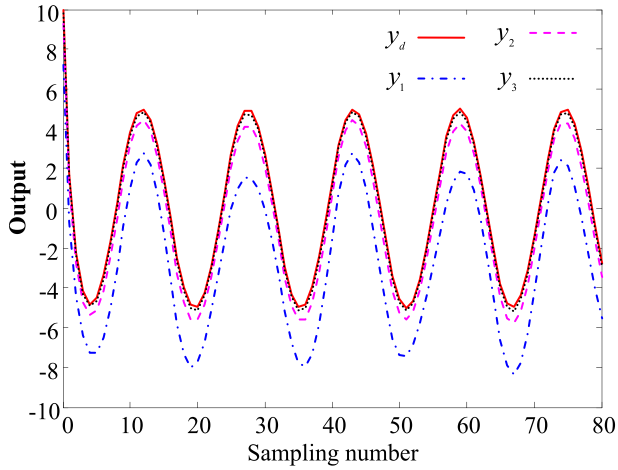

FIGURE 1. Tracking behavior of GOILC (5) for Case 1-1.

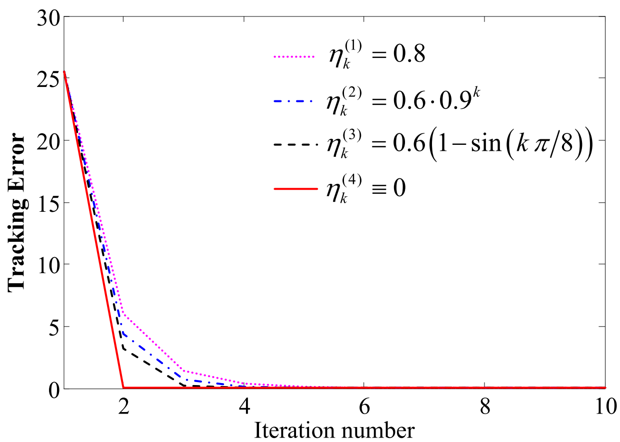

FIGURE 2. Tracking error tendencies of GOILC (5) for Case 1-1.

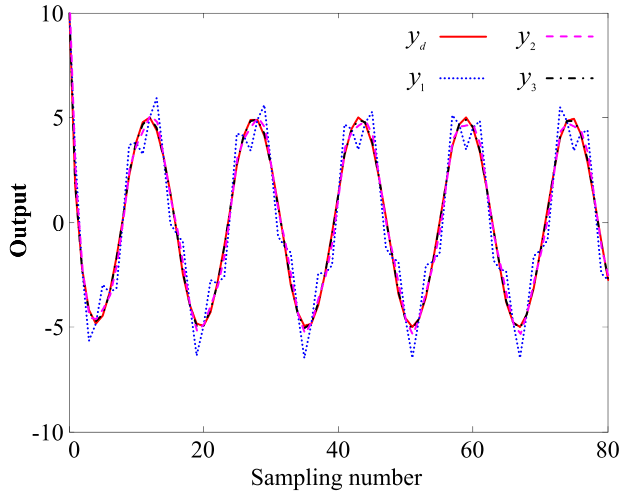

FIGURE 3. Tracking behavior of GOILC (5) for Case 1-2.

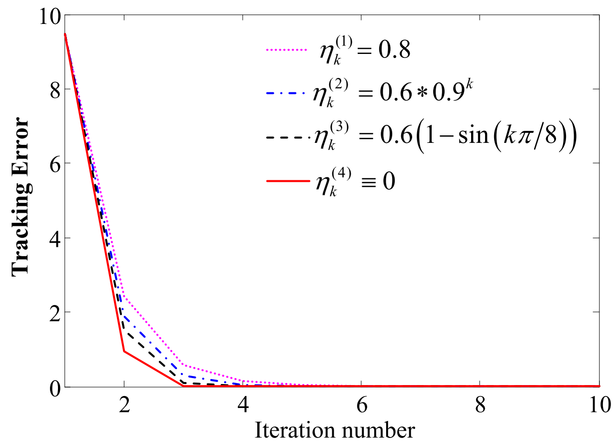

FIGURE 4. Tracking error tendencies of GOILC (5) for Case 1-2.

dotted ones show the output at the 1st, 2nd, and 3rd iterations, respectively.

From Fig.1, it is confirmed that the output y_k of the system (112) driven by the learning of GOILC (5) can track the desired trajectory with quite good performance.

Fig.2 exhibits the comparable tracking errors, where the dotted, the dash-dotted, the dashed, and the solid curves are the tracking-error tendencies of GOILC (5) for the tuning-factor orders $\eta_k^{(1)}$, $\eta_k^{(2)}$, $\eta_k^{(3)}$ and $\eta_k^{(4)}$ respectively.

It is evident from Fig.2 that those four tracking errors are linearly monotonically convergent, and the lighter tuning-factor sequence makes the convergence more rapid. Remarkably, the solid tracking error for the tuning-factor sequence $\eta_k^{(4)} \equiv 0$ converges expeditious and becomes zero at the second iteration. This agrees with the result in Theorem 2.

Case 1-2: Matrix $\mathbf{E}_1$ is singular.

Formulate the first-iteration control input as $\boldsymbol{u}_1 = \mathbf{H}^{-1}\boldsymbol{y}_1$, where the first-iteration output $\boldsymbol{y}_1$ is arranged as

$y_1(n+1) = y_d(n+1) + 1.5\sin(0.5(n+1)\pi), n \in D^+$.

It is understandable that $e_1(2j) = 0$, for $j = 0, 1, 2, \cdots, 40$, which infers that the matrix $\mathbf{E}_1$ is singular.

Fig.3 expresses the tracking behavior for the tuning factor being fixed as $\eta_k = 0.8$, where the solid curve depicts the desired trajectory while the dotted, the dashed, and the dash-dotted ones give a picture of the output at the first, the second and the third iterations, respectively. It delivers that the GOILC (5) pushes the system (112) to track the desired trajectory in a perfect mode.

Fig. 4 compares the tracking errors made by GOILC (5) with different tuning-factor sequences, where the dotted, the dash-dotted, the dashed, and the solid curves are the tracking errors for the tuning-factor sequences being $\eta_k^{(1)}$, $\eta_k^{(2)}$, $\eta_k^{(3)}$ and $\eta_k^{(4)}$ respectively. It is detected that all the tracking errors linearly monotonically converge to a nullity as iteration carries on. More to the point, the dash tracking error for the smaller tuning-factor sequence $\eta_k^{(3)}$ transfers a faster convergence than that of the larger tuning-factor sequence $\eta_k^{(2)}$. As an extreme case when the tuning-factor sequence $\eta_k^{(4)} \equiv 0$, the solid tracking error vanishes at the third iteration.

Part 2: Validation and efficacy of Quasi P-type GOILC (60)

This part demonstrates the convergence feature of QGOILC (60) for distinct uncertainty degrees with a fixed tuning-factor $\eta_k = 0.02$. Additionally, to take advantage of the first iterative control input provided in *Part* 1, the simulations are composed for the cases when $\mathbf{E}_1$ is non-singular and singular, correspondingly. The parameter uncertainties for the system (112) are randomly made as

$$\Delta a_1 = 0.3(rand - 0.5)a_1,\ \Delta a_2 = 0.4(rand - 0.5)a_2,$$
$$\Delta a_3 = 0.5(rand - 0.5)a_3,\ \Delta b_1 = 0.15(rand - 0.5)b_1,$$
$$\Delta b_2 = 0.25(rand - 0.5)b_2,\ \Delta b_3 = 0.35(rand - 0.5)b_3,$$
$$\Delta c_1 = 0.6(rand - 0.5)c_1,\ \Delta c_2 = 0.7(rand - 0.5)c_2,$$
$$\Delta c_3 = 0.8(rand - 0.5)c_3.$$

Here, every single variable *rand* stands for a self-determining random number between 0 and 1. Let

Uncertainty 0:

$\Delta a_i = 0$, $\Delta b_i = 0$ and $\Delta c_i = 0$, for $i = 1, 2, 3$.

Then $\Delta m^{(0)} = 0$.

Case 2-1: Matrix $\mathbf{E}_1$ is non-singular.

The control input $\boldsymbol{u}_1$ is preferred as the same as in *Case1-1*.

Uncertainty1:

$$\Delta a_1 = 0.0036, \Delta a_2 = 0.0011, \Delta a_3 = -0.0028,$$
$$\Delta b_1 = 0.0096, \Delta b_2 = 0.0339, \Delta b_3 = -0.0593,$$
$$\Delta c_1 = 0.0501, \Delta c_2 = -0.0928, \Delta c_3 = -0.0485.$$

It is calculated that the relative uncertainty scale of $\mathbf{M}$ is $\Delta m^{(1)} = ||\Delta\mathbf{M}\cdot\mathbf{M}^{-1}||_2 = 0.4764$.

Uncertainty 2:

$$\Delta a_1 = -0.0028, \Delta a_2 = 0.0093, \Delta a_3 = 0.0372,$$
$$\Delta b_1 = 0.0045, \Delta b_2 = 0.0153, \Delta b_3 = 0.0605,$$
$$\Delta c_1 = 0.0450, \Delta c_2 = 0.0239, \Delta c_3 = -0.0572.$$

It is tested that $\Delta m^{(2)} = ||\Delta\mathbf{M}\cdot\mathbf{M}^{-1}||_2 = 0.7824$.

Case 2-2: Matrix $\mathbf{E}_1$ is singular.

The first-iteration input $\boldsymbol{u}_1$ is taken as alike as that of in *Case 1-2*.

Uncertainty 3:

$$\Delta a_1 = -0.0012, \Delta a_2 = 0.0055, \Delta a_3 = -0.0327,$$
$$\Delta b_1 = 0.0074, \Delta b_2 = -0.0060, \Delta b_3 = 0.0436,$$
$$\Delta c_1 = -0.0289, \Delta c_2 = -0.0633\ , \Delta c_3 = 0.2333.$$

By determining, we get $\Delta m^{(3)} = ||\Delta\mathbf{M}\cdot\mathbf{M}^{-1}||_2 = 0.3739$.

Uncertainty 4:

$$\Delta a_1 = -0.0051, \Delta a_2 = 0.0098, \Delta a_3 = -0.0267,$$
$$\Delta b_1 = -0.0039, \Delta b_2 = 0.0092, \Delta b_3 = 0.0697,$$
$$\Delta c_1 = 0.0042, \Delta c_2 = 0.1717, \Delta c_3 = -0.1313.$$

It is confirmed $\Delta m^{(4)} = ||\Delta\mathbf{M}\cdot\mathbf{M}^{-1}||_2 = 3.6421$.

Fig.5 put the comparable strictly monotonic convergences of *Case 2-1* on view for the QGOILC (60) with *Uncertainty 0*, *Uncertainty* 1 and *Uncertainty* 2. While Fig.6 competes for the strictly monotonic convergence of *Case 2-2* for the QGOILC (60) with *Uncertainty 0*, *Uncertainty* 3 and *Uncertainty* 4.

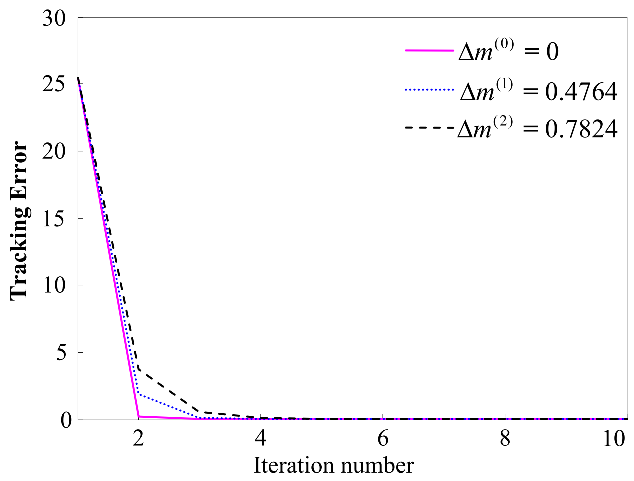

FIGURE 5. Tracking error tendencies of QGOILC (60) for Case 2-1.

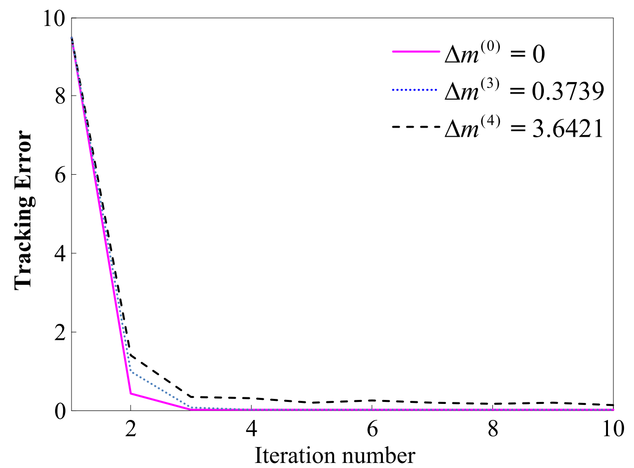

FIGURE 6. Tracking error tendencies of QGOILC (60) for Case 2-2.

From Fig.5 and Fig.6, it is realized that the dashed and the dotted tracking-error curves for the QGOILC (60) with uncertainties converge slower than the solid tracking-error curve for null uncertainties. Moreover, the dotted tracking-error curve for uncertainties with the smaller relative scale converges quicker than that of the dashed tracking-error curve for uncertainties with the larger relative uncertainty degree. It is seen in Fig.6 that the dashed tracking error curve with uncertainty $\Delta m = 3.6421$ needs more iterations to fulfill the convergence requirements. i.e., a smaller relative uncertainty scale means a faster convergence of the tracking error. This is common.

VI. CONCLUSION

A P-type gain-optimized iterative learning control (GOILC) scheme come into being in this paper by arguing the learning-gain vector for the sequential performance index combined by the linear-quadratic form of the tracking error and the incremental inputs weighted by the iteration-wise tuning factor that put forward the concerning importance ratio of the incremental input relating to the tracking error. The paper initially modified the differential-algebraic structure of the singular system as an input-output transmission by the lifted-vector method in order to obtain the convergence in a descriptive way. By means of the matrix permutation transformation and the property of the quadratic function, the sequential learning-gain vector is explicitly formulated by the system Markov matrix and the iteration-wise tracking error. Additionally, the linearly monotonic convergence is derived under the assumption that the initial state of the dynamic subsystem is resettable. The theoretical investigation and the numerical simulations suggest that the tracking error is linearly monotonically convergent and the faster convergence is achievable by a smaller tuning factor. Further, when the system parameters have uncertainties in additive forms, the P-type GOILC scheme may turn to a quasi P-type GOILC, which substitutes the exact Markov parameters matrix with the approximated one. The linearly monotonic convergence is induced on the premise that the initial state is resettable, but the parameter uncertainties are confined in an appropriate range. Numerical simulations manifest that, in most circumstances, the tracking-error tendency for the parameters uncertainties with a smaller relative scale renders a faster convergence rate than that of the tracking-error curve for the uncertainties with a larger relative uncertainty degree, i.e., a smaller relative uncertainties scale means a quicker convergence of the tracking error. This is common but not definitely confirmative as the relative uncertainty degree is a multi-uncertainty assessment. It is worthy to emphasize that the direct discrete-time domain methodology is an effective way to visualize the robustness of the GOILC to the uncertainties of parameters and the comprehensive convergence outcomes could update the existing robustness. Nevertheless, the job is done under the ideal assumption that the initial state is resettable to zero and the output is quantifiable with no disturbance. Indeed, the convergence characteristic for the exploited GOILC to count the external disturbance is a crucial topic. The issue will be taken into consideration in the future. In addition some future challenges are as follows:

1. To apply the methodology to a linear discrete-time varying singular system.
2. To apply the proposed algorithm to a non-linear singular system.
3. When the order of the learning scheme is larger or smaller than the relative degree of the system.

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