IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received February 2, 2021, accepted February 27, 2021, date of publication March 4, 2021, date of current version March 11, 2021.

Digital Object Identifier 10.1109/ACCESS.2021.3063746

S-Asymptotically ω-Periodic Solutions of Fractional-Order Complex-Valued Recurrent Neural Networks With Delays

YUANYUAN HOU[1] AND LIHUA DAI[1,2]
[1]Department of Mathematics and Statistics, Pu'er University, Pu'er 665000, China
[2]School of Mathematics and Statistics, Southwest University, Chongqing 400715, China

Corresponding author: Lihua Dai (hlm2136816@163.com)

This work was supported by the Science Research Fund of Education Department of Yunnan Province of China under Grant 2018JS517.

ABSTRACT In this paper, we consider the problem of the S-asymptotically ω-periodic synchronization of fractional-order complex-valued recurrent neural networks with time delays. Firstly, we can not explicitly decompose the fractional-order complex-valued systems into equivalent fractional-order real-valued systems, by means of the contraction mapping principle and some important features of Mittag-Leffler functions, we obtain some sufficient conditions for the existence and uniqueness of S-asymptotically ω-periodic solutions for this class of neural networks. Then, by constructing an appropriate Lyapunov functional, the theory of fractional differential equation, and some inequality techniques, sufficient conditions are obtained to guarantee the global Mittag-Leffler synchronization of the drive-response systems. Finally, two examples are given to illustrate the effectiveness and feasibility of our main results.

INDEX TERMS Fractional-order derivative, S-asymptotically ω-periodic solution, Mittag-Leffler function, complex-valued neural network, synchronization.

I. INTRODUCTION

In the past few years, the dynamical behaviors of complex-valued neural networks (CVNNs) have been extensively studied and analyzed, their application has been extended to optoelectronics, image, remote sensing, quantum neuron devices and systems, spatiotemporal analysis of the physiological nervous system, and artificial neural information processing [1], [2]. Complex-valued neural networks are not only the simple extension of real-valued neural networks, but also are quite different from real-valued neural networks and have more complicated properties than real-valued neural networks. Hence, it is very important to study the dynamical properties of CVNNs, such as the existence and stability of the equilibrium, periodic solutions, almost periodic solutions, which have been studied by many scholars [3]–[11]. For example, authors of [8] considered the robust passivity and stability analysis of uncertain complex-valued impulsive neural network. Since the concept of the drive-response synchronization for coupled chaotic systems was proposed in Pecora and Carrol [12], chaos synchronization has become a hot topic and much attention ([13]–[18]) has been paid to control and chaos synchronization because of its potential applications in secure communication, automatic control, biological systems, information science. Recently, some scholars have studied the synchronization of the complex-valued neural networks [19]–[22].

Note that Caputo fractional derivative can more accurately depict the memory and hereditary characteristics of various materials and processes, which has been successfully applied in various fields, such as electromagnetic waves, dielectric polarization, mathematical biology and neural networks [23]–[31]. In recent years, some outstanding results about dynamical analysis and synchronization control for fractional-order complex-valued neural networks have been reported [32]–[38]. For example, authors of [33] considered the problem of synchronization of fractional-order complex-valued neural networks with time delays via a decomposing method; in [38], authors studied the quasi-projective synchronization of fractional-order complex-valued recurrent neural networks by a non-decomposition method.

The associate editor coordinating the review of this manuscript and approving it for publication was Tu Ngoc Nguyen.

Besides, the theory of S-asymptotically ω-functions with values in Banach spaces was initiated in [39]. Later, some researchers have considered the existence of S-asymptotically ω solutions of fractional-order differential equations, and fractional-order integro-differential equations [40]. Since the Caputo fractional-order derivative, it is shown that ω-periodic or autonomous fractional-order neural networks cannot generate exactly ω-periodic signals. Thus, during recent years, many researchers have paid increasing attention to deal with the S-asymptotically ω-periodic solution of fractional-order non-autonomous neural networks [41]–[43]. Compared with the previous results, rare results are available for S-asymptotically ω-periodic solutions of fractional-order complex-valued neural networks.

However, up to now, there is no result about the S-asymptotically ω-periodic synchronization of the fractional-order complex-valued recurrent neural networks with time delays, which is still an open challenge. Therefore, it is necessary to study the synchronization of fractional-order complex-valued recurrent neural networks.

With the inspiration from the previous research, in order to fill the gap in the research field of fractional-order complex-valued recurrent neural networks, the work of this article comes from two main motivations. (1) In practical applications, ω-periodic motion is an interesting and significant dynamical property for fractional-order differential equations. However, some results show that ω-periodic or autonomous fractional-order neural networks cannot generate exactly ω-periodic signals. So, in past decade, many authors studied S-asymptotically ω-periodic oscillations of fractional-order non-autonomous neural networks [41]–[43]. (2) Recently, many literatures [33]–[38] had studied the fractional-order complex-valued neural networks. It is noteworthy that the scholars have not begun to consider the S-asymptotically ω-periodic oscillation for fractional-order complex-valued neural networks, thus it is worth studying S-asymptotically ω-periodic motion of fractional-order complex-valued neural network models via a non-decomposition
method.

Compared with the previous literatures, the main contributions of this paper are listed as follows.

(1) Firstly, to the best of our knowledge, this is the first time to study the S-asymptotically ω-periodic synchronization for complex-valued neural networks.

(2) Secondly, in this paper, without separating the complex-valued neural networks into two real-valued systems. Therefore, the results are less conservative and more general [33]–[35], and we improve the norm.

(3) Thirdly, our method of this paper can be used to study the S-asymptotically ω-periodic synchronization for other types of fractional-order complex-valued neural networks.

(4) Finally, examples and numerical simulations are given to verify the effectiveness of the conclusion.

A. NOTATIONS

Throughout this paper, for fractional-order derivative, we always choose the Caputo fractional derivative operator ${}_0^C D_t^\alpha$. Let $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{C}$ and $\mathbb{C}^n$ denote the set of real numbers, the n-dimensional real vector space, the set of complex numbers, and the n-dimensional complex vector space, respectively. Let $\bar{x} = x^R - ix^I$ be the conjugate of $x \in \mathbb{C}$. For a complex number $x = x^R + ix^I$, $i = \sqrt{-1}$ is the imaginary unit, x^R and x^I are the real and imaginary parts of x, respectively. For every $x \in \mathbb{C}$, the norm of x is defined as $\|x\|_{\mathbb{C}} = \sqrt{x\bar{x}} = \sqrt{(x^R)^2 + (x^I)^2}$, and for $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n$, we define $\|x\|_0 = \max_{l \in \mathcal{J}}\{\|x_l\|_{\mathbb{C}}\}$. $C([0, +\infty), \mathbb{C}^n)$ is a set composed of all continuous functions from $[0, +\infty)$ into $\mathbb{C}^n$. The notation $SAP_\omega(\mathbb{C}^n)$ stands for the subspace of $C([0, +\infty), \mathbb{C}^n)$ consisting of the S-asymptotically ω-periodic functions.

B. MODEL DESCRIPTION

Motivated by the above statement, in this paper, we consider the following fractional-order complex-valued recurrent neural network with time-varying delays:

$$
\begin{aligned}
{}_0^C D_t^\alpha x_l(t) = &-d_l(t)x_l(t) + \sum_{j=1}^{n} a_{lj}(t)f_j\big(x_j(t)\big) \\
&+ \sum_{j=1}^{n} b_{lj}(t)g_j\big(x_j(t - \tau_{lj}(t))\big) + I_l(t), \qquad (1)
\end{aligned}
$$

where $0 < \alpha < 1$, $t \geq 0$, $l \in \{1, 2, \ldots, n\} =: \mathcal{J}$, n is the number of neurons in layers; $x_l(t) \in \mathbb{C}$ and $I_l(t) \in \mathbb{R}$ are the state and bias of the l-th neuron at time t, respectively; $d_l(t) > 0$ is the self-feedback connection weight; $a_{lj}(t), b_{lj}(t) \in \mathbb{C}$ are the connection weight and the delay connection weight of neural network at time t, respectively; $\tau_{lj}(t)$ is the transmission delays; $f_j, g_j : \mathbb{C} \to \mathbb{C}$ are the activation functions.

For the convenience, we will adopt the following notation:

$$
\begin{aligned}
a_{lj}^+ &= \sup_{t \in \mathbb{R}} \|a_{lj}(t)\|_{\mathbb{C}}, \quad b_{lj}^+ = \sup_{t \in \mathbb{R}} \|b_{lj}(t)\|_{\mathbb{C}}, \\
\tau &= \max_{l,j \in \mathcal{J}} \big\{ \sup_{t \in \mathbb{R}} \tau_{lj}(t) \big\}.
\end{aligned}
$$

The initial conditions of the system (1) are of the form

$$
x_l(s) = \varphi_l(s), \quad s \in [-\tau, 0], \quad l \in \mathcal{J},
$$

where $\varphi_l \in C([-\tau, 0], \mathbb{C})$.

This paper is organized as follows: In Section 2, we introduce some definitions and preliminary lemmas. In Section 3, we establish some sufficient conditions for the existence of S-asymptotically ω-periodic solutions of (1). In Section 4, the global Mittag-Leffler synchronization is investigated. In Section 5, some numerical examples are provided to verify the effectiveness of the theoretical results. Finally, in Section 6, the conclusions are drawn.

II. PRELIMINARIES

In this section, we recall some definitions and make some preparations. In this paper, we will adopt Caputo fractional derivative.

Definition 1: [23] The fractional integral with fractional order $\alpha > 0$ of function $f(t)$ is defined as

$$ {}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, $$

where $\Gamma(\cdot)$ is Gamma function defined by $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}e^{-t}\mathrm{d}t$.

Definition 2: [23] Caputo fractional derivative of function $f(t)$ is defined by

$$ {}_0^CD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha}\mathrm{d}s, \quad 0 < \alpha < 1. $$

Definition 3: [23] The one parameter Mittag-Leffler function is defined as

$$ E_\alpha(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(k\alpha+1)}, $$

the two parameter Mittag-Leffler function is defined as

$$ E_{\alpha,\beta}(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(k\alpha+\beta)}, $$

where $x \in \mathbb{C}$, $\alpha, \beta > 0$. Furthermore, $E_{1,1}(x) = E_1(x) = e^x$.

Definition 4: [23] The Laplace transform for $f(t)$ is defined as

$$ F(s) = \mathcal{L}\{f(t); s\} = \int_0^{+\infty} e^{-st} f(t)\mathrm{d}t, \quad s \in \mathbb{C}. $$

Lemma 1: [23] $\frac{\mathrm{d}}{\mathrm{d}x}\left[x^\alpha E_{\alpha,\alpha+1}(\lambda x^\alpha)\right] = x^{\alpha-1}E_{\alpha,\alpha}(\lambda x^\alpha)$, where $\alpha, \lambda, x \in \mathbb{C}$.

Lemma 2: [31] If $\lambda > 0$ and $\alpha \in (0,1]$, then $\lim_{t\to+\infty} t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha) = \frac{1}{\lambda}$ and $t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha) \leq \frac{1}{\lambda}$ for $t \geq 0$.

Lemma 3: [43] If $a, \lambda > 0$ and $\alpha \in (0,1]$, then $\lim_{t\to+\infty} E_\alpha(-\lambda t^\alpha) = 0$ and

$$ \lim_{t\to+\infty} \int_0^a (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)\mathrm{d}s = 0. $$

Definition 5: A function $f \in C([0,+\infty), \mathbb{C}^n)$ is called S-asymptotically periodic if there exists $\omega > 0$ such that $\lim_{t\to+\infty} \|f(t+\omega) - f(t)\|_0 = 0$. In this case, we say that ω is an asymptotic period of f and that f is S-asymptotically ω-periodic.

Consider a fractional delayed equation

$$ {}_0^CD_t^\alpha x(t) = f(t, x_t), \quad t \geq 0, \tag{2} $$

where $0 < \alpha \leq 1$ and $x_t(\sigma) = x(t+\sigma)$, $\sigma \in [-\tau, 0]$ f is a continuous vector function satisfying $f(t,0) = 0$.

Lemma 4: [30] (Modified fractional Razumikhin theorem). The zero solution of fractional delayed equation (2) will be asymptotically stable if there exist three constants $\beta_1 > 0$, $\beta_2 > 0$, $\beta_3 > 0$ and a quadratic Lyapunov function $V(x)$ satisfying

$$ \beta_1\|x\|^2 \leq V(x) \leq \beta_2\|x\|^2, $$

and its α-order derivative along equation (2) satisfies

$$ {}_0^CD_t^\alpha V(x(t)) \leq -\beta_3\|x\|^2, $$

whenever

$$ V(x(t+\sigma)) \leq \xi V(x(t)), \quad \sigma \in [-\tau, 0], $$

for some $\xi > 1$.

Lemma 5: [24] Let $V(t)$ be a continuous function on $[0,+\infty)$ and satisfying

$$ {}_0^CD_t^\alpha V(t) \leq -\xi V(t), $$

where $0 < \alpha < 1$ and $\xi > 0$. Then

$$ V(t) \leq -\xi V(0)E_\alpha\left[-\xi t^\alpha\right], \quad t \geq 0. $$

Lemma 6: [36] Suppose $x(t) \in \mathbb{C}^n$ is a differentiable vector-valued function, then for any $t \geq 0$ and $0 < \alpha < 1$,

$$ {}_0^CD_t^\alpha\left(\overline{x(t)}x(t)\right) \leq \overline{x(t)}\,{}_0^CD_t^\alpha x(t) + \left({}_0^CD_t^\alpha\overline{x(t)}\right)x(t). $$

Throughout this paper, we assume that the following conditions hold:

- (H_1) τ_{lj} is ω-periodic function; $d_l \in C(\mathbb{R},\mathbb{C})$ is S-asymptotically ω-periodic function; $a_{lj}, b_{ljk}, I_l \in C(\mathbb{R},\mathbb{C})$ are S-asymptotically ω-periodic functions, where $l, j \in \mathcal{J}$.
- (H_2) There exist positive constants L_j^f, L_j^g such that for any $x, y \in \mathbb{C}$,

$$ \left\|f_j(x) - f_j(y)\right\|_{\mathbb{C}} \leq L_j^f\left\|x-y\right\|_{\mathbb{C}}, $$
$$ \left\|g_j(x) - g_j(y)\right\|_{\mathbb{C}} \leq L_j^g\left\|x-y\right\|_{\mathbb{C}}, $$

  where $l \in \mathcal{J}$.
- (H_3) For $l \in \mathcal{J}$, $\Upsilon_l < d_l^-$, where

$$ \Upsilon_l = \sum_{j=1}^n\left[a_{lj}^+L_j^f + b_{lj}^+L_j^g\right]. $$

III. S-ASYMPTOTICALLY ω-PERIODICITY

In this section, we will study the existence S-asymptotically ω-periodic solutions of system (1).

A. VOLTERRA INTEGRAL EXPRESSION OF FRACTIONAL-ORDER COMPLEX-VALUED NEURAL NETWORKS

Let $\mathbb{E} = \left\{x = (x_1, x_2, \ldots, x_n)^T \in SAP_\omega(\mathbb{C}^n)\right\}$ with the norm $\left\|x\right\|_{\mathbb{E}} = \max_{l\in\mathcal{J}}\left\{\sup_{t\in\mathbb{R}}\|x_l(t)\|_{\mathbb{C}}\right\}$, then $\mathbb{E}$ is a Banach space.

From (H_3), we can choose a positive constant $\lambda > d_l^- - \Upsilon_l$ such that

$$ 0 < \delta := \max_{l\in\mathcal{J}}\left\{\frac{\lambda - d_l^- + \Upsilon_l}{\lambda}\right\} < 1. \tag{3} $$

Now, for any given $\phi \in \mathbb{E}$, we consider the following system:

$$ {}_0^C D_t^\alpha x_l(t) = -\lambda x_l(t) + \mathcal{H}_l^\phi(t), \quad l \in \mathcal{J}, \tag{4} $$

where

$$ \mathcal{H}_l^\phi(t) = (\lambda - d_l(t))\phi_l(t) + \sum_{j=1}^n a_{lj}(t) f_j\big(\phi_j(t)\big) + \sum_{j=1}^n b_{lj}(t) g_j\big(\phi_j(t-\tau_{lj}(t))\big) + I_l(t). $$

Taking the Laplace transform in the two sides of system (4), it can be obtained

$$ s^\alpha x_l(s) - s^{\alpha-1}\phi_l(0) = -\lambda x_l(s) + \mathcal{L}\Big[\mathcal{H}_l^\phi(t)\Big](s), $$

that is

$$ x_l(s) = \frac{s^{\alpha-1}}{s^\alpha+\lambda}\phi_l(0) + \frac{1}{s^\alpha+\lambda}\mathcal{L}\Big[\mathcal{H}_l^\phi(t)\Big](s), $$

by taking the inverse Laplace transform in the two sides of the above equality, we can obtain

$$ x_l(t) = E_\alpha\big[-\lambda t^\alpha\big]\phi_l(0) + \int_0^t (t-s)^{\alpha-1} \times E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big]\mathcal{H}_l^\phi(s)\mathrm{d}s. $$

B. THE EXISTENCE OF S-ASYMPTOTICALLY ω-PERIODIC SOLUTIONS

Theorem 1: Let (H_1)-(H_3) hold. Then system (1) has a unique S-asymptotically ω-periodic solution.

Proof: Let $\Phi : SAP_\omega(\mathbb{C}^n) \to C\big([0,+\infty), \mathbb{C}^n\big)$ be the operator defined by

$$ (\Phi\phi)_l(t) = E_\alpha\big[-\lambda t^\alpha\big]\phi_l(0) + \int_0^t (t-s)^{\alpha-1} \times E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big]\mathcal{H}_l^\phi(s)\mathrm{d}s, \tag{5} $$

where $l \in \mathcal{J}$.

First, we show that $\Phi : \mathbb{E} \to \mathbb{E}$. For any $\phi \in \mathbb{E}$ with $\|\phi\|_{\mathbb{E}} < +\infty$, then for any $\varepsilon > 0$, there is a positive number $t_1 > 0$ such that

$$ \big\|\phi_l(t+\omega) - \phi_l(t)\big\|_{\mathbb{C}} < \varepsilon, \quad t > t_1. $$

From (5), for any $t > t_1$, $l \in \mathcal{J}$, it follows that

$$ \begin{aligned} (\Phi\phi)_l(t+\omega) &= E_\alpha\big[-\lambda(t+\omega)^\alpha\big]\phi_l(0) + \int_0^{t+\omega} (t+\omega-s)^{\alpha-1} \times E_{\alpha,\alpha}\big[-\lambda(t+\omega-s)^\alpha\big]\mathcal{H}_l^\phi(s)\mathrm{d}s \\ &= E_\alpha\big[-\lambda(t+\omega)^\alpha\big]\phi_l(0) + \int_{-\omega}^{t} (t-s)^{\alpha-1} \times E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big]\mathcal{H}_l^\phi(s+\omega)\mathrm{d}s. \end{aligned} $$

Consequently, for any $t > t_1$, $l \in \mathcal{J}$, we have

$$ \begin{aligned} &(\Phi\phi)_l(t+\omega) - (\Phi\phi)_l(t) \\ &= E_\alpha\big[-\lambda(t+\omega)^\alpha\big]\phi_l(0) - E_\alpha\big[-\lambda t^\alpha\big]\phi_l(0) \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big] \times \Big[(\lambda - d_l(s+\omega))\phi_l(s+\omega) - (\lambda - d_l(s))\phi_l(s) \\ &\quad + \sum_{j=1}^n \Big[a_{lj}(s+\omega) f_j\big(\phi_j(s+\omega)\big) - a_{lj}(s) f_j\big(\phi_j(s)\big)\Big] \\ &\quad + \sum_{j=1}^n \Big[b_{lj}(s+\omega) g_j\big(\phi_j(s+\omega-\tau_{lj}(s))\big) - b_{lj}(s) g_j\big(\phi_j(s-\tau_{lj}(s))\big) + I_l(s+\omega) - I_l(s)\Big]\mathrm{d}s \\ &\quad + \int_{-\omega}^0 (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big] \times \Big[(\lambda - d_l(s+\omega))\phi_l(s+\omega) + \sum_{j=1}^n a_{lj}(s+\omega) \\ &\quad \times f_j\big(\phi_j(s+\omega)\big) + \sum_{j=1}^n b_{lj}(s+\omega) \times g_j\big(\phi_j(s+\omega-\tau_{lj}(s))\big) + I_l(s+\omega)\Big]\mathrm{d}s \\ &=: \hat{K}_l^1(t) + \hat{K}_l^2(t) + \hat{K}_l^3(t) + \hat{K}_l^4(t) + \hat{K}_l^5(t) + \hat{K}_l^6(t) + \hat{K}_l^7(t) + \hat{K}_l^8(t) + \hat{K}_l^9(t) + \hat{K}_l^{10}(t) + \hat{K}_l^{11}(t) + \hat{K}_l^{12}(t), \end{aligned} $$

where

$$ \begin{aligned} \hat{K}_l^1(t) &= E_\alpha\big[-\lambda(t+\omega)^\alpha\big]\phi_l(0) - E_\alpha\big[-\lambda t^\alpha\big]\phi_l(0), \\ \hat{K}_l^2(t) &= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big] \times (\lambda - d_l(s+\omega))\big[\phi_l(s+\omega) - \phi_l(s)\big]\mathrm{d}s, \\ \hat{K}_l^3(t) &= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big] \times \big[d_l(s+\omega) - d_l(s)\big]\phi_l(s)\mathrm{d}s, \\ \hat{K}_l^4(t) &= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big] \times \sum_{j=1}^n \big[a_{lj}(s+\omega) - a_{lj}(s)\big] f_j\big(\phi_j(s+\omega)\big)\mathrm{d}s, \\ \hat{K}_l^5(t) &= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big] \times \sum_{j=1}^n a_{lj}(s)\Big[f_j\big(\phi_j(s+\omega)\big) - f_j\big(\phi_j(s)\big)\Big]\mathrm{d}s, \\ \hat{K}_l^6(t) &= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big] \times \sum_{j=1}^n \big[b_{lj}(s+\omega) - b_{lj}(s)\big] \times g_j\big(\phi_j(s+\omega-\tau_{lj}(s))\big)\mathrm{d}s, \end{aligned} $$

$$\hat{K}_l^7(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big] \times \sum_{j=1}^n b_{lj}(s)\Big[g_j\big(\phi_j(s+\omega-\tau_{lj}(s))\big) - g_j\big(\phi_j(s-\tau_{lj}(s))\big)\Big]\mathrm{d}s,$$

$$\hat{K}_l^8(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big] \times \big[I_l(s+\omega) - I_l(s)\big]\mathrm{d}s,$$

$$\hat{K}_l^9(t) = \int_{-\omega}^0 (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big] \times (\lambda - d_l(s+\omega))\phi_l(s+\omega)\mathrm{d}s$$

$$\hat{K}_l^{10}(t) = \int_{-\omega}^0 (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big] \times \sum_{j=1}^n a_{lj}(s+\omega) f_j\big(\phi_j(s+\omega)\big)\mathrm{d}s$$

$$\hat{K}_l^{11}(t) = \int_{-\omega}^0 (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big] \times \sum_{j=1}^n b_{lj}(s+\omega) g_j\big(\phi_j(s+\omega-\tau_{lj}(s))\big)\mathrm{d}s,$$

$$\hat{K}_l^{12}(t) = \int_{-\omega}^0 (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big] I_l(s+\omega)\mathrm{d}s.$$

From Lemma 3, for each $\varepsilon > 0$, $l \in \mathcal{J}$, it has $t_2 > t_1$ such that

$$\big\|\hat{K}_l^1(t)\big\|_{\mathbb{C}} = \Big|E_\alpha\big[-\lambda(t+\omega)^\alpha\big] - E_\alpha\big[-\lambda t^\alpha\big]\Big|\big\|\phi_l(0)\big\|_{\mathbb{C}} < \varepsilon, \ \forall t > t_2.$$

Noting that $E_{\alpha,\alpha}\big[-\lambda t^\alpha\big] \geq 0$ for $t \geq 0$. By Lemma 1, it deduces

$$\begin{aligned}\big\|\hat{K}_l^2(t)\big\|_{\mathbb{C}} &\leq \bigg\|\int_0^{t_1} (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big] \times (\lambda - d_l(s+\omega))\big[\phi_l(s+\omega) - \phi_l(s)\big]\mathrm{d}s\bigg\|_{\mathbb{C}} \\ &\quad + \bigg\|\int_{t_1}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big] \times (\lambda - d_l(s+\omega))\big[\phi_l(s+\omega) - \phi_l(s)\big]\mathrm{d}s\bigg\|_{\mathbb{C}} \\ &\leq 2\lambda\|\phi_l\|_{\mathbb{C}} \int_0^{t_1} (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big]\mathrm{d}s + \lambda\varepsilon \int_{t_1}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big]\mathrm{d}s \\ &= 2\lambda\|\phi\|_{\mathbb{E}} \int_0^{t_1} (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big]\mathrm{d}s + \lambda\varepsilon(t-t_1)^\alpha E_{\alpha,\alpha+1}\big[-\lambda(t-t_1)^\alpha\big], \\ &\quad t > t_1, \ l \in \mathcal{J}. \end{aligned} \tag{6}$$

From Lemmas 2-3, there exists $t_3 > t_2$ such that

$$\big\|\hat{K}_l^2(t)\big\|_{\mathbb{C}} < 2\varepsilon, \quad t > t_3, \ l \in \mathcal{J}. \tag{7}$$

In a similar way, there exists $t_4 > t_3$ such that

$$\begin{cases} \big\|\hat{K}_l^3(t)\big\|_{\mathbb{C}} < \dfrac{2\|\phi\|_{\mathbb{E}}}{\lambda}\varepsilon, \\ \big\|\hat{K}_l^4(t)\big\|_{\mathbb{C}} < \dfrac{2}{\lambda}\displaystyle\sum_{j=1}^n \big(L_j^f\|\phi\|_{\mathbb{E}} + \|f_j(0)\|_{\mathbb{C}}\big)\varepsilon, \\ \big\|\hat{K}_l^5(t)\big\|_{\mathbb{C}} < \dfrac{2}{\lambda}\displaystyle\sum_{j=1}^n a_{lj}^+ L_j^f\varepsilon, \\ \big\|\hat{K}_l^6(t)\big\|_{\mathbb{C}} < \dfrac{2}{\lambda}\displaystyle\sum_{j=1}^n \big(L_j^g\|\phi\|_{\mathbb{E}} + \|g_j(0)\|_{\mathbb{C}}\big)\varepsilon, \\ \big\|\hat{K}_l^7(t)\big\|_{\mathbb{C}} < \dfrac{2}{\lambda}\displaystyle\sum_{j=1}^n b_{lj}^+ L_j^g\varepsilon, \end{cases} \tag{8}$$

$$\begin{cases} \big\|\hat{K}_l^8(t)\big\|_{\mathbb{C}} < \frac{2}{\lambda}\varepsilon, \\ \big\|\hat{K}_l^9(t)\big\|_{\mathbb{C}} < \lambda\|\phi\|_{\mathbb{E}}\varepsilon, \\ \big\|\hat{K}_l^{10}(t)\big\|_{\mathbb{C}} < \displaystyle\sum_{j=1}^n a_{lj}^+\big(L_j^f\|\phi\|_{\mathbb{E}} + \|f_j(0)\|_{\mathbb{C}}\big)\varepsilon, \\ \big\|\hat{K}_l^{11}(t)\big\|_{\mathbb{C}} < \displaystyle\sum_{j=1}^n b_{lj}^+\big(L_j^g\|\phi\|_{\mathbb{E}} + \|g_j(0)\|_{\mathbb{C}}\big)\varepsilon, \\ \big\|\hat{K}_l^{12}(t)\big\|_{\mathbb{C}} < I_l^+\varepsilon, \quad t > t_4, \ l \in \mathcal{J}. \end{cases} \tag{9}$$

By (6)-(9), it has $\mathcal{K} > 0$ large enough ensuring that

$$\big\|(\Phi\phi)_l(t+\omega) - (\Phi\phi)_l(t)\big\|_{\mathbb{C}} < \mathcal{K}\varepsilon, \quad t > t_4, \ l \in \mathcal{J},$$

which implies that $\Phi\phi \in SAP_\omega(\mathbb{C}^n)$.

Next, we show that Φ is a contraction mapping in $\mathbb{E}$. For any $\phi, \psi \in \mathbb{E}$, $l \in \mathcal{J}$, it gets from (5) and Lemma 2 that

$$\begin{aligned}&\big\|(\Phi\phi)_l(t) - (\Phi\psi)_l(t)\big\|_{\mathbb{C}} \\ &= \bigg\|\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big] \times \bigg[\big[(\lambda - d_l(s))\phi_l(s) - (\lambda - d_l(s))\psi_l(s)\big] + \sum_{j=1}^n a_{lj}(s)\Big[f_j\big(\phi_j(s)\big) - f_j\big(\psi_j(s)\big)\Big] \\ &\quad + \sum_{j=1}^n b_{ljk}(s)\Big[g_j\big(\phi_j(s-\tau_{lj}(s))\big) - g_j\big(\psi_j(s-\tau_{lj}(s))\big)\Big]\bigg]\mathrm{d}s\bigg\|_{\mathbb{C}} \\ &\leq \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big] \times \bigg[(\lambda - d_l^-)\big\|\phi_l(s) - \psi_l(s)\big\|_{\mathbb{C}} + \sum_{j=1}^n \big\|a_{lj}(s)\big\|_{\mathbb{C}}\Big\|f_j\big(\phi_j(s)\big) - f_j\big(\psi_j(s)\big)\Big\|_{\mathbb{C}} \\ &\quad + \sum_{j=1}^n \big\|b_{lj}(s)\big\|_{\mathbb{C}}\Big\|g_j\big(\phi_j(s-\tau_{lj}(s))\big) - g_j\big(\psi_j(s-\tau_{lj}(s))\big)\Big\|_{\mathbb{C}}\bigg]\mathrm{d}s\end{aligned}$$

$$\leq \Big[\lambda - d_l^- + \sum_{j=1}^n a_{lj}^+ L_j^f + \sum_{j=1}^n b_{lj}^+ L_j^g\Big]\|\phi-\psi\|_{\mathbb{E}}$$
$$\times \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[-\lambda(t-s)^\alpha\big]\mathrm{d}s$$
$$\leq \Big[\lambda - d_l^- + \sum_{j=1}^n a_{lj}^+ L_j^f + \sum_{j=1}^n b_{lj}^+ L_j^g\Big]$$
$$\times \|\phi-\psi\|_{\mathbb{E}} t^\alpha E_{\alpha,\alpha+1}\big[-\lambda t^\alpha\big]$$
$$\leq \frac{1}{\lambda}\Big[\lambda - d_l^- + \sum_{j=1}^n a_{lj}^+ L_j^f + \sum_{j=1}^n b_{lj}^+ L_j^g\Big]$$
$$\times \|\phi-\psi\|_{\mathbb{E}}, \quad t \geq 0. \tag{10}$$

It follows from (3) and (10) that

$$\|\Phi\phi - \Phi\psi\|_{\mathbb{E}} \leq \delta\|\phi-\psi\|_{\mathbb{E}},$$

which implies that Φ is a contraction mapping. Therefore, Φ has a unique fixed point in $\mathbb{E}$, that is, (1) has a unique S-asymptotically ω-periodic solution. The proof is complete. ■

Remark 1: In the calculation process of Theorem 1, by using the relevant properties of Mittag-Leffler functions, we obtain the existence and uniqueness of S-asymptotically ω-periodic solution of system (1).

Remark 2: Theorem 1 to S-asymptotically ω-periodic solution existence criteria for considered network models by employing non-decomposing method. When $\alpha = 1$, our result is still true for an integer-order case.

IV. S-ASYMPTOTICALLY ω-PERIODIC SYNCHRONIZATION

In this section, by designing a very general nonlinear controller, constructing an appropriate Lyapunov functional, we shall study the S-asymptotically ω-periodic synchronization problem of system (1).

For this purpose, we consider the system (1) as a drive system, and a response system is designed as:

$$ {}_0^C D_t^\alpha y_l(t) = -c_l(t)y_l(t) + \sum_{j=1}^n a_{lj}(t) f_j\big(y_j(t)\big)$$
$$+ \sum_{j=1}^n b_{lj}(t) g_j\big(y_j(t-\tau_{lj}(t))\big)$$
$$+ I_l(t) + U_l(t), \tag{11}$$

where $l \in \mathcal{J}$, $y_l(t) \in \mathbb{C}$ denotes the state of the response system, and $U_l(t) \in \mathbb{C}$ is a state-feedback controller, the rest notation is the same as those in system (1).

Set $z(t) = y(t) - x(t)$, where $z(t) = \big(z_1(t), \ldots, z_n(t)\big)^T$. For $l \in \mathcal{J}$, subtracting (1) from (11) yields the following error system:

$$ {}_0^C D_t^\alpha z_l(t) = -d_l(t)z_l(t) + \sum_{j=1}^n a_{lj}(t)\hat{f}_j\big(z_j(t)\big)$$
$$+ \sum_{j=1}^n b_{lj}(t)\hat{g}_j\big(z_j(t-\tau_{lj}(t))\big) + U_l(t), \tag{12}$$

where

$$\hat{f}_j\big(z_j(t)\big) = f_j\big(y_j(t)\big) - f_j\big(x_j(t)\big),$$
$$\hat{g}_j\big(z_j(t-\tau_{lj}(t))\big) = g_j\big(y_j(t-\tau_{lj}(t))\big) - g_j\big(x_j(t-\tau_{lj}(t))\big).$$

In order to realize the S-asymptotically ω-periodic synchronization of the drive-response system, we design the following state-feedback controller:

$$U_l(t) = -e_l(t)z_l(t) + \sum_{j=1}^n c_{lj}(t)h_j\big(z_j(t)\big), \tag{13}$$

where $e_l \in C(\mathbb{R}, \mathbb{R}^+)$, $c_{lj} \in C(\mathbb{R}, \mathbb{C})$, $h_j : \mathbb{C} \to \mathbb{C}$, $l, j \in \mathcal{J}$.

Definition 6: The response system (11) and the drive system (1) are said to achieve global Mittag-Leffler synchronization, if there exist positive constants M, ξ and β such that

$$\|y(t)-x(t)\|_{\mathbb{C}^n} \leq M\|\psi-\phi\|_1\Big(E_\alpha\big[-\xi t^\alpha\big]\Big)^\beta, \quad t \geq 0,$$

where

$$\|y(t)-x(t)\|_{\mathbb{C}^n} = \max_{l\in\mathcal{J}}\Big\{\|y_l(t)-x_l(t)\|_{\mathbb{C}}\Big\},$$
$$\|\psi-\phi\|_1 = \max_{l\in\mathcal{J}}\Big\{\sup_{s\in[-\tau,0]}\|\psi_l(s)-\phi_l(s)\|_{\mathbb{C}}\Big\}.$$

Theorem 2: Let (H_1)-(H_3) hold. Suppose further that

(H_4) There exist positive constants L_j^h such that for any $x, y \in \mathbb{C}$,
$$\|h_j(x)-h_j(y)\|_{\mathbb{C}} \leq L_j^h\|x-y\|_{\mathbb{C}}$$
and $h_j(\mathbf{0}) = \mathbf{0}$, where $j \in \mathcal{J}$.

(H_5) For $\mu > 1$, we have $\rho - \mu\zeta > 0$, where
$$\rho = \min_{l\in\mathcal{J}}\Big\{2d_l^- + 2e_l^- - 3 - \sum_{j=1}^n\Big(a_{jl}^+ L_j^f\Big)^2 - \sum_{j=1}^n\Big(c_{jl}^+ L_j^h\Big)^2\Big\},$$
$$\zeta = \max_{l\in\mathcal{J}}\Big\{\sum_{j=1}^n\Big(b_{jl}^+ L_j^g\Big)^2\Big\}.$$

Then the drive system (1) and response system (11) are globally Mittag-Leffler synchronized based on the controller (13).

Proof: We consider the Lyapunov function as follow:

$$V(t) = \max_{l\in\mathcal{J}}\Big\{\overline{z_l(t)}z_l(t)\Big\}. \tag{14}$$

From Lemma 6, calculating the Caputo derivative of $V(t)$ along the trajectory of (12) derives that

$$ {}_0^C D_t^\alpha V(t)$$
$$\leq \max_{l\in\mathcal{J}}\Big\{\overline{z_l(t)}\,{}_0^C D_t^\alpha z_l(t) + \Big({}_0^C D_t^\alpha \overline{z_l(t)}\Big)z_l(t)\Big\}$$
$$= \max_{l\in\mathcal{J}}\Big\{\overline{z_l(t)}\Big[-d_l(t)z_l(t) + \sum_{j=1}^n a_{lj}(t)\hat{f}_j\big(z_j(t)\big)$$
$$+ \sum_{j=1}^n b_{lj}(t)\hat{g}_j\big(z_j(t-\tau_{lj}(t))\big) - e_l(t)z_l(t)$$

$$
\begin{aligned}
&+\sum_{j=1}^{n} c_{lj}(t)h_j\big(z_j(t)\big)\Big]+\Big[-d_l(t)\overline{z_l(t)}\\
&+\sum_{j=1}^{n}\overline{a_{lj}(t)\hat{f}_j\big(z_j(t)\big)}\\
&+\sum_{j=1}^{n}\overline{b_{lj}(t)\hat{g}_j\big(z_j(t-\tau_{lj}(t))\big)}\\
&-e_l(t)\overline{z_l(t)}+\sum_{j=1}^{n}\overline{c_{lj}(t)h_j\big(z_j(t)\big)}\Big]z_l(t)\Big\}\\
=&\max_{l\in\mathcal{J}}\Big\{\big(-2d_l(t)-2e_l(t)\big)\overline{z_l(t)}z_l(t)\\
&+\sum_{j=1}^{n}\Big[\overline{z_l(t)}a_{lj}(t)\hat{f}_j\big(z_j(t)\big)\\
&+\overline{a_{lj}(t)\hat{f}_j\big(z_j(t)\big)}z_l(t)\Big]\\
&+\sum_{j=1}^{n}\Big[\overline{z_l(t)}b_{lj}(t)\hat{g}_j\big(z_j(t-\tau_{lj}(t))\big)\\
&+\overline{b_{lj}(t)\hat{g}_j\big(z_j(t-\tau_{lj}(t))\big)}\\
&\times z_l(t)\Big]+\sum_{j=1}^{n}\Big[\overline{z_l(t)}c_{lj}(t)h_j\big(z_j(t)\big)\\
&+\overline{c_{lj}(t)h_j\big(z_j(t)\big)}z_l(t)\Big]\Big\}\\
\le&\max_{l\in\mathcal{J}}\Big\{\big(-2d_l(t)-2e_l(t)\big)\overline{z_l(t)}z_l(t)\\
&+\sum_{j=1}^{n}\overline{a_{lj}(t)\hat{f}_j\big(z_j(t)\big)}\\
&\times\Big(a_{lj}(t)\hat{f}_j\big(z_j(t)\big)\Big)+\overline{z_l(t)}z_l(t)\\
&+\sum_{j=1}^{n}\overline{b_{lj}(t)\hat{g}_j\big(z_j(t-\tau_{lj}(t))\big)}\\
&\times\Big(b_{lj}(t)\hat{g}_j\big(z_j(t-\tau_{lj}(t))\big)\Big)+\overline{z_l(t)}z_l(t)\\
&+\sum_{j=1}^{n}\overline{c_{lj}(t)h_j\big(z_j(t)\big)}\Big(c_{lj}(t)h_j\big(z_j(t)\big)\Big)\\
&+\overline{z_l(t)}z_l(t)\Big\}\\
\le&\max_{l\in\mathcal{J}}\Big\{\big(3-2d_l(t)-2e_l(t)\big)\overline{z_l(t)}z_l(t)\\
&+\sum_{j=1}^{n}\Big(a_{lj}^{+}L_j^{f}\Big)^2\overline{z_j(t)}z_j(t)+\sum_{j=1}^{n}\Big(b_{lj}^{+}L_j^{g}\Big)^2\\
&\times\overline{z_l(t-\tau_{lj}(t))}z_l(t-\tau_{lj}(t))\\
&+\sum_{j=1}^{n}\Big(c_{lj}^{+}L_j^{h}\Big)^2\overline{z_j(t)}z_j(t)\Big\}\\
\le&\max_{l\in\mathcal{J}}\Big\{\Big[3-2d_l^{-}-2e_l^{-}+\sum_{j=1}^{n}\Big(a_{jl}^{+}L_j^{f}\Big)^2\\
&+\sum_{j=1}^{n}\Big(c_{jl}^{+}L_j^{h}\Big)^2\Big]\overline{z_l(t)}z_l(t)\\
&+\sum_{j=1}^{n}\Big(b_{jl}^{+}L_j^{g}\Big)^2\overline{z_l(t-\tau)}z_l(t-\tau)\Big\}\\
=&-\rho V(t)+\zeta V(t-\tau).
\end{aligned}\tag{15}
$$

Combining (15) and Lemma 4 has

$$
\begin{aligned}
{}_0^C D_t^{\alpha} V(t) &\le -\rho V(t)+\mu\zeta V(t)\\
&=-(\rho-\mu\zeta)V(t)
\end{aligned}\tag{16}
$$

for $\mu>1$. Combining (16) with Lemma 5 derives

$$
V(t)\le V(0)E_{\alpha}\big[-(\rho-\mu\zeta)t^{\alpha}\big],\quad 0<\alpha<1.
$$

From the Lyapunov function (14), we have $V(0)\le\|\psi-\phi\|_1^2$. Furthermore, we have

$$
\|y(t)-z(t)\|_{\mathbb{C}^n}\le\|\psi-\phi\|_1\Big(E_{\alpha}\big[-(\rho-\mu\zeta)t^{\alpha}\big]\Big)^{\frac{1}{2}}.
$$

Therefore, the drive system (1) and the response system (11) are globally Mittag-Leffler synchronized based on the controller (13). This completes the proof. ■

Remark 3: As far as we know, the Lyapunov method provides a very effective tool to realize synchronization analysis of nonlinear systems. However, it is very complicated to calculate the fractional-order derivative of an auxiliary function. The mathematical difficulties we encounter in this paper are as follows. (1) How to calculate Caputo derivative of Lyapunov function that contains complex-valued function and its conjugate. To solve this problem, we use Lemma 6 to avoid calculating the fractional-order derivatives of the Lyapunov functional. (2) How to deal with time-delay terms. For this problem, we adopt Modified fractional Razumikhin theorem.

V. ILLUSTRATIVE EXAMPLE

In this section, we give two examples to illustrate the feasibility and effectiveness of our results obtained in Sections 3 and 4.

Example 1: Let $n = 2$. Consider the following fractional-order complex-valued neural network:

$$
\begin{aligned}
{}_0^C D_t^{\alpha} x_l(t) =& -d_l(t)x_l(t)+\sum_{j=1}^{2} a_{lj}(t)f_j\big(x_j(t)\big)\\
&+\sum_{j=1}^{2} b_{lj}(t)g_j\big(x_j(t-\tau_{lj}(t))\big)+I_l(t),
\end{aligned}\tag{17}
$$

the corresponding response system is given by

$$
\begin{aligned}
{}_0^C D_t^{\alpha} y_l(t) =& -d_l(t)y_l(t)+\sum_{j=1}^{2} a_{lj}(t)f_j\big(y_j(t)\big)\\
&+\sum_{j=1}^{2} b_{lj}(t)g_j\big(y_j(t-\tau_{lj}(t))\big)\\
&+I_l(t)+U_l(t),
\end{aligned}\tag{18}
$$

where $l=1,2$, $\alpha=0.5$, the coefficients are follows:

$$
f_j(x_j)=\frac{1}{3}\sin x_j^R+i\frac{1}{4}\sin x_j^I,
$$

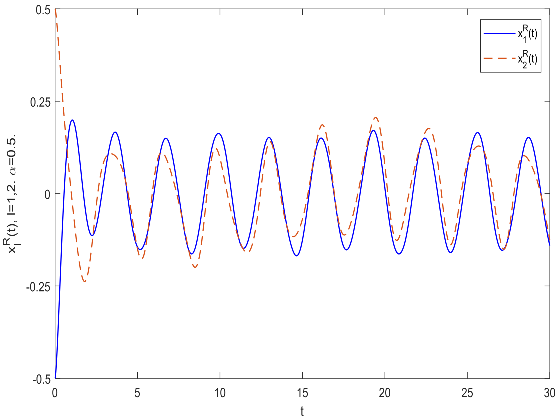

FIGURE 1. Time revolution of the real parts of system (17) without control.

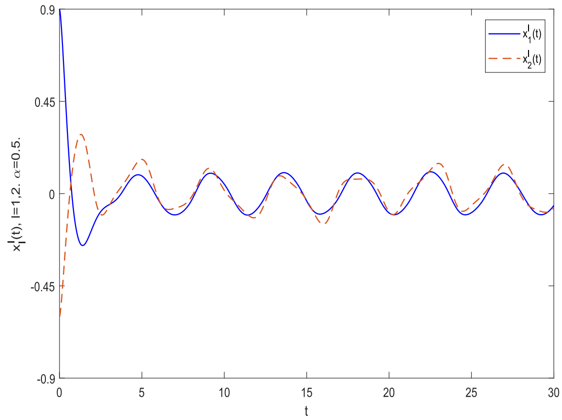

FIGURE 2. Time revolution of the imaginary parts of system (17) without control.

$$
\begin{aligned}
g_j(x_j) &= \frac{1}{4}\tanh x_j^R + i\frac{1}{6}\cos x_j^I,\\
h_j(x_j) &= \frac{1}{6}\arctan x_j^R + i\frac{1}{6}\tanh x_j^I,\\
d_1(t) &= 3 + |\cos(\sqrt{2}t)|, \quad d_2(t) = 4 - 2\sin(\sqrt{3}t),\\
a_{lj}(t) &= 0.5\cos(\sqrt{3}t) + i0.2\cos(2t),\\
b_{lj}(t) &= 0.8\sin(\sqrt{2}t) + i0.5\sin t,\\
e_1(t) &= 1 + |\sin(\sqrt{3}t)|, \quad e_2(t) = 2 - 0.5\cos(\sqrt{2}t),\\
c_{lj}(t) &= 1.5\cos(2t) + i0.8\sin(\sqrt{2}t),\\
I_l(t) &= 0.2\sin(\sqrt{3}t) + i0.3\cos(\sqrt{2}t),\\
\tau_{lj}(t) &= \frac{1}{2}|\sin t|, \quad l, j = 1, 2.
\end{aligned}
$$

By a simple calculation, we have

$$
\begin{aligned}
&d_1^- = 3, \quad d_2^- = 2, \quad e_1^- = 1, \quad e_2^- = 1.5,\\
&L_j^f \leq 0.4167, \quad L_j^g \leq 0.3005, \quad L_j^h \leq 0.2357,\\
&a_{lj}^+ \leq 0.5385, \quad b_{lj}^+ \leq 0.9434, \quad c_{lj}^+ \leq 1.7,\\
&\Upsilon_1 = \Upsilon_2 \approx 1.0158, \quad \Upsilon_1 < d_1^-, \quad \Upsilon_2 < d_2^-.
\end{aligned}
$$

Take $\mu = 4$, we have

$$
\begin{aligned}
\rho = \min_{l=1,2}\Bigg\{&2d_l^- + 2e_l^- - 3 - \sum_{j=1}^{2}\left(a_{jl}^+ L_j^f\right)^2\\
&- \sum_{j=1}^{2}\left(c_{jl}^+ L_j^h\right)^2\Bigg\} \approx 3.5782,
\end{aligned}
$$

$$
\zeta = \max_{l=1,2}\left\{\sum_{j=1}^{2}\left(b_{jl}^+ L_j^g\right)^2\right\} \approx 0.1607.
$$

Thus, we have

$$
\rho - \mu\zeta = 2.9354 > 0.
$$

So, all the assumptions of Theorems 1 and 2 are satisfied. Therefore by Theorem 2, system (17) and (18) are globally Mittag-Leffler synchronized based on the controller (13). The drive system (17) have the initial value $x_1(0) = 0.5 - 0.1i$, $x_2(0) = -0.8+0.5i$. The response system (18) have the initial value $y_1(0) = 0.5 - 0.1i$, $y_2(0) = -0.8 + 0.5i$. The error

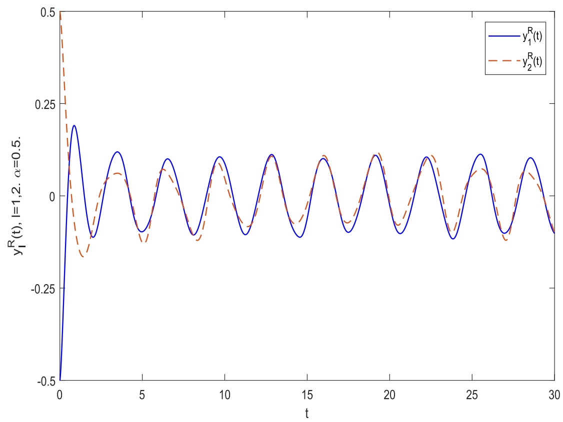

FIGURE 3. Time revolution of the real parts of system (18) under controller (13).

system have the initial value $z_1(0) = -0.2 + 0.4i$, $z_2(0) = 0.3 - 0.5i$. By using the Simulink toolbox in MATLAB, Figures 1-4 depict the time revolution of real parts and imaginary parts of x_1, x_2, y_1 and y_2, respectively. Figure 5 shows simulation results of error system. From simulation results in Figures 1-5, it is clearly seen that the drive-response systems (17)-(18) achieve synchronization.

Example 2: Let $n = 3$. Consider the following fractional-order complex-valued neural network:

$$
\begin{aligned}
{}_0^C D_t^\alpha x_l(t) = &-d_l(t)x_l(t) + \sum_{j=1}^{3} a_{lj}(t) f_j\big(x_j(t)\big)\\
&+ \sum_{j=1}^{3} b_{lj}(t) g_j\big(x_j(t - \tau_{lj}(t))\big) + I_l(t), \qquad (19)
\end{aligned}
$$

the corresponding response system is given by

$$
\begin{aligned}
{}_0^C D_t^\alpha y_l(t) = &-d_l(t)y_l(t) + \sum_{j=1}^{3} a_{lj}(t) f_j\big(y_j(t)\big)\\
&+ \sum_{j=1}^{3} b_{lj}(t) g_j\big(y_j(t - \tau_{lj}(t))\big)\\
&+ I_l(t) + U_l(t), \qquad (20)
\end{aligned}
$$

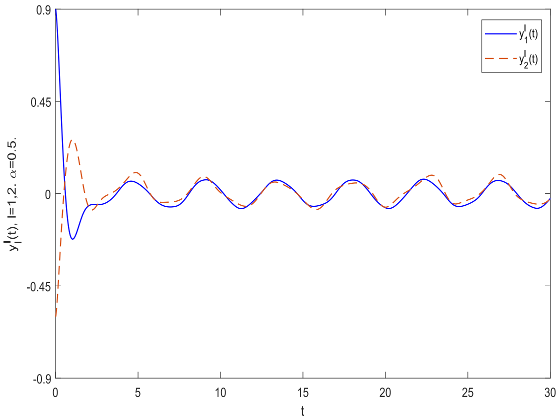

FIGURE 4. Time revolution of the imaginary parts of system (18) under controller (13).

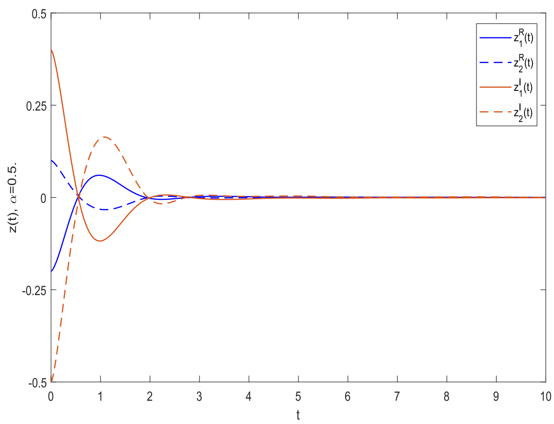

FIGURE 5. State response curve of the real and imaginary parts of synchronization error.

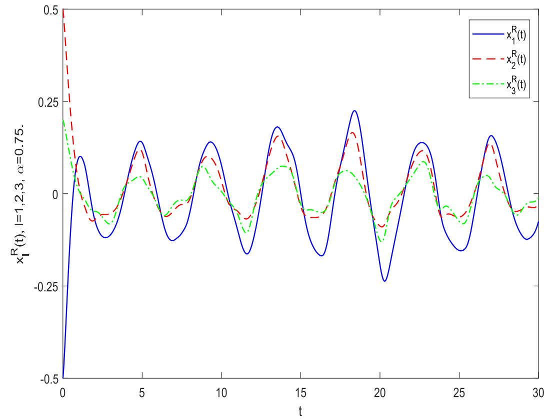

FIGURE 6. Time revolution of the real parts of system (19) without control.

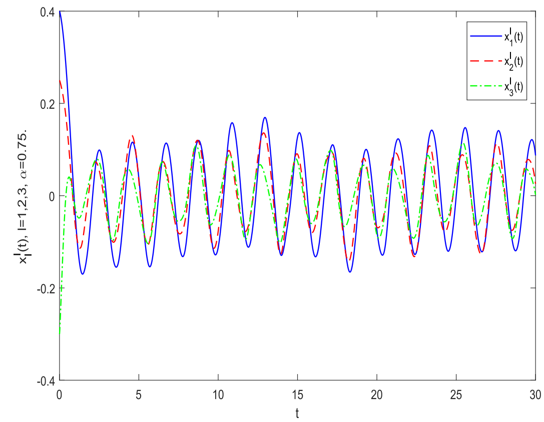

FIGURE 7. Time revolution of the imaginary parts of system (19) without control.

where $l = 1, 2, 3$, $\alpha = 0.75$, the coefficients are follows:

$$
\begin{aligned}
f_j(x_j) &= \frac{1}{5}\tanh x_j^R + i\frac{1}{8}\sin(x_j^R + x_j^I),\\
g_j(x_j) &= \frac{1}{4}\sin(x_j^R + x_j^I) + i\frac{1}{5}\tanh x_j^I,\\
h_j(x_j) &= \frac{1}{8}\sin x_j^R + i\frac{1}{10}\sin(x_j^R + x_j^I),\\
d_1(t) &= 2 + |\sin(\sqrt{3}t)|, \quad d_2(t) = 5 - 2\cos(\sqrt{2}t),\\
d_3(t) &= 7 - 3\cos(\sqrt{5}t), \quad e_1(t) = 1.5 + |\cos(2t)|,\\
e_2(t) &= 2 + |\sin(2t)|, \quad e_3(t) = 3 - 0.5\sin(\sqrt{3}t),\\
a_{lj}(t) &= 0.8\sin(2t) + i0.5\cos(\sqrt{3}t),\\
b_{lj}(t) &= 0.9\cos(\sqrt{3}t) + i0.6\sin(\sqrt{2}t),\\
c_{lj}(t) &= \sin(\sqrt{2}t) + i1.5\cos(\sqrt{3}t),\\
I_l(t) &= 0.4\cos(\sqrt{2}t) + i0.5\cos(3t),\\
\tau_{lj}(t) &= 1 + \sin(2t), \quad l, j = 1, 2, 3.
\end{aligned}
$$

By a simple calculation, we have

$$
\begin{aligned}
&d_1^- = 2, \quad d_2^- = 3, \quad d_3^- = 4,\\
&e_1^- = 1.5, \quad e_2^- = 2, \quad e_3^- = 2.5,\\
&L_j^f \le 0.2358, \quad L_j^g \le 0.3202, \quad L_j^h \le 0.1601,\\
&a_{lj}^+ \le 0.9434, \quad b_{lj}^+ \le 1.0817, \quad c_{lj}^+ \le 1.8028,\\
&\Upsilon_1 = \Upsilon_2 = \Upsilon_3 \approx 1.7064,\\
&\Upsilon_1 < d_1^-, \quad \Upsilon_2 < d_2^-, \quad \Upsilon_3 < d_3^-.
\end{aligned}
$$

Take $\mu = 5$, we have

$$
\begin{aligned}
\rho &= \min_{l=1,2,3}\Bigg\{2d_l^- + 2e_l^- - 3 - \sum_{j=1}^{3}\left(a_{jl}^+ L_j^f\right)^2\\
&\quad - \sum_{j=1}^{3}\left(c_{jl}^+ L_j^h\right)^2\Bigg\} \approx 3.6016,\\
\zeta &= \max_{l=1,2,3}\left\{\sum_{j=1}^{3}\left(b_{jl}^+ L_j^g\right)^2\right\} \approx 0.3599.
\end{aligned}
$$

Thus, we have

$$\rho - \mu\zeta = 1.8021 > 0.$$

So, all the assumptions of Theorems 1 and 2 are satisfied. Therefore by Theorem 2, system (19) and (20) are globally Mittag-Leffler synchronized based on the controller (13). The drive system (19) have the initial value $x_1(0) = -0.5 + 0.4i$, $x_2(0) = 0.5 + 0.25i$, $x_3(0) = 0.4 - 0.3i$. The response system (20) have the initial value $y_1(0) = -0.5 + 0.4i$,

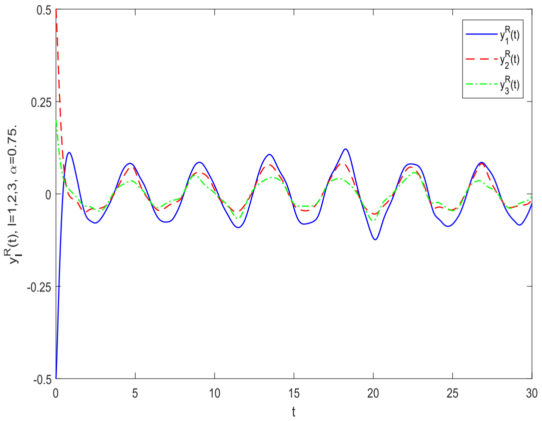

FIGURE 8. **Time revolution of the real parts of system** (20) **under controller** (13)**.**

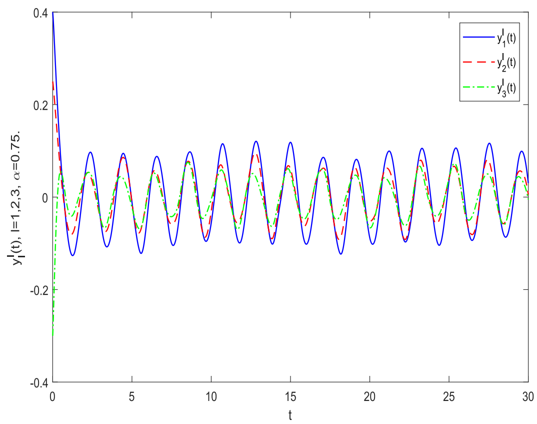

FIGURE 9. **Time revolution of the imaginary parts of system** (20) **under controller** (13)**.**

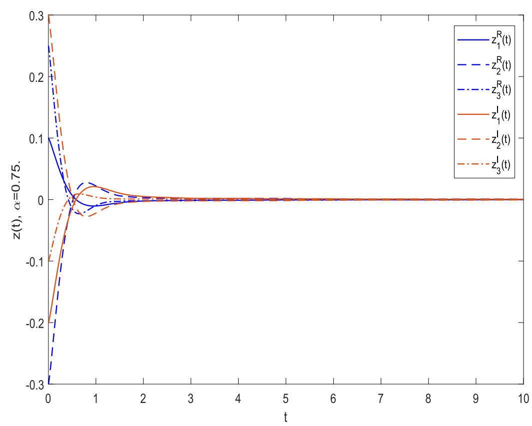

FIGURE 10. **State response curve of the real and imaginary parts of synchronization error.**

$y_2(0) = 0.5 + 0.25i$, $y_3(0) = 0.4 - 0.3i$. By using the Simulink toolbox in MATLAB, Figures 6-9 depict the time revolution of real parts and imaginary parts of x_1, x_2, x_3, y_1, y_2 and y_3, respectively. Figure 10 has the initial value $z_1(0) = 0.1 - 0.2i$, $z_2(0) = -0.3 + 0.3i$, $z_3(0) = 0.25 - 0.1i$ and shows the synchronization errors between x and y. From simulation results in Figures 6-10, it is clearly seen that the drive-response systems (19)-(20) achieve synchronization.

Remark 4: Fractional-order complex-valued system includes fractional-order real-valued system as its special cases. In fact, in Example V.1 and Example V.2, if all the coefficients are functions from $\mathbb{R}$ to $\mathbb{R}$, and all the activation functions are functions from $\mathbb{R}$ to $\mathbb{R}$, then the state $x_l(t) \equiv x_l^R(t) \in \mathbb{R}$, in this case, systems (17)-(20) are fractional-order real-valued system. Then, similar to the proofs of 1 and 2 and under the same corresponding conditions, one can show that the similar results of Theorems 1 and 2 are still valid (see [41]–[43]).

VI. CONCLUSION

In this paper, we consider the problem of the S-asymptotically ω-periodic synchronization of fractional-order complex-valued recurrent neural networks. By using the Banach fixed point theorem and some important features of Mittag-Leffler functions, we obtain some sufficient conditions for the existence of S-asymptotically ω-periodic solutions for the neural networks by direct method, and we improve the norm. Then, by constructing an appropriate Lyapunov functional, the theory of fractional differential equation, and some inequality techniques, a novel sufficient condition has been derived to ensure the global Mittag-Leffler synchronization for the considered fractional-order neural networks. In order to demonstrate the usefulness of the presented results, some numerical examples are given. The works of this paper improve and extend the old results in literatures [33]–[35], and propose a good research thinking to study S-asymptotically ω-periodic solutions and the global Mittag-Leffler synchronization of fractional-order complex-valued recurrent neural networks with time-varying delays. In future work, S-asymptotically ω-periodic solutions in the quaternion field can be considered.

REFERENCES

[1] A. Hirose, *Complex-Valued Neural Networks: Theories and Applications*. Singapore: World Scientific, 2003.
[2] I. Aizenberg, *Complex-Valued Neural Networks With Multi-Valued Neurons*. Heidelberg, Germany: Springer, 2011.
[3] J. Hu and J. Wang, "Global stability of complex-valued recurrent neural networks with time-delays," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 23, no. 6, pp. 853–865, Jun. 2012.
[4] B. Zhou and Q. Song, "Boundedness and complete stability of complex-valued neural networks with time delay," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 24, no. 8, pp. 1227–1238, Aug. 2013.
[5] T. Fang and J. Sun, "Stability of complex-valued recurrent neural networks with time-delays," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 25, no. 9, pp. 1709–1713, Sep. 2014.
[6] J. Pan, X. Liu, and W. Xie, "Exponential stability of a class of complex-valued neural networks with time-varying delays," *Neurocomputing*, vol, 164, pp. 293–299, Sep. 2015.
[7] K. Subramanian and P. Muthukumar, "Global asymptotic stability of complex-valued neural networks with additive time-varying delays," *Cogn. Neurodyn.*, vol. 11, no. 3, pp. 293–306, 2017.
[8] G. Rajchakit and R. Sriraman, "Robust passivity and stability analysis of uncertain complex-valued impulsive neural networks with time-varying delays," *Neural Process. Lett.*, pp. 1–26, Jan. 2021, doi: 10.1007/s11063-020-10401-w.
[9] P. Chanthorn, G. Rajchakit, U. Humphries, P. Kaewmesri, R. Sriraman, and C. P. Lim, "A delay-dividing approach to robust stability of uncertain stochastic complex-valued Hopfield delayed neural networks," *Symmetry*, vol. 12, no. 5, p. 683, 2020.

[10] P. Chanthorn, G. Rajchakit, J. Thipcha, C. Emharuethai, R. Sriraman, C. P. Lim, and R. Ramachandran, "Robust stability of complex-valued stochastic neural networks with time-varying delays and parameter uncertainties," *Mathematics*, vol. 8, no. 5, p. 742, 2020.

[11] M. Yan, J. Qiu, X. Chen, X. Chen, C. Yang, and A. Zhang, "Almost periodic dynamics of the delayed complex-valued recurrent neural networks with discontinuous activation functions," *Neural Comput. Appl.*, vol. 30, no. 11, pp. 3339–3352, 2018.

[12] L. M. Pecora and T. L. Carrol, "Synchronization in chaotic systems," *Phys. Rev. Lett.*, vol. 64, no. 4, pp. 821–824, 1990.

[13] X. Huang and J. Cao, "Generalized synchronization for delayed chaotic neural networks: A novel coupling scheme," *Nonlinearity*, vol. 19, no. 12, pp. 2797–2811, 2006.

[14] L. Pan, J. Cao, and J. Hu, "Synchronization for complex networks with Markov switching via matrix measure approach," *Appl. Math. Model.*, vol. 39, no. 18, pp. 5636–5649, Sep. 2015.

[15] Y. Li and C. Li, "Matrix measure strategies for stabilization and synchronization of delayed BAM neural networks," *Nonlinear Dyn.*, vol. 84, no. 3, pp. 1759–1770, May 2016.

[16] Y. Liu, L. Sun, J. Lu, and J. Liang, "Feedback controller design for the synchronization of Boolean control networks," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 27, no. 9, pp. 1991–1996, Sep. 2016.

[17] X. Lv, X. Li, J. Cao, and M. Perc, "Dynamical and static multisynchronization of coupled multistable neural networks via impulsive control," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 12, pp. 6062–6072, Dec. 2018.

[18] C. Huang, J. Lu, G. Zhai, J. Cao, G. Lu, and M. Perc, "Stability and stabilization in probability of probabilistic Boolean networks," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 32, no. 1, pp. 241–251, Jan. 2021.

[19] X. Ding, J. Cao, A. Alsaedi, F. E. Alsaadi, and T. Hayat, "Robust fixed-time synchronization for uncertain complex-valued neural networks with discontinuous activation functions," *Neural Netw.*, vol. 90, pp. 42–55, Jun. 2017.

[20] D. Liu, S. Zhu, and E. Ye, "Synchronization stability of memristor-based complex-valued neural networks with time delays," *Neural Netw.*, vol. 96, pp. 115–127, Dec. 2017.

[21] Y. Yuan, Q. Song, Y. Liu, and F. E. Alsaadi, "Synchronization of complex-valued neural networks with mixed two additive time-varying delays," *Neurocomputing*, vol. 332, pp. 149–158, Mar. 2019.

[22] Z. Zhang, A. Li, and S. Yu, "Finite-time synchronization for delayed complex-valued neural networks via integrating inequality method," *Neurocomputing*, vol. 318, pp. 248–260, Nov. 2018.

[23] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Boston, MA, USA: Elsevier, 2006.

[24] H.-L. Li, C. Hu, Y.-L. Jiang, L. Zhang, and Z. Teng, "Global Mittag–Leffler stability for a coupled system of fractional-order differential equations on network with feedback controls," *Neurocomputing*, vol. 214, pp. 233–241, Nov. 2016.

[25] G. Rajchakit, P. Chanthorn, M. Niezabitowski, R. Raja, D. Baleanu, and A. Pratap, "Impulsive effects on stability and passivity analysis of memristor-based fractional-order competitive neural networks," *Neurocomputing*, vol. 417, pp. 290–301, Dec. 2020.

[26] G. Rajchakit, P. Chanthorn, P. Kaewmesri, R. Sriraman, and C. P. Lim, "Global Mittag–Leffler stability and stabilization analysis of fractional-order quaternion-valued memristive neural networks," *Mathematics*, vol. 8, no. 3, p. 422, Mar. 2020.

[27] U. Humphries, G. Rajchakit, P. Kaewmesri, P. Chanthorn, R. Sriraman, R. Samidurai, and C. P. Lim, "Global stability analysis of fractional-order quaternion-valued bidirectional associative memory neural networks," *Mathematics*, vol. 8, no. 5, p. 801, May 2020.

[28] A. Pratap, R. Raja, C. Sowmiya, O. Bagdasar, J. Cao, and G. Rajchakit, "Robust generalized Mittag–Leffler synchronization of fractional order neural networks with discontinuous activation and impulses," *Neural Netw.*, vol. 103, pp. 128–141, Jul. 2018.

[29] G. Rajchakit, A. Pratap, R. Raja, J. Cao, J. Alzabut, and C. Huang, "Hybrid control scheme for projective lag synchronization of Riemann–Liouville sense fractional order memristive BAM neural networks with mixed delays," *Mathematics*, vol. 7, no. 8, p. 759, Aug. 2019.

[30] S. Liu, R. Yang, X.-F. Zhou, W. Jiang, X. Li, and X.-W. Zhao, "Stability analysis of fractional delayed equations and its applications on consensus of multi-agent systems," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 73, pp. 351–362, Jul. 2019.

[31] T. Zhang and L. Xiong, "Periodic motion for impulsive fractional functional differential equations with piecewise Caputo derivative," *Appl. Math. Lett.*, vol. 101, Mar. 2020, Art. no. 106072.

[32] P. Chanthorn, G. Rajchakit, S. Ramalingam, C. P. Lim, and R. Ramachandran, "Robust dissipativity analysis of Hopfield-type complex-valued neural networks with time-varying delays and linear fractional uncertainties," *Mathematics*, vol. 8, no. 4, p. 595, Apr. 2020.

[33] H. Bao, J. H. Park, and J. Cao, "Synchronization of fractional-order complex-valued neural networks with time delay," *Neural Netw.*, vol. 81, pp. 16–28, Sep. 2016.

[34] W. Zhang, J. Cao, D. Chen, and F. Alsaadi, "Synchronization in fractional-order complex-valued delayed neural networks," *Entropy*, vol. 20, no. 1, p. 54, Jan. 2018.

[35] J. Chen, B. Chen, and Z. Zeng, "Global asymptotic stability and adaptive ultimate Mittag–Leffler synchronization for a fractional-order complex-valued memristive neural networks with delays," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 49, no. 12, pp. 2519–2535, Dec. 2019.

[36] S. Yang, J. Yu, C. Hu, and H. Jiang, "Quasi-projective synchronization of fractional-order complex-valued recurrent neural networks," *Neural Netw.*, vol. 104, pp. 104–113, Aug. 2018.

[37] H.-L. Li, C. Hu, J. Cao, H. Jiang, and A. Alsaedi, "Quasi-projective and complete synchronization of fractional-order complex-valued neural networks with time delays," *Neural Netw.*, vol. 118, pp. 102–109, Oct. 2019.

[38] X. You, Q. Song, and Z. Zhao, "Global Mittag–Leffler stability and synchronization of discrete-time fractional-order complex-valued neural networks with time delay," *Neural Netw.*, vol. 122, pp. 382–394, Feb. 2020.

[39] H. R. Henríquez, M. Pierri, and P. Táboas, "On S-asymptotically ω-periodic functions on Banach spaces and applications," *J. Math. Anal. Appl.*, vol. 343, no. 2, pp. 1119–1130, Jul. 2008.

[40] M. Pierri, "On S-asymptotically ω-periodic functions and applications," *Nonlinear Anal., Theory, Methods Appl.*, vol. 75, no. 2, pp. 651–661, Jan. 2012.

[41] B. Chen and J. Chen, "Global asymptotical ω-periodicity of a fractional-order non-autonomous neural networks," *Neural Netw.*, vol. 68, pp. 78–88, Aug. 2015.

[42] A. Wu and Z. Zeng, "Boundedness, Mittag–Leffler stability and asymptotical ω-periodicity of fractional-order fuzzy neural networks," *Neural Netw.*, vol. 74, pp. 73–84, Feb. 2016.

[43] H. Qu, T. Zhang, and J. Zhou, "Global stability analysis of S-asymptotically ω-periodic oscillation in fractional-order cellular neural networks with time variable delays," *Neurocomputing*, vol. 399, pp. 390–398, Jul. 2020.

YUANYUAN HOU received the B.S. degree from Hunan University, in 2008, and the master's degree from Yunnan University, Kunming, China, in 2016, both in software engineering. She is currently a Teacher with the School of Mathematics and Statistics, Pu'er University, China. Her current research interests include neural networks and software engineering.

LIHUA DAI received the B.S. degree in mathematics and applied mathematics from Yunnan Normal University, Kunming, China, in 2013, and the M.E. degree in systems theory from Yunnan University, Kunming, in 2016. She is currently pursuing the Ph.D. degree with the School of Mathematics and Statistics, Southwest University, Chongqing, China. Her current research interests include neural networks, nonlinear dynamic systems, and mathematical biology.