Received February 8, 2021, accepted March 2, 2021, date of publication March 9, 2021, date of current version March 18, 2021.

Digital Object Identifier 10.1109/ACCESS.2021.3065007

Feedback Stabilization for Discrete-Time Singularly Perturbed Systems via Limited Information With Data Packet Dropout

WEI LIU, YANYAN WANG, AND YITAO TIAN
School of Mathematics and Statistics, Zhoukou Normal University, Zhoukou 466001, China
Corresponding author: Yanyan Wang (yywang918@163.com)

This work was supported in part by the National Natural Science Foundation of China under Grant 61703447, in part by the Research Foundation of the Henan Higher Education Institutions of China under Grant 21A110027, and in part by the Foundation for Key Teachers of the Henan Higher Education Institutions of China under Grant 2019GGJS217.

ABSTRACT This paper is concerned with the feedback stabilization of discrete time singularly perturbed systems under information constraints. First, the designed coder and decoder are connected via a limited communication channel with data packet dropout, which is assumed to obey the independent and identically distributed Bernoulli processes. Under the above conditions, the transmission error between estimated state and input state will tend to zero exponentially. Meanwhile, the upper bound of the packet loss rate can also be obtained when the communication channel capacity is limited. Then, under the proposed coder-decoder pair, a sufficient condition for the asymptotic stability of the closed-loop system is given by linear matrix inequalities. Furthermore, the upper bound of the small perturbation parameter for the stability of systems can be explicitly estimated with a workable computational way. Finally, two examples are given to illustrate the effectiveness of the proposed method.

INDEX TERMS Singularly perturbed systems, input-to-state stability (ISS), linear matrix inequality (LMI), data packet dropout, limited information.

I. INTRODUCTION

In recent years, the singularly perturbed system has attracted more and more attention due to its wide application in engineering practice, such as tunnel diode circuit nonlinear circuit, time invariant RLC network, aircraft control system, armature control dc motor, nuclear reaction, etc. The most essential characteristic for dealing with this kind of system is to alleviate the high dimensionality and ill-condition phenomenon from the interaction of slow and fast dynamics. In the case of the small parameter $\varepsilon = 0$, the corresponding singularly perturbed system will be reduced to a singular system. Meanwhile, many results have been obtained, for example, the asynchronous $H\infty$ filtering for fuzzy singular Markovian switching systems with retarded time-varying delays is considered in [1], it is shown that novel admissibility and filtering conditions have been given to guarantee the stochastic admissibility and the $H\infty$ performance level. Reference [2] considers the issue of asynchronous feedback control for fuzzy singular systems with mode-dependent time-varying delays via T-S fuzzy control technique under observer-based event-driven characteristic. The results show that, by parallel distributed compensation technique, the asynchronous fuzzy feedback controller can be devised and the asynchronous fuzzy controller modes can be depicted by a hidden Markovian model. However, It should be mentioned that a direct application of the approach for singular systems may lead to ill-conditioned results and induce the numerically stiff problem for the presence of the small parasitic parameter ε. Therefore, the singularly perturbed system is basically different from the other normal systems due to the appearance of small perturbed parameters. For the continuous time singularly perturbed systems, so far, it has been basically mature, and many results have been reported [3]–[8]. For example, by the Riccati equations approach, Shi, *et al.* in [6] investigates the robust disturbance attenuation with stability for singularly perturbed linear systems with matched condition. In [8], the absolute

The associate editor coordinating the review of this manuscript and approving it for publication was Haipeng Yao.

stability problem for Lur'e singularly perturbed systems is considered, where the stability criterion in terms of linear matrix inequalities (LMIs) is obtained.

For the research of the discrete time singularly perturbed systems, many results have also been obtained [9]–[16]. Based on the two-step methodology and the Riccati equation, the H_∞ controller design for fast sampling discrete-time singularly perturbed systems is given in [9], [10]. It is noticed that the method involves a large number of calculations, and thus it is difficult to apply in a high dimension system. Recently, the LMI technique has also been proposed to solve the H_∞ problem for different kinds of discrete-time linear singularly perturbed systems [11]–[15], which effectively eliminates the regularity restrictions that are attached to the Riccati-based solutions. Reference [16] considers the problem of designing a hybrid composite dynamic output feedback controller for the fast sampling discrete-time singularly perturbed system, in which sufficient conditions are presented to calculate the gain matrices of the dynamic output feedback controller based on the two low-order subsystems.

On the other hand, with the progress of computer technology and network communication technology, networked control systems have entered a stage of rapid development and developed many results; see the survey paper [17] and the references therein. In classic control theory, it is usually assumed that the control signals can be transmitted with infinite precision. However, it is just not the case in practical situations, e.g. for networked control systems, in which the controlled plant and controller are installed in different places, the control signals can be transmitted via a limited communication channel. In this case, due to the bandwidth constraint, time delay and data packet dropout may occur during data communication, these are very important problems in networked control systems. Recently, some methods have been studied to deal with the above-mentioned problems [18]–[24]. In [21], the admissibility analysis and stabilization for implicit Markovian jump systems (IMJSs) with retarded discrete-distributed delays are considered, in which admissibilization conditions are presented in terms of LMIs. Reference [22] addresses the problem of non-fragile delay feedback control for neutral stochastic Markovian jump systems with time-varying delays. Zhou and Zhang in [23] study the H-infinity fault detection for time-delays delta operator systems with random two-channels packet losses and limited communication. The result shows that the sufficient conditions for the asymptotical stability of the residual systems with performance based on delta operator systems can be obtained by using the Lyapunov functional technique. Moreover, some results of the singularly perturbed network control system have also been obtained [25]–[27]. For example, the model-based networked control for singularly perturbed systems with nonlinear uncertainties is studied in [25]–[26]. Due to the limited communication channel, when the signal is transmitted through the network, the signal needs to be sampled and quantized. So far, various quantization methods have been developed [28]–[31]. It is shown in [29] that a recursive coder–decoder state estimation scheme is proposed when considering a state estimation problem for a linear continuous-time system via a limited capacity communication channel. In [30], the problem of achieving ISS with respect to completely unknown disturbances for linear control systems with quantized state measurements is considered, in which a new dynamic quantized control design methodology is presented.

Recently, the feedback stabilization problem with limited information for singularly perturbed systems has also been considered [32]–[38]. For example, Tian *et al.* in [38] considers the quantized feedback problem of discrete time singularly perturbed systems with information constraints, the result shows that the controlled system can be stabilized under the proper quantization scheme. However, it is noted that the packet loss problem has not been considered in the above results. To our best knowledge, the stabilization problem for discrete-time singularly perturbed systems with packet loss and quantization has seldom received attention. Is still possible to reach stabilization for such a control system? Unfortunately, there is no clear conclusion at present. Therefore, further research on this topic is necessary.

Motivated by the above, this paper studies the feedback stabilization of discrete-time singularly perturbed systems under information constrains, in which the dropout packet loss is considered. It is assumed that the controlled system and the controller are connected via a limited communication channel, in which the control signal coded and sent may be lost due to the limited bandwidth. The results show that the proposed procedures can stabilize the system at a certain level of data packet loss rate. Compared with the existing results, the developed method in this paper has the following advantages: 1) An auxiliary system is added in the design of coder and decoder, thus the difficulty for the meaningless limit in the discrete system is overcome; 2) The nonlinear disturbances include the fast and slow states, which make the considered system more generally; 3) By constructing the Lyapunov function, the sufficient conditions for the input-to-state stability of the closed-loop system are given in terms of LMIs, and more complex equations are not involved; 4) The upper bound of the small perturbation parameter can be solved by GEVP, and the LMI Toolbox can be used to tested numerically efficiently.

II. PROBLEM FORMULATION

Consider the following discrete-time singularly perturbed systems with nonlinear perturbed given by

$$\begin{aligned} x_1(k+1) &= (I+\varepsilon A_{11})x_1(k) + \varepsilon A_{12}x_2(k) \\ &\quad + \varepsilon B_{u1}u(k) + \varepsilon H_1 f_1(x_1, x_2, u(k)), \quad \text{(1a)} \\ x_2(k+1) &= A_{21}x_1(k) + A_{22}x_2(k) \\ &\quad + B_{u2}u(k) + H_2 f_2(x_1, x_2, u(k)), \quad \text{(1b)} \end{aligned}$$

where $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}(n_1+n_2=n)$, represent the state vectors of the slow and fast modes, respectively; $u \in R^s$ is the input vector; ε is a positive singularly perturbed parameter;

In the systems (1a) and (1b), all matrices are constant matrices with appropriate dimensions. $f_i(x_1, x_2, u(k))$ $(i = 1, 2)$ is assumed to satisfy the following Lipschitz condition for all $(x_1, x_2, u), (\tilde{x}_1, \tilde{x}_2, \tilde{u}) \in R^{n_1} \times R^{n_2} \times R^s$,

$$\begin{aligned}\|f_i(x_1, x_2, u(k)) - f_i(\tilde{x}_1, \tilde{x}_2, \tilde{u}(k))\| \le \|F_{i1}(x_1 - \tilde{x}_1) \\ + F_{i2}(x_2 - \tilde{x}_2) + F_1(u - \tilde{u})\|, \quad i = 1, 2. \quad (2)\end{aligned}$$

where $F_{ij}(i = 1, 2; j = 1, 2)$ and F_1 are known constant matrices with appropriate dimensions.

We also assume that the initial condition $x(0)$ of the system (1) lies in a known set χ.

Define

$$\begin{aligned} x &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, E_0 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, E_\varepsilon = \begin{pmatrix} \varepsilon I & 0 \\ 0 & I \end{pmatrix}, \\ A &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}, B_u = \begin{pmatrix} B_{u1} \\ B_{u2} \end{pmatrix}, \\ f(x, u) &= \begin{pmatrix} f_1(x_1, x_2, u(k)) \\ f_2(x_1, x_2, u(k)) \end{pmatrix}, \\ F &= \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}. \end{aligned}$$

The system (1) can be rewritten as

$$x(k+1) = A_\varepsilon x(k) + B_{u\varepsilon} u(k) + H_\varepsilon f(x(k), u(k)), \quad (3)$$

where $A_\varepsilon = E_0 + E_\varepsilon A$, $B_{u\varepsilon} = E_\varepsilon B_u$, $H_\varepsilon = E_\varepsilon H$.

It is easy to verify that the nonlinear item $f(x(k), u(k))$, satisfies the following conditions for all $(x(k), u(k)) \in R^n \times R^s$:

$$\begin{aligned}\|f(x(k), u(k)) - f(\tilde{x}(k), \tilde{u}(k))\| \\ \le \|F(x - \tilde{x}) + F_1(u - \tilde{u})\|. \quad (4)\end{aligned}$$

This paper considers the stabilization problem of the system (1) via a limited capacity channel with data packet dropout. First, when the signal is transmitted via a limited communication network, it is assumed that there exists the data packet dropout. In particular, given a sampling period p, the coded signal $h(jp)$ is obtained from the state $x(k)$ of the system, which is selected from the encoded table $\mathcal{H}$ of size q and transmitted by the network at the time jp. Because the transmission channel capacity is limited, the coded signal may be lost during transmission. Therefore, we add a register in front of the decoder. Registers produce a set of discrete variables $\delta(jp)$ whose values are 0 or 1. $\delta(jp) = 0$ and $\delta(jp) = 1$ represent the loss and successful transmission of coded signals, respectively. At the remote reception, a decoder is designed to decode the received code-words and produce the estimate state $\hat{x}(k)$ of the controlled system. This situation is illustrated in Fig. 1.

Remark 1: The switch in the Fig. 1 is closed (that is, the location of S_1) indicating that the coded signal $h(jp)$ is transmitted successfully. At this point, the state estimation $\hat{x}(k)$ and the control input $u(k)$ will be updated accordingly. When the switch is open (i.e. located in S_2), the coded signal $h(jp)$ is lost. At this point, the estimated state $\hat{x}(k)$ and the control input $u(k)$ will not be updated; in this case, the data of the previous moment will continue to be used.

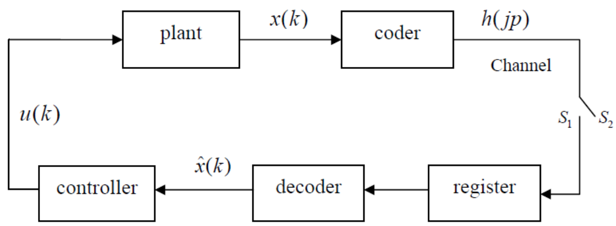


FIGURE 1. State encoding via a communication channel.

The encoder-decoder used in this paper is mainly derived from the literature [29]. Specially, the coder and the decoder are of the following form:

Coder:

$$h(jp) = \Im_j(x(\cdot)|_0^{jp}); \quad (5)$$

Decoder:

$$\hat{x}(k)|_{jp}^{(j+1)p} = \aleph_j(h(p), h(2p), \cdots\cdots, h(jp)), \quad (6)$$

where $\Im_j$ and $\aleph_j (j = 1, 2, \cdots, n)$ are coder and decoder functions to be designed, and $\delta(jp) \in \{0, 1\}$ obey independent and identically distributed Bernoulli processes with

$$\text{Prob}\{\delta(jp) = 0\} = \delta, \quad \text{Prob}\{\delta(jp) = 1\} = 1 - \delta. \quad (7)$$

Remark 2: In the discussion of this article, without loss of generality, it is assumed that the first coder signal is transmitted successfully. Meanwhile, we also assume that, at each transfer time jp, the decoder always knows the value of the previous switch variable $\delta(jp - p)$. This means that the encoder is always able to obtain information that the encoded signal whether successfully transmitted.

Definition 1: ([39]) Consider the discrete-time nonlinear system:

$$x(k+1) = f(x(k), u(k)), \quad (8)$$

where state $x(\cdot)$ is in R^n, $f: R^n \times R^q \to R^n$ is continuous and locally Lipschitz in x and u. The input u is a bounded function for all $k \ge 0$. Then the system (8) is said to be input-to-state stable (ISS) if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any initial state $x(0)$, the solution $x(k)$ exists for all $k \ge 0$ and satisfies:

$$\|x(k)\| \le \beta(\|x(0)\|, k) + \gamma \sup_{0 \le \tau \le k} \|u(\tau)\|. \quad (9)$$

In the following, we give some lemmas and theorems which play an important role in the design of our stabilization procedure.

Lemma 1: [39] Let $V: R^n \to R$ be a continuously differentible function such that

$$\begin{aligned}&\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|), \\ &V(x(k+1)) - V(x(k)) \le -W(x(k)), \|x\| \ge \rho(\|u\|) > 0,\end{aligned}$$

where α_1 and α_2 are class $\mathcal{K}_\infty$ functions, ρ is a class $\mathcal{K}$ function, and $W(x)$ is a continuous and positive definite function on R^n. Then, the system (8) is input- to-state stable with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

Lemma 2: Given a scalar $\alpha \geq 1$, if there exist a scalar $\mu_1 > 0$, matrices $0 < P_{11} \in R^{n_1 \times n_1}$, P_{21} and $0 < P_{22} \in R^{n_2 \times n_2}$ such that the following linear matrix inequality holds

$$\bar{\Lambda} = \Lambda_0 + \Lambda_1 < 0, \tag{10}$$

where

$$\Lambda_0 = \begin{pmatrix} \Theta_1 & \Theta_2 & P_{11}H_1 & P_{21}^T H_2 \\ * & \Theta_3 & 0 & P_{22}H_2 \\ * & * & -\mu_1 I & 0 \\ * & * & * & -\mu_1 I \end{pmatrix},$$
$$\begin{aligned} \Theta_1 &= P_{11}^T A_{11} + A_{11}^T P_{11} + A_{21}^T P_{21} + P_{21}^T A_{21} + \mu_1 F_{11}^T F_{11} \\ &\quad + \mu_1 F_{21}^T F_{21}, \\ \Theta_2 &= P_{11} A_{12} + P_{21} A_{22} + \mu_1 F_{11}^T F_{12} + \mu_1 F_{21}^T F_{22} \\ &\quad - \alpha^2 P_{21}^T + A_{21}^T P_{22}, \\ \Theta_3 &= \mu_1 F_{12}^T F_{12} + \mu_1 F_{22}^T F_{22} + A_{22}^T P_{22} + P_{22} A_{22} \\ &\quad - \alpha^2 P_{22} - P_{22}, \\ \Lambda_1 &= (A_1 \;\; H)^T P_2 (A_1 \;\; H), \\ A_1 &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} - I \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 0 \\ 0 & P_{22} \end{pmatrix}, \end{aligned} \tag{11}$$

then there exists an $\varepsilon_1^* > 0$, such that, for any two solutions $z_1(k)$ and $z_2(k)$ of the system (3), the following inequality

$$\|z_1(k+1) - z_2(k+1)\|_\infty \leq \mu\alpha \|z_1(k) - z_2(k)\|_\infty \tag{12}$$

holds for $\varepsilon \in (0, \varepsilon_1^*]$ and $k \geq 0$, where

$$\begin{aligned} \mu &= \sqrt{n \frac{\lambda_{\max}(\alpha^2 P_\varepsilon)}{\lambda_{\min}(\alpha^2 P_\varepsilon)} (1 - \frac{\lambda_{\min}(-\Pi_\varepsilon)}{\lambda_{\max}(\alpha^2 P_\varepsilon)})}, \\ P_\varepsilon &= \begin{pmatrix} \varepsilon^{-1} P_{11} & P_{21}^T \\ P_{21} & P_{22} \end{pmatrix}. \end{aligned} \tag{13}$$

Proof: First, since $P_{11} > 0$ and $P_{22} > 0$, there exists a scalar $\varepsilon_{11} > 0$ such that $P_{11} - \varepsilon P_{21}^T P_{22}^{-1} P_{21} > 0$ for all $\varepsilon \in (0, \varepsilon_{11}]$. By Schur Complement Lemma, it yields

$$P_\varepsilon = \begin{pmatrix} \varepsilon^{-1} P_{11} & P_{21}^T \\ P_{21} & P_{22} \end{pmatrix} > 0, \varepsilon \in (0, \varepsilon_{11}].$$

Next, let

$$\begin{aligned} z(k) &= \alpha^{-k}(z_1(k) - z_2(k)), \\ \phi(k) &= \alpha^{-k}(f(z_1(k), u(k)) - f(z_2(k), u(k))), \end{aligned}$$

then, we have

$$\begin{aligned} z(k+1) &= \alpha^{-(k+1)}(z_1(k+1) - z_2(k+1)) \\ &= \alpha^{-(k+1)}[A_\varepsilon(z_1(k) - z_2(k)) + H_\varepsilon(f(z_1(k), u(k)) \\ &\quad - f(z_2(k), u(k))] \\ &= \alpha^{-1} A_\varepsilon z(k) + \alpha^{-1} H_\varepsilon \phi(k), \end{aligned} \tag{14}$$

where $\|\phi\| \leq \|Fz\|$.

Hence, we can choose the candidate Lyapunov function as:

$$V_1(z(k)) = \alpha^2 z^T(k) P_\varepsilon z(k).$$

For constants $\alpha \geq 1$ and $\mu_1 > 0$, the derivative of V_1 along the system (13) yields

$$\begin{aligned} \Delta V_1(z(k)) &= V_1(k+1) - V_1(k) \\ &= \alpha^2 z^T(k+1) P_\varepsilon z(k+1) - \alpha^2 z^T(k) P_\varepsilon z(k) \\ &\leq \alpha^2 z^T(k+1) P_\varepsilon z(k+1) - \alpha^2 z^T(k) P_\varepsilon z(k) \\ &\quad + \mu_1(z^T F^T F z - \phi^T \phi) \\ &\leq \begin{pmatrix} z^T & \phi^T \end{pmatrix} \Pi_\varepsilon \begin{pmatrix} z \\ \phi \end{pmatrix}, \end{aligned}$$

where

$$\Pi_\varepsilon = \bar{\Lambda} + \varepsilon \Lambda_2, \Lambda_2 = (A \;\; H)^T \begin{pmatrix} P_{11} & P_{21}^T \\ P_{21} & 0 \end{pmatrix} (A \;\; H). \tag{15}$$

It follows from (10) that there exists a sufficiently small scalar $\varepsilon_{12} > 0$ such that $\Pi_\varepsilon < 0$ for any given $\varepsilon \in (0, \varepsilon_{12}]$, thus $\Delta V_1(z(k)) < 0$. Let $\varepsilon_1^* = \min\{\varepsilon_{11}, \varepsilon_{12}\}$, then, we can get $P_\varepsilon > 0$ and $\Delta V_1(z(k)) < 0$ for any given $\varepsilon \in (0, \varepsilon_1^*]$. Denoting $\lambda_{0\varepsilon} = \lambda_{\min}(-\Pi_\varepsilon)$, $\lambda_{1\varepsilon} = \lambda_{\min}(\alpha^2 P_\varepsilon)$, we have $\lambda_{0\varepsilon} > 0$, $\lambda_{1\varepsilon} > 0$ and

$$\Delta V_1(z(k)) \leq -\lambda_{0\varepsilon}(||z(k)||^2 + ||\phi||^2) \leq -\lambda_{0\varepsilon}||z(k)||^2$$

for any $\varepsilon \in (0, \varepsilon_1^*]$. In addition, let $\lambda_{2\varepsilon} = \lambda_{\max}(\alpha^2 P_\varepsilon) > 0$, one has

$$V_1(z(k)) = \alpha^2 z^T(k) P_\varepsilon z(k) \leq \lambda_{2\varepsilon} \|z(k)\|^2.$$

Then, we have $\Delta V_1(z(k)) \leq -\lambda_{0\varepsilon} \|z(k)\|^2 \leq -\lambda_{0\varepsilon}\lambda_{2\varepsilon}^{-1} V_1(z(k))$. This indicates

$$V_1(k+1) \leq (1 - \lambda_{0\varepsilon}\lambda_{2\varepsilon}^{-1}) V_1(k),$$

therefore

$$\|z(k+1)\|_\infty \leq \sqrt{n \frac{\lambda_{2\varepsilon}}{\lambda_{1\varepsilon}} (1 - \frac{\lambda_{0\varepsilon}}{\lambda_{2\varepsilon}})} \|z(k)\|_\infty,$$

let $\mu = \sqrt{n \frac{\lambda_{2\varepsilon}}{\lambda_{1\varepsilon}} (1 - \frac{\lambda_{0\varepsilon}}{\lambda_{2\varepsilon}})}$, due to

$$z(k) = \alpha^{-k}(z_1(k) - z_2(k)),$$

we get

$$\|z_1(k+1) - z_2(k+1)\|_\infty \leq \mu\alpha \|z_1(k) - z_2(k)\|_\infty$$

for all $k \geq 0$. This completes the proof. ∎

Remark 3: The estimation of an upper bound $\varepsilon_1^* > 0$ to guarantee that the inequality (13) holds for all $\varepsilon \in (0, \varepsilon_1^*]$ is also an interesting topic. It follows from Lemma 2 that the upper bound $\varepsilon_1^* = \lambda_1^{-1} > 0$ can be obtained by solving the following minimization problem

$$\begin{aligned} \min \lambda_1 \text{s.t.} W &< \lambda_1 P_{11}, \begin{pmatrix} W & P_{21}^T \\ P_{21} & P_{22} \end{pmatrix} > 0, \\ \bar{\Lambda} &< 0, \Lambda_2 < -\lambda_1 \bar{\Lambda}, \end{aligned} \tag{16}$$

where W is a positive definite matrix, $\bar{\Lambda}$ and Λ_2 are defined in (11) and (15), respectively, which can be solved effectively by applying GEVP solver in LMI control toolbox.

III. MAIN RESULTS

In this subsection, we have two goals. First, an appropriate encoder-decoder pair is designed to guarantee that the transmission error tends to zero exponentially. Meanwhile, the channel capacity and the maximum packet loss rate can be obtained; Second, a suitable feedback control law with transmission error as the input is designed, such that the sufficient condition for the ISS of the system (3) is obtained and then the asymptotical stability also can be achieved based on the ISS property.

Definition 2: The systems (1) is said to be stabilization via a limited channel, if there exists a coder-decoder (5) and (6) such that

$$\lim_{k\to\infty} E\left\{\|x\left(k\right)\|_{\infty}\right\} = 0,) \tag{17}$$

where E is a mathematical expectation.

A. UNIFORM STATE QUANTIZATION

Next, we first recall the uniform state quantization method in [29].

Suppose that the number N taken by the coder satisfies $N = q^n$, where q is a positive integer. For any given constant $r > 0$, the super-cube box $B(0, r) = \{x \in R^n | \|x\|_\infty \le r\}$ is partitioned into q^n equal super-cube boxes. The i component x_i of x is divided into q disjoint intervals by the following way:

$$I_1^i(r) := \left\{x_i : -r \le x_i < -r + \frac{2r}{q}\right\},$$
$$I_2^i(r) := \left\{x_i : -r + \frac{2r}{q} \le x_i < -r + \frac{4r}{q}\right\},$$
$$\cdots$$
$$I_q^i(r) := \left\{x_i : r - \frac{2r}{q} \le x_i \le r\right\}.$$

Then, for any given $x \in B(0, r)$, there exist a group of integers $i_j \in \{1, 2, \cdots, q\}, j = (1, 2, \cdots, n)$ such that

$$x \in I_{i_1}^1(r) \times I_{i_2}^2(r) \times \cdots \times I_{i_n}^n(r) \subset B(0, r).$$

The center of the super-cube box $I_{i_1}^1(r) \times I_{i_2}^2(r) \times \cdots \times I_{i_n}^n(r)$, containing the original state x is defined as

$$C_r(i_1, i_2, \cdots, i_n) = \left[-r + \frac{2i_1 - 1}{q}r\ ,\ \ -r + \frac{2i_2 - 1}{q}r, \right.$$
$$\left. \cdots,\ \ -r + \frac{2i_n - 1}{q}r\right]^T.$$

B. STATE FEEDBACK STABILIZATION

Now we are in the position to design the coder-decoder pair. For convenience, we denote

$$a(0) = m_0 = \sup_{x_0 \in \chi} \|x_0\|_\infty, a(p) = \mu^p\alpha^p\frac{a(0)}{q},$$
$$a((j+1)p) = \mu^p\alpha^p\left(1 - \left(1 - \frac{1}{q}\right)\delta\left(jp\right)\right)a(jp), j \ge 1. \tag{18}$$

Coder: For

$$x(jp) - \bar{x}(jp) \in I_{i_1}^1 \times I_{i_2}^2 \times \cdots \times I_{i_n}^n \in B_{a(jp)};$$
$$h(jp) = \{i_1, i_2, \cdots, i_n\}, \tag{19}$$

where $\bar{x}(k)$ is defined as following:

$$\begin{cases}\bar{x}(0) = 0,\\ \bar{x}(k) = \hat{x}(k), k \ne jp,\\ \bar{x}(k+1) = A_\varepsilon\hat{x}(k) + B_{u\varepsilon}u(k) + H_\varepsilon f(\hat{x}(k), u(k)),\\ k = jp - 1\end{cases} \tag{20}$$

Decoder: For $h(jp) = \{i_1, i_2, \cdots, i_n\}$,

$$\begin{cases}\hat{x}(k+1) = A_\varepsilon\hat{x}(k) + B_{u\varepsilon}u(k) + H_\varepsilon f(\hat{x}(k), u(k)),\\ \hat{x}(0) = 0,\\ \hat{x}(jp) = \bar{x}(jp) + \delta\left(jp\right)C_{a(jp)}\left\{i_1, i_2, \cdots, i_n\right\},\\ u(k) = K\hat{x}(k).\end{cases} \tag{21}$$

Theorem 1: For the given data packet dropout rate δ, if

$$\left(\delta + \frac{1-\delta}{q}\right)\mu^p\alpha^p < 1 \tag{22}$$

for any $\mu > 0, \alpha \ge 1$, then the transmission error between the state x of controlled system (3) and its estimate state $\hat{x}$ decays to zero exponentially for any $\varepsilon \in (0,\ \ \varepsilon_1^*]$ under the proposed coder-decoder pair (20)–(21).

Proof: we first show that the decoding condition satisfies

$$x(jp) - \bar{x}(jp) \in B(0, a(jp)) \tag{23}$$

for all $j \ge 0$, according to (18) and (21), we conclude that (23) holds for $j = 0$. For the case $j = 1$, the information transmission is successful, that is $\delta\left(p\right) = 1$. It follows from the inequality (13) and $\bar{x}(0) = 0$ that

$$\|x(p) - \bar{x}(p)\|_\infty \le \mu^p\alpha^p\left\|x(0) - \hat{x}(0)\right\|_\infty$$
$$\le \mu^p\alpha^p\frac{a(0)}{q} = a(p).$$

Now we assume that (23) holds for $1, 2, \cdots, j$, and then we show that it is also true for $j + 1$:

$$\begin{aligned}&\|x(j+1)p - \bar{x}(j+1)p\|_\infty\\ &\quad\le \mu^p\alpha^p\left\|x(jp) - \hat{x}(jp)\right\|_\infty\\ &\quad\le \mu^p\alpha^p\left\{\delta\left(jp\right)\left\|x(jp) - \bar{x}(jp) - C_{a(jp)}\left\{i_1, i_2, \cdots, i_n\right\}\right\|_\infty\right.\\ &\qquad\left. + \left(1 - \delta\left(jp\right)\right)\left\|x\left(jp\right) - \bar{x}\left(jp\right)\right\|_\infty\right\}\\ &\quad\le \mu^p\alpha^p\delta\left(jp\right)\frac{a(jp)}{q} + \mu^p\alpha^p\left(1 - \delta\left(jp\right)\right)a(jp)\\ &\quad= \left(1 - \left(1 - \frac{1}{q}\right)\delta\left(jp\right)\right)\mu^p\alpha^p a(jp) = a((j+1)p).\end{aligned} \tag{24}$$

Applying the method of mathematical induction, it can be shown that (23) holds for all $j \ge 1$.

Furthermore, from (18) and (24), we obtain that

$$\|e(k)\|_\infty = \left\|x(k) - \hat{x}(k)\right\|_\infty$$
$$\le \mu^{k-jp}\alpha^{k-jp}\left\|x(jp) - \hat{x}(jp)\right\|_\infty \le a((j+1)p),$$

where $e(k) = x(k) - \hat{x}(k)$ is the transmission error and $k \in z^+$, $jp < k \le (j+1)p$. By the definition of $a(jp)$, we have

$$\|e(k)\|_\infty \le a((j+1)p)$$

$$= \left(1-\left(1-\frac{1}{q}\right)\delta(jp)\right)\mu^p\alpha^p a(jp)$$
$$= \cdots = \left(\mu^p\alpha^p\left(1-\left(1-\frac{1}{q}\right)\delta(jp)\right)\right)^{j+1} a(0),$$

Furthermore, we obtain that

$$E\{\|e(k)\|_\infty\} \leq E\{a_1((j+1)p)\}$$
$$= \cdots = \left(\mu^p\alpha^p\left(\delta+\frac{1-\delta}{q}\right)\right)^{j+1} a(0)$$
$$= \left(\mu^p\alpha^p\left(\delta+\frac{1-\delta}{q}\right)\right)^{j+1} m_0.$$

Let $\rho = -\ln\left(\left(\delta+\frac{1-\delta}{q}\right)\mu^p\alpha^p\right)$, then

$$E\{\|e(k)\|_\infty\} \leq \left(\mu^p\alpha^p\left(\delta+\frac{1-\delta}{q}\right)\right)^{j+1} m_0$$
$$\leq e^{-(j+1)\rho}m_0 \leq e^{-k\rho}m_0.$$

This completes the proof. ■

Remark 4: Condition (22) implies that $\delta\mu^p\alpha^p < 1$ and $q > \mu^p\alpha^p$, thus the upper bound of admissible data packet dropout rate can be given as follows

$$\delta_{\max} < \frac{q-\mu^p\alpha^p}{\mu^p\alpha^p(q-1)}.$$

Remark 5: Theorem 1 shows that the asymptotic stability of the transmission can be guaranteed if a proper relationship holds between sampling period, system growth rate and packet loss rate.

Next, we will consider the state feedback control. One has from (1) and (21) that

$$x(k+1) = (A_\varepsilon + B_{u\varepsilon}K)x(k) + B_{u\varepsilon}Ke$$
$$+H_\varepsilon f(x(k), K(x(k)+e(k))) \quad (25)$$

with the constraint

$$\|f(x, K(x+e))\| \leq \|Fx + F'Kx + F'Ke\|$$
$$= \|(F+F'K)x + F'Ke\|.$$

The following will confirm that there exists a proper feedback control law such that the resulting closed-loop system (25) is made ISS with the transmission error e as the input.

Theorem 2: If there exist a scalar $\mu_2 > 0$, matrices Y, and a lower triangular matrix

$$X = \begin{pmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{pmatrix}$$

with $0 < X_{11} \in R^{n_1\times n_1}$ and $0 < X_{22} \in R^{n_2\times n_2}$ such that the following matrix inequality holds

$$\bar{\Omega} = \begin{pmatrix} \Omega_1 & \mu_2^{-1}H & X^TA_2+Y^TB_{u2}^T & X^TF^T+Y^TF'^T \\ * & -\mu_2^{-1}I & \mu_2^{-1}\tilde{H}_2^T & 0 \\ * & * & -X_{22} & 0 \\ * & * & * & -\mu_2^{-1}I \end{pmatrix} < 0, \quad (26)$$

where $\Omega_1 = X^TA_1^T + A_1X + Y^TB_u^T + B_uY, A_2 = \left(A_{21}\ A_{22}-I\right), \tilde{H}_2 = \left(0\ H_2\right)$, then there exists an $\varepsilon_2^* > 0$, such that the closed-loop system (25) is ISS with respect to transmission error e for $\varepsilon \in (0, \varepsilon_2^*]$. Meanwhile, the controller gain matrix can be chosen as $K = YX^{-1}$.

Proof: First, substituting $K = YX^{-1}$ into (26), then, the inequality (26) is equivalent to

$$\begin{pmatrix} \Omega_2 & \mu_2^{-1}H & X^T(A_2+B_{u2}K)^T & X^T(F+F'K)^T \\ * & -\mu_2^{-1}I & \mu_2^{-1}\tilde{H}_2^T & 0 \\ * & * & -X_{22} & 0 \\ * & * & * & -\mu_2^{-1}I \end{pmatrix} < 0, \quad (27)$$

where $\Omega_2 = X^T(A_1+B_uK)^T + (A_1+B_uK)X$. By the Schur Complement Lemma, the inequality (27) can be rewritten as

$$\begin{pmatrix} \Omega_3 & \mu_2^{-1}H & X^T(A_2+B_{u2}K)^T \\ * & -\mu_2^{-1}I & \mu_2^{-1}\tilde{H}_2^T \\ * & * & -X_{22} \end{pmatrix} < 0, \quad (28)$$

where

$$\Omega_3 = X^T(A_1+B_uK)^T + (A_1+B_uK)X$$
$$+\mu_2X^T(F+F'K)^T(F+F'K)X.$$

Pre- and post- multiplying the inequality (28) by $diag\left(X^{-T}, \mu_2, I\right)$ and $diag(X^{-1}, \mu_2, I)$, respectively; Denote $X^{-1} = \bar{P}_1 = \begin{pmatrix} \bar{P}_{11} & 0 \\ \bar{P}_{21} & \bar{P}_{22} \end{pmatrix}, Y = K\bar{P}_1^{-1}, X_{22}^{-1} = \bar{P}_{22}$, then we have

$$\begin{pmatrix} \Omega_4 & \bar{P}_1^TH & (A_2+B_{u2}K)^T \\ * & -\mu_2I & \tilde{H}_2^T \\ * & * & -\bar{P}_{22}^{-1} \end{pmatrix} < 0, \quad (29)$$

where $\Omega_4 = (A_1+B_uK)^T\bar{P}_1 + \bar{P}_1(A_1+B_uK) + \mu_2(F+F'K)^T(F+F'K)$. By the Schur Complement Lemma, one has

$$\begin{pmatrix} \Omega_4 & \bar{P}_1^TH \\ * & -\mu_2I \end{pmatrix} + \begin{pmatrix} (A_2+B_{u2}K)^T \\ \tilde{H}_2^T \end{pmatrix}\bar{P}_{22}\left(A_2+B_{u2}K\ \ \tilde{H}_2\right) < 0. \quad (30)$$

Let $A_2 = \left(A_{21}\ A_{22}-I\right), \bar{P}_2 = diag\left(0\ \bar{P}_{22}\right)$, we can get

$$\bar{\Omega} = \Phi_1 + \Phi_2\begin{pmatrix} \Omega_4 & \bar{P}_1^TH \\ * & -\mu_2I \end{pmatrix} + \begin{pmatrix} (A_1+B_uK)^T \\ H^T \end{pmatrix}$$
$$\times \bar{P}_2\left(A_1+B_uK\ \ H\right) < 0. \quad (31)$$

Choosing the Lyapunov function candidate as follows

$$V(k) = x^T\bar{P}_\varepsilon x,$$

where

$$\bar{P}_\varepsilon = \begin{pmatrix} \varepsilon^{-1}\bar{P}_{11} & \bar{P}_{21}^T \\ \bar{P}_{21} & \bar{P}_{22} \end{pmatrix}, \bar{P}_2 = \begin{pmatrix} 0 & 0 \\ 0 & \bar{P}_{22} \end{pmatrix}, P_3 = \begin{pmatrix} \bar{P}_{11} & \bar{P}_{21}^T \\ \bar{P}_{21} & 0 \end{pmatrix}.$$

Similar to the proof of Lemma 2, there exists a scalar $\varepsilon_{21} > 0$, for any $\varepsilon \in (0, \varepsilon_{21}]$, we have $\bar{P}_\varepsilon > 0$. Thus, we can guarantee the definiteness of V for any $\varepsilon \in (0, \varepsilon_{21}]$. Then, the derivative of V along (25) yields

$$\begin{aligned}\Delta V &= V(k+1) - V(k) \\ &\leq x^T(k+1)\bar{P}_\varepsilon x(k+1) - x^T(k)\bar{P}_\varepsilon x(k) \\ &\quad + \mu_2 \left[\begin{pmatrix} x^T & e^T \end{pmatrix} \begin{pmatrix} (F+F'K)^T \\ (F'K)^T \end{pmatrix} \right. \\ &\quad \left. \begin{pmatrix} F+F'K & F'K \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix} - f^T f \right] \\ &= \begin{pmatrix} x^T & f^T \end{pmatrix} (\Phi_1 + \Phi_2 + \varepsilon\bar{\Omega}_0) \begin{pmatrix} x \\ f \end{pmatrix} \\ &\quad + 2x^T(A_\varepsilon + B_{u\varepsilon}K)^T \bar{P}_\varepsilon B_{u\varepsilon} K e \\ &\quad + 2e^T K^T B_{u\varepsilon}^T \bar{P}_\varepsilon H_\varepsilon f + e^T K^T B_{u\varepsilon}^T \bar{P}_\varepsilon B_{u\varepsilon} K e \\ &\quad + \mu_2 e^T (F'K)^T F' K e \\ &\quad + 2\mu_2 e^T (F'K)^T (F+F'K) X,\end{aligned}$$

where

$$\begin{aligned}\Omega_\varepsilon &= \bar{\Omega} + \varepsilon\bar{\Omega}_0 = \Phi_1 + \Phi_2 + \varepsilon\bar{\Omega}_0, \\ \Phi_1 &= \begin{pmatrix} M_1 & \bar{P}_1^T H \\ * & -\mu_2 I \end{pmatrix}, \\ M_1 &= (A_1 + B_u K)^T \bar{P}_1 + \bar{P}_1 (A_1 + B_u K) \\ &\quad + \mu_2 (F + F'K)^T (F + F'K), \\ \Phi_2 &= \begin{pmatrix} A_1 + B_u K & H \end{pmatrix}^T \bar{P}_2 \begin{pmatrix} A_1 + B_u K & H \end{pmatrix}, \\ \bar{\Omega}_0 &= \begin{pmatrix} A + B_u K & H \end{pmatrix}^T \bar{P}_3 \begin{pmatrix} A + B_u K & H \end{pmatrix}.\end{aligned}$$

It follows from (26) that there exists a sufficiently small scalar $\varepsilon_{22} > 0$ such that $\bar{\Omega} + \varepsilon\bar{\Omega}_0 < 0$ for any $\varepsilon \in (0, \varepsilon_{22}]$. Let $\varepsilon_2^* = \min\{\varepsilon_{21}, \varepsilon_{22}\}$, $\bar{\alpha} = \lambda_{\inf\limits_{\varepsilon \in (0, \varepsilon_2^*]}} (-\Omega_\varepsilon)$, we have $\bar{\alpha} > 0$. Therefore, for any $\varepsilon \in (0, \varepsilon_2^*]$,

$$\Delta V \leq -\bar{\alpha} \|x\|^2 + \beta \|x\| \|e\| + \tau \|e\|^2 + \gamma \|e\| \|f\|,$$

where f satisfies

$$\begin{aligned}\|f\| \leq \|Fx + F_1 u\| &= \|(F + F_1 K) x + F_1 K e\| \\ &\leq \|F + F_1 K\| \|x\| + \|F_1 K\| \|e\|,\end{aligned}$$

thus

$$\begin{aligned}\Delta V &\leq -\bar{\alpha} \|x\|^2 + (\beta + \gamma \|F + F_1 K\|) \|x\| \|e\| \\ &\quad + (\tau + \gamma \|F_1 K\|) \|e\|^2 \leq -\bar{\alpha}(1-\theta) \|x\|^2\end{aligned}$$

for any the equation can be derived, as shown at the bottom of next page
where

$$\begin{aligned}0 < \theta < 1, \beta &= \sup_{\varepsilon \in (0, \varepsilon_2^*]} 2 \left\| (A_\varepsilon + B_{u\varepsilon} Y)^T \bar{P}_\varepsilon B_{u\varepsilon} K \right\| \\ &\quad + \mu_2 \left\| (F'K)^T (F + F'K) \right\|, \\ \tau &= \sup_{\varepsilon \in (0, \varepsilon_2^*]} 2 \left\| K^T B_{u\varepsilon}^T \bar{P}_\varepsilon B_{u\varepsilon} K \right\| \\ &\quad + \mu_2 \left\| (F'K)^T F' K \right\|, \\ \gamma &= \sup_{\varepsilon \in (0, \varepsilon_2^*]} 2 \left\| K^T B_{u\varepsilon}^T \bar{P}_\varepsilon H_\varepsilon \right\|.\end{aligned}$$

Hence, the conditions of Lemma 1 are satisfied, and we obtained that the closed-loop system (25) is ISS with respect to the transmission error e. Note that the transmission error decays to zero by Theorem 1, thus it follows that the stabilization can be reached. This completes the proof. ∎

Remark 6: Similar to GEVP (16), there exist a scalar $\lambda_2 > 0$ and a positive matrix $\bar{W} > 0$, such that an upper bound $\varepsilon_2^* = \lambda_2^{-1}$ for ISS of the closed-loop system (25) can be obtained by solving the following GEVP:

$$\min \lambda_2 \text{s.t.} \bar{W} < \lambda_2 \bar{P}_{11}, \begin{pmatrix} \bar{W} & \bar{P}_{21}^T \\ \bar{P}_{21} & \bar{P}_{22} \end{pmatrix} > 0, \bar{\Omega} < 0, \bar{\Omega}_0 < -\lambda_2 \bar{\Omega}. \tag{32}$$

Remark 7: In this paper, we show that under the similar condition as those of [32], the proposed procedures can stabilize the system at a certain level of data packet loss rate, which make the approach more practical and applicable. Compared with [40], when sufficient conditions for asymptotic stability of system (1) are derived, the corresponding control law does not involve more complex equations. Next, simulation results will show the effectiveness of the approaches.

IV. NUMERICAL EXAMPLES

In engineering practice, some control problems are motivated by numerous applications where communication between the plant and the controller is limited due to bandwidth capacity or security constraints, such as underwater vehicles, nuclear reactor, unmanned aerial vehicles, etc. In this case, data packet dropout may occur during data communication, thus the signals need to be sampled and quantized. A proper quantized feedback control law is necessary. Next, we, in this section, will present two examples to illustrate the effectiveness of our results.

Example 4.1: Consider the following nuclear reactor model, which was first established in [12].

$$\dot{x}_1 = -\lambda x_1 + \lambda x_2, \tag{33}$$

$$\dot{x}_2 = \frac{\beta}{v} x_1 + \frac{\beta}{v} x_2 + \frac{\rho}{v}, \tag{34}$$

where x_1 and x_2 are the normalized precursors' concentration and neutron density, respectively. ρ, λ, β and v are the reactivity, precursors' decay constant, delayed-neutron yield and neutron generation-time, respectively. The parameters are $\lambda = 0.001$, $\beta = 0.0064$ and $v = 0.08$. Let $\rho = u + f_1(x_1, x_2)$, u and f_1 here are linear and nonlinear inputs, respectively. According to [12], we discretize the model with a sampling period $t = 0.05s$ and a zero-order holder. The derived discrete-time singularly perturbed system can be written in the form (4) with the following parameter:

$$A = \begin{pmatrix} -0.3417 & 0.3417 \\ 0.2733 & 0.7267 \end{pmatrix}, B_u = \begin{pmatrix} 9.0021 \\ 42.7983 \end{pmatrix},$$

$$H = \begin{pmatrix} 9.0021 & 0 \\ 0 & 42.7983 \end{pmatrix}.$$

Let $f_1(x_1, x_2) = 10^{-3} \times \sin(4x_1 + x_2)$, then it is easy to show that f_1 satisfies the condition (4) with

$$F_{11} = F_{21} = 0.0004, F_{12} = F_{22} = 0.0001.$$

For $\alpha = 5.1$, by solving the LMI (10), the feasible solutions can be obtained as follow

$$P_{11} = 0.0305, P_{12} = 0.0004, P_{22} = 0.0018, \mu_1 = 4.6651.$$

For the state feedback control, by applying Theorem 2, the following solutions can be obtained from (26)

$$X = \begin{pmatrix} 25.3506 & 0 \\ -16.1240 & 70.8638 \end{pmatrix},$$
$$Y = \begin{pmatrix} -0.3697 & -0.7849 \end{pmatrix}, \mu_2 = 35.5872.$$

Thus, the state feedback control gain matrix can be chosen as

$$K = YX^{-1} = (-0.0216 \quad -0.0111).$$

Furthermore, by solving the GEVP (16) and (32), an upper bound $\varepsilon^* = 3.9841$ can be derived, which means that system is stabilizable for $\varepsilon \in (0, \varepsilon^*]$. If we choose $\varepsilon = 0.1$, the sampling period $p = 0.4$ and $q = 120$, we can get $\mu = 19.7128$ by (13). Thus, the upper bound of the data dropout rate $\delta_{\max} \leq 0.1506$ can be obtained. When no packet loss occurs, the simulation of corresponding coder-decoder-controller procedure (19) - (21) is shown in Fig. 2, Fig. 3 is a simulation of the closed-loop system with $\delta = 0.1$.

Example 4.2: Consider a linearized model of the F-8 aircraft, which is borrowed from [3]. The equations of motion given by the following equation:

$$\begin{pmatrix} \dot{v}(t) \\ \dot{\theta}(t) \\ \dot{\alpha}(t) \\ \dot{q}(t) \end{pmatrix} = \begin{pmatrix} X_v & \frac{-g}{V_0} & \frac{X_\alpha}{V_0} & 0 \\ 0 & 0 & 0 & 1 \\ Z_v V_0 & 0 & Z_\alpha & 1 \\ M_v V_0 & 0 & M_\alpha & M_q \end{pmatrix} \begin{pmatrix} v(t) \\ \theta(t) \\ \alpha(t) \\ q(t) \end{pmatrix} + \begin{pmatrix} \frac{X_\delta}{V_0} \\ 0 \\ Z_\delta \\ M_\alpha \end{pmatrix} \delta(t), \tag{35}$$

where θ, α, q and δ are, respectively; the incremental pitch angle, angle of attack, pitch rate and elevator position, while $v = (V - V_0)/V_0$ is the normalised incremental velocity. The parameters $X_v, X_\alpha, X_\delta, Z_v, Z_\alpha, Z_\delta, M_v, M_\alpha, M_\delta, M_q, g$ and V_0 can be found in [3]. Let $v(t) = x_1(t), \theta(t) = x_2(t), \alpha(t) = x_3(t), q(t) = x_4(t)$ and $\delta(t) = u(t)$. According to [3], by a proper scaling, this model is presented in the following singularly perturbed continuous form:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \varepsilon\dot{x}_3(t) \\ \varepsilon\dot{x}_4(t) \end{pmatrix} = \begin{pmatrix} -0.1954 & -0.6765 & -0.9172 & 0.1090 \\ 1.4783 & 0 & 0 & 0 \\ -0.0516 & 0 & -0.3680 & 0.4380 \\ 0.0136 & 0 & -2.1026 & -0.2146 \end{pmatrix}$$

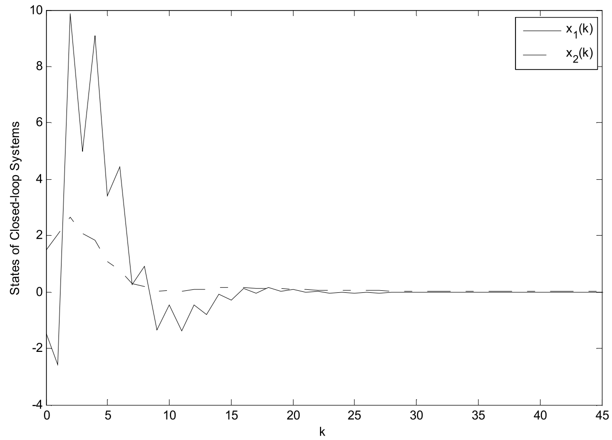


FIGURE 2. The state response of the closed-loop system with $\delta = 0$.

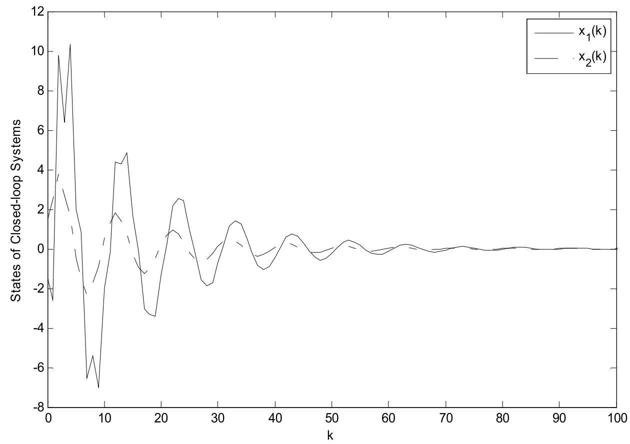


FIGURE 3. The state response of the closed-loop system with $\delta = 0.1$.

$$\times \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} -0.0231 \\ -16.9450 \\ -0.0482 \\ -3.8110 \end{pmatrix} u(t),$$

Here, we discrete the model with a sampling period $t = 0.03$ and a zero-order holder, then the discrete-time singularly perturbed model can be obtained with the following parameters:

$$A_{11} = \begin{pmatrix} -0.2090 & -0.6744 \\ 1.4737 & -0.0150 \end{pmatrix}, A_{12} = \begin{pmatrix} -0.9126 & 0.1024 \\ -0.0203 & 0.0023 \end{pmatrix},$$
$$A_{21} = \begin{pmatrix} -0.0015 & 0 \\ 0.0005 & 0 \end{pmatrix}, A_{22} = \begin{pmatrix} 0.9886 & 0.0130 \\ -0.0625 & 0.9932 \end{pmatrix},$$
$$B_1 = \begin{pmatrix} 0.1432 \\ -16.9431 \end{pmatrix}, B_2 = \begin{pmatrix} -0.0022 \\ -0.1139 \end{pmatrix}, \alpha = 2.1.$$

$$\|x\| \geq \frac{(\beta + \gamma \|F + F_1 K\|) \sqrt{(\beta + \gamma \|F + F_1 K\|)^2 + 4\alpha\theta(\tau + \gamma \|F_1 K\|)}}{2\alpha\theta},$$

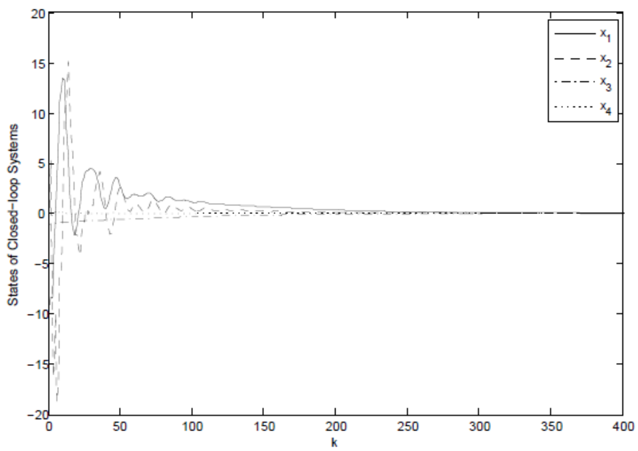


FIGURE 4. The state response of the closed-loop system with $\delta = 0$.

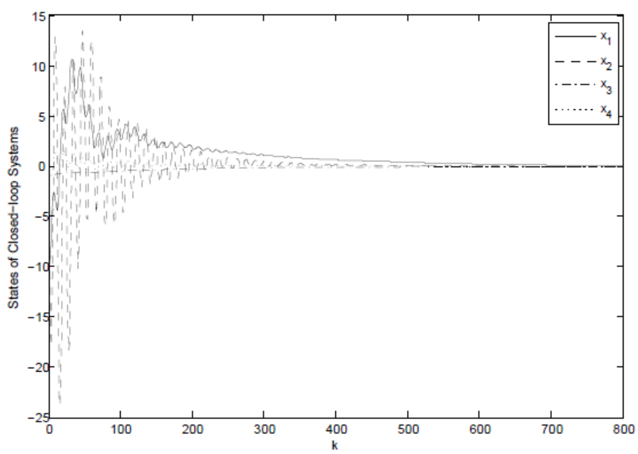


FIGURE 5. The state response of the closed-loop system with $\delta = 0.05$.

By utilizing Theorem 2, we can get the following solutions from (26)

$$X = \begin{pmatrix} 9.5243 & 2.6144 & 0 & 0 \\ 2.6144 & 9.5428 & 0 & 0 \\ 21.3270 & -10.3270 & 2.3780 & -0.5309 \\ 172.9782 & -53.8740 & -0.5309 & 15.8838 \end{pmatrix},$$

$$Y = \begin{pmatrix} 0.6112 & 0.4786 & -0.0372 & 0.0597 \end{pmatrix}.$$

Hence, the state feedback control gain matrix can be chosen as

$$K = YX^{-1} = \begin{pmatrix} 0.0260 & 0.0453 & -0.0149 & 0.0033 \end{pmatrix}.$$

Furthermore, by solving the GEVP (16) and (32), an upper bound $\varepsilon_2 = 0.2241$ can be obtained, which means the system is stabilizable for $\varepsilon \in (0, \varepsilon^*]$. In addition, when choosing $\varepsilon = 0.1$, the sampling period $p = 0.3$ and $q = 10$, the upper bound of the data dropout rate $\delta_{\max} \leq 0.1173$ can be obtained. The simulations for the corresponding coder-decoder-controller procedure (19)-(21) are shown in Figs 4 and 5, respectively.

As shown in the simulations, compared with the case of no data packet dropout, lower convergence rate of the system state can be obtained when a certain rate of data packet dropout occurs. Logically, it is reasonable. If one wants to get better performance, it needs to pay a higher price as cost. It is a tradeoff.

Remark 8: Through the above two numerical examples, the simple validity of our method is proved. Note that in reference [12], the upper bound of the minimum perturbed parameter needs to be given in advance. This makes the actual operation very difficult, because it is very difficult to find the appropriate upper bound. By contrast, our method avoids this problem and can accurately obtain the upper bound.

V. CONCLUSION

This paper discusses the feedback control problem of the discrete time singularly perturbed system with limited communication channel and data packet dropout, in which the uniform quantization method is adopted. First, we assume that there exists a certain rate of data packet dropout during the transmission. Then, the auxiliary system is added to set up the proper coder-decoder pair, so that the transmission error converges to zero exponentially. By using the LMI technique and Lyapunov function method, a sufficient condition for the input-to-state stability of the closed-loop system is obtained, and the asymptotic stability of the system is also guaranteed based on the ISS property. Finally, the upper bound of small perturbed parameters can also be obtained by a workable way.

Declaration of Conflicting Interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Compliance with ethical standards

Conflict of interest: The authors declare they have no conflict of interest.

REFERENCES

[1] G. Zhuang, S.-F. Su, J. Xia, and W. Sun, "HMM-based asynchronous H_∞ filtering for fuzzy singular Markovian switching systems with retarded time-varying delays," *IEEE Trans. Cybern.*, vol. 51, no. 3, pp. 1189–1203, Mar. 2021, doi: 10.1109/TCYB.2020.2977127.

[2] G. Zhuang, W. Sun, S.-F. Su, and J. Xia, "Asynchronous feedback control for delayed fuzzy degenerate jump systems under observer-based event-driven characteristic," *IEEE Trans. Fuzzy Syst.*, early access, Sep. 30, 2020, doi: 10.1109/TFUZZ.2020.3027336.

[3] P. V. Kokotovic, H. K. Khalil, and J. O'Reilly, *Singular Perturbation Methods in Control: Analysis and Design*. London, U.K.: Academic, 1986.

[4] Y. Zhang, Z. Yang, Z. Z. Yu, X. L. Hu, Y. Z. Wang, and H. Shen, "Delay dependent H_∞ control of wind energy conversion systems via singular perturbation theory," *Trans. Inst. Meas. Control*, vol. 43, no. 1, pp. 194–204, 2021, doi: 10.1177/0142331220952302.

[5] H. Liu, F. Sun, and Z. Sun, "Stability analysis and synthesis of fuzzy singularly perturbed systems," *IEEE Trans. Fuzzy Syst.*, vol. 13, no. 2, pp. 273–284, Apr. 2005.

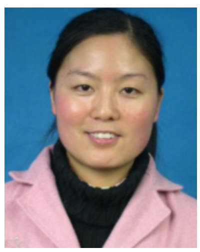
[6] P. Shi, S.-P. Shue, and R. K. Agarwal, "Robust disturbance attenuation with stability for a class of uncertain singularly perturbed systems," *Int. J. Control*, vol. 70, no. 6, pp. 873–891, Jan. 1998.

[7] Y. Han and X. Liu, "Continuous higher-order sliding mode control with time-varying gain for a class of uncertain nonlinear systems," *ISA Trans.*, vol. 62, pp. 193–201, May 2016.

[8] C. Yang, Q. Zhang, J. Sun, and T. Chai, "Lur'e Lyapunov function and absolute stability criterion for Lur'e singularly perturbed systems," *IEEE Trans. Autom. Control*, vol. 56, no. 11, pp. 2666–2671, Nov. 2011.

[9] K. B. Datta and A. RaiChaudhuri, "H_2/H_∞ control of discrete singularly perturbed systems: The state feedback case," *Automatica*, vol. 38, no. 10, pp. 1791–1797, Oct. 2002.
[10] T. V. Vu and M. E. Sawan, "H_∞ control for singularly perturbed sampled data systems," in *Proc. IEEE Int. Symp. Circuits Syst.*, Chicago, IL, USA, May 1993, pp. 2506–2509.
[11] J. Dong and G.-H. Yang, "Robust H_∞ control for standard discrete-time singularly perturbed systems," *IET Control Theory Appl.*, vol. 1, no. 4, pp. 1141–1148, 2007.
[12] J. Dong and G.-H. Yang, "H_∞ control for fast sampling discrete-time singularly perturbed systems," *Automatica*, vol. 44, pp. 1385–1393, May 2008.
[13] S. Y. Xu and G. Feng, "New results on H_∞ control of discrete singularly perturbed systems," *Automatica*, vol. 45, no. 10, pp. 2339–2343, 2009.
[14] H. Shen, Y. Z. Men, Z. G. Wu, and J. H. Park, "Nonfragile H_∞ control for fuzzy Markovian jump systems under fast sampling singular perturbation," *IEEE Trans. Syst., Man, Cybern.*, vol. 48, no. 12, pp. 2058–2069, Dec. 2018.
[15] C. Y. Yang and L. N. Zhou, "H_∞ control and ε-bound estimation of discrete-time singularly perturbed systems," *Circuits, Syst., Signal Process.*, vol. 35, pp. 2640–2654, Sep. 2016.
[16] W. Liu, Z. Wang, H. Dai, and M. Naz, "Dynamic output feedback control for fast sampling discrete-time singularly perturbed systems," *IET Control Theory Appl.*, vol. 10, no. 15, pp. 1782–1788, Oct. 2016.
[17] R. A. Gupta and M.-Y. Chow, "Networked control system: Overview and research trends," *IEEE Trans. Ind. Electron.*, vol. 57, no. 7, pp. 2527–2535, Jul. 2010.
[18] H. Zhang, G. Feng, H. Yan, and Q. Chen, "Sampled-data control of nonlinear networked systems with time-delay and quantization," *Int. J. Robust Nonlinear Control*, vol. 26, no. 5, pp. 919–933, Mar. 2016.
[19] Y. L. Wang and G. H. Yang, "H_∞ control of networked control systems with time delay and packet disordering," *IET Control Theory Appl.*, vol. 1, no. 5, pp. 1344–1354, 2007.
[20] T. Shi, P. Shi, and S. Wang, "Robust sampled-data model predictive control for networked systems with time-varying delay," *Int. J. Robust Nonlinear Control*, vol. 29, no. 6, pp. 1758–1768, Apr. 2019.
[21] G. Zhuang, J. Xia, J.-E. Feng, W. Sun, and B. Zhang, "Admissibilization for implicit jump systems with mixed retarded delays based on reciprocally convex integral inequality and Barbalat's lemma," *IEEE Trans. Syst., Man, Cybern. Syst.*, early access, Jan. 16, 2020, doi: 10.1109/TSMC.2020.2964057.
[22] G. Zhuang, S. Xu, J. Xia, Q. Ma, and Z. Zhang, "Non-fragile delay feedback control for neutral stochastic Markovian jump systems with time-varying delays," *Appl. Math. Comput.*, vol. 355, pp. 21–32, Aug. 2019.
[23] J. Zhou and D. Zhang, "H-infinity fault detection for delta operator systems with random two-channels packet losses and limited communication," *IEEE Access*, vol. 7, pp. 94448–94459, 2019, doi: 10.1109/ACCESS.2019.2928306.
[24] Z.-D. Lu, G.-T. Ran, G.-L. Zhang, and F.-X. Xu, "Event-triggered H∞ fuzzy filtering for networked control systems with quantization and delays," *IEEE Access*, vol. 6, pp. 20231–20241, 2018, doi: 10.1109/ACCESS.2018.2819244.
[25] H. Yu, G. Lu, and Y. Zheng, "On the model-based networked control for singularly perturbed systems with nonlinear uncertainties," *Syst. Control Lett.*, vol. 60, no. 9, pp. 739–746, Sep. 2011.
[26] Z. M. Wang and W. Liu, "Output feedback networked control of singular perturbation," in *Proc. 9th World Congr. Intell. Control Automat.*, Jun. 2011, pp. 645–650.
[27] S. H. J. Heijmans, D. Nešić, R. Postoyan, and W. P. M. H. Heemels, "Singularly perturbed networked control systems," *IFAC-PapersOnLine*, vol. 51, no. 23, pp. 106–111, 2018.
[28] F.-L. Qu, Z.-H. Guan, D.-X. He, and M. Chi, "Event-triggered control for networked control systems with quantization and packet losses," *J. Franklin Inst.*, vol. 352, no. 3, pp. 974–986, Mar. 2015.
[29] A. V. Savkin and T. M. Cheng, "Detectability and output feedback stabilizability of nonlinear networked control systems," *IEEE Trans. Autom. Control*, vol. 52, no. 4, pp. 730–735, Apr. 2007.
[30] D. Liberzon and D. Nesic, "Input-to-state stabilization of linear systems with quantized state measurements," *IEEE Trans. Autom. Control*, vol. 52, no. 5, pp. 767–781, May 2007.
[31] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. Autom. Control*, vol. 45, no. 7, pp. 1279–1289, Jul. 2000.
[32] Y. Wang, W. Liu, and J. Xu, "Feedback stabilization of singularly perturbed systems under information constraints," *Asian J. Control*, vol. 18, no. 5, pp. 1739–1748, Sep. 2016.
[33] W. Liu, Z. Wang, and M. Ni, "Controlled synchronization for chaotic systems via limited information with data packet dropout," *Automatica*, vol. 49, no. 8, pp. 2576–2579, Aug. 2013.
[34] W. Liu, Z. M. Wang, and M. K. Ni, "Quantized feedback stabilization for a class of linear systems with nonlinear disturbances," *Nonlinear Anal., Hybrid Systems*, vol. 8, no. 5, pp. 248–256, 2013.
[35] H. J. Gao and T. W. Chen, "H-infinity estimation for uncertain systems with limited communication capacity," *IEEE Trans. Autom. Control*, vol. 52, no. 11, pp. 2070–2084, Nov. 2007.
[36] L. Zhou and G. Lu, "Detection and stabilization for discrete-time descriptor systems via a limited capacity communication channel," *Automatica*, vol. 45, no. 10, pp. 2272–2277, Oct. 2009.
[37] H. Shen, Y. Men, Z.-G. Wu, J. Cao, and G. Lu, "Network-based quantized control for fuzzy singularly perturbed semi-Markov jump systems and its application," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 66, no. 3, pp. 1130–1140, Mar. 2019.
[38] Y. Tian, W. Liu, and X. Que, "Feedback stabilization of discrete time singularly perturbed systems with communication constraints," *Asian J. Control*, vol. 22, no. 4, pp. 1484–1493, Jul. 2020.
[39] Z.-P. Jiang and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, no. 6, pp. 857–869, Jun. 2001.
[40] Z. H. Shao and M. E. Sawan, "Stabilization of uncertain singularly perturbed systems," *IEEE Proc.-Control Theory Appl.*, vol. 153, no. 1, pp. 99–103, 2006.
[41] H. Singh, D. S. Naidu, J. A. Heinen, and R. H. Brown, "Robust stability of singularly perturbed state feedback systems using unified approach," *IEE Proc.-Control Theory Appl.*, vol. 148, no. 5, pp. 391–396, Sep. 2001.

WEI LIU received the M.S. and Ph.D. degrees in mathematics from East China Normal University, China, in 2010 and 2016, respectively. Since 2010, he has been with Zhoukou Normal University, Zhoukou, China. His research interests include singularly perturbed systems, robust control, networked control systems, and nonlinear systems.



YANYAN WANG received the M.S. degree in mathematics from Central China Normal University, China, in 2008. Since 2008, she has been with Zhoukou Normal University, Zhoukou, China. Her research interests include singularly perturbed systems and networked control systems.

YITAO TIAN received the B.S. degree in mathematics from Zhoukou Normal University, China, in 2016, and the M.S. degree in applied mathematics from Zhengzhou University, China, in 2019. Her research interests include singularly perturbed systems and networked control systems.