IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received January 27, 2021, accepted February 16, 2021, date of publication March 9, 2021, date of current version May 13, 2021.
*Digital Object Identifier 10.1109/ACCESS.2021.3064844*

# Stochastic Discrete-Time Delay Feedback Stabilization for Nonlinear Systems Under the Sublinear Expectation Framework

**YUYUAN LI[1], XIUWEI YIN[2], JIANQIU LU[3], YANHUI TONG[1], AND YANAN JIANG[4]**
[1]School of Electronic and Electrical Engineering, Shanghai University of Engineering Science, Shanghai 201620, China
[2]Department of Statistics, Anhui Normal University, Wuhu 241000, China
[3]Department of Statistics, Donghua University, Shanghai 201620, China
[4]College of Information Science and Technology, Donghua University, Shanghai 201620, China

Corresponding author: Yanan Jiang (y.n.jiang@qq.com)

This work was supported in part by the Shanghai University of Engineering Science under Grant 0232-E3-0507-20-05084, and in part by the Natural Science Foundation of Anhui Province under Grant 2008085QA20.

**ABSTRACT** This paper is concerned with the stochastic stabilization problem for nonlinear systems by $G$-Brownian motion with feedback control based on discrete-time state observations with a time delay. By constructing an auxiliary system, which is a continuous-time stochastic system driven by $G$-Brownian motion, we establish a sufficient criterion in terms of the observation gap $\tau$ and the time delay $\tau_0$ to ensure the quasi-surely exponential stability of the stochastic controlled system. Moreover, implementation of the control strategy is provided. Finally, an example is given to demonstrate the theoretical result.

**INDEX TERMS** Discrete-time state observations, $G$-Brownian motion, quasi-surely exponential stability, time delay.

## I. INTRODUCTION

It has been known that noise can be used to stabilize a given unstable system or make a stable system even more stable. On this topic of stochastic stabilization, the pioneering work is due to Khasminskii [1] who stabilized a linear system by using two white noise sources. Then the theory was developed to deal with stochastic stabilization and destabilization of nonlinear systems [2], [3], a class of functional systems [4] and hybrid systems [5], [6]. It is noted that the stochastic feedback controls in these papers depend on continuous-time state observations, which are expensive and impractical. To tackle this drawback, Mao proposed a new feedback control based on discrete-time state observations in 2013 [7] and later studied the almost sure exponential stabilization for linear and nonlinear systems by discrete-time stochastic feedback control [8]. However, delays often occur due in particular to the measurement and transport phenomena in many control applications. That is, there always exists a time lag between the time when the state observation is made and the time when the feedback control reaches the system. It is therefore more realistic to consider the effect of time delay when designing the discrete-time feedback control. So far, there only exist a small number of literatures on this problem, see [9], [10] for moment exponential stabilization of hybrid stochastic differential equations and [11] for almost surely exponential stabilization of nonlinear stochastic differential equations. But all of the results are derived in the classical framework, that is, the expectation is linear.

On the other hand, due to the powerful applications in many real world problems, such as the model uncertainty, risk measures and the superhedging in finance, the $G$-expectation and $G$-Brownian motion theory has made a significant progress since Peng initiated it in 2007 [12]. Under the $G$-framework, Peng [13] introduced the stochastic differential equations driven by $G$-Brownian motion ($G$-SDEs, in short). Since then, some interesting works on the properties of the $G$-SDEs have been reported; for more details, one can see e.g. [14]–[23] and the references therein. In particular, an emphasis has been placed on the stability analysis and stabilization. Zhang and Chen [15] proved when a quasi-surely exponentially stable linear system was disturbed by $G$-Brownian motion, the stochastic perturbed system preserved this nice property. In [17], the authors considered the moment exponential stability and quasi-sure stability of $G$-SDEs with Lyapunov-type conditions. Ren *et al.* [22] designed feedback control based

The associate editor coordinating the review of this manuscript and approving it for publication was Feiqi Deng.

on discrete-time state observations in the drift part to stabilize a class of stochastic differential equations driven by $G$-Brownian motion. $G$-Brownian motion with discrete-time stochastic feedback control was presented to stabilize a given unstable system quasi-sure exponentially in [23]. However, to our best knowledge, there is little literature on stochastic stabilization of nonlinear systems via discrete-time feedback control with a time delay under the $G$-expectation framework.

In this study, we aim to explore the stochastic stabilization for an unstable nonlinear system via discrete-time feedback control with a time delay induced by $G$-Brownian motion. It should be mentioned that our work is not a simple generalization of the existing results (see, e.g., [9], [10]). Firstly, the discrete-time feedback controls with a time delay were designed in the drift part of hybrid stochastic systems and the moment exponential stability criteria were established in [9], [10]. While here the discrete-time feedback control with a time delay induced by $G$-Brownian motion can be considered as the diffusion part of the controlled system and the quasi-surely exponential stability will be discussed. Moreover, due to the nonlinearity of the $G$-expectation, we have to employ a new approach. Besides, under the $G$-expectation framework, several new $G$-analysis techniques have to be employed to developed our theory. It should also be pointed out that the analysis in this paper becomes much more complicated because of the difficulties arisen from dealing with the observation duration $\tau$ and the time delay $\tau_0$ compared with [23].

All of the points made above show the motivation and innovations of our study. The rest of this paper is organized as follows. In Section 2, we give some notations, definitions, useful propositions and the $G$-Itô formula for later use. In Section 3, after stating the problem, the main results are established. Section 4 covers the method for designing the control function and an illustrate example. Finally, this article is concluded in Section 5.

## II. PRELIMINARIES

In this section, we first recall some facts in the framework of $G$-expectation, the readers can refer to Peng [24] for more details.

We denote by $\langle x, y\rangle = x^{\mathrm{T}}y$ the scalar product and $|x| = \sqrt{x^{\mathrm{T}}x}$ the Euclidean norm of the $n$-dimensional Euclidean space $\mathbb{R}^n$.

Let $\Omega$ be the space of all $\mathbb{R}$-valued continuous functions $\omega$ with $\omega_0 = 0$. $\mathcal{B}(\Omega)$ denotes the Borel $\sigma$-algebra of $\Omega$. For every $\omega \in \Omega$, let $B_t(\omega) := \omega(t)$ be the canonical process. For $t > 0$, let $\Omega_t = \{\omega_{t\wedge\cdot} : \omega \in \Omega\}$. Define

$$L_{ip}(\Omega_t) := \{\varphi(B_{t_1\wedge t}, \ldots, B_{t_n\wedge t}) : n \geq 1,\ \varphi \in C_{Lip}(\mathbb{R}^n)\}$$

and

$$L_{ip}(\Omega) := \bigcup_{t>0} L_{ip}(\Omega_t).$$

Then we can further define the related $G$-expectation on $L_{ip}(\Omega)$ by

$$\begin{aligned}&\bar{\mathbb{E}}[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})]\\&\quad := \bar{\mathbb{E}}[\psi(B_{t_1}, B_{t_2}, \ldots, B_{t_{n-1}})],\end{aligned}$$

where $\bar{\mathbb{E}}\psi(x_1, \ldots, x_{n-1}) = \bar{\mathbb{E}}[\varphi(x_1, \ldots, x_{n-1}, B_{t_n} - B_{t_{n-1}})]$. $G(\alpha) = \frac{1}{2}(\bar{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-)$ is a sublinear and monotonic function, where $\alpha \in \mathbb{R}$ and $0 \leq \underline{\sigma}^2 \leq \bar{\sigma}^2 < +\infty$. Under the $G$-expectation $\bar{\mathbb{E}}$, the $\{B_t, t \geq 0\}$ is said to be $G$-Brownian motion. Furthermore, $\bar{\mathbb{E}}(B_1^2) = \bar{\sigma}^2$, $-\bar{\mathbb{E}}(-B_1^2) = \underline{\sigma}^2$.

*Definition 1:* ( [25]) According to the representation theorem of $G$-expectation, $\bar{\mathbb{E}}$ can be constructed as follows

$$\bar{\mathbb{E}}[X] = \sup_{\mathbb{P}\in\mathcal{P}_G} \mathbb{E}^{\mathbb{P}}[X],\ X \in L_G^1(\Omega),$$

where $\mathcal{P}_G$ is a weakly compact family of martingale measures on $(\Omega, \mathcal{B}(\Omega))$. As to $\mathcal{P}_G$, we naturally define the associated Choquet capacity

$$\bar{\mathbb{C}}(A) := \sup_{\mathbb{P}\in\mathcal{P}_G} \mathbb{P}(A),\ A \in \mathcal{B}(\Omega).$$

*Definition 2:* ( [25]) A set $O \in \mathcal{B}(\Omega)$ is said to be polar if $\bar{\mathbb{C}}(O) = 0$ and a property is said to hold quasi-surely (q.s., in short) if it holds outside a polar set.

*Proposition 1:* ( [24]) For any $\eta \in \mathcal{M}_G^2(0, T)$, we have

$$\bar{\mathbb{E}}\int_0^T \eta(t)dB(t) = 0$$

and

$$\bar{\mathbb{E}}\Big[\int_0^T \eta(t)dB(t)\Big]^2 = \bar{\mathbb{E}}\int_0^T \eta^2(t)d\langle B\rangle_t \leq \bar{\sigma}^2\bar{\mathbb{E}}\int_0^T \eta^2(t)dt,$$

where $\langle B\rangle$ is the quadratic variation of $B$.

*Proposition 2:* ( [14]) Let $\eta \in \mathcal{M}_G^2(0, T)$, then

$$\bar{\mathbb{E}}\left(\sup_{s\leq u\leq t}\Big|\int_s^u \eta(u)dB_u\Big|^2\right) \leq 4\bar{\sigma}^2\bar{\mathbb{E}}\int_s^t \eta^2(u)du$$

and

$$\bar{\mathbb{E}}\left(\sup_{s\leq u\leq t}\Big|\int_s^u \eta(u)d\langle B\rangle_u\Big|^2\right) \leq \bar{\sigma}^4(t-s)\bar{\mathbb{E}}\int_s^t \eta^2(u)du$$

hold for any $0 \leq t \leq T$.

*Lemma 1:* ( [24], $G$-Itô formula) Let $X = (X^1, \ldots, X^n)$ be an n-dimensional process on $[0, T]$ with the form

$$X_t^\nu = X_0^\nu + \int_0^t \alpha_s^\nu ds + \int_0^t \eta_s^\nu d\langle B\rangle_s + \int_0^t \beta_s^\nu dB_s,$$

where $\alpha^\nu, \eta^\nu \in M_G^1(0, T)$ and $\beta^\nu \in M_G^2(0, T)$, $\nu = 1, \ldots, n$. Then for each $t \in [0, T]$ and $\Phi \in C^{1,2}([0, T] \times \mathbb{R}^n)$, we have

$$\begin{aligned}&\Phi(t, X_t) - \Phi(0, X_0)\\&\quad = \sum_{\nu=1}^n \int_0^t \partial_{x^\nu}\Phi(s, X_s)\beta_s^\nu dB_s\\&\qquad + \int_0^t [\partial_s\Phi(s, X_s) + \partial_{x^\nu}\Phi(s, X_s)\alpha_s^\nu]ds\end{aligned}$$

$$+\int_0^t\{\sum_{\nu=1}^n \partial_{x^\nu}\Phi(s,X_s)\eta_s^\nu$$
$$+\frac{1}{2}\sum_{\mu,\nu=1}^n \partial^2_{x^\mu x^\nu}\Phi(s,X_s)\beta_s^\mu\beta_s^\nu\}d\langle B\rangle_s.$$

Let us proceed to give some extra notations for later use. For $\tau > 0$, let $C([-\tau,0];\mathbb{R}^n)$ the family of continuous functions $\zeta : [-\tau,0] \to \mathbb{R}^n$ with the norm $\|\zeta\| = \sup_{-\tau\le u\le 0}|\zeta(u)|$. Denote by $L_G^2(\Omega_t, C([-\tau,0];\mathbb{R}^n))$ the family of $\mathcal{B}(\Omega_t)$-measurable $C([-\tau,0];\mathbb{R}^n)$-valued random variables $\xi$ such that $\bar{\mathbb{E}}\|\xi\|^2 < +\infty$.

## III. MAIN RESULTS

Consider the following unstable system

$$dx(t) = f(x(t))dt, \quad t \ge 0, \tag{1}$$

with initial data $x(0) = x_0 \in \mathbb{R}^n$, where $f : \mathbb{R}^n \to \mathbb{R}^n$. We aim to use the feedback control based on discrete-time state observations with a time delay induced by $G$-Brownian motion $g(x(\delta_t))d\langle B\rangle_t + \sigma(x(\delta_t))dB_t$ to stabilize the system, where $\delta_t = [\frac{t}{\tau}]\tau - \tau_0$, of which $\tau$ is the duration between two consecutive state observations and $\tau_0$ stands for the lag time. Therefore, the controlled system becomes

$$dx(t) = f(x(t))dt + g(x(\delta_t))d\langle B\rangle_t + \sigma(x(\delta_t))dB_t \tag{2}$$

with initial date $x_0 = \xi \in L_G^2(\Omega_t, C([-\tau_0,0];\mathbb{R}^n))$, where $g,\ \sigma : \mathbb{R}^n \to \mathbb{R}^n$. Let us impose the following assumption.

*Assumption 1:* Assume that the functions $f, g$ and $\sigma$ are continuous functions, and there exist three positive constants $K_1, K_2$ and $K_3$ such that

$$|f(x)-f(y)| \le K_1|x-y|,$$
$$|g(x)-g(y)| \le K_2|x-y|,$$
$$|\sigma(x)-\sigma(y)| \le K_3|x-y|,$$

for all $(x,y) \in \mathbb{R}^n\times\mathbb{R}^n$. Moreover, for the purpose of stability analysis, let $f(0) = g(0) = \sigma(0) = 0$.

It is known that under Assumption 1, the equation (2) has a unique solution $x(t)$ on $t \ge -\tau_0$ and $\bar{\mathbb{E}}|x(t)|^2 < \infty$ (see [19]).

To achieve the stabilization goal, we need an auxiliary system

$$\begin{cases} dy(t) = f(y(t))dt + g(y(t))d\langle B\rangle_t + \sigma(y(t))dB_t, \\ y(0) = y_0 \in L_G^2(\Omega_{t_0}). \end{cases} \tag{3}$$

It has been showed in [17] that equation (3) has a unique solution and its second moment is finite.

The key technique employed in this paper is to compare the discrete-time delay feedback controlled system (2) with the continuous system (3). It will be shown that if system (3) is quasi-surely exponentially stable, then system (2) is also quasi-surely exponentially stable if $\tau$ and $\tau_0$ are small enough. To ensure the quasi-surely exponential stability of system (3), we introduce the following notations and impose another assumption.

Denote by $V \in C^{2,1}(\mathbb{R}^n\times\mathbb{R}_+,\mathbb{R}_+)$ the family of nonnegative functions $V(x,t)$ defined on $\mathbb{R}^n\times\mathbb{R}_+$ such that they are continuously twice differentiable in $x$ and once in $t$. For $V \in C^{2,1}(\mathbb{R}^n\times\mathbb{R}_+,\mathbb{R}_+)$, we define $LV : \mathbb{R}^n\times\mathbb{R}_+ \to \mathbb{R}$ by

$$LV(x,t) := V_t(x,t) + \langle V_x(x,t), f(x)\rangle + G\big(\langle V_x(x,t), 2g(x)\rangle + \langle V_{xx}(x,t)\sigma(x),\sigma(x)\rangle\big),$$

where

$$V_t(x,t) = \frac{\partial V(x,t)}{\partial t}, \quad V_x(x,t) = \Big(\frac{\partial V(x,t)}{\partial x_i}\Big)_{n\times 1},$$

$$V_{xx}(x,t) = \Big(\frac{\partial^2 V(x,t)}{\partial x_i\partial x_j}\Big)_{n\times n}.$$

*Assumption 2:* Assume that there exist a function $V \in C^{2,1}(\mathbb{R}^n\times\mathbb{R}_+;\mathbb{R}_+)$ and constants $q > 0$, $\bar{c_1} \ge c_1 > 0$ and $c_2 \in \mathbb{R}$, $c_3 \ge 0$, $c_3 > c_2$, such that

$$c_1|x|^q \le V(x,t) \le \bar{c_1}|x|^q,$$
$$LV(x,t) \le c_2 V(x,t),$$

and

$$G(-|V_x(x,t)\sigma(x)|^2) \le -c_3V^2(x,t).$$

*Lemma 2:* Let Assumptions 1 and 2 hold. Suppose $\theta \in (0,1)$ is sufficiently small such that

$$p := \theta q < 1, \quad \gamma := \theta((1-\theta)c_3 - c_2) > 0.$$

Then the solution $y(t) := y(t;0,y(0))$ of equation (3) satisfies

$$\bar{\mathbb{E}}|y(t)|^p \le M\bar{\mathbb{E}}|y(0)|^p e^{-\gamma t},$$

where $M = \left(\frac{\bar{c}_1}{c_1}\right)^\theta$.

*Proof:* Obviously, the result holds for $y(0) = 0$. Thus, we only need to consider the case that $y_0 \ne 0$, then we have $y(t) \ne 0$ quasi-sure for all $t \ge 0$ (see [26]). Without loss of generality, we may as well assume $y_0$ is deterministic. Otherwise, we can use the property of the conditional expectation. Let $U(y,t) = (V(y,t))^\theta$, then the $G$-Itô formula implies that for $t \ge 0$,

$$e^{\gamma t}\bar{\mathbb{E}}U(y(t),t) = U(y(0),0) + \bar{\mathbb{E}}\int_0^t\big[\gamma e^{\gamma s}U(y(s),s) + e^{\gamma s}LU(y(s),s)\big]ds. \tag{4}$$

It follows from Assumption 2 that

$$\begin{aligned} LU(y,t) &= \theta V^{\theta-1}(y,t)\Big(V_t(y,t) + \langle V_y(y,t), f(y)\rangle\Big) \\ &\quad + G\Big(\theta V^{\theta-1}(y,t)\Big[\langle V_y(y,t), 2g(y)\rangle \\ &\quad + \langle V_{yy}(y,t)\sigma(y),\sigma(y)\rangle\Big] \\ &\quad - \theta(1-\theta)V^{\theta-2}(y,t)|V_y(y,t)\sigma(y)|^2\Big) \\ &\le \theta V^{\theta-1}(x,t)LV(y,t) \\ &\quad + \theta(1-\theta)V^{\theta-2}(y,t)G(-|V_y(y,t)\sigma(y)|^2) \\ &\le \theta(c_2-(1-\theta)c_3)V^\theta(y,t) \\ &= -\gamma U(y,t). \end{aligned}$$

Substituting this into (4) yields

$$e^{\gamma t}\bar{\mathbb{E}}U(y(t),t) \le U(y(0),0).$$

Therefore, we can further derive from Assumption 2 that for all $t \ge 0$,

$$\bar{\mathbb{E}}|y(t)|^p \le \left(\frac{\bar{c}_1}{c_1}\right)^\theta |y(0)|^p e^{-\gamma t} = M|y(0)|^p e^{-\gamma t}.$$

We complete the proof.

*Lemma 3:* Under Assumption 1, let $0 < p < 1$ and $x(t) := x(t;0,\xi)$ be the solution of (2). Then

$$\bar{\mathbb{E}}\Big(\sup_{0\le u\le \tau_0} |x(t+u)-x(t)|^p\Big) \le H_1(\tau_0,p,t)\|\xi\|^p \quad (5)$$

for all $t \ge 0$, where

$$H_1(\tau_0,p,t) = \Big[6\tau_0(K_1^2\tau_0 + K_2^2\bar{\sigma}^4\tau_0 + 4K_3^2\bar{\sigma}^2) \times e^{(2K_1+2\bar{\sigma}^2K_2+\bar{\sigma}^2K_3^2)(t+\tau_0)}\Big]^{p/2}.$$

*Proof:* According to the method of conditional expectation, we only need to show the lemma for deterministic initial data $\xi \in C([-\tau_0,0];\mathbb{R}^n)$. By the $G$-Itô formula, we have

$$\begin{aligned}|x(t)|^2 &= |x_0|^2 + 2\int_0^t \langle x(s), f(x(s))\rangle ds \\ &\quad + 2\int_0^t \langle x(s), \sigma(x(\delta_s))\rangle dB_s \\ &\quad + \int_0^t \Big[2\langle x(s), g(x(\delta_s))\rangle + |\sigma(x(\delta_s))|^2\Big] d\langle B\rangle_s.\end{aligned}$$

We can derive from Assumption 1 that

$$\begin{aligned}\bar{\mathbb{E}}|x(t)|^2 &\le |x_0|^2 + 2K_1\int_0^t \bar{\mathbb{E}}|x(s)|^2 ds \\ &\quad + (2\bar{\sigma}^2K_2 + \bar{\sigma}^2K_3^2)\int_{t_0}^t \Big(\sup_{-\tau_0\le u\le s}\bar{\mathbb{E}}|x(u)|^2\Big) ds.\end{aligned}$$

Observe that

$$\sup_{-\tau_0\le u\le t}\bar{\mathbb{E}}|x(u)|^2 \le \|\xi\|^2 + \sup_{0\le u\le t}\bar{\mathbb{E}}|x(u)|^2.$$

Consequently,

$$\begin{aligned}&\sup_{-\tau_0\le u\le t}\bar{\mathbb{E}}|x(u)|^2 \\ &\quad \le 2\|\xi\|^2 + (2K_1 + 2K_2\bar{\sigma}^2 + K_3^2\bar{\sigma}^2) \\ &\qquad \times \int_0^t \Big(\sup_{-\tau_0\le u\le s}\bar{\mathbb{E}}|x(u)|^2\Big) ds.\end{aligned}$$

The Gronwall inequality shows that

$$\sup_{-\tau_0\le u\le t}\bar{\mathbb{E}}|x(u)|^2 \le 2\|\xi\|^2 e^{(2K_1+2K_2\bar{\sigma}^2+K_3^2\bar{\sigma}^2)t}. \quad (6)$$

According to Proposition 2, we also derive that

$$\begin{aligned}&\bar{\mathbb{E}}\Big(\sup_{0\le u\le \tau_0}|x(t+u)-x(t)|^2\Big) \\ &\le 3\tau_0\int_t^{t+\tau_0}\bar{\mathbb{E}}|f(x(s))|^2 ds + 3\bar{\sigma}^4\tau_0\int_t^{t+\tau_0}\bar{\mathbb{E}}|g(x(\delta_s))|^2 ds \\ &\quad + 12\bar{\sigma}^2\int_t^{t+\tau_0}\bar{\mathbb{E}}|\sigma(x(\delta_s))|^2 ds \\ &\le 3(K_1^2\tau_0 + K_2^2\bar{\sigma}^4\tau_0 + 4K_3^2\bar{\sigma}^2) \\ &\quad \times \int_t^{t+\tau_0}\Big(\sup_{-\tau_0\le u\le s}\bar{\mathbb{E}}|x(u)|^2\Big) ds \\ &\le 6\tau_0(K_1^2\tau_0 + K_2^2\bar{\sigma}^4\tau_0 + 4K_3^2\bar{\sigma}^2)\|\xi\|^2 \\ &\quad \times e^{(2K_1+2\bar{\sigma}^2K_2+\bar{\sigma}^2K_3^2)(t+\tau_0)}.\end{aligned} \quad (7)$$

Therefore, a simple application of the Hölder inequality yields

$$\begin{aligned}&\bar{\mathbb{E}}\Big(\sup_{0\le u\le \tau_0}|x(t+u)-x(t)|^p\Big) \\ &\quad \le \Big[6\tau_0(K_1^2\tau_0 + K_2^2\bar{\sigma}^4\tau_0 + 4K_3^2\bar{\sigma}^2) \\ &\qquad \times e^{(2K_1+2\bar{\sigma}^2K_2+\bar{\sigma}^2K_3^2)(t+\tau_0)}\Big]^{p/2}\|\xi\|^p \\ &\quad = H_1(\tau_0,p,t)\|\xi\|^p.\end{aligned}$$

The proof is complete.

*Theorem 1:* Let Assumptions 1 and 2 hold. Then there exists a domain $\mathbb{D} \subset \mathbb{R}_+^2$, such that for all $(\tau,\tau_0) \in \mathbb{D}$, the solution $x(t;0,\xi)$ of equation (2) satisfies

$$\limsup_{t\to\infty}\frac{\log(|x(t;0,\xi)|)}{t} < 0, \ q.s. \quad (8)$$

for all initial date $\xi \in L_G^2(\Omega_t, C([-\tau_0,0];\mathbb{R}^n))$.

*Proof:* Likewise, we only consider the case of deterministic $\xi \in C([-\tau_0,0];\mathbb{R}^n)$. Due to the techniques used in the proof, we need some extra data $x_\theta = \{x(\theta), -2\tau_0 \le \theta < -\tau_0\}$ for the system (2) and we may as well set $x(\theta) = y(0)$ for all $-2\tau_0 \le \theta < -\tau_0$. To make it clearer, we divide the proof into three steps.

**Step I.** Let $h = [\frac{\tau_0}{\tau}] + 1$. For any $t \ge 0$, there exists an integer $n \ge 0$ such that $n\tau \le t < (n+1)\tau$. If $n \ge h$, then $\delta_t = n\tau - \tau_0 \ge 0$. Therefore, it follows from Assumption 1, Propositions 1 and 2 that

$$\begin{aligned}&\bar{\mathbb{E}}|x(t)-x(\delta_t)|^2 \\ &\le 3\bar{\mathbb{E}}\Big|\int_{n\tau-\tau_0}^t f(x(s))ds\Big|^2 + 3\bar{\mathbb{E}}\Big|\int_{n\tau-\tau_0}^t g(x(\delta_s))d\langle B\rangle_s\Big|^2 \\ &\quad + 3\bar{\mathbb{E}}\Big|\int_{n\tau-\tau_0}^t \sigma(x(\delta_s))dB_s\Big|^2 \\ &\le 3K_1^2(t-n\tau+\tau_0)\int_{n\tau-\tau_0}^t \bar{\mathbb{E}}|x(s)|^2 ds \\ &\quad + [3K_2^2\bar{\sigma}^4(t-n\tau+\tau_0) + 3K_3^2\bar{\sigma}^2]\int_{n\tau-\tau_0}^t \bar{\mathbb{E}}|x(\delta_s)|^2 ds \\ &\le 6K_1^2(\tau+\tau_0)\int_{n\tau-\tau_0}^t \bar{\mathbb{E}}|x(s)-x(\delta_s)|^2 ds \\ &\quad + [6K_1^2(\tau+\tau_0) + 3K_2^2\bar{\sigma}^4(\tau+\tau_0) + 3K_3^2\bar{\sigma}^2] \\ &\quad \times \int_{n\tau-\tau_0}^t \bar{\mathbb{E}}|x(\delta_s)|^2 ds.\end{aligned} \quad (9)$$

Note that

$$\int_{n\tau-\tau_0}^{t} \bar{\mathbb{E}}|x(\delta_s)|^2 ds \le \sum_{k=0}^{h} \int_{(n-k)\tau}^{(n-k+1)\tau} \bar{\mathbb{E}}|x(\delta_s)|^2 ds = \tau \sum_{k=0}^{h} \bar{\mathbb{E}}|x((n-k)\tau-\tau_0)|^2. \quad (10)$$

Substitute (10) into (9) yields

$$\begin{aligned}&\bar{\mathbb{E}}|x(t)-x(\delta_t)|^2\\ &\le 6K_1^2(\tau+\tau_0)\int_{n\tau-\tau_0}^{t}\bar{\mathbb{E}}|x(s)-x(\delta_s)|^2ds\\ &\quad+\tau[6K_1^2(\tau+\tau_0)+3K_2^2\bar{\sigma}^4(\tau+\tau_0)\\ &\quad+3K_3^2\bar{\sigma}^2]\sum_{k=0}^{h}\bar{\mathbb{E}}|x((n-k)\tau-\tau_0)|^2.\end{aligned}$$

It follows from the Gronwall inequality that

$$\bar{\mathbb{E}}|x(t)-x(\delta_t)|^2 \le H_2(\tau_0,\tau)\sum_{k=0}^{h}\bar{\mathbb{E}}|x((n-k)\tau-\tau_0)|, \quad (11)$$

where

$$H_2(\tau_0,\tau) := \tau[6K_1^2(\tau+\tau_0)+3K_2^2\bar{\sigma}^4(\tau+\tau_0)+3K_3^2\bar{\sigma}^2] \times e^{6K_1^2(\tau+\tau_0)^2}.$$

On the other hand, if $0 \le n \le h-1$, we have

$$\begin{aligned}&\bar{\mathbb{E}}|x(t)-x(\delta_t)|^2\\ &\le 3\bar{\mathbb{E}}\Big|\int_0^t f(x(s))ds\Big|^2+3\bar{\mathbb{E}}\Big|\int_0^t g(x(\delta_s))d\langle B\rangle_s\Big|^2\\ &\quad+3\bar{\mathbb{E}}\Big|\int_0^t \sigma(x(\delta_s))dB_s\Big|^2\\ &\le 3K_1^2t\int_0^t\bar{\mathbb{E}}|x(s)|^2ds\\ &\quad+3[K_2^2\bar{\sigma}^4t+K_3^2\bar{\sigma}^2]\int_0^t\bar{\mathbb{E}}|x(\delta_s)|^2ds\\ &\le 6K_1^2(\tau+\tau_0)\int_{n\tau-\tau_0}^{t}\bar{\mathbb{E}}|x(s)-x(\delta_s)|^2ds\\ &\quad+[6K_1^2(\tau+\tau_0)+3K_2^2\bar{\sigma}^4(\tau+\tau_0)+3K_3^2\bar{\sigma}^2]\\ &\quad\times\int_{n\tau-\tau_0}^{t}\bar{\mathbb{E}}|x(\delta_s)|^2ds.\end{aligned}$$

We can similarly obtain the estimate (11). Therefore, (11) holds for any $n\tau \le t < (n+1)\tau$.

**Step II.** For any $t \ge 0$, we can choose $n \ge 0$ such that $n\tau \le t < (n+1)\tau$, then

$$\begin{aligned}&\int_0^t e^{\beta s}\bar{\mathbb{E}}|x(s)-x(\delta_s)|^2ds\\ &=\int_{n\tau}^{t}e^{\beta s}\bar{\mathbb{E}}|x(s)-x(\delta_s)|^2ds\\ &\quad+\sum_{l=0}^{n-1}\int_{(n-l-1)\tau}^{(n-l)\tau}e^{\beta s}\bar{\mathbb{E}}|x(s)-x(\delta_s)|^2ds\\ &\le H_2(\tau_0,\tau)\int_{n\tau}^{t}e^{\beta s}\sum_{k=0}^{h}\bar{\mathbb{E}}|x((n-k)\tau-\tau_0)|ds\\ &\quad+H_2(\tau_0,\tau)\\ &\quad\times\sum_{l=0}^{n-1}\int_{(n-l-1)\tau}^{(n-l)\tau}e^{\beta s}\sum_{k=n-l-1-h}^{n-l-1}\bar{\mathbb{E}}|x(k\tau-\tau_0)|^2ds\\ &=: \mathrm{I}_{11}+\mathrm{I}_{12}.\end{aligned}$$

For the term $\mathrm{I}_{11}$, we have

$$\begin{aligned}\mathrm{I}_{11}&=H_2(\tau_0,\tau)\sum_{k=0}^{h}e^{\beta k}\int_{n\tau-k\tau}^{t-k\tau}e^{\beta s}\bar{\mathbb{E}}|x((n-k)\tau-\tau_0)|ds\\ &=H_2(\tau_0,\tau)\sum_{k=0}^{h}e^{\beta k}\int_{n\tau-k\tau}^{t-k\tau}e^{\beta s}\bar{\mathbb{E}}|x(\delta_s)|^2ds. \quad (12)\end{aligned}$$

Similarly,

$$\begin{aligned}\mathrm{I}_{12}&=H_2(\tau_0,\tau)\\ &\quad\times\sum_{l=0}^{n-1}\int_{(n-l-1)\tau}^{(n-l)\tau}e^{\beta s}\sum_{k=n-l-1-h}^{n-l-1}\bar{\mathbb{E}}|x(k\tau-\tau_0)|^2ds\\ &=H_2(\tau_0,\tau)\sum_{l=0}^{n-1}\sum_{k=0}^{h}e^{\beta k\tau}\\ &\quad\times\int_{(n-l-1-k)\tau}^{(n-l-k)\tau}e^{\beta s}\bar{\mathbb{E}}|x((n-l-1-k)\tau-\tau_0)|^2ds\\ &=H_2(\tau_0,\tau)\sum_{l=0}^{n-1}\sum_{k=0}^{h}e^{\beta k\tau}\int_{(n-l-1-k)\tau}^{(n-l-k)\tau}e^{\beta s}\bar{\mathbb{E}}|x(\delta_s)|^2ds. \quad (13)\end{aligned}$$

Combining (12) with (13), we get

$$\begin{aligned}&\int_0^t e^{\beta s}\bar{\mathbb{E}}|x(s)-x(\delta_s)|^2ds\\ &\le H_2(\tau_0,\tau)\sum_{k=0}^{h}e^{\beta k\tau}\int_{-k\tau}^{t-k\tau}e^{\beta s}\bar{\mathbb{E}}|x(\delta_s)|^2ds\\ &\le H_2(\tau_0,\tau)\sum_{k=0}^{h}e^{\beta k\tau}\\ &\quad\times\left[\int_0^t e^{\beta s}\bar{\mathbb{E}}|x(\delta_s)|^2ds+\int_{-k\tau}^{0}e^{\beta s}\bar{\mathbb{E}}|x(\delta_s)|^2ds\right]\\ &\le\frac{(e^{(h+1)\beta\tau}-1)H_2(\tau_0,\tau)}{e^{\beta\tau}-1}\times\int_0^t e^{\beta s}\bar{\mathbb{E}}|x(\delta_s)|^2ds\\ &\quad+\frac{\tau}{2}(h+h^2)e^{\beta h\tau}\|\xi\|^2.\end{aligned}$$

Letting $\beta \to 0$, we obtain

$$\begin{aligned}&\int_0^t\bar{\mathbb{E}}|x(s)-x(\delta_s)|^2ds\\ &\le H_2(\tau_0,\tau)(h+1)\Big[\int_0^t\bar{\mathbb{E}}|x(\delta_s)|^2ds+\frac{1}{2}(\tau+\tau_0)\|\xi\|^2\Big]\\ &\le H_2(\tau_0,\tau)(h+1)\\ &\quad\times\left[2\|\xi\|^2\int_0^t e^{(2K_1+2K_2\bar{\sigma}^2+K_3^2\bar{\sigma}^2)s}ds+\frac{1}{2}(\tau+\tau_0)\|\xi\|^2\right]\end{aligned}$$

$$\leq \Big[\frac{2e^{(2K_1+2K_2\bar{\sigma}^2+K_3^2\bar{\sigma}^2)t}}{2K_1+2K_2\bar{\sigma}^2+K_3^2\bar{\sigma}^2}+\frac{1}{2}(\tau+\tau_0)\Big] \times H_2(\tau_0,\tau)(h+1)\|\xi\|^2 := H_3(\tau,\tau_0,t)\|\xi\|^2. \quad (14)$$

By the $G$-Itô formula, we have

$$\begin{aligned}|x(t)-y(t)|^2 &= 2\int_0^t \langle x(s)-y(s), f(x(s))-f(y(s))\rangle ds\\ &+2\int_0^t \langle x(s)-y(s), \sigma(x(\delta_s)-\sigma(y(s))\rangle dB_s\\ &+2\int_0^t \langle x(s)-y(s), g(x(\delta_s)-g(y(s),s)\rangle d\langle B\rangle_s\\ &+\int_0^t |\sigma(x(\delta_s)-\sigma(y(s),s)|^2 d\langle B\rangle_s.\end{aligned}$$

Then it follows from Assumption 1 that

$$\begin{aligned}\bar{\mathbb{E}}|x(t)-y(t)|^2 &\leq 2K_1\int_0^t \bar{\mathbb{E}}|x(s)-y(s)|^2\, ds\\ &+K_3^2\bar{\sigma}^2\int_0^t \bar{\mathbb{E}}|x(\delta_s)-y(s)|^2 ds\\ &+2K_2\bar{\sigma}^2\int_0^t \bar{\mathbb{E}}|x(s)-y(s)||x(\delta_s)-y(s)|ds\\ &\leq (2K_1+K_2\bar{\sigma}^2)\int_0^t \bar{\mathbb{E}}|x(s)-y(s)|^2\, ds\\ &+(K_2\bar{\sigma}^2+K_3^2\bar{\sigma}^2)\int_0^t \bar{\mathbb{E}}|x(\delta_s)-y(s)|^2 ds\\ &\leq (2K_1+3K_2\bar{\sigma}^2+2K_3^2\bar{\sigma}^2)\int_0^t \bar{\mathbb{E}}|x(s)-y(s)|^2 ds\\ &+2(K_2\bar{\sigma}^2+K_3^2\bar{\sigma}^2)\int_0^t \bar{\mathbb{E}}|x(\delta_s)-x(s)|^2 ds.\end{aligned}$$

According to the Gronwall inequality and inequality 14, we have

$$\begin{aligned}&\bar{\mathbb{E}}|x(t)-y(t)|^2\\ &\leq 2(K_2\bar{\sigma}^2+K_3^2\bar{\sigma}^2)H_3(\tau,\tau_0,t)\|\xi\|^2\\ &\quad+2(K_2\bar{\sigma}^2+K_3^2\bar{\sigma}^2)(2K_1+3K_2\bar{\sigma}^2+3K_3^2\bar{\sigma}^2)\|\xi\|^2\\ &\quad\times e^{(2K_1+3K_2\bar{\sigma}^2+3K_3^2\bar{\sigma}^2)t}\int_0^t H_2(\tau_0,\tau)(h+1)\\ &\quad\times\Big[\frac{2e^{(2K_1+2K_2\bar{\sigma}^2+K_3^2\bar{\sigma}^2)s}}{2K_1+2K_2\bar{\sigma}^2+K_3^2\bar{\sigma}^2}+\frac{\tau+\tau_0}{2}\Big]\\ &\quad\times\times e^{(2K_1+3K_2\bar{\sigma}^2+3K_3^2\bar{\sigma}^2)s}ds\\ &\leq \Big[2(K_2\bar{\sigma}^2+K_3^2\bar{\sigma}^2)H_3(\tau,\tau_0,t)\\ &\quad+2(K_2\bar{\sigma}^2+K_3^2\bar{\sigma}^2)(2K_1+3K_2\bar{\sigma}^2+3K_3^2\bar{\sigma}^2)\\ &\quad\times e^{(2K_1+3K_2\bar{\sigma}^2+3K_3^2\bar{\sigma}^2)t}\\ &\quad\times\Big(\frac{(\tau+\tau_0)t}{2}H_2(\tau_0,\tau)(h+1)e^{(2K_1+3K_2\bar{\sigma}^2+3K_3^2\bar{\sigma}^2)t}\\ &\quad+\frac{2tH_2(\tau_0,\tau)(h+1)}{2K_1+2K_2\bar{\sigma}^2+K_3^2\bar{\sigma}^2}e^{(4K_1+5K_2\bar{\sigma}^2+4K_3^2\bar{\sigma}^2)t}\Big)\Big]\|\xi\|^2\\ &=: H_4(\tau,h,\tau_0,t)\|\xi\|^2.\end{aligned}$$

**Step III**. For sufficiently small $\varepsilon > 0$, let

$$\Theta_\varepsilon := \gamma^{-1}\log(M\varepsilon^{-1}),$$

where $\gamma, M$ are defined in Lemma 2. Choose $\bar{k}$ such that

$$\frac{\Theta_\varepsilon-\tau_0}{\tau}\leq \bar{k} < \frac{\Theta_\varepsilon-\tau_0}{\tau}+1.$$

It follows from Lemma 2 that

$$\begin{aligned}&\bar{\mathbb{E}}|x(\bar{k}\tau+\tau_0)|^p\\ &\leq \bar{\mathbb{E}}|y(\bar{k}\tau+\tau_0)|^p+\bar{\mathbb{E}}|x(\bar{k}\tau+\tau_0)-y(\bar{k}\tau+\tau_0)|^p\\ &\leq \Big(Me^{-\gamma(\bar{k}\tau+\tau_0)}+[H_4(\tau,h,\tau_0,\bar{k}\tau+\tau_0)]^{\frac{p}{2}}\Big)\|\xi\|^p.\end{aligned}$$

By elementary inequality and Lemma 3, we get

$$\begin{aligned}&\bar{\mathbb{E}}\Big(\sup_{0\leq u\leq\tau_0}|x(\bar{k}\tau+\tau_0+u)|^p\Big)\\ &\leq \bar{\mathbb{E}}|x(\bar{k}\tau+\tau_0)|^p\\ &\quad+\bar{\mathbb{E}}\Big(\sup_{0\leq u\leq\tau_0}|x(\bar{k}\tau+\tau_0+u)-x(\bar{k}\tau+\tau_0)|^p\Big)\\ &\leq \Big(Me^{-\gamma(\bar{k}\tau+\tau_0)}+[H_4(\tau,p,\tau_0,\bar{k}\tau+\tau_0)]^{\frac{p}{2}}\\ &\quad+H_1(\tau_0,\bar{k}\tau+\tau_0)\Big)\|\xi\|^p\\ &\leq \Big(\varepsilon+[H_4(\tau,h,\tau_0,\Theta_\varepsilon+\tau)]^{\frac{p}{2}}+H_1(\tau_0,\Theta_\varepsilon+\tau)\Big)\|\xi\|^p. \quad (15)\end{aligned}$$

Obviously, there exists a subset $\mathbb{D}\subset\mathbb{R}_+^2$ such that

$$\Psi(\varepsilon,p) := \varepsilon+[H_4(\tau,h,\tau_0,\Theta_\varepsilon+\tau)]^{\frac{p}{2}}+H_1(\tau_0,\Theta_\varepsilon+\tau) < 1 \quad (16)$$

for any $(\tau,\tau_0)\in\mathbb{D}$. Therefore, we have

$$\begin{aligned}\bar{\mathbb{E}}\|x_{\bar{k}\tau+2\tau_0}\|^p &= \bar{\mathbb{E}}\Big(\sup_{-\tau_0\leq u\leq 0}|x(\bar{k}\tau+2\tau_0+u)|^p\Big)\\ &\leq e^{-\lambda\Delta}\|\xi\|^p,\end{aligned}$$

where $\Delta=\bar{k}\tau+2\tau_0$ and $\lambda=\Delta^{-1}\log(\Psi^{-1}(\varepsilon,p))$. Similarly,

$$\bar{\mathbb{E}}\|x_{k\Delta}\|^p\leq e^{-k\lambda\Delta}\|\xi\|^p,\ k=1,2,\cdots.$$

Note that

$$\begin{aligned}&\bar{\mathbb{E}}\Big(\sup_{0\leq t\leq\Delta}|x(t)|^2\Big)\\ &\leq 4\|\xi\|^2+4\bar{\mathbb{E}}\Big|\int_0^\Delta f(x(s))ds\Big|^2\\ &\quad+4\bar{\mathbb{E}}\Big(\sup_{0\leq t\leq\Delta}\Big|\int_0^t g(x(\delta_s))d\langle B\rangle_s\Big|^2\Big)\\ &\quad+4\bar{\mathbb{E}}\Big(\sup_{0\leq t\leq\Delta}\Big|\int_0^t \sigma(x(\delta_s))dB_s\Big|^2\Big)\\ &\leq 4\|\xi\|^2+4K_1^2\Delta\int_0^{\Delta_1}\bar{\mathbb{E}}|x(s)|^2 ds\end{aligned}$$

$$
\begin{aligned}
&+(4K_2^2\bar{\sigma}^2\Delta+16K_3^2\bar{\sigma}^2)\int_0^{\Delta}\bar{\mathbb{E}}|x(\delta_s)|^2ds\\
\le\ & 4\|\xi\|^2+4(K_1^2\Delta+K_2^2\bar{\sigma}^2\Delta+4K_3^2\bar{\sigma}^2)\\
&\times\int_0^{\Delta}\sup_{-\tau_0\le u\le s}\bar{\mathbb{E}}|x(u)|^2ds\\
\le\ & 4\|\xi\|^2+\frac{8(K_1^2\Delta+K_2^2\bar{\sigma}^2\Delta+4K_3^2\bar{\sigma}^2)}{2K_1+2K_2\bar{\sigma}^2+K_3^2\bar{\sigma}^2}\\
&\times e^{(2K_1+2K_2\bar{\sigma}^2+K_3^2\bar{\sigma}^2)\Delta}\|\xi\|^2\\
=:\ & H_5(\tau,\tau_0)\|\xi\|^2.
\end{aligned}
$$

By using Hölder inequality, we get

$$
\bar{\mathbb{E}}\Big(\sup_{0\le t\le\Delta}|x(t)|^p\Big)\le H_5^{\frac{p}{2}}(\tau,\tau_0)\|\xi\|^p.
$$

Repeating the above procedure, we have

$$
\begin{aligned}
&\bar{\mathbb{E}}\Big(\sup_{k\Delta\le t\le(k+1)\Delta}|x(t)|^p\Big)\\
&\le H_5^{\frac{p}{2}}(\tau,\tau_0)\bar{\mathbb{E}}\|x_{k\Delta}\|^p\\
&\le H_5^{\frac{p}{2}}(\tau,\tau_0)e^{-k\lambda\Delta}\bar{\mathbb{E}}\|\xi\|^p,\ k=1,2,\cdots.
\end{aligned}
$$

An application of the Markov's inequality (see [17]) yields

$$
\begin{aligned}
&\bar{\mathbb{C}}\Big(\sup_{k\Delta\le t\le(k+1)\Delta}|x(t)|^p\ge e^{-\frac{1}{2}k\lambda\Delta}\Big)\\
&\qquad\le H_5^{\frac{p}{2}}(\tau,\tau_0)e^{-\frac{1}{2}k\lambda\Delta}\bar{\mathbb{E}}\|\xi\|^p
\end{aligned}
$$

for $k=0,1,\cdots$. Consequently, we get

$$
\sum_{k=1}^{\infty}\bar{\mathbb{C}}\Big(\sup_{k\Delta\le t\le(k+1)\Delta}|x(t)|^p\ge e^{-\frac{1}{2}k\lambda\Delta}\Big)<+\infty.
$$

Therefore, it follows from the Borel-Cantelli lemma under sublinear expectation (see [27]) that

$$
\limsup_{t\to\infty}\frac{\log(|x(t)|)}{t}\le-\frac{\lambda}{2p},\qquad q.s.
$$

The desired result is obtained.

## IV. AN EXAMPLE

Before giving a specific example to demonstrate the effectiveness of our theory, let us first clear and summary the implementation as the following two steps:

- Under Assumptions 1 and 2, choose a constant $\theta\in(0,1)$ and define $p,\gamma,M$ as in Lemma 2.
- Choose another constant $\varepsilon\in(0,1)$ and compute $\Theta_\varepsilon$. If we fix the time lag $\tau_0$, then we can get the upper bound $\tau_*$ for the observation duration $\tau$ by solving the equation

$$
\varepsilon+[H_4(\tau,h,\tau_0,\Theta_\varepsilon+\tau)]^{\frac{p}{2}}+H_1(\tau_0,\Theta_\varepsilon+\tau)=1. \tag{17}
$$

Then the controlled $G$-SDDE (2) is exponentially stable quasi-surely as long as the states are observed frequently enough in the sense that $\tau<\tau_*$.

*Example 1:* For an unstable system

$$
dx(t)=0.05x(t)dt
$$

with initial date $x(0)=1$. We aim to design the linear discrete-time feedback control with a time delay induced by $G$-Brownian motion $-x(\delta_t)d\langle B\rangle_t+0.6x(\delta_t)dB_t$ to make the stochastic controlled system

$$
dx(t)=0.05x(t)dt-x(\delta_t)d\langle B\rangle_t+0.6x(\delta_t)dB_t \tag{18}
$$

quasi-surely exponentially stable, where $B_t$ is one-dimensional G-Brownian motion and $B_1\sim N(0,[\frac{1}{2},1])$. Obviously, Assumption 1 is satisfied with

$$
K_1=0.05,\quad K_2=1,\quad K_3=0.6.
$$

We take the Lyapunov function $V(x,t)=x^2$ and hence Assumption 2 holds with

$$
c_1=\bar{c_1}=1,\quad q=2,\quad c_2=-0.27,\quad c_3=0.36.
$$

We choose $\theta=0.49$, then $p=0.98$ and $\gamma=0.44$. We further choose $\varepsilon=0.95$ and it is easy to compute $\Theta_\varepsilon=0.1158$. Equation (17) becomes

$$
[H_4(\tau,h,\tau_0,0.1158+\tau)]^{\frac{p}{2}}+H_1(\tau_0,0.1158+\tau)=0.05,
$$

which has the unique positive root $\tau_*=1.84\times10^{-4}$ if the time delay $\tau_0=1\times10^{-5}$. Therefore, for the feedback time delay $\tau_0=1\times10^{-5}$, the stochastic controlled system (18) is exponentially stable quasi-surely as long as $\tau<1.84\times10^{-4}$.

By the Euler-Maruyama method, the numerical simulation of the upper expectation $\bar{\mathbb{E}}|X(t)|$ and the lower expectation $\mathcal{E}|X(t)|$ of the solution to stochastic system (18) with $\tau_0=10^{-5}$ and $\tau=10^{-6}$ is plotted in Figure 1, where we use the algorithm from [28] to approximate the $G$-expectation. We observe from Figure 1 that $\bar{\mathbb{E}}|X(t)|$ is stable, then the solution to (18) is quasi surely stable. The computer simulation supports our theoretical results clearly.

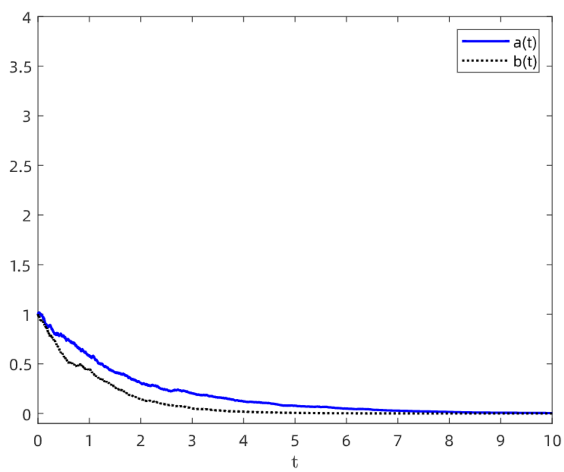



**FIGURE 1. The computer simulation of the upper expectation $a(t)=\bar{\mathbb{E}}|X(t)|$ and the lower expectation $b(t)=\mathcal{E}|X(t)|$ of the solution to the $G$-SDDE (18) with $\tau_0=10^{-5}$ and $\tau=10^{-6}$ using the Euler-Maruyama method.**

## V. CONCLUSION

In this paper, we have proved that an unstable nonlinear system can be stabilized by $G$-Brownian motion with feedback control based on discrete-time state observations with a time delay. Sufficient conditions in terms of the observation gap $\tau$ and the time delay $\tau_0$ have been developed to guarantee the quasi-surely exponential stability of the stochastic controlled system. An example has been given to show the implementation and illustrate the theoretical results.

## ACKNOWLEDGMENT

The authors would like to thank anonymous referees and editors for their helpful comments and suggestions.

## REFERENCES

[1] R. Z. Khasminskii, *Stochastic Stability of Differential Equations*. Alphen aan den Rijn, The Netherlands: Sijthoff and Noordhoff, 1980.

[2] M. Scheutzow, "Stabilization and destabilization by noise in the plane," *Stochastic Anal. Appl.*, vol. 11, no. 1, pp. 97–113, Jan. 1993, doi: 10.1080/07362999308809304.

[3] X. Mao, "Stochastic stabilization and destabilization," *Syst. Control Lett.*, vol. 23, no. 4, pp. 279–290, Oct. 1994, doi: 10.1016/0167-6911(94)90050-7.

[4] J. A. D. Appleby and X. Mao, "Stochastic stabilisation of functional differential equations," *Syst. Control Lett.*, vol. 54, no. 11, pp. 1069–1081, Nov. 2005, doi: 10.1016/j.sysconle.2005.03.003.

[5] X. Mao, G. G. Yin, and C. Yuan, "Stabilization and destabilization of hybrid systems of stochastic differential equations," *Automatica*, vol. 43, no. 2, pp. 264–273, Feb. 2007, doi: 10.1016/j.automatica.2006.09.006.

[6] F. Deng, Q. Luo, and X. Mao, "Stochastic stabilization of hybrid differential equations," *Automatica*, vol. 48, no. 9, pp. 2321–2328, Sep. 2012, doi: 10.1016/j.automatica.2012.06.044.

[7] X. Mao, "Stabilization of continuous-time hybrid stochastic differential equations by discrete-time feedback control," *Automatica*, vol. 49, no. 12, pp. 3677–3681, Dec. 2013, doi: 10.1016/j.automatica.2013.09.005.

[8] X. Mao, "Almost sure exponential stabilization by discrete-time stochastic feedback control," *IEEE Trans. Autom. Control*, vol. 61, no. 6, pp. 1619–1624, Jun. 2016, doi: 10.1109/TAC.2015.2471696.

[9] Q. Qiu, W. Liu, L. Hu, X. Mao, and S. You, "Stabilization of stochastic differential equations with Markovian switching by feedback control based on discrete-time state observation with a time delay," *Statist. Probab. Lett.*, vol. 115, pp. 16–26, Aug. 2016, doi: 10.1016/j.spl.2016.03.024.

[10] Q. Zhu and Q. Zhang, "Pth moment exponential stabilisation of hybrid stochastic differential equations by feedback controls based on discrete-time state observations with a time delay," *IET Control Theory Appl.*, vol. 11, no. 12, pp. 1992–2003, Aug. 2017, doi: 10.1049/iet-cta.2017.0181.

[11] L. Liu and Z. Wu, "Intermittent stochastic stabilization based on discrete-time observation with time delay," *Syst. Control Lett.*, vol. 137, Mar. 2020, Art. no. 104626, doi: 10.1016/j.sysconle.2020.104626.

[12] S. Peng, "$G$-expectation, $G$-Brownian motion and related stochastic calculus of Itô type," in *Stochastic Analysis and Applications*, vol. 2. Heidelberg, Germany: Springer, 2007, pp. 541–567, doi: 10.1007/978-3-540-70847-6-25.

[13] S. Peng, "Multi-dimensional $G$-Brownian motion and related stochastic calculus under $G$-expectation," *Stochastic Processes Appl.*, vol. 118, no. 12, pp. 2223–2253, Dec. 2008, doi: 10.1016/j.spa.2007.10.015.

[14] F. Gao, "Pathwise properties and homeomorphic flows for stochastic differential equations driven by $G$-Brownian motion," *Stochastic Processes Appl.*, vol. 119, no. 10, pp. 3356–3382, Oct. 2009, doi: 10.1016/j.spa.2009.05.010.

[15] D. Zhang and Z. Chen, "Exponential stability for stochastic differential equation driven by $G$-Brownian motion," *Appl. Math. Lett.*, vol. 25, no. 11, pp. 1906–1910, Nov. 2012, doi: 10.1016/j.aml.2012.02.063.

[16] P. Luo and F. Wang, "Stochastic differential equations driven by $G$-Brownian motion and ordinary differential equations," *Stochastic Processes Appl.*, vol. 124, no. 11, pp. 3869–3885, Nov. 2014, doi: 10.1016/j.spa.2014.07.004.

[17] X. Li, X. Lin, and Y. Lin, "Lyapunov-type conditions and stochastic differential equations driven by $G$-Brownian motion," *J. Math. Anal. Appl.*, vol. 439, no. 1, pp. 235–255, Jul. 2016, doi: 10.1016/j.jmaa.2016.02.042.

[18] Y. Li and L. Yan, "Stability of delayed hopfield neural networks under a sublinear expectation framework," *J. Franklin Inst.*, vol. 355, no. 10, pp. 4268–4281, Jul. 2018, doi: 10.1016/j.jfranklin.2018.04.007.

[19] C. Fei, W. Fei, and L. Yan, "Existence and stability of solutions to highly nonlinear stochastic differential delay equations driven by $G$-Brownian motion," *Appl. Math.-A J. Chin. Universities*, vol. 34, no. 2, pp. 184–204, Jun. 2019, doi: 10.1007/s11766-019-3619-x.

[20] S. Deng, C. Fei, W. Fei, and X. Mao, "Stability equivalence between the stochastic differential delay equations driven by $G$-Brownian motion and the Euler–Maruyama method," *Appl. Math. Lett.*, vol. 96, pp. 138–146, Oct. 2019, doi: 10.1016/j.aml.2019.04.022.

[21] C. Fei, W. Fei, X. Mao, and L. Yan, "Delay-dependent asymptotic stability of highly nonlinear stochastic differential delay equations driven by $G$-Brownian motion," 2020, *arXiv:2004.13229*. [Online]. Available: http://arxiv.org/abs/2004.13229

[22] Y. Ren, W. Yin, and R. Sakthivel, "Stabilization of stochastic differential equations driven by $G$-Brownian motion with feedback control based on discrete-time state observation," *Automatica*, vol. 95, pp. 146–151, Sep. 2018, doi: 10.1016/j.automatica.2018.05.039.

[23] W. Yin, J. Cao, and Y. Ren, "Quasi-sure exponential stabilization of stochastic systems induced by $G$-Brownian motion with discrete time feedback control," *J. Math. Anal. Appl.*, vol. 474, no. 1, pp. 276–289, Jun. 2019, doi: 10.1016/j.jmaa.2019.01.045.

[24] S. Peng, *Nonlinear Expectations and Stochastic Calculus under Uncertainty*. Berlin, Germany: Springer-Verlag, 2019.

[25] L. Denis, M. Hu, and S. Peng, "Function spaces and capacity related to a sublinear expectation: Application to $G$-Brownian motion paths," *Potential Anal.*, vol. 34, no. 2, pp. 139–161, Feb. 2011, doi: 10.1007/s11118-010-9185-x.

[26] W. Yin, J. Cao, and Y. Ren, "Quasi-sure exponential stability and stabilisation of stochastic delay differential equations under $G$-expectation framework," *Int. J. Control*, pp. 1–12, Mar. 2020, doi: 10.1080/00207179.2020.1740794.

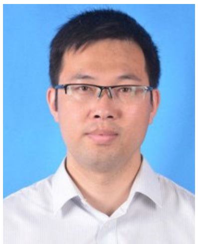
[27] Z. Chen, P. Wu, and B. Li, "A strong law of large numbers for non-additive probabilities," *Int. J. Approx. Reasoning*, vol. 54, no. 3, pp. 365–377, Apr. 2013, doi: 10.1016/j.ijar.2012.06.002.

[28] C. Fei and W. Fei, "Consistency of least squares estimation to the parameter for stochastic differential equations under distribution uncertainty," *Acta Math. Sci.*, vol. 39A, no. 6, pp. 1499–1513, Oct. 2019, doi: 121.43.60.238/sxwlxbA/CN/Y2019/V39/I6/1499.

**YUYUAN LI** received the B.S. degree in mathematics and applied mathematics from Guangxi Normal University, Guilin, China, in 2011, and the Ph.D. degree in control science and engineering from Donghua University, Shanghai, China, in 2018. She is currently a Lecturer with the School of Electronic and Electrical Engineering, Shanghai University of Engineering Science, Shanghai. Her research interests include stability analysis and control of stochastic systems.

**XIUWEI YIN** received the B.S. and M.S. degrees from the School of Mathematics and Computer Science, Anhui Normal University, China, in 2012 and 2015, respectively, and the Ph.D. degree from the School of Information Science and Technology, Donghua University, Shanghai, China, in 2018. He is currently a Lecturer with the School of Mathematics and Statistics, Anhui Normal University. His research interest includes analysis and applications of stochastic (partial) differential equations.

**JIANQIU LU** received the Ph.D. degree in mathematics and statistics from the University of Strathclyde, Glasgow, U.K., in 2018. She joined the Department of Statistics, Donghua University, Shanghai, China, in 2019. Her main research interests include stochastic systems analysis and control, including stability, stabilization, and control theory of hybrid stochastic systems.



**YANHUI TONG** received the B.S. degree in automation and navigation and the M.S. degree in guidance and control from Northeastern University, Shenyang, China, in 2007 and 2009, respectively, and the Ph.D. degree in control science and engineering from the Space Control and Inertial Technology Center, Harbin Institute of Technology, China, in 2013. From November 2013 to September 2017, he worked with the Shanghai Institute of Electro-Mechanical Engineering, Shanghai Academy of Spaceflight Technology, China, as a Control System Engineer. He is currently a Lecturer with the School of Electronic and Electrical Engineering, Shanghai University of Engineering Science, China. His research interests include switched systems control, model predictive control, and their applications in flight control systems and industrial systems.

**YANAN JIANG** received the B.S. degree in statistical from Zhoukou Normal University, Zhoukou, China, in 2015, and the M.S. degree in computational mathematics from Shanghai Normal University, Shanghai, China, in 2019. She is currently pursuing the Ph.D. degree in control science and engineering with Donghua University, China. Her research interest includes theory of stochastic differential equations and their application.