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Mathematically Proving Bell Nonlocality Motivated by the GHZ Argument

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ABSTRACT Bell nonlocality of quantum states is an important resource in quantum information and then has various applications. It is usually detected by the violation of some Bell's inequalities and the all-versus-nothing test. In the present paper, we aim to establish some mathematical methods for proving Bell nonlocality without inequalities, inspired by the work [Phys. Rev. Lett., 89, 080402 (2002)] regarding the GHZ paradox. For self-containedness, we recall the mathematical definition of Bell nonlocality proposed in [Sci. China-Phys. Mech. Astron. 62, 030311 (2019)] and then give some basic properties on it. Then we derive some necessary conditions for a multipartite state to be Bell local and obtain some sufficient conditions for a state to be Bell nonlocal in terms of "expectations" of local observables without invoking Bell inequalities. Unlike the standard approach to nonlocality detection based on violation of Bell inequalities, the obtained criteria are formulated in terms of certain relations for expectation values of local observables that are constructed from the well-known GHZ paradoxes.

INDEX TERMS Bell nonlocality, LHV model, GHZ argument, GHZ paradox, POVM measurement.

I. INTRODUCTION

Quantum nonlocality was first discovered by Einstein, Podolsky and Rosen (EPR) in 1935, including quantum entanglement, quantum steering and Bell nonlocality [1]. They formulated an apparent paradox of quantum theory (EPR paradox) and given a "thought" experiment that argues the wave function description in quantum mechanics is incomplete. According to the EPR paradox on local realism, quantum theory allows a curious phenomenon: the so-called "spooky action at a distance". In the next year 1936, Schrödinger [2] firstly introduced the terminology "entanglement" and "steering" to describe such quantum "spooky action". Debates on whether quantum theory is complete and how to understand quantum entanglement lasted for the following 20 years and were finally concluded by Bohm [3] and Bell [4], [5].

Quantum entanglement, originated from the EPR paradox, is the essence of quantum formalism and holistic property of compound quantum systems involves nonclassical correlations between subsystems and then has many applications for many quantum processes, including quantum cryptography, quantum teleportation, dense coding and so on.

Bell nonlocality originated from the Bell's 1964 paper. Bell in his paper showed that when some entangled state is suitably measured, the probabilities for the outcomes violate an inequality, named the Bell inequality. This property of quantum states found by Bell is the so-called Bell nonlocality and was reviewed by Brunner *et al.* [6]. It is an important resource in quantum information and then has been widely discussed by Home and Selleri [7], Khalfin and Tsirelson [8], Tsirelson [9], Werner and Wolf [10], Genovese [11], and [12]–[19]. Recently, Eliëns *et al.* [20] proposed to analyse a Bell scenario as a tensor network and obtained a perspective permitting to test and quantify Bell nonlocality resorting to very efficient algorithms originating from compressed sensing. Furthermore, Férot and Roscilde [21] proposed and validated an efficient variational scheme, based on the solution of inverse Ising problems. Ming *et al.* [22] investigated the measure of quantumness in experimentally observed neutrino oscillations via the nonlocal advantage of quantum coherence, quantum steering, and Bell nonlocality.

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Quantum steering, also called EPR steering, is an intermediary property between Bell nonlocality and entanglement and was first observed by Schrödinger [2] in the context of the EPR paradox ([23], [24]). It is also an important resource in quantum information and then has been recently discussed in [25]–[33]. Especially, mathematical definitions of Bell nonlocality and EPR steerability of bipartite states were formulated and their characterizations were given in [18] by Cao and Guo.

Usually, Bell nonlocality for quantum states is detected by violation of some Bell's inequalities, such as Clause-Horne-Shimony-Holt inequality for two qubits. A proof of nonlocality without inequalities for two particles had been given earlier by Heywood and Redhead [34] which was much simplified by Brown and Svetlichny [35]. Greenberger, Horne, and Zeilinger (GHZ) [36] gave a proof of nonlocality but without using inequalities, in which a minimum of three particles was required in their proof. Mermin [37] provided a simple unified form for the major no-hidden-variables theorems by two examples. Hardy in [38] and [39]) proposed the two-particle 2-dimensional 2-setting Hardy paradox and gave the maximum probability of Bell's nonlocality, which is about 0.09. In 1997, Boschi *et al.* [40] discovered two-particle 2-dimensional k-setting Hardy paradox. Aravind [41] established a Bell's theorem without inequalities and only two distant observers. Dong *et al.* obtained in [19] some methods for detecting Bell nonlocality based on the Hardy Paradox. Chen *et al.* [42] proved that Bell nonlocal states can be constructed from some steerable states. They also established in [43] a mapping criteria between nonlocality and steerability in qudit-qubit systems and between steerability and entanglement in qubit-qudit systems. Jiang *et al.* [44] proposed a generalized Hardy's paradox, Yang *et al.* [45] proposed stronger Hardy-type paradox based on the Bell inequality and its experimental test.

One of the most important insights into multipartite (actually tripartite) entanglement is provided by the Greenberger-Horne-Zeilinger (GHZ) argument (also called GHZ paradox in the literature) [36]. In its formulation given by Mermin [37], the GHZ argument is both an intrinsic contradiction arising when dealing with noncontextual variables (a Kochen-Specker (KS) theorem) and a Bell-EPR theorem that rules out local hidden-variable models [46]. The GHZ paradox reveals a stronger quantum nonlocality, known as GHZ nonlocality, and provides an ''all or nothing argument'' on quantum nonlocality [47]–[50]. Earlier of GHZ paradoxes have been generalized to the case of multipartite and multilevel systems [51]–[55]. Especially, Cerf *et al.* [46] constructed GHZ contradictions for three or more parties sharing an entangled state when the dimension of each subsystem is an even integer d. They examined the criteria that a GHZ paradox must satisfy in order to be genuinely M partite and d dimensional. In the summary part, the authors pointed out that an interesting extension of their work would be to construct Mermin-like inequalities for qudits from the constructed paradoxes, which would lay the grounds for an experimental testing of multipartite multidimensional nonlocality. The statistical strength of nonlocality proofs was first defined and discussed by van Dam *et al.* in [51] in terms of the amount of evidence against local realism provided by the corresponding experiments. The measure proposed in [51] tells us how many trials of the experiment we should perform in order to observe a substantial violation of local realism. Cabello and Moreno [52] proved a necessary and sufficient condition for the existence of proofs of Bell's theorem using only single-qubit measurements. Ryu *et al.* [53] constructed a generalized GHZ contradiction for multipartite and high-dimensional systems and proved that D-dimensional GHZ theorem for an N-partite system holds as long as N is not divisible by all nonunit divisors of D, smaller than N.

Historically, the GHZ paradox provided by Mermin [37] is based on the predictions of quantum mechanics (PQM) and the following ''assignment assumption''(AA):

If some functional relation $f(A, B, C, \ldots, D) = 0$ holds as an operator identity among the observables of a mutually commuting set, then since the results of the simultaneous measurements of $A, B, C, \ldots, D$ will be one of the sets $a, b, c, \ldots, d$ of simultaneous eigenvalues of $A, B, C, \ldots, D$, the results of those measurements must also satisfy $f(a, b, c, \ldots, d) = 0$, whatever the state of the system prior to the measurement.

For example, $A^m B^n C^\ell = I \Longrightarrow a^m b^n c^\ell = 1$, according to the AA.

Followed Mermin's work, various discussions about the GHZ paradoxes (or GHZ contradictions) are essentially based on the AA above and the PQM to construct a contradiction (a paradox), showing that the AA is not valid, instead of establishing a general method for proving Bell nonlocality with the inexistence of the local hidden variable models (LHVMs).

Different from the existing discussions about the GHZ paradox, we aim to establish some theoretical methods for mathematically proving Bell nonlocality with the inexistence of the local hidden variable models (LHVMs), which is motivated by the work [46] . For self-containedness, we recall the mathematical definition of Bell nonlocality concluded in [18] and give some basic properties on it in Section 2. In Section 3, we first derive logically some necessary conditions for a multipartite state to be Bell local and then obtain some sufficient conditions for a state to be Bell nonlocal.

II. CONCEPT AND BASIS PROPERTIES

Given n quantum systems $S_1, S_2, \ldots, S_n$ described by Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$, we consider the composite system $S_1 S_2 \cdots S_n$ described by the Hilbert space $\mathcal{H}^{(n)} := \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ where $\dim(\mathcal{H}_k) = d_k < +\infty$. We use $\mathcal{D}(\mathcal{H}^{(n)})$ or $\mathcal{D}(S_1 S_2 \cdots S_n)$ to denote the set of all quantum states of the system $\mathcal{H}^{(n)}$, I_k to denote the identity operator on $\mathcal{H}_k$ and use $[d]$ to denote the set $\{1, 2, \ldots, d\}$. For a linear operator T on a finite dimensional Hilbert space, we use $\mathrm{tr}(T)$ and $\sigma(T)$ to the trace and the spectrum of T, respectively.

Recall that a positive operator-valued measure (POVM) of a quantum system X is a set $M = \{M_1, M_2, \ldots, M_d\} \equiv \{M_k\}_{k=1}^{d}$ of positive operators acting on the Hilbert space $\mathcal{H}_X$ such that $\sum_{k=1}^{d} M_k = I_X$. For each index $j \in [n]$, let $M^{x_j} = \{M_{a_j|x_j}\}_{a_j=1}^{o_j} (x_j \in [m_j])$ be m_j POVMs of system S_j, where o_j denotes the number of measurement operators of M^{x_j}, a_j's denote the labels of the outcomes, and $M_{a_j|x_j}$ is the a_ith-measurement operator of the POVM M^{x_j}. Then we obtain $m_1 m_2 \cdots m_n$ POVMs of $S_1 S_2 \cdots S_n$:

$$M^{x_1,\ldots,x_n} = \left\{M_{a_1|x_1} \otimes \cdots \otimes M_{a_n|x_n} : a_i \in [o_i]\right\}, \tag{1}$$

where $x = (x_1, \ldots, x_n) \in [m_1] \times \cdots \times [m_n]$, which lead to a set of POVMs:

$$\{M^{x_1,\ldots,x_n} : x_i \in [m_i]\} \equiv \{M^{x_1,\ldots,x_n}\}_{x_1,\ldots,x_n}, \tag{2}$$

called a *measurement assemblage (MA)* of $\mathcal{H}^{(n)}$ and denoted by

$$\mathcal{M} = \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n, \tag{3}$$

where

$$\mathcal{M}_j = \left\{M^{x_j} : x_j \in [m_j]\right\}$$

forms an MA of $\mathcal{H}_j$ for each $j \in [n]$.

According the hypothesises of quantum mechanics, when the system $S_1 S_2 \cdots S_n$ laying in a state ρ is measured with a POVM $M^{x_1,\ldots,x_n}$ labeled by $x = (x_1, \ldots, x_n)$ and given by Eq. (1), the conditional probability of obtaining the result $a = (a_1, \ldots, a_n)$ reads

$$P_\rho\{a|x\} = \mathrm{tr}[(M_{a_1|x_1} \otimes \cdots \otimes M_{a_n|x_n})\rho]. \tag{4}$$

See Figure 1 below for an explanation of this experiment setting.

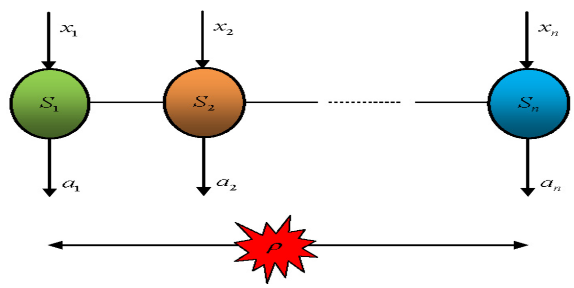

FIGURE 1. Sketch of an experiment for Bell nonlocality, in which ρ denotes the shared state of systems $S_1, S_2, \ldots, S_n$, $x_1, x_2, \ldots, x_n$ denote the labels of POVM measurements acting on $S_1, S_2, \ldots, S_n$, respectively, and $a_1, a_2, \ldots, a_n$ denote the corresponding outcomes.

Generalizing the mathematical definition of Bell locality of bipartite states given by Cao and Guo in [18], we fix the following definition of Bell locality of n-partite states.

Definition 1: A state $\rho \in \mathcal{D}(\mathcal{H}^{(n)})$ is said to be *Bell local* for a given MA $\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_n$ of $\mathcal{H}^{(n)}$ given by Eqs. (1) and (2), there exists a probability distribution (PD) $\pi = \{\pi_\lambda\}_{\lambda=1}^{d}$ such that

$$P_\rho\{a|x\} = \sum_{\lambda=1}^{d} \pi_\lambda P_1(a_1|x_1, \lambda) \cdots P_n(a_n|x_n, \lambda) \tag{5}$$

for all $x = (x_1, \ldots, x_n) \in [m_1] \times \cdots \times [m_n]$ and all $a = (a_1, \ldots, a_n) \in [o_1] \times \cdots \times [o_n]$, where $P_i(a_i|x_i, \lambda) \geq 0$, $\sum_{a_i=1}^{o_i} P_i(a_i|x_i, \lambda) = 1 (\forall i \in [n], \forall \lambda \in [d])$.

A state ρ is said to be *Bell local* if it is Bell local for every MA $\mathcal{M}$ given by Eqs. (1) and (2). If ρ is not Bell local, then we say that it is *Bell nonlocal*.

Furthermore, a pure state $|\psi\rangle$ of $\mathcal{H}^{(n)}$ is said to be Bell local (resp. Bell nonlocal) means that the corresponding density operator $\rho = |\psi\rangle\langle\psi|$ is Bell local (resp. Bell nonlocal).

Moreover, Eq. (5) is called a *local hidden variable (LHV) model* of ρ for $\mathcal{M}$.

By Definition 1, we have the following conclusions.

(1) A state ρ is Bell local for an MA $\mathcal{M}$ of $\mathcal{H}^{(n)}$ if and only if there exists an LHV model of ρ for $\mathcal{M}$.

(2) A state ρ is Bell local if and only if for every MA $\mathcal{M}$ of $\mathcal{H}^{(n)}$, there exists an LHV model of ρ for $\mathcal{M}$.

(3) A state ρ is Bell nonlocal if and only if there exists an MA $\mathcal{M}$ such that an LHV model of ρ for $\mathcal{M}$ does not exist.

Denote by $\mathcal{BL}(\mathcal{H}^{(n)}, \mathcal{M})$, $\mathcal{BL}(\mathcal{H}^{(n)})$, $\mathcal{BNL}(\mathcal{H}^{(n)}, \mathcal{M})$, and $\mathcal{BNL}(\mathcal{H}^{(n)})$ the sets of all Bell local states for $\mathcal{M}$, Bell local states, Bell nonlocal states for $\mathcal{M}$, and Bell nonlocal states of the system $\mathcal{H}^{(n)}$, respectively. Then

$$\mathcal{BL}(\mathcal{H}^{(n)}) = \bigcap_{\mathcal{M}} \mathcal{BL}(\mathcal{H}^{(n)}, \mathcal{M}), \tag{6}$$

$$\mathcal{BNL}(\mathcal{H}^{(n)}) = \bigcup_{\mathcal{M}} \mathcal{BNL}(\mathcal{H}^{(n)}, \mathcal{M}) \tag{7}$$

where $\mathcal{M}$ runs over all of the MAs of $\mathcal{H}^{(n)}$. Sometimes, we write $\mathcal{BL}(\mathcal{H}^{(n)})$ and $\mathcal{BNL}(\mathcal{H}^{(n)})$ as $\mathcal{BL}(S_1 S_2 \cdots S_n)$ and $\mathcal{BNL}(S_1 S_2 \cdots S_n)$, respectively.

Remark 2: By Definition 2, one can check that when $\rho \in \mathcal{BL}(\mathcal{H}^{(n)})$, $\mathrm{tr}_\Delta(\rho) \in \mathcal{BL}(\otimes_{i\in[n]\setminus\Delta}\mathcal{H}_i)$ for any nonempty proper subset Δ of $[n]$, where $\mathrm{tr}_\Delta = \prod_{i\in\Delta} \mathrm{tr}_{S_i}$ and $\otimes_{i\in[n]\setminus\Delta}\mathcal{H}_i$ denotes the tensor product of $\mathcal{H}_i$ for all $i \in [n]\setminus\Delta$. Conversely, if $\mathrm{tr}_\Delta(\rho) \in \mathcal{BNL}(\otimes_{i\in[n]\setminus\Delta}\mathcal{H}_i)$ for some $\Delta \subset [n]$, then $\rho \in \mathcal{BNL}(\mathcal{H}^{(n)})$. Roughly speaking, *total Bell locality $\Longrightarrow$ partial Bell locality; partial Bell nonlocality $\Longrightarrow$ total Bell nonlocality*. For example, $\rho^{ABC} \otimes \rho^{XY} \in \mathcal{BL}(ABCXY) \Longrightarrow \rho^{ABC} \in \mathcal{BL}(ABC)$ and $\rho^{XY} \in \mathcal{BL}(XY)$; and $\rho^{ABC} \in \mathcal{BNL}(ABC)$, or $\rho^{XY} \in \mathcal{BNL}(XY) \Longrightarrow \rho^{ABC} \otimes \rho^{XY} \in \mathcal{BNL}(ABCXY)$.

Remark 3: Every fully separable state

$$\rho = \sum_{\lambda=1}^{r} \pi_\lambda \rho_\lambda^{S_1} \otimes \rho_\lambda^{S_2} \otimes \cdots \otimes \rho_\lambda^{S_n}$$

of $\mathcal{H}^{(n)}$ has always an LHV model (5) for every MA $\mathcal{M}$ denoted by Eq. (2), where $P_j(a_j|x_j, \lambda) = \mathrm{tr}(M_{a_j|x_j}\rho_\lambda^{S_j})$ and then it is Bell local.

To give an illustration of Bell locality with expectations of local observables, we let $\mathcal{A} = \{A_i\}_{i=1}^{\ell}$, $\mathcal{B} = \{B_j\}_{j=1}^{m}$ and $\mathcal{C} = \{C_k\}_{k=1}^{n}$ be families of observables of $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_C$, respectively. Assume that operators A_i (resp., B_j and C_k) have just e_A (resp., e_B and e_C) different eigenvalues. Thus, we have the following spectral decompositions:

$$A_i = \sum_{r=1}^{e_A} a_r^{(i)} M_{r|i}, \quad B_j = \sum_{s=1}^{e_B} b_s^{(j)} N_{s|j}, \; C_k = \sum_{t=1}^{e_C} c_t^{(k)} L_{t|k},$$

leading to MAs:

$$\mathcal{M}_A = \{M^1, M^2, \ldots, M^\ell\},$$
$$\mathcal{N}_B = \{N^1, N^2, \ldots, N^m\},$$
$$\mathcal{L}_C = \{L^1, L^2, \ldots, L^n\},$$

consisting of projective measurements, where

$$M^i = \{M_{r|i}\}_{r=1}^{d_A}, N^j = \{N_{s|j}\}_{s=1}^{d_B}, L^k = \{L_{t|k}\}_{t=1}^{d_C}.$$

Let $\rho \in \mathcal{BL}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$. Then by definition, there is a PD $\{\pi_\lambda\}_{\lambda=1}^d$ s.t.

$$\mathrm{tr}[(M_{r|i} \otimes L_{s|j} \otimes N_{t|k})\rho] = \sum_{\lambda=1}^{d} \pi_\lambda P_A(r|i,\lambda) P_B(s|j,\lambda) P_C(t|k,\lambda) \quad (8)$$

for all possible i, j, k, r, s and t, where $\{P_A(r|i,\lambda)\}_{r=1}^{o_A}$, $\{P_B(s|j,\lambda)\}_{s=1}^{o_B}$ and $\{P_C(t|k,\lambda)\}_{t=1}^{o_C}$ are PDs. Thus, the expectation of $A_i \otimes B_j \otimes C_k$ w.r.t. ρ reads

$$\begin{aligned}
\langle A_i \otimes B_j \otimes C_k \rangle_\rho
&= \sum_{r=1}^{e_A} \sum_{s=1}^{e_B} \sum_{t=1}^{e_C} a_r^{(i)} b_s^{(j)} c_t^{(k)} \langle M_{r|i} \otimes N_{s|j} \otimes L_{t|k} \rangle_\rho \\
&= \sum_{r=1}^{e_A} \sum_{s=1}^{e_B} \sum_{t=1}^{e_C} a_r^{(i)} b_s^{(j)} c_t^{(k)} tr[(M_{r|i} \otimes N_{s|j} \otimes L_{t|k})\rho] \\
&= \sum_{\lambda=1}^{d} \pi_\lambda \left(\sum_{r=1}^{e_A} a_r^{(i)} P_A(r|i,\lambda) \right) \left(\sum_{s=1}^{e_B} b_s^{(j)} P_B(s|j,\lambda) \right) \left(\sum_{t=1}^{e_C} c_t^{(k)} P_C(t|k,\lambda) \right) \\
&= \sum_{\lambda=1}^{d} \pi_\lambda \langle a^{(i)} \rangle_\lambda \cdot \langle b^{(j)} \rangle_\lambda \cdot \langle c^{(k)} \rangle_\lambda,
\end{aligned}$$

where

$$\langle a^{(i)} \rangle_\lambda = \sum_{r=1}^{e_A} a_r^{(i)} P_A(r|i,\lambda),$$
$$\langle b^{(j)} \rangle_\lambda = \sum_{s=1}^{e_B} b_s^{(j)} P_B(s|j,\lambda),$$
$$\langle c^{(k)} \rangle_\lambda = \sum_{t=1}^{e_C} c_s^{(k)} P_C(t|k,\lambda),$$

which are the expectations of random variables

$$a^{(i)} \sim \sigma(A_i) = \{a_1^{(i)}, a_2^{(i)}, \ldots, a_{e_A}^{(i)}\}$$

with PD $\{P_A(1|i,\lambda), P_A(2|i,\lambda), \ldots, P_A(e_A|i,\lambda)\}$;

$$b^{(j)} \sim \sigma(B_j) = \{b_1^{(j)}, b_2^{(j)}, \ldots, b_{e_B}^{(j)}\}$$

with PD $\{P_B(1|j,\lambda), P_B(2|j,\lambda), \ldots, P_B(e_B|j,\lambda)\}$;

$$c^{(k)} \sim \sigma(C_k) = \{c_1^{(k)}, c_2^{(k)}, \ldots, c_{e_C}^{(k)}\}$$

with PD $\{P_C(1|k,\lambda), P_C(2|k,\lambda), \ldots, P_C(e_C|k,\lambda)\}$.

Clearly, for all $\lambda = 1, 2, \ldots, d$, it holds that

$$\min \sigma(A_i) \le \langle a^{(i)} \rangle_\lambda \le \max \sigma(A_i),$$
$$\min \sigma(B_j) \le \langle b^{(j)} \rangle_\lambda \le \max \sigma(B_j),$$
$$\min \sigma(C_k) \le \langle c^{(k)} \rangle_\lambda \le \max \sigma(C_k)$$

for all possible i, j, k.

This leads to the following proposition, which reveals an important property of Bell local states and serves the proofs of our results.

Proposition 4: Let ρ be a Bell local state of $\mathcal{H}_{ABC} := \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Then expectations of local observables $\{A_i \otimes B_j \otimes C_k\}_{ijk}$ w.r.t. ρ can be represented as convex combinations of expectations of local random variables with different distributions: $\forall i, j, k$,

$$\langle A_i \otimes B_j \otimes C_k \rangle_\rho = \sum_{\lambda=1}^{d} \pi_\lambda \langle a^{(i)} \rangle_\lambda \cdot \langle b^{(j)} \rangle_\lambda \cdot \langle c^{(k)} \rangle_\lambda. \quad (9)$$

A similar conclusion is also valid for n-partite system $\mathcal{H}^{(n)}$.

III. MAIN RESULTS

The main aim of this paper is to derive some necessary conditions for a multipartite state to be Bell local in terms of "measurement expectations" of local observables without invoking Bell inequalities and then obtain some sufficient conditions for a state to be Bell nonlocal. Our conclusions will be given according to different cases of n (the number of subsystems).

Theorem 5: Let $\rho \in \mathcal{BL}(\mathcal{H}_{ABC})$. Then for all families $\mathcal{A} = \{A_1, A_2\}$, $\mathcal{B} = \{B_1, B_2\}$ and $\mathcal{C} = \{C_1, C_2\}$ of ± 1-valued observables of A, B and C, respectively, it holds that

$$\langle A_1 \otimes B_2 \otimes C_2 \rangle_\rho = \langle A_2 \otimes B_1 \otimes C_2 \rangle_\rho = \langle A_2 \otimes B_2 \otimes C_1 \rangle_\rho = -1$$

implies that

$$\langle A_1 \otimes B_1 \otimes C_1 \rangle_\rho = -1. \quad (10)$$

Proof: For any families $\mathcal{A} = \{A_1, A_2\}$, $\mathcal{B} = \{B_1, B_2\}$ and $\mathcal{C} = \{C_1, C_2\}$ of ± 1-valued observables of A, B and C, respectively, we obtain the following projective MAs:

$$\mathcal{M}_A = \{M^1, M^2\}, \mathcal{N}_B = \{N^1, N^2\}, \quad \mathcal{L}_C = \{L^1, L^2\}$$

where

$$M^i = \{M_{+|i}^i, M_{-|i}^i\}, M_{+|i}^i - M_{-|i}^i = A_i,$$
$$N^j = \{N_{+|j}^j, N_{-|j}^j\}, N_{+|j}^j - N_{-|j}^j = B_j,$$
$$L^k = \{L_{+|k}^k, L_{-|k}^k\}, L_{+|k}^k - L_{-|k}^k = C_k.$$

Since $\rho \in \mathcal{BL}(\mathcal{H}_{ABC})$, by Definition 1, there is a PD $\{\pi_\lambda\}_{\lambda=1}^d$ s.t.

$$\mathrm{tr}[(M_{r|i} \otimes L_{s|j} \otimes N_{t|k})\rho] = \sum_{\lambda=1}^{d} \pi_\lambda P_A(r|i,\lambda) P_B(s|j,\lambda) P_C(t|k,\lambda) \quad (11)$$

for all $i, j, k = 1, 2$; $r, s, t \in \{+, -\}$, where

$$\{P_A(r|i,\lambda)\}_{r=1}^{o_A}, \{P_B(s|j,\lambda)\}_{s=1}^{o_B} \text{ and } \{P_C(t|k,\lambda)\}_{t=1}^{o_C}$$

are PDs. Clearly, we may assume that $\pi_\lambda > 0$ for all $\lambda = 1, 2, \ldots, d$. Thus, $\forall i, j, k = 1, 2$,

$$\langle A_i \otimes B_j \otimes C_k\rangle_\rho = \sum_{\lambda=1}^{d} \pi_\lambda \langle a^{(i)}\rangle_\lambda \cdot \langle b^{(j)}\rangle_\lambda \cdot \langle c^{(k)}\rangle_\lambda, \quad (12)$$

where

$$\langle a^{(i)}\rangle_\lambda = P_A(+|i,\lambda) - P_A(-|i,\lambda),$$
$$\langle b^{(j)}\rangle_\lambda = P_B(+|j,\lambda) - P_B(-|j,\lambda),$$
$$\langle c^{(k)}\rangle_\lambda = P_C(+|k,\lambda) - P_C(-|k,\lambda).$$

Let

$$\langle A_1B_2C_2\rangle_\rho = \langle A_2B_1C_2\rangle_\rho = \langle A_2B_2C_1\rangle_\rho = -1.$$

Then

$$\sum_{\lambda=1}^{d} \pi_\lambda \langle a^{(1)}\rangle_\lambda \cdot \langle b^{(2)}\rangle_\lambda \cdot \langle c^{(2)}\rangle_\lambda,$$
$$\sum_{\lambda=1}^{d} \pi_\lambda \langle a^{(2)}\rangle_\lambda \cdot \langle b^{(1)}\rangle_\lambda \cdot \langle c^{(2)}\rangle_\lambda,$$
$$\sum_{\lambda=1}^{d} \pi_\lambda \langle a^{(2)}\rangle_\lambda \cdot \langle b^{(2)}\rangle_\lambda \cdot \langle c^{(1)}\rangle_\lambda$$

are all equal to -1. Hence, for all $\lambda = 1, 2, \ldots, d$, we have

$$\begin{aligned}\langle a^{(1)}\rangle_\lambda \cdot \langle b^{(2)}\rangle_\lambda \cdot \langle c^{(2)}\rangle_\lambda &= \langle a^{(2)}\rangle_\lambda \cdot \langle b^{(1)}\rangle_\lambda \cdot \langle c^{(2)}\rangle_\lambda \\ &= \langle a^{(2)}\rangle_\lambda \cdot \langle b^{(2)}\rangle_\lambda \cdot \langle c^{(1)}\rangle_\lambda \\ &= -1.\end{aligned}$$

Finding product of the three quantities above yields that

$$\langle a^{(1)}\rangle_\lambda \langle b^{(1)}\rangle_\lambda \langle c^{(1)}\rangle_\lambda \left[\langle a^{(2)}\rangle_\lambda \cdot \langle b^{(2)}\rangle_\lambda \cdot \langle c^{(2)}\rangle_\lambda\right]^2 = -1$$

and so $\langle a^{(1)}\rangle_\lambda \langle b^{(1)}\rangle_\lambda \langle c^{(1)}\rangle_\lambda = -1$ for all $\lambda = 1, 2, \ldots, d$. It follows from Eq. (12) that

$$\langle A_1 \otimes B_1 \otimes C_1\rangle_\rho = \sum_{\lambda=1}^{d} \pi_\lambda \langle a^{(1)}\rangle_\lambda \cdot \langle b^{(1)}\rangle_\lambda \cdot \langle c^{(1)}\rangle_\lambda = -1.$$

The implication (10) is proved. The proof is completed.

Corollary 6: Let $\rho \in \mathcal{BL}(\mathcal{H}_{ABC})$. Then for all families $\mathcal{A} = \{A_1, A_2\}$, $\mathcal{B} = \{B_1, B_2\}$ and $\mathcal{C} = \{C_1, C_2\}$ of ± 1-valued observables of A, B and C, respectively, it holds that

$$\langle A_1 \otimes B_2 \otimes C_2\rangle_\rho = \langle A_2 \otimes B_1 \otimes C_2\rangle_\rho = \langle A_2 \otimes B_2 \otimes C_1\rangle_\rho = 1$$

implies

$$\langle A_1 \otimes B_1 \otimes C_1\rangle_\rho = 1. \quad (13)$$

Proof. When $\langle A_1 \otimes B_2 \otimes C_2\rangle_\rho = \langle A_2 \otimes B_1 \otimes C_2\rangle_\rho = \langle A_2 \otimes B_2 \otimes C_1\rangle_\rho = 1$, we have

$$\langle (-A_1) \otimes B_2 \otimes C_2\rangle_\rho = -1,$$
$$\langle A_2 \otimes (-B_1) \otimes C_2\rangle_\rho = -1,$$
$$\langle A_2 \otimes B_2 \otimes (-C_1)\rangle_\rho = -1$$

and Theorem 5 yields that

$$\langle (-A_1) \otimes (-B_1) \otimes (-C_1)\rangle_\rho = -1$$

and so $\langle A_1 \otimes B_1 \otimes C_1\rangle_\rho = 1$. The proof is completed.

Using Theorem 5 yields the following conclusion, which gives a method for detecting Bell nonlocality without invoking Bell inequalities.

Corollary 7: Let $\rho \in \mathcal{D}(\mathcal{H}_{ABC})$. If there are families $\mathcal{A} = \{A_1, A_2\}$, $\mathcal{B} = \{B_1, B_2\}$ and $\mathcal{C} = \{C_1, C_2\}$ of ± 1-valued observables of A, B and C, respectively, such that

$$\begin{cases}\langle A_1 \otimes B_2 \otimes C_2\rangle_\rho = -1;\\ \langle A_2 \otimes B_1 \otimes C_2\rangle_\rho = -1;\\ \langle A_2 \otimes B_2 \otimes C_1\rangle_\rho = -1;\\ \langle A_1 \otimes B_1 \otimes C_1\rangle_\rho = 1,\end{cases} \quad (14)$$

then $\rho \in \mathcal{BNL}(\mathcal{H}_{ABC})$.

Theoretically, this corollary gives a method for proving a state to be Bell nonlocal in the sense of Definition 1. Practically, if an experiment shows that $\langle A_1 \otimes B_2 \otimes C_2\rangle_\rho$, $\langle A_2 \otimes B_1 \otimes C_2\rangle_\rho$ and $\langle A_2 \otimes B_2 \otimes C_1\rangle_\rho$ are close to -1 while $\langle A_1 \otimes B_1 \otimes C_1\rangle_\rho$ is close to 1, then we may conclude that ρ is Bell nonlocal according to Definition 1.

Moreover, if we call the state ρ satisfying the condition (14) for some $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ "GHZ nonlocal", then Corollary 7 can be rewritten as "If a state ρ of a system ABC is GHZ nonlocal, then it must be Bell nonlocal." Thus, the QHZ nonlocality is stronger than the Bell nonlocality.

Example 8: For the three-qubit GHZ state $|G_{2\times 3}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, put

$$\rho = |G_{2\times 3}\rangle\langle G_{2\times 3}|, \mathcal{A} = \mathcal{B} = \mathcal{C} = \{\sigma^x, \sigma^y\},$$

that is, $A_1 = B_1 = C_1 = \sigma^x$ and $A_2 = B_2 = C_2 = \sigma^y$. Then $\langle A_1 \otimes B_1 \otimes C_1\rangle_\rho = 1$ and

$$\langle A_1 \otimes B_2 \otimes C_2\rangle_\rho = \langle A_2 \otimes B_1 \otimes C_2\rangle_\rho = \langle A_2 \otimes B_2 \otimes C_1\rangle_\rho = -1.$$

It follows from Corollary 7 that $\rho = |G_{2\times 3}\rangle\langle G_{2\times 3}|$ is Bell nonlocal and so is $|G_{2\times 3}\rangle$.

For the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + e^{i\theta}|111\rangle)(\theta \in \mathbb{R})$, choosing a unitary operator U such that $U|0\rangle = |0\rangle$ and $U|1\rangle = e^{-i\theta/3}|1\rangle$ yields that $(U \otimes U \otimes U)|\psi\rangle = |G_{2\times 3}\rangle$. Thus, $|\psi\rangle$ is Bell nonlocal. Generally, for a three-qubit state $|\varphi\rangle = \frac{1}{\sqrt{2}}(|abc\rangle + e^{i\theta}|a'b'c'\rangle)(\theta \in \mathbb{R})$, where $\{|a\rangle, |a'\rangle\}$, $\{|b\rangle, |b'\rangle\}$ and $\{|c\rangle, |c'\rangle\}$ are orthonormal bases for $\mathbb{C}^2$. By choosing three unitary operators U, V, W such that $U|a\rangle = V|b\rangle = W|c\rangle = |0\rangle$ and $U|a'\rangle = V|b'\rangle = W|c'\rangle = e^{-i\theta/3}|1\rangle$, we get $(U \otimes V \otimes W)|\varphi\rangle = |G_{2\times 3}\rangle$ and so $|\varphi\rangle$ is also Bell nonlocal. For example, the states

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|010\rangle + e^{i\theta}|101\rangle)(\theta \in \mathbb{R}),$$
$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|110\rangle + e^{i\theta}|001\rangle)(\theta \in \mathbb{R})$$

are both Bell nonlocal.

Similar to the proof of Theorem 5, one can prove the following theorem.

Theorem 9: Let $\rho \in \mathcal{BL}(\mathcal{H}^{(5)})$. Then for all families $\mathcal{A}_k = \{X_k, Y_k\}$ of ± 1-valued observables of $\mathcal{H}_k (k = 1, 2 \ldots, 5)$ with

$$\langle X_1 \otimes X_2 \otimes X_3 \otimes Y_4 \otimes Y_5 \rangle_\rho = -1, \tag{15}$$
$$\langle Y_1 \otimes X_2 \otimes X_3 \otimes X_4 \otimes Y_5 \rangle_\rho = -1, \tag{16}$$
$$\langle Y_1 \otimes Y_2 \otimes X_3 \otimes X_4 \otimes X_5 \rangle_\rho = -1, \tag{17}$$
$$\langle X_1 \otimes Y_2 \otimes Y_3 \otimes X_4 \otimes X_5 \rangle_\rho = -1, \tag{18}$$
$$\langle X_1 \otimes X_2 \otimes Y_3 \otimes Y_4 \otimes X_5 \rangle_\rho = -1, \tag{19}$$

it holds that $\langle X_1 \otimes X_2 \otimes X_3 \otimes X_4 \otimes X_5 \rangle_\rho = -1$.

When $\mathcal{H}^{(5)} = (\mathbb{C}^2)^{\otimes 5}$, by using Theorem 9 for $\{X_k, Y_k\} = \{\sigma^x, \sigma^y\}$ and $\rho = |G_{2\times 5}\rangle\langle G_{2\times 5}|$, we see that the GHZ state

$$|G_{2\times 5}\rangle = \frac{1}{\sqrt{2}}(|00000\rangle + |11111\rangle)$$

is Bell nonlocal since conditions (15)-(19) hold but $\langle X_1 \otimes X_2 \otimes X_3 \otimes X_4 \otimes X_5 \rangle_\rho = 1$.

Theorem 10: Let $d_k = \dim(\mathcal{H}_k) \geq 4 (k = 1, 2, \ldots, 5)$ and $\rho \in \mathcal{BL}(\mathcal{H}^{(5)})$. Then for all families $\{A_k, B_k\}$ of unitary operators on $\mathcal{H}_k$ with spectrums:

$$\sigma(A_k) = \sigma(B_k) = \{1, -1, i, -i\} \quad (k = 1, 2, \ldots, 5)$$

and satisfying

$$\langle A_1^3 \otimes B_2 \otimes B_3 \otimes B_4 \otimes B_5 \rangle_\rho = -1, \tag{20}$$
$$\langle B_1 \otimes A_2^3 \otimes B_3 \otimes B_4 \otimes B_5 \rangle_\rho = -1, \tag{21}$$
$$\langle B_1 \otimes B_2 \otimes A_3^3 \otimes B_4 \otimes B_5 \rangle_\rho = -1, \tag{22}$$
$$\langle B_1 \otimes B_2 \otimes B_3 \otimes A_4^3 \otimes B_5 \rangle_\rho = -1, \tag{23}$$
$$\langle B_1 \otimes B_2 \otimes B_3 \otimes B_4 \otimes A_5^3 \rangle_\rho = -1, \tag{24}$$

it holds that $\langle A_1 \otimes A_2 \otimes A_3 \otimes A_4 \otimes A_5 \rangle_\rho = -1$.

Proof: Let the families $\{A_k, B_k\} (k = 1, 2, \ldots, 5)$ satisfy the described conditions. Then A_k, B_k has the following spectral decompositions:

$$A_k = M_{1|1}^{(k)} + iM_{2|1}^{(k)} - M_{3|1}^{(k)} - iM_{4|1}^{(k)} = \sum_{a_k=1}^{4} \omega_4^{a_k-1} M_{a_k|1}^{(k)},$$
$$B_k = M_{1|2}^{(k)} + iM_{2|2}^{(k)} - M_{3|2}^{(k)} - iM_{4|2}^{(k)} = \sum_{a_k=1}^{4} \omega_4^{a_k-1} M_{a_k|2}^{(k)},$$

where $\omega_4 = e^{2\pi i/4}$, and

$$X^k := \left\{M_{1|1}^{(k)}, M_{2|1}^{(k)}, M_{3|1}^{(k)}, M_{4|1}^{(k)}\right\}$$
$$Y^k := \left\{M_{1|2}^{(k)}, M_{2|2}^{(k)}, M_{3|2}^{(k)}, M_{4|2}^{(k)}\right\}$$

form two projective measurements of the kth subsystem $\mathcal{H}_k$. Hence, we obtain a measurement assemblage $\mathcal{M}_k = \{X^k, Y^k\}$ of the kth subsystem $\mathcal{H}_k$. Since ρ is Bell local, there exists a PD $\{\pi_\lambda\}_{\lambda=1}^{d}$ such that

$$\mathrm{tr}[(M_{a_1|x_1}^{(1)} \otimes \cdots \otimes M_{a_5|x_5}^{(5)})\rho] = \sum_{\lambda=1}^{d} \pi_\lambda P_1(a_1|x_1, \lambda) \cdots P_5(a_5|x_5, \lambda) \tag{25}$$

for all $x_k = 1, 2$ and all $a_k = 1, 2$.

Clearly, we may assume that $\pi_\lambda > 0$ for all $\lambda \in [d]$. The LHV model (25) implies that

$$\begin{aligned}
&\langle A_1^3 \otimes B_2 \otimes B_3 \otimes B_4 \otimes B_5 \rangle_\rho \\
&= \sum_{a_1, a_2, \ldots, a_5=1}^{4} \omega_4^{3(a_1-1)+a_2+a_3+a_4+a_5-4} \cdot \\
&\quad \mathrm{tr}[(M_{a_1|1}^{(1)} \otimes M_{a_2|2}^{(2)} \otimes \cdots \otimes M_{a_5|2}^{(5)})\rho] \\
&= \sum_{\lambda=1}^{d} \pi_\lambda \sum_{a_1, a_2, \ldots, a_5=1}^{4} \omega_4^{3(a_1-1)+a_2+a_3+a_4+a_5-4} \cdot \\
&\quad P_1(a_1|1, \lambda) P_2(a_2|2, \lambda) \cdots P_5(a_5|2, \lambda) \\
&= \sum_{\lambda=1}^{d} \pi_\lambda \langle A_1^3 \rangle_\lambda \langle B_2 \rangle_\lambda \langle B_3 \rangle_\lambda \langle B_4 \rangle_\lambda \langle B_5 \rangle_\lambda,
\end{aligned}$$

where

$$\langle A_1^3 \rangle_\lambda = \sum_{a_1=1}^{4} \omega_4^{3(a_1-1)} P_1(a_1|1, \lambda),$$
$$\langle B_k \rangle_\lambda = \sum_{a_k=1}^{4} \omega_4^{a_k-1} P_k(a_k|2, \lambda) (k = 2, 3, 4, 5).$$

This shows that

$$\langle A_1^3 \otimes A_2 \otimes A_3 \otimes A_4 \otimes A_5 \rangle_\rho = \sum_{\lambda=1}^{d} \pi_\lambda \langle A_1^3 \rangle_\lambda \langle A_2 \rangle_\lambda \langle A_3 \rangle_\lambda \langle A_4 \rangle_\lambda \langle A_5 \rangle_\lambda. \tag{26}$$

Similarly,

$$\langle A_1 \otimes A_2 \otimes A_3 \otimes A_4 \otimes A_5 \rangle_\rho = \sum_{\lambda=1}^{d} \pi_\lambda \langle A_1 \rangle_\lambda \langle A_2 \rangle_\lambda \langle A_3 \rangle_\lambda \langle A_4 \rangle_\lambda \langle A_5 \rangle_\lambda, \tag{27}$$

where

$$\langle A_k \rangle_\lambda = \sum_{a_k=1}^{4} \omega_4^{a_k-1} P_k(a_k|1, \lambda) (k = 1, 2, \ldots, 5). \tag{28}$$

Note that $|\langle A_1^3 \rangle_\lambda \langle B_2 \rangle_\lambda \langle B_3 \rangle_\lambda \langle B_4 \rangle_\lambda \langle B_5 \rangle_\lambda| \leq 1$ for all $\lambda \in [d]$ and -1 is an extreme point of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$, we see from Eqs. (20) and (26) that

$$\langle A_1^3 \rangle_\lambda \langle B_2 \rangle_\lambda \langle B_3 \rangle_\lambda \langle B_4 \rangle_\lambda \langle B_5 \rangle_\lambda = -1, \quad \forall \lambda \in [d]. \tag{29}$$

z Similarly, one can prove that $\forall \lambda \in [d]$,

$$\langle B_1 \rangle_\lambda \langle A_2^3 \rangle_\lambda \langle B_3 \rangle_\lambda \langle B_4 \rangle_\lambda \langle B_5 \rangle_\lambda = -1, \tag{30}$$
$$\langle B_1 \rangle_\lambda \langle B_2 \rangle_\lambda \langle A_3^3 \rangle_\lambda \langle B_4 \rangle_\lambda \langle B_5 \rangle_\lambda = -1, \tag{31}$$
$$\langle B_1 \rangle_\lambda \langle B_2 \rangle_\lambda \langle B_3 \rangle_\lambda \langle A_4^3 \rangle_\lambda \langle B_5 \rangle_\lambda = -1, \tag{32}$$
$$\langle B_1 \rangle_\lambda \langle B_2 \rangle_\lambda \langle B_3 \rangle_\lambda \langle B_4 \rangle_\lambda \langle A_5^3 \rangle_\lambda = -1. \tag{33}$$

By finding the product of the left-hand sides of Eqs. (29-33), we obtain that

$$\langle A_1^3 \rangle_\lambda \langle A_2^3 \rangle_\lambda \langle A_3^3 \rangle_\lambda \langle A_4^3 \rangle_\lambda \langle A_5^3 \rangle_\lambda \cdot \prod_{k=1}^{4} (\langle B_k \rangle_\lambda)^4 = -1. \tag{34}$$

Hence, $|\langle A_k^3 \rangle_\lambda| = |\langle B_k \rangle_\lambda| = 1 (k = 1, 2, \ldots)$ and so $\langle A_k^3 \rangle_\lambda$ as well as $\langle B_k \rangle_\lambda$ are extreme points of the unit disk $\mathbb{D}$. Note that both $\langle A_k^3\rangle_\lambda$ and $\langle B_k\rangle_\lambda$ are convex combinations of points $1, \omega_4, \omega_4^2$ and ω_4^4, we conclude that both $\langle A_k^3\rangle_\lambda$ and $\langle B_k\rangle_\lambda$ are elements of $1, \omega_4, \omega_4^2$ and ω_4^4, i.e.,

$$P_k(a_k'|1,\lambda) = 1, P_k(a_k|1,\lambda) = 0(a_k \neq a_k'),$$
$$P_k(a_k''|2,\lambda) = 1, P_k(a_k|2,\lambda) = 0(a_k \neq a_k'')$$

for some a_k' and a_k''. Thus, when $k = 1, 2, \ldots, 5$,

$$\langle A_k^3\rangle_\lambda = \omega_4^{3(a_k'-1)} = (\langle A_k\rangle_\lambda)^3, \langle B_k\rangle_\lambda = \omega_4^{a_k''-1}.$$

It follows from (34) that we have

$$(\langle A_1\rangle_\lambda\langle A_2\rangle_\lambda\langle A_3\rangle_\lambda\langle A_4\rangle_\lambda\langle A_5\rangle_\lambda)^3 = -1. \tag{35}$$

Since $\langle A_k\rangle_\lambda \in \{1, \omega_4, \omega_4^2, \omega_4^3\}$, we have

$$\langle A_1\rangle_\lambda\langle A_2\rangle_\lambda\langle A_3\rangle_\lambda\langle A_4\rangle_\lambda\langle A_5\rangle_\lambda \in \{1, \omega_4, \omega_4^2, \omega_4^3\}.$$

Consequently, we see from (35) that

$$\langle A_1\rangle_\lambda\langle A_2\rangle_\lambda\langle A_3\rangle_\lambda\langle A_4\rangle_\lambda\langle A_5\rangle_\lambda = \omega_4^2 = -1.$$

The proof is completed.

Corollary 11: Let $d_k = \dim(\mathcal{H}_k) \geq 4(k = 1, 2, \ldots, 5)$ *and* $\rho \in \mathcal{BL}(\mathcal{H}^{(5)})$*. Then for all families* $\{A_k, B_k\}$ *of unitary operators on* $\mathcal{H}_k$ *with the spectrums:*

$$\sigma(A_k) = \sigma(B_k) = \{1, -1, i, -i\}(k = 1, 2, \ldots, 5)$$

and satisfying

$$\langle A_1^3 \otimes B_2 \otimes B_3 \otimes B_4 \otimes B_5\rangle_\rho = 1, \tag{36}$$
$$\langle B_1 \otimes A_2^3 \otimes B_3 \otimes B_4 \otimes B_5\rangle_\rho = 1, \tag{37}$$
$$\langle B_1 \otimes B_2 \otimes A_3^3 \otimes B_4 \otimes B_5\rangle_\rho = 1, \tag{38}$$
$$\langle B_1 \otimes B_2 \otimes B_3 \otimes A_4^3 \otimes B_5\rangle_\rho = 1, \tag{39}$$
$$\langle B_1 \otimes B_2 \otimes B_3 \otimes B_4 \otimes A_5^3\rangle_\rho = 1, \tag{40}$$

it holds that $\langle A_1 \otimes A_2 \otimes A_3 \otimes A_4 \otimes A_5\rangle_\rho = 1$.

Proof: The proof is completed by using Theorem 10 for the families $\{-A_k, B_k\}(k = 1, 2, \ldots, 5)$.

Corollary 12: For a state $\rho \in \mathcal{D}(\mathcal{H}^{(5)})$*, if there exist families* $\{A_k, B_k\}$ *of unitary operators on* $\mathcal{H}_k$ *with the spectrums:*

$$\sigma(A_k) = \sigma(B_k) = \{1, -1, i, -i\} \quad (k = 1, 2, \ldots, 5)$$

and satisfying the conditions (36)-(40) and

$$\langle A_1 \otimes A_2 \otimes A_3 \otimes A_4 \otimes A_5\rangle_\rho = -1,$$

then $\rho \in \mathcal{BNL}(\mathcal{H}^{(5)})$.

To use Corollary 12 and check the Bell nonlocality of the generalized GHZ state

$$|G_{4\times 5}\rangle = \frac{1}{2}\sum_{k=0}^{3}|kkkkk\rangle, \tag{41}$$

let us introduce generalizations [46] of the Pauli matrices

$$\sigma^x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma^y = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
$$\sigma^z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as follows. Let $\mathcal{H}$ be a d-dimensional Hilbert space with an ONB $\{|e_k\rangle\}_{k=0}^{d-1}$. Define

$$\sigma_d^x = X = \sum_{k=0}^{d-1}|e_{k\oplus 1}\rangle\langle e_k|, \quad k \oplus 1 = k + 1(mod\ d), \tag{42}$$

$$\sigma_d^y = Y = e^{p\pi i/d}\sum_{k=0}^{d-1}(\omega_d)^k|e_{k\ominus 1}\rangle\langle e_k|, \quad k \ominus 1 = k - 1(mod\ d), \tag{43}$$

$$\sigma_d^z = Z = \sum_{k=0}^{d-1}(\omega_d)^k|e_k\rangle\langle e_k|, \tag{44}$$

where $\omega_d = e^{2\pi i/d}$ denotes the principal dth-root of unity, $p = 0$ for d odd and $p = 1$ for d even.

It is easy to check that X, Y and Z are all unitary operators on $\mathcal{H}$ with the same spectrum

$$\sigma(X) = \sigma(Y) = \sigma(Z) = \{1, \omega_d, (\omega_d)^2, \ldots, (\omega_d)^{d-1}\}.$$

We call X, Y and Z the generalized Pauli operators (GPOs) of order d. Under the basis $\{|e_k\rangle\}_{k=0}^{d-1}$ used in the definition of X, Y and Z, GPOs have their matrix representations. For example, when $d = 2$, we have

$$\sigma_2^x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^x,$$
$$\sigma_2^y = Y = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sigma^y,$$
$$\sigma_2^z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^z.$$

When $d = 3$, we have

$$\sigma_3^x = X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\sigma_3^y = Y = \begin{pmatrix} 0 & \omega_3 & 0 \\ 0 & 0 & \omega_3^2 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\sigma_3^z = Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_3 & 0 \\ 0 & 0 & \omega_3^2 \end{pmatrix},$$

When $H = \mathbb{C}^4$, $d = 4$, we have

$$\sigma_4^x = X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$
$$\sigma_4^y = Y = e^{\pi i/4}\begin{pmatrix} 0 & \omega_4 & 0 & 0 \\ 0 & 0 & \omega_4^2 & 0 \\ 0 & 0 & 0 & \omega_4^3 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$
$$\sigma_4^z = Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega_4 & 0 & 0 \\ 0 & 0 & \omega_4^2 & 0 \\ 0 & 0 & 0 & \omega_4^3 \end{pmatrix}.$$

Moreover, it is easy to check that the generalized Pauli operators X, Y and Z have the following properties:
(1) $XY = e^{2\pi pi/d}Z$;
(2) $X^d = Y^d = Z^d = I_d$;
(3) $\forall a, b \in \mathbb{Z}$,

$$Y^bX^a = e^{2\pi abi/d}X^aY^b, Z^bX^a = e^{2\pi abi/d}X^aZ^b.$$

In the case that $H = \mathbb{C}^2$ and the ONB $\{|e_0\rangle, |e_1\rangle\}$ takes as the usual basis $\{|0\rangle, |1\rangle\}$ for $\mathbb{C}^2$, the generalized Pauli operators X, Y, Z reduce to $\sigma^x, i\sigma^y, -\sigma^z$.

When $H = \mathbb{C}^4$, $d = 4$, X, Y and Z have the same spectrum $\{1, i, -1, -i\}$ and lead to the following six local unitary operators on $(\mathbb{C}^4)^{\otimes 5}$:

$$\begin{cases} V_0 = X_1 \otimes X_2 \otimes X_3 \otimes X_4 \otimes X_5; \\ V_1 = (X_1)^3 \otimes Y_2 \otimes Y_3 \otimes Y_4 \otimes Y_5; \\ V_2 = Y_1 \otimes (X_2)^3 \otimes Y_3 \otimes Y_4 \otimes Y_5; \\ V_3 = Y_1 \otimes Y_2 \otimes (X_3)^3 \otimes Y_4 \otimes Y_5; \\ V_4 = Y_1 \otimes Y_2 \otimes Y_3 \otimes (X_4)^3 \otimes Y_5; \\ V_5 = Y_1 \otimes Y_2 \otimes Y_3 \otimes Y_4 \otimes (X_5)^3, \end{cases} \tag{45}$$

where $X_k = X, Y_k = Y$, acting on the kth subsystem. We call these operators the 4×5-GHZ operators.

By using Corollary 12 for $A_k = \sigma_4^x, B_k = \sigma_4^y (k = 1, 2, \ldots, 5)$, we see that if a state ρ of $(\mathbb{C}^4)^{\otimes 5}$ satisfies $\langle V_k \rangle_\rho = -1 (k = 1, 2, \ldots, 5)$ but $\langle V_0 \rangle_\rho \neq -1$, then it is Bell nonlocal. Especially, when the ONB $\{|e_k\rangle\}_{k=0}^3$ takes as the usual basis $\{|k\rangle\}_{k=0}^3$ for $\mathbb{C}^4$, we can prove the Bell nonlocality of $|G_{4\times5}\rangle$ given by (41). In this case, it is easy to check that $|G_{4\times5}\rangle$ is a common eigenstate of the above operators $V_0, V_1, \ldots, V_5$ with eigenvalues $v_0 = 1, v_1 = \ldots = v_5 = -1$, respectively. Thus, $\langle V_k \rangle_\rho = -1 (k = 1, 2, \ldots, 5)$ but $\langle V_0 \rangle_\rho = 1$. It follows from Corollary 12 that $|G_{4\times5}\rangle$ is Bell nonlocal.

Similar to the proof of Theorem 10, one can deduce the following theorem.

Theorem 13: Let $d = \dim(\mathcal{H}_k)$ be an even integer $\geq 2 (k \in [d+1])$ and $\rho \in \mathcal{BL}(\mathcal{H}^{(d+1)})$. Then for all families $\{A_k, B_k\}$ of unitary operators on $\mathcal{H}_k$ with the spectrums:

$$\sigma(A_k) = \sigma(B_k) = \{1, \omega_d, (\omega_d)^2, \ldots, (\omega_d)^{d-1}\},$$

where $k \in [d+1]$, satisfying

$$\langle A_1^{d-1} \otimes B_2 \otimes B_3 \otimes \cdots \otimes B_d \otimes B_{d+1} \rangle_\rho = -1, \tag{46}$$
$$\langle B_1 \otimes A_2^{d-1} \otimes B_3 \otimes \cdots \otimes B_d \otimes B_{d+1} \rangle_\rho = -1, \tag{47}$$
$$\langle B_1 \otimes B_2 \otimes A_3^{d-1} \otimes \cdots \otimes B_d \otimes B_{d+1} \rangle_\rho = -1, \tag{48}$$
$$\cdots\cdots, \tag{49}$$
$$\langle B_1 \otimes B_2 \otimes B_3 \otimes \cdots \otimes A_d^{d-1} \otimes B_{d+1} \rangle_\rho = -1, \tag{50}$$
$$\langle B_1 \otimes B_2 \otimes B_3 \otimes \cdots B_d \otimes A_{d+1}^{d-1} \rangle_\rho = -1, \tag{51}$$

it holds that $\langle A_1 \otimes A_2 \otimes A_3 \cdots \otimes A_{d+1} \rangle_\rho = -1$.

Corollary 14: When $d = \dim(\mathcal{H}_k)$ is an even integer $\geq 2 (k \in [d+1])$ and $\rho \in \mathcal{D}(\mathcal{H}^{(d+1)})$, if there exist families $\{A_k, B_k\}$ of unitary operators on $\mathcal{H}_k$ with the spectrums:

$$\sigma(A_k) = \sigma(B_k) = \{1, \omega_d, (\omega_d)^2, \ldots, (\omega_d)^{d-1}\}$$

for all $k \in [d+1]$ and satisfying conditions (46)-(51) and

$$\langle A_1 \otimes A_2 \otimes A_3 \cdots \otimes A_{d+1} \rangle_\rho \neq -1,$$

then $\rho \in \mathcal{BNL}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{d+1})$.

The 4×5-GHZ operators (45) have the following general forms given in [46]:

$$\begin{cases} V_0 = X_1 \otimes X_2 \otimes X_3 \otimes X_4 \otimes X_5 \otimes \cdots \otimes X_d \otimes X_{d+1}; \\ V_1 = (X_1)^{d-1} \otimes Y_2 \otimes Y_3 \otimes Y_4 \otimes Y_5 \otimes \cdots \otimes Y_d \otimes Y_{d+1}; \\ V_2 = Y_1 \otimes (X_2)^{d-1} \otimes Y_3 \otimes Y_4 \otimes Y_5 \otimes \cdots \otimes Y_d \otimes Y_{d+1}; \\ V_3 = Y_1 \otimes Y_2 \otimes (X_3)^{d-1} \otimes Y_4 \otimes Y_5 \otimes \cdots \otimes Y_d \otimes Y_{d+1}; \\ \cdots\cdots; \\ V_d = Y_1 \otimes Y_2 \otimes Y_3 \otimes Y_4 \otimes Y_5 \otimes \cdots \otimes (X_d)^{d-1} \otimes Y_{d+1}; \\ V_{d+1} = Y_1 \otimes Y_2 \otimes Y_3 \otimes Y_4 \otimes Y_5 \otimes \cdots \otimes Y_d \otimes (X_{d+1})^{d-1}, \end{cases} \tag{52}$$

where d is an even positive integer ≥ 2 and $X_k = \sigma_d^x$, $Y_k = \sigma_d^y$. When we take the ONB $\{|e_k\rangle\}_{k=0}^{d-1}$ as the usual basis $\{|k\rangle\}_{k=0}^{d-1}$ for $\mathbb{C}^d$, it is easy to check that the generalized GHZ state

$$|G_{d\times(d+1)}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k\rangle^{\otimes(d+1)}$$

is a common eigenstate of the operators $V_0, V_1, \ldots, V_{d+1}$ with corresponding eigenvalues $v_0 = 1, v_1 = \ldots = v_{d+1} = -1$, respectively. By using Corollary 14, for $A_k = \sigma_d^x, B_k = \sigma_d^y (k = 1, 2, \ldots, d+1)$, we see that if a state ρ of $(\mathbb{C}^d)^{\otimes(d+1)}$ satisfies $\langle V_k \rangle_\rho = -1 (k \in [d+1])$ but $\langle V_0 \rangle_\rho \neq -1$, then it is Bell nonlocal. Especially, $|G_{d\times(d+1)}\rangle$ is Bell nonlocal.

Theorem 15: Let $d = \dim(\mathcal{H}_k)$ be an even integer $\geq 2 (k \in [d+2])$ and $\rho \in \mathcal{BL}(\mathcal{H}^{(d+2)})$. Then for all families $\{A_k, B_k\}$ of unitary operators on $\mathcal{H}_k$ with the spectrums:

$$\sigma(A_k) = \sigma(B_k) = \{1, \omega_d, (\omega_d)^2, \ldots, (\omega_d)^{d-1}\}$$

for all $k \in [d+2]$ satisfying the following $d+3$ equations

$$\langle A_1^{d-1} \otimes B_2 \otimes B_3 \otimes \cdots \otimes B_{d+1} \otimes B_{d+2} \rangle_\rho = -1, \tag{53}$$
$$\langle B_1 \otimes A_2^{d-1} \otimes B_3 \otimes \cdots \otimes B_{d+1} \otimes B_{d+2} \rangle_\rho = -1, \tag{54}$$
$$\langle B_1 \otimes B_2 \otimes A_3^{d-1} \otimes \cdots \otimes B_{d+1} \otimes B_{d+2} \rangle_\rho = -1, \tag{55}$$
$$\cdots\cdots, \tag{56}$$
$$\langle B_1 \otimes B_2 \otimes B_3 \otimes \cdots \otimes A_{d+1}^{d-1} \otimes B_{d+2} \rangle_\rho = -1, \tag{57}$$
$$\langle B_1 \otimes B_2 \otimes B_3 \otimes \cdots \otimes B_d \otimes B_{d+1} \otimes A_{d+2}^{d-1} \rangle_\rho = -1, \tag{58}$$
$$\langle B_1^{d-1} \otimes A_2 \otimes A_3 \otimes \cdots \otimes A_{d+1} \otimes A_{d+2} \rangle_\rho = -1, \tag{59}$$

it holds that $\langle A_1 \otimes B_2 \otimes B_3 \otimes \cdots \otimes B_{d+2} \rangle_\rho = -1$.

Indeed, when the dimension d is a small even number, to find different GHZ operators is possible. For example, we have the following two results.

Theorem 16: Let $\rho \in \mathcal{BL}(\mathcal{H}^{(4)})$. Then for all families $\mathcal{A}_k = \{X_k, Y_k\}$ of ± 1-valued observables of $\mathcal{H}_k (k = 1, 2 \ldots, 4)$, with

$$\langle X_1 \otimes Y_2 \otimes Y_3 \otimes X_4 \rangle_\rho = -1, \tag{60}$$
$$\langle X_1 \otimes Y_2 \otimes X_3 \otimes Y_4 \rangle_\rho = -1, \tag{61}$$
$$\langle X_1 \otimes X_2 \otimes Y_3 \otimes Y_4 \rangle_\rho = -1, \tag{62}$$

it holds that $\langle X_1 \otimes X_2 \otimes X_3 \otimes X_4 \rangle_\rho = -1$.

When $\mathcal{H}^{(4)} = (\mathbb{C}^2)^{\otimes 4}$, by using Theorem 16 for $\{X_k, Y_k\} = \{\sigma^x, \sigma^y\}$ and $\rho = |G_{2\times 4}\rangle\langle G_{2\times 4}|$, we see that the GHZ state

$$|G_{2\times 4}\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$$

is Bell nonlocal since conditions (60)-(62) hold but $\langle X_1 \otimes Y_2 \otimes Y_3 \otimes X_4\rangle_\rho = 1$.

IV. CONCLUSION

Usually, Bell nonlocality is detected by violation of some Bell's inequalities, such as CHSH inequality for two qubits. It can be revealed by a method without invoking Bell inequalities, e.g. by constructing a Hardy paradox or a GHZ paradox to prove Bell locality of the constructed states.

Hardy Paradox says essentially that there exists a state ρ of system AB and ± 1-valued observables $A_k, B_k (k = 1, 2)$ of systems A and B, respectively, with $t_1 < t_2$ satisfying

$$P_\rho(A_1 < B_1) = P_\rho(B_1 < A_2) = P_\rho(A_2 < B_2) = 0,$$
$$P_\rho(A_1 < B_2) \neq 0 \tag{63}$$

Contrast to the existing all-versus-nothing test of nonlocality, Dong *et al.* obtained [19] some methods for detecting Bell nonlocality based on the Hardy Paradox and proved mathematically if there are $\{t_1, t_2\}$-valued observables $A_k, B_k (k = 1, 2)$ of systems A and B, respectively, with $t_1 < t_2$ such that Eq. (63) holds, then the state ρ must be Bell nonlocal according to the definition of Bell nonlocality proposed by Cao and Guo in [18].

GHZ paradox says indeed that there exists a tripartite state ρ and some families $\mathcal{A} = \{A_1, A_2\}$, $\mathcal{B} = \{B_1, B_2\}$ and $\mathcal{C} = \{C_1, C_2\}$ of ± 1-valued observables of A, B and C, respectively, such that

$$\langle A_1 \otimes B_2 \otimes C_2\rangle_\rho = \langle A_2 \otimes B_1 \otimes C_2\rangle_\rho = \langle A_2 \otimes B_2 \otimes C_1\rangle_\rho = 1,$$
$$\langle A_1 \otimes B_1 \otimes C_1\rangle_\rho = -1. \tag{64}$$

Distinguishing with the existing discussions about the GHZ paradox, but motivated by them, we have established some methods for detecting Bell nonlocality in a multipartite system without invoking Bell inequalities. All of our results have been given as theorems and corollaries in terms of "expectations" of local observables without invoking Bell inequalities. For instance, based on the mathematical definition of Bell locality, we have proved logically that when Eq. (64) is satisfied by a tripartite state ρ and some families $\mathcal{A} = \{A_1, A_2\}$, $\mathcal{B} = \{B_1, B_2\}$ and $\mathcal{C} = \{C_1, C_2\}$ of ± 1-valued observables of A, B and C, respectively, the state ρ must be Bell nonlocal.

This may lead to a new idea for proving Bell nonlocality without using Bell inequalities.

CONFLICT OF INTEREST STATEMENT

The author declares that the research was conducted in the absence of every commercial or financial relationships that could be construed as a potential conflict of interest.

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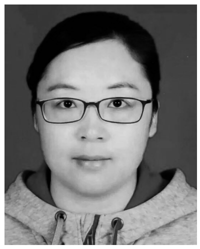

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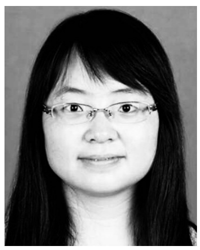

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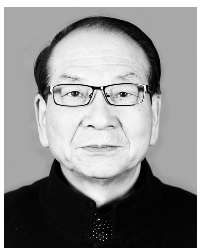

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