IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received February 11, 2021, accepted February 19, 2021, date of publication March 1, 2021, date of current version March 24, 2021.

Digital Object Identifier 10.1109/ACCESS.2021.3062873

Anderson Corollary Based on New Approximation Method for Continuous Interval Systems

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This work was supported in part by the Dirección General de Universidades, Investigación e Innovación of Castilla-La Mancha through the ProSeaWind Project under Grant SBPLY/19/180501/000102, in part by Science and Engineering Research Board (SERB), Department of Science and Technology (DST), Government of India, under Grant ECR/2017/000212, in part by the ENDURUNS Project, European Research Council (ERC) through the European Union's Horizon 2020 Research and Innovation Program under Agreement H2020-MG-2018-2019-2020n.824348, and in part by the TEQIP CRS Program under Project ID1-5766329561.

ABSTRACT In this research, a new technique is developed for reducing the order of high-order continuous interval systems. The model denominator is derived using Anderson corollary and Routh table. Numerator is derived by matching the formulated Markov parameters (MPs) and time moments (TMs). Distinctive features of the proposed approach are: (i) New and simpler expressions for MPs and TMs; (ii) Retaining not only TMs but also MPs while deriving the model; (iii) Minimizing computational complexity while preserving the essential characteristics of system; (iv) Ensuring to produce a stable model for stable system; (v) No need to invert the system transfer function denominator while obtaining the TMs and MPs; and (vi) No need to solve a set of complex interval equations while deriving the model. Two single-input-single-output test cases are considered to illustrate the proposed technique. Comparative analysis is also presented based on the results obtained. The simplicity and effectiveness of the proposed technique are established from the simulation outcomes achieved.

INDEX TERMS Interval systems, Kharitonov polynomials, Markov parameter, time moments, modelling, Routh approximation.

I. INTRODUCTION

For real world applications, the description of physical systems, in terms of mathematical models, produces high-order transfer functions generally. These transfer functions are relatively complex for in-depth analysis, computer simulations, and controller design [1], [2]. Therefore, the analysis and controller design of such systems become a challenging task. The simplification of such high-order transfer functions into low-order models can be considered as a possible solution. The simplification should be processed so that that the low-order models should retain the dominant characteristics of high-order systems. The simplification offers remarkable features , e.g., reduction in computational effort during simulating the behavior of system, feasible controller design, better understanding of the dynamic behavior of the system, etc., [3].

In literature, a large number of order reduction techniques are available for non-interval systems both in continuous and discrete time domains [4]. These methods include aggregation matrix [5], power decomposition method [6], pole retention technique [7], time moment matching technique [8], Routh stability criterion [9], Pade approximation [10], Hurwitz polynomial based approximation [11], stability preservation method [12], etc. In spite of the availability of several reduction techniques, only a few among

The associate editor coordinating the review of this manuscript and approving it for publication was Wen-Sheng Zhao.

these are extended for order reduction of interval systems. The main reason behind this are involvement of complex interval arithmetic and difficulty in stability analysis of model. Kharitonov [13] proposed, a breakthrough result to verify the robust stability of interval system. This result, known as Kharitonov theorem, attracted many researchers in the field of modelling and system analysis of continuous and discrete interval systems.

Many practical systems in engineering industries possess uncertainties in parameters during entire range of operating conditions. These uncertainties in the system parameters occur due to sensor noises, nonlinear effects, actuator constraints, internal and external disturbances, aging effect, manual errors, etc. The consideration of uncertainties in model of the system itself turns out to a transfer function having interval parameters. The transfer function having interval parameters is known as interval systems. Some practical systems, mathematically modelled as interval systems, are cold rolling mill, DC shunt motor, oblique wing aircraft, and Riverol-Pilipovik water treatment. The interval transfer functions of cold rolling mill, DC shunt motor, oblique wing aircraft, and Riverol-Pilipovik water treatment are given, respectively, in (1), (2), (3) and (4).

The interval transfer function of $G_{11}(s)$, $G_{12}(s)$, $G_{21}(s)$, and $G_{22}(s)$ are given as below

In the transfer functions given in (1)-(4), the coefficients of numerator and denominator polynomials are varying in definite intervals.

The pioneering work for order reduction of continuous interval system is proposed by Bandyopadhyay *et al.* [14] based on Routh-Pade technique. Here, denominator of the model is obtained by direct truncation of Routh table and numerator is derived using matching of coefficients of power series expansion of the system to model. But, in [15], it is shown that this method generates unstable interval models for stable high-order interval systems in few cases. To overcome this limitation, the formula given in [14], is modified while constructing the elements of Routh table. The resulting reduced-order interval models are assumed to be stable. Further, Yang [16] proved that the method developed in [15] also does not ensure stability of reduced model in all cases. Finally, Dolgin [17] inserted two additional conditions while constructing the elements of Routh table to overcome the problem of unstable denominator polynomial.

Recently, other methods have also been developed for order reduction of interval systems. A reduction technique based on Routh approximation (RA) using Kharitonov polynomial is presented in [18]. In [19], a direct RA method for reduction of interval systems is observed to generate a stable reduced model. The article [20] proposed reduction employing optimization techniques like particle swarm optimization. Also, mixed technique is suggested in [21], where the numerator of the model is obtained using classical reduction methods like Cauer second form, Pade approximation, differentiation and moment matching method, and the denominator is calculated by differentiation method. A variable substitution method is developed in [22], where overshoot of the non-linear system is considered as an major criteria for controller design. An linear matrix criteria was developed for singular fractional order system with order $0 < \alpha < 1$ from non singular decomposition method and stability theory [23]. Zhang and Yang [24] developed a new control strategy that guarantees the prescribed tracking performance for a class of uncertain nonlinear single system with unknown control direction. Kumar *et al.* [25] developed a technique to derive denominator and numerator of an interval model from Routh approximation using Kharitonov polynomials. In article [26], a class of control problems for multi-input-multi-output (MIMO) unknown Euler-Lagrange systems with output constrained are investigated, and also

$$G(s) = \frac{[0.5, 2.6] + [3, 16]\,s + [4.2, 21]\,s^2}{[0.05, 0.15] + [1, 2.5]\,s + [3, 8]\,s^2\,[1, 1]\,s^3} \tag{1}$$

$$G(s) = \frac{[50000, 50000]}{[2025, 2475]s + [1200, 2800]s^2 + [9.6, 33.6]s^3} \tag{2}$$

$$G(s) = \frac{[900, 1660] + [54, 74]s}{[-1, 1] + [301, 339]s + [504, 808]s^2 + [28, 46]s^3 + 10s^4} \tag{3}$$

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \tag{4}$$

$$\begin{aligned} G_{11}(s) &= \frac{0.0045(0.104s) + 1}{(0.012s^2)} \\ G_{12}(s) &= 0 \\ G_{21}(s) &= \frac{[900, 1660] + [54, 74]s}{[-1, 1] + [301, 339]s + [504, 808]s^2 + [28, 46]s^3 + 10s^4} \\ G_{22}(s) &= \frac{10(-3s + 1)}{s^2 + 5s + 1} \end{aligned}$$

fault- tolerant control technique is effective to compensate for the actuator faults which ensure reliability of the dynamic system like inverted pendulum. Hote *et al.* [27] proposed a non-interval reduced model for a high-order interval system. The denominator of the model is derived from Anderson corollary and Routh approximation whereas the numerator is calculated by matching steady-state value of system to that of model. The reduced model obtained from this technique does not contain interval values of steady-state of high-order system. Recently, Singh *et al.* [28] formulated expressions for calculation of MPs and TMs. Further, Routh-Pade approximation for simplification of high-order interval systems employing the derived MPs and TMs is proposed in [28].

In this investigation, two simple generalized expressions for calculating the Markov parameters (MPs) and time moments (TMs) of continuous interval systems are proposed. Unlike the other methods, this method does not require to invert the denominator of transfer functions of the system and model nor an additional step of solving a set of interval equations for calculating MPs and TMs. Firstly, denominator of model is derived using Anderson corollary and direct truncation of Routh table. Secondly, the coefficients of numerator polynomial of the model are achieved by equating some MPs and TMs of the system with those of the model. Considering the matching of TMs, the steady-state response is improved, while the matching of MPs improves transient response matching. The key highlights of the proposed method are:

- Simpler expressions for MPs and TMs are developed.
- Both MPs and TMs are retained while deriving the model.
- The method is simple and involves relatively lesser computations.
- The proposed model preserves the essential characteristics of system.
- It generates stable model for stable system.
- There is no requirement of inversion of denominator of transfer function of the system to calculate TMs.
- There is no need to solve a set of complex interval equations for deriving the MPs and TMs.

The overview of article is as follows: problem formulation is discussed in Section 2; the generalized expressions of MPs and TMs of interval system are derived in Section 3; Section 4 discusses procedure to derive model; Section 5 includes the demonstration of proposed technique with the help of two test systems; and, finally Section 6 provides conclusions.

II. PROPOSED TECHNIQUE

The high-order continuous interval system (HOCIS) can be expressed by equation (5)

$$G_n(s) = \frac{P_n(s)}{Q_n(s)} = \frac{P_0 + P_1 s + P_2 s^2 + \cdots + P_{n-1} s^{n-1}}{Q_0 + Q_1 s + Q_2 s^2 + \cdots + Q_n s^n} \quad (5)$$

where

$$P_i = [P_i^-, P_i^+] \quad (6)$$

for $i = 0, 1, 2, \ldots, n-1$ and

$$Q_i = [Q_i^-, Q_i^+] \quad (7)$$

for $i = 0, 1, 2, \ldots, n$.

$P_n(s)$ and $Q_n(s)$ are, respectively, the numerator and denominator interval polynomials of HOCIS. The parameters P_i^- and Q_i^- are the lower limits, and P_i^+ and Q_i^+ are the upper limits of interval coefficients. Expansions of HOCIS, (5), about $s = 0$ and $s = \infty$, respectively, are expressed as

$$G_n(s) = \alpha_0 + \alpha_1 s + \cdots + \alpha_k s^k + \cdots \quad (8)$$

$$G_n(s) = \beta_1 s^{-1} + \beta_2 s^{-2} + \cdots + \beta_k s^{-k} + \cdots \quad (9)$$

where, $\alpha_i = [\alpha_i^-, \alpha_i^+]$ for $i = 0, 1, 2, \ldots$ and $\beta_i = [\beta_i^-, \beta_i^+]$ for $i = 1, 2, 3, \ldots$ are TMs and MPs of interval system respectively.

The adequate kth-order reduced order continuous interval model (ROCIM) of the system given in (5) is represented by (10)

$$G_k(s) = \frac{p_k(s)}{q_k(s)} = \frac{u_0 + u_1 s + \cdots + u_{k-1} s^{k-1}}{v_0 + v_1 s + \cdots + v_k s^k} \quad (10)$$

where $k < n$, being

$$u_i = [u_i^-, u_i^+] \quad (11)$$

for $i = 0, 1, 2, \ldots, k-1$ and

$$v_i = [v_i^-, v_i^+] \quad (12)$$

for $i = 0, 1, 2, \ldots, k$.

$p_k(s)$ and $q_k(s)$ are, respectively, interval polynomials of ROCIM. The model (10), is expressed in terms of TMs and MPs by equations (13) and (14)

$$G_k(s) = \hat{\alpha}_0 + \hat{\alpha}_1 s + \cdots + \hat{\alpha}_k s^k + \cdots \text{(expansion about } s = 0) \quad (13)$$

$$G_k(s) = \hat{\beta}_1 s^{-1} + \hat{\beta}_2 s^{-2} + \cdots + \hat{\beta}_k s^{-k} + \cdots \text{(expansion about } s = \infty) \quad (14)$$

where

$$\hat{\alpha}_i = [\hat{\alpha}_i^-, \hat{\alpha}_i^+] \quad (15)$$

for $i = 0, 1, 2, \ldots$ and

$$\hat{\beta}_i = [\hat{\beta}_i^-, \hat{\beta}_i^+] \quad (16)$$

for $i = 1, 2, 3, \ldots$

III. PROPOSED GENERALIZED EXPRESSIONS OF TMs AND MPs

In order to simplify the problem of calculating TMs and MPs of the interval system, the HOCIS, given in (5), is rewritten as equation (17)

$$G_n(s) = \frac{P_n(s)}{\overline{Q}_n(s)} = \frac{[P_0^-, P_0^+] + [P_1^-, P_1^+]s + [P_2^-, P_2^+]s^2 + \cdots + [P_{n-1}^-, P_{n-1}^+]s^{n-1}}{\overline{Q}_0 + \overline{Q}_1 s + \overline{Q}_2 s^2 + \cdots + \overline{Q}_n s^n} \quad (17)$$

It is to be noted that the coefficients of the denominator polynomial $\overline{Q}_n(s)$ are fixed values instead of interval coefficients of $Q_n(s)$. These fixed values are mid-points of the intervals, which are calculated using equation (18)

$$\overline{Q}_i = (Q_i^- + Q_i^+)/2 \tag{18}$$

where $i = 0, 1, 2, \ldots, n$.

Therefore, about $s = 0$ and $s = \infty$, power series expansions of (17), respectively, are given as

$$\begin{aligned} G_n(s) &= \frac{P_n(s)}{\overline{Q}_n(s)} \\ &= \alpha_0 + \frac{[P_1^-, P_1^+] - \alpha_0\overline{Q}_1}{\overline{Q}_0}s \\ &\quad + \frac{[P_2^-, P_2^+] - \alpha_0\overline{Q}_2 - \alpha_1\overline{Q}_1}{\overline{Q}_0}s^2 + \cdots \end{aligned} \tag{19}$$

$$\begin{aligned} G_n(s) &= \frac{P_n(s)}{\overline{Q}_n(s)} \\ &= \beta_1 s^{-1} + \frac{[P_{n-2}^-, P_{n-2}^+] - \beta_1\overline{Q}_{n-1}}{\overline{Q}_n}s^{-2} \\ &\quad + \frac{[P_{n-3}^-, P_{n-3}^+] - \beta_1\overline{Q}_{n-2} - \beta_2\overline{Q}_{n-1}}{\overline{Q}_n}s^{-2} + \cdots \end{aligned} \tag{20}$$

where, $\alpha_0 = \dfrac{[P_0^-, P_0^+]}{\overline{Q}_0}$ and $\beta_1 = \dfrac{[P_{n-1}^-, P_{n-1}^+]}{\overline{Q}_n}$

Using (8) and (19), the generalized expression for TMs of continuous interval system (5) can be written by equation (21)

$$\alpha_m = \frac{P_m - \sum_{i=0}^{m-1} \alpha_i \overline{Q}_{m-i}}{\overline{Q}_0}, \quad m = 0, 1, 2, 3, \ldots \tag{21}$$

By comparing (9) and (20), the generalized expression for MPs of continuous interval system, given in (5), are expressed by equation (22)

$$\beta_m = \frac{P_{n-m} - \sum_{i=1}^{m-1} \beta_i \overline{Q}_{n-m+i}}{\overline{Q}_n}, \quad m = 1, 2, 3, \ldots \tag{22}$$

In a similar manner, TMs and MPs (Appendix A) of ROCIM (10) turn out to be given by equations (23) and (24)

$$\overset{\wedge}{\alpha}_m = \frac{u_m - \sum_{i=0}^{m-1} \overset{\wedge}{\alpha}_i \, \overline{v}_{m-i}}{\overline{v}_0}, \quad m = 0, 1, 2, 3, \ldots \tag{23}$$

$$\overset{\wedge}{\beta}_m = \frac{P_{k-m} - \sum_{i=1}^{m-1} \overset{\wedge}{\beta}_i \, \overline{v}_{k-m+i}}{\overline{v}_k}, \quad m = 1, 2, 3, \ldots \tag{24}$$

IV. PRINCIPAL RESULTS

To illustrate the effectiveness and applicability of proposed TMs and MPs, Anderson corollary based improved Routh-Pade approximation is proposed in this section.

A. PROPOSED LEMMA FOR APPROXIMATION OF THIRD-ORDER INTERVAL SYSTEM

The desired first-order and second-order continuous interval models of a third-order continuous interval system is given by equation (25)

$$\begin{aligned} G_3(s) &= \frac{P_3(s)}{Q_3(s)} \\ &= \frac{[P_0^-, P_0^+] + [P_1^-, P_1^+]s + [P_2^-, P_2^+]s^2}{[Q_0^-, Q_0^+] + [Q_1^-, Q_1^+]s + [Q_2^-, Q_2^+]s^2 + [Q_3^-, Q_3^+]s^3} \end{aligned} \tag{25}$$

where equation can (25) be represented by equation (26)

$$G_1(s) = \frac{p_1(s)}{q_1(s)} = \frac{u_0}{v_1 s + v_0} \tag{26}$$

$$G_2(s) = \frac{p_2(s)}{q_2(s)} = \frac{u_1 s + u_0}{v_2 s^2 + v_1 s + v_0} \tag{27}$$

The procedure to obtain the desired and reduced models are discussed below.

1) PROCEDURE TO CALCULATE THE DENOMINATOR

The coefficients of denominator polynomial are obtained employing Anderson corollary [27]. The procedure is described as follows:

Step 1: Apply Anderson corollary [27] on HOCIS. After application, (25) can be expressed by equation (28)

$$G(s) = \frac{P_0^- + P_1^+ s + P_2^+ s^2}{Q_0^+ + Q_1^- s + Q_2^- s^2 + Q_3^+ s^3} \tag{28}$$

Step 2: Construct Routh table for denominator polynomial of (28) as presented in Table 1.

TABLE 1. Routh table.

s^3	Q_3^+	Q_1^-
s^2	Q_2^-	Q_0^+
s^1	$\dfrac{Q_2^- Q_1^- - Q_3^+ Q_0^+}{Q_2^-} = X$	
s^0	Q_0^+	

Step 3: Obtain denominator polynomial from $(n+1-k)$ and $(n+2-k)$ rows of the Routh table (Table 1).

The denominator polynomials $q_1(s)$ and $q_2(s)$ of the first-order and second-order models, $G_1(s)$ and $G_2(s)$, respectively, become equations (29) and (30)

$$\begin{aligned} q_1(s) &= v_1 s + v_0 \\ &= Xs + Q_0^+ \\ &= \frac{Q_2^- Q_1^- - Q_3^+ Q_0^+}{Q_2^-}s + Q_0^+ \end{aligned} \tag{29}$$

$$\begin{aligned} q_2(s) &= v_2 s^2 + v_1 s + v_0 \\ &= Q_2^- s^2 + Xs + Q_0^+ \\ &= Q_2^- s^2 + \frac{Q_2^- Q_1^- - Q_3^+ Q_0^+}{Q_2^-}s + Q_0^+ \end{aligned} \tag{30}$$

2) PROCEDURE TO CALCULATE THE NUMERATOR

The coefficients of unknown numerator polynomial are achieved by equating some initial TMs and MPs of HOCIS and ROCIM as

$$\alpha_i = \hat{\alpha}_i \quad \text{for } i = 0, 1, 2, 3, \ldots, (\mu - 1) \tag{31}$$

and

$$\beta_i = \hat{\beta}_i \quad \text{for } i = 1, 2, 3, \ldots, \lambda \tag{32}$$

where, $\mu + \lambda = k$ and $\mu \geq 1$. At least one time moment of HOCIS and ROCIM is matched by considering $\mu \geq 1$, which guarantees a better matching of steady-state response between original system and model, and also matching of Markov parameters improve the transient-state response.

The new and simple expressions for calculating time moments $(\alpha_i, \hat{\alpha}_i)$ and Markov parameters $(\beta_i, \hat{\beta}_i)$ of HOCIS and ROCIM are proposed in Section III.

B. PROPOSED LEMMA FOR APPROXIMATION OF SECOND-ORDER INTERVAL SYSTEM

Let the desired first-order continuous interval model of a second-order continuous interval system to be given by equation (33)

$$G_2(s) = \frac{P_2(s)}{Q_2(s)} = \frac{[P_0^-, P_0^+] + [P_1^-, P_1^+]s}{[Q_0^-, Q_0^+] + [Q_1^-, Q_1^+]s + [Q_2^-, Q_2^+]s^2} \tag{33}$$

be represented by equation (34)

$$G_1(s) = \frac{p_1(s)}{q_1(s)} = \frac{u_0}{v_1 s + v_0} \tag{34}$$

The procedure to obtain the desired first-order model is discussed below.

1) PROCEDURE TO CALCULATE THE DENOMINATOR

The coefficients of denominator polynomial are obtained using Anderson corollary [27]. The procedure is illustrated below.

Step 1: Apply Anderson corollary [27] on HOCIS. Hence, (33) becomes as equation (35)

$$G(s) = \frac{P_0^- + P_1^+ s}{Q_0^+ + Q_1^- s + Q_2^- s^2} \tag{35}$$

Step 2: Construct the Routh table for (35). The Routh table for (35) is provided in Table 2.

TABLE 2. Routh table.

s^2	Q_2^+	Q_0^-
s^1	Q_1^-	
s^0	Q_0^-	

Step 3: Obtain the denominator polynomial from Routh table (Table 2).

The denominator $q_1(s)$ of the first-order model can be achieved from Routh table (Table 2) as given by equation (36)

$$q_1(s) = v_1 s + v_0 = Q_1^- s + Q_0^- \tag{36}$$

2) PROCEDURE TO CALCULATE THE NUMERATOR

The coefficient of numerator, $u_0(s)$, is derived by equating initial TM of HOCIS and ROCIM such that

$$\alpha_0 = \hat{\alpha}_0 \tag{37}$$

By using equations (21) and (23) from Section III, equation (37) becomes as

$$\frac{P_0}{\overline{Q}_0} = \frac{u_0}{\overline{v}_0} \tag{38}$$

the matching of initial time moment will improves the steady-state responses between HOCIS and ROCIM. The formulas for calculating the time moments $(\alpha_0, \hat{\alpha}_0)$ of HOCIS and ROCIM are given Section III.

V. TEST CASES

Two test systems are taken into consideration to demonstrate procedure and performance of the proposed technique.

A. TEST CASE 1

Consider a single-input-single-output (SISO) third-order interval system given by equation (39)

$$G_3(s) = \frac{[2, 3]s^2 + [17.5, 18.5]s + [15, 16]}{[2, 3]s^3 + [17, 18]s^2 + [35, 36]s + [20.5, 21.5]} \tag{39}$$

The desired first-order and second-order interval models of (39) are given by equations (40) and (41)

$$G_1(s) = \frac{p_1(s)}{q_1(s)} = \frac{u_0}{v_1 s + v_0} \tag{40}$$

and

$$G_2(s) = \frac{p_2(s)}{q_2(s)} = \frac{u_1 s + u_0}{v_2 s^2 + v_1 s + v_0} \tag{41}$$

1) CALCULATION OF DENOMINATOR POLYNOMIAL

The coefficients of denominator polynomial are calculated as follows

Step 1: Utilizing (28), the high-order interval system (39) modifies to equation (42)

$$G(s) = \frac{15 + 18.5s + 3s^2}{21.5 + 35s + 17s^2 + 3s^3} \tag{42}$$

Step 2: The Routh table for (42) is provided in Table 3.

Step 3: The denominator of the first-order interval model, (40), obtained using (29), is given by equation (43)

$$q_1(s) = 31.2s + 21.5 \tag{43}$$

The denominator of the second-order interval model, (41), calculated using (30), is written as equation (44)

$$q_2(s) = 17s^2 + 31.2s + 21.5 \tag{44}$$

TABLE 3. Routh table.

s^3	3	35
s^2	17	21.5
s^1	31.2	
s^0	21.5	

2) CALCULATION OF NUMERATOR POLYNOMIAL

The initial TMs and MPs of the HOCIS, (39), calculated from (21) and (22), are given by equations (45-47)

$$\alpha_0 = [0.714, 0.762] \tag{45}$$

$$\alpha_1 = [-0.455, -0.326] \tag{46}$$

$$\beta_1 = [0.8, 1.2] \tag{47}$$

The numerator polynomial of first-order model, (40), can be obtained by equating first TM of HOCIS and ROCIM by equation (48)

$$\hat{\alpha}_0 = \alpha_0 \tag{48}$$

Using (23) and (45), the numerator polynomial obtained is given by equation (49)

$$[u_0^-, u_0^+] = [14.99, 16.002] \tag{49}$$

Therefore, the desired first-order interval model, $G_1(s)$, obtained using (43) and (49), becomes equation (50)

$$G_1(s) = \frac{[14.99, 16.002]}{31.2s + 21.5} \tag{50}$$

Similarly, the numerator polynomial of desired second-order model, (41), can be calculated by matching initial TMs and MPs, such that $\hat{\alpha}_0 = \alpha_0$ and $\hat{\beta}_1 = \beta_1$. Using (23), (45), (24) and (47), the numerator coefficients of (41) turn out to be given by equation (51-52)

$$[u_0^-, u_0^+] = [14.99, 16.002] \tag{51}$$

$$[u_1^-, u_1^+] = [13.6, 20.4] \tag{52}$$

Therefore, the desired second-order interval model, $G_2(s)$, obtained using (51), (52) and (44) takes the form given by equation (53)

$$G_2(s) = \frac{[12.5, 16.8]s + [15.35, 16.38]}{17s^2 + 31.2s + 21.5} \tag{53}$$

To prove the efficacy of proposed technique, the obtained proposed model of (39) is compared with the other approximants obtained using existing methods. The second-order approximants of (39) obtained using methods from Kumar *et al.* [25], Hote *et al.* [27], Singh *et al.* [28], Bandyopadhyay *et al.* [29], Sastry *et al.* [30], and Kumar *et al.* [31] are given in equations (54)-(59), respectively.

$$G_{MK}(s) = \frac{[1.172, 1.36]s+[1.0269, 1.11]}{[1, 1]s^2+[2.35, 2.62]s+[1.41, 1.52]} \tag{54}$$

$$G_H(s) = \frac{15}{17s^2+31.2s+21.5} \tag{55}$$

$$G_{SD}(s) = \frac{[8.27, 24.05]s+[14.35, 16.77]}{[17, 18]s^2+[29.47, 35.7]s+[20.5, 21.5]} \tag{56}$$

TABLE 4. Time moments and Markov parameter of system and models.

System/ models	Authors	First two TMs	First MP
(39)	System	α_0=[0.714, 0.762] α_1=[−0.455, −0.326]	β_1=[0.8, 1.2]
(53)	Proposed method	$\hat{\alpha}_0$=[0.714, 0.762] $\hat{\alpha}_1$=[−0.525, −0.255]	$\hat{\beta}_1$=[0.74, 1.001]
(54)	Kumar *et al.* [25]	$\hat{\alpha}_0$=[0.703, 0.761] $\hat{\alpha}_1$=[−0.491, −0.259]	$\hat{\beta}_1$=[1.172, 1.368]
(55)	Hote *et al.* [27]	$\hat{\alpha}_0$=0.697 $\hat{\alpha}_1$=−1.012	$\hat{\beta}_1$=0.8823
(56)	Singh *et al.* [28]	$\hat{\alpha}_0$=[0.683, 0.799] $\hat{\alpha}_1$=[−0.851, 0.085]	$\hat{\beta}_1$=[0.473, 1.374]
(57)	Bandyopadhyay *et al.* [29]	$\hat{\alpha}_0$=[0.635, 0.843] $\hat{\alpha}_1$=[−0.658, −0.121]	$\hat{\beta}_1$=[1.009, 1.255]
(58)	Sastry *et al.* [30]	$\hat{\alpha}_0$=[0.672, 0.881] $\hat{\alpha}_1$=[−0.774, −0.113]	$\hat{\beta}_1$=[0.941, 1.349]
(59)	Kumar *et al.* [31]	$\hat{\alpha}_0$=[0.675, 0.807] $\hat{\alpha}_1$=[−0.715, −0.074]	$\hat{\beta}_1$=[0.639, 1.169]

$$G_B(s) = \frac{[1.01, 1.26]s+[0.841, 1.12]}{[1.0, 1.0]s^2+[2.02, 2.45]s+[1.15, 1.51]} \tag{57}$$

$$G_{Sa}(s) = \frac{[0.94, 1.35]s+[0.891, 1.167]}{[1, 1]s^2+[2.02, 2.45]s+[1.15, 1.51]} \tag{58}$$

$$G_K(s) = \frac{[11.12, 20.38]s+[14.17, 16.94]}{[17.01,18.04]s^2+[31.38,33.61]s+[20.31, 21.71]} \tag{59}$$

Therefore, the initial TMs and MPs for system, proposed model, and models obtained due to various existing techniques are calculated and provided in Table 4. From the data tabulated in Table 4, the TMs of system are [0.714,0.762], and [−0.455,−0.326], whereas those of the proposed model are [0.714,0.762] and [−0.525,−0.255]. Therefore, it is clear that TMs of system and proposed model are matching very closely. However, the TMs of models obtained using other existing methods Kumar *et al.* [25], Hote *et al.* [27], Singh *et al.* [28], Bandyopadhyay *et al.* [29], Sastry *et al.* [30], and Kumar *et al.* [31] are, respectively, ([0.703,0.761], [−0.491,−0.259]), (0.697, −1.012), ([0.683,0.799], [−0.851,0.085]), ([0.635,0.843], [−0.658,−0.121]), ([0.672,0.881], [−0.774,−0.113]), and ([0.675,0.807], [−0.715,−0.074]). From these values, it is evident that TMs due to the techniques proposed by Kumar *et al.* [25], Hote *et al.* [27], Singh *et al.* [28], Bandyopadhyay *et al.* [29], Sastry *et al.* [30], and Kumar *et al.* [31] are having larger deviation from those of the system. Also, it can be noted here that the TMs of the model proposed by Hote *et al.* [27] is having non-interval values. Therefore, from this analysis, it is clearly proven that the proposed method is able to produce interval TMs, which are closer to those of the system when compared to the TMs obtained for other techniques.

The Markov parameter of HOCIS and ROCIMs obtained by different techniques from literature are provided in Table 4. It is clearly observed that Markov parameter of proposed model is [0.74,1.001], which is closer to the Markov parameter of system, i.e., [0.8,1.2]. However, the Markov

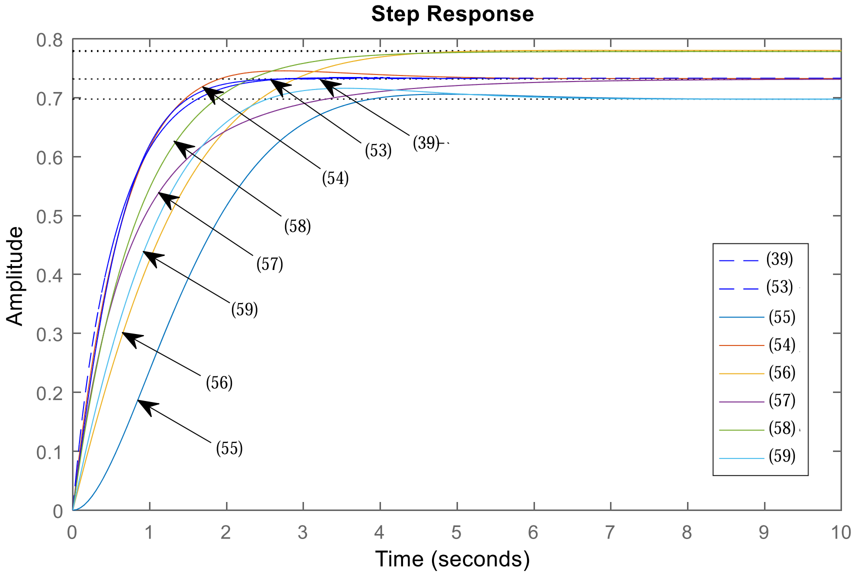

FIGURE 1. Step responses of models and original system.

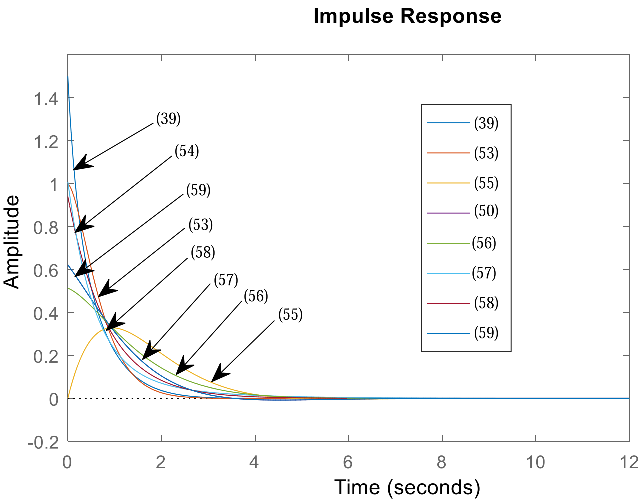

FIGURE 2. Impulse responses of models and original system.

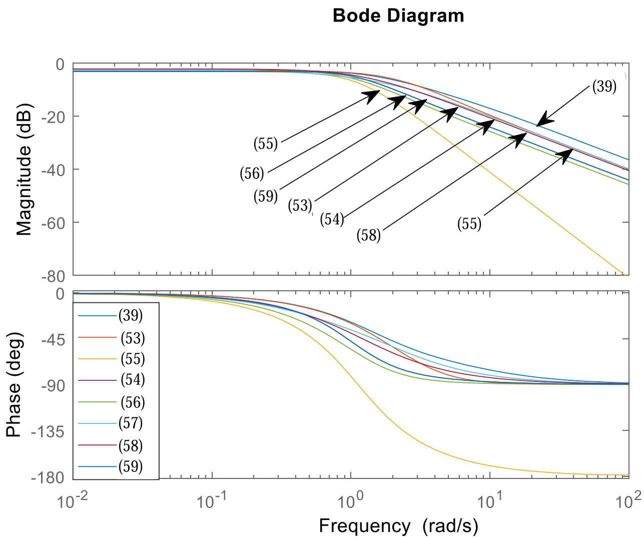

FIGURE 3. Bode responses of models and original system.

parameter of models due to the techniques proposed by Kumar *et al.* [25], Hote *et al.* [27], Singh *et al.* [28], Bandyopadhyay *et al.* [29], Sastry *et al.* [30], and Kumar *et al.* [31] are, respectively, given as [1.172,1.368], [0.8823], [0.473,1.374], [1.009,1.255], [0.941,1.349], and [0.639,1.169]. This data clearly shows that the MP of other techniques due to Kumar *et al.* [25], Hote *et al.* [27], Singh *et al.* [28], Bandyopadhyay *et al.* [29], Sastry *et al.* [30], and Kumar *et al.* [31] are having more deviation. Moreover, in case of model proposed by Hote *et al.* [27], it is clear that it is producing non-interval MP. Hence, it is clearly seen that the proposed method is generating interval MP, which are closer to that of the system as compared to other techniques considered. Figure 1 shows the step responses of system (39), model (53) obtained by proposed method, and models (54)-(59) derived by methods existing in literature due to Kumar *et al.* [25], Hote *et al.* [27], Singh *et al.* [28], Bandyopadhyay *et al.* [29], Sastry *et al.* [30] and Kumar *et al.* [31]. From Figure 1, it is clearly observable that the step response of proposed model (53) is matching closely to that of the original system (39) than step responses of the other models (54)-(59). The same holds true for both impulse and frequency responses as given in Figures 2 and 3. From the above discussion, it can be concluded that the proposed approximant is more efficient in performance than the other approximants in terms of transient and steady state response matching.

B. TEST SYSTEM 2

A second-order interval system is given by equation (60)

$$G(s) = \frac{[2.0, 3.0]\,s + [15.0, 16.0]}{[2.0, 3.0]\,s^2 + [12.0, 13.0]\,s + [10.0, 11.0]} \quad (60)$$

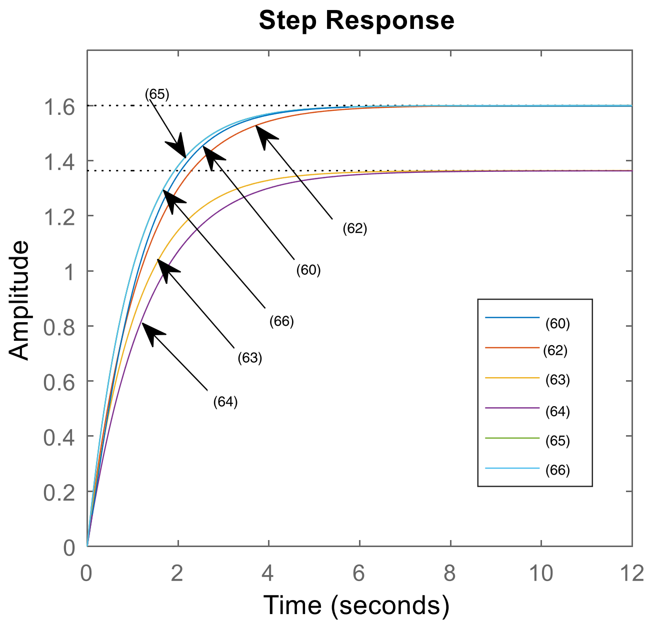

FIGURE 4. Step responses of models and original system.

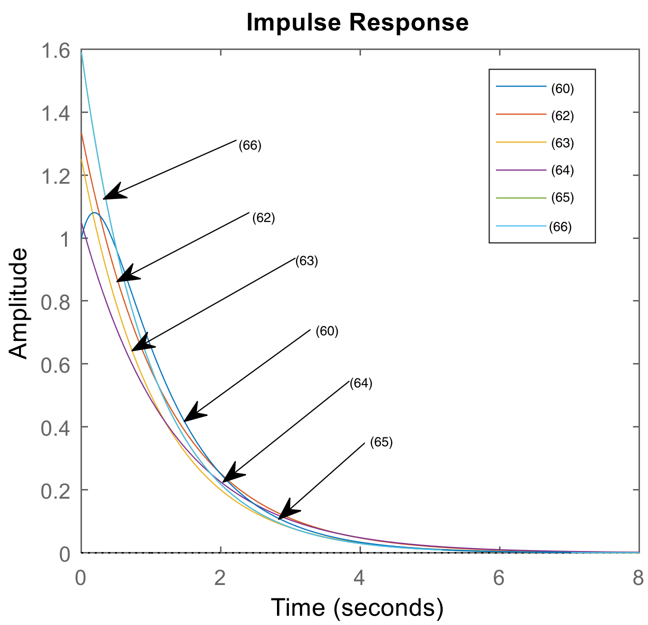

FIGURE 5. Impulse responses of models and original system.

and its desired first-order transfer function be of the form according to equation (61)

$$G_1(s) = \frac{u_0}{v_1 s + v_0} \tag{61}$$

The first-order interval model for (60) using (37) and (38) turns out to be given equation (62)

$$G_1(s) = \frac{[13.64,\, 16]}{12s + 10} \tag{62}$$

While the first-order models obtained due to the techniques proposed by Hote *et al.* [27], Sastry *et al.* [30], Bandyopadhyay *et al.* [14] and Singh and Chandra [32] are given by

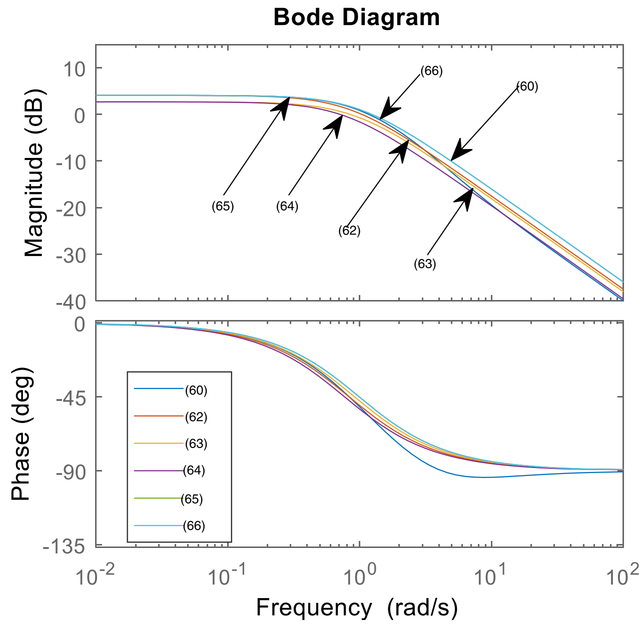

FIGURE 6. Bode responses of models and original system.

TABLE 5. Time moments of system and models.

System/ models	Authors	Time moment
(60)	System	α_0=[1.36, 1.6]
(62)	Proposed method	$\hat{\alpha}_0$=[1.36, 1.6]
(63)	Hote *et al.* [27]	$\hat{\alpha}_0$=0.697
(64)	Sastry *et al.* [30]	$\hat{\alpha}_0$=[1.15, 1.91]
(65)	Bandyopadhyay *et al.* [14]	$\hat{\alpha}_0$=[1.06, 2.07]
(66)	Singh and Chandra [32]	$\hat{\alpha}_0$=[1.06, 2.07]

equation (63-66)

$$G_1(s) = \frac{15}{12s + 11} \tag{63}$$

$$G_{Sa}(s) = \frac{[1.05,\, 1.467]}{[1,\, 1]\, s + [0.77,\, 0.917]} \tag{64}$$

$$G_B(s) = \frac{[12.58,\, 19.072]}{[12,\, 13]\, s + [9.23,\, 11.92]} \tag{65}$$

$$G_{SC}(s) = \frac{[12.58,\, 19.072]}{[12,\, 13]\, s + [9.23,\, 11.92]} \tag{66}$$

The first TM of system (60) and models (62)-(66) are calculated and tabulated in Table 5. Table 5 shows that the first TM of the original system (60) and model (62) obtained by the proposed method are exactly the same. However, the first TM of other models (63)-(66) obtained using existing methods are deviating from the TM of original system (60).

Figures 4-6 represent comparison of the step, impulse and frequency responses of the original system (60) with the proposed model (62) and other models (63)-(66). It is clear that response of proposed model (62) is very close to original system (60) than other models (63)-(66). The steady-state response of proposed model is same as of original system.

This shows that proposed method is efficient in producing better approximant for interval systems.

VI. CONCLUSION

This research work shows a computationally efficient and simpler algorithm for reducing high-order interval systems. In addition to proposing the new efficient algorithm for reducing continuous interval systems, simpler generalized expressions for calculating MPs and TMs are also proposed in such a manner so that there is no requirement of inversion of transfer function. Also, a solution of a set of interval equations can successfully be evaded while obtaining these parameters. The denominator of proposed model is derived using Anderson corollary and Routh approximation while numerator is obtained by equating initial TMs and MPs of system and model. The steady-state and transient-state responses of proposed model match closely to those of system. Two SISO test systems are considered to demonstrate the proposed technique. The simulation results prove that the proposed technique offers an excellent alternative approach for reducing the order of continuous interval systems. The future research directions of this technique lies in design of control using reduced order modeling.

APPENDIX A

Rewriting ROCIM (10) as equation (67)

$$G_k(s) = \frac{p_k(s)}{\overline{q}_k(s)} = \frac{[u_0^-, u_0^+] + [u_1^-, u_1^+]s + [u_2^-, u_2^+]s^2 + \cdots + [u_{k-1}^-, u_{k-1}^+]s^{k-1}}{\overline{v}_0 + \overline{v}_1 s + \overline{v}_2 s^2 + \cdots + \overline{v}_k s^k} \quad (67)$$

The power series expansions of (67) about $s = 0$ is given by equation (68)

$$G_k(s) = \hat{\alpha}_0 + \frac{[u_1^-, u_1^+] - \hat{\alpha}_0\, \overline{v}_1}{\overline{v}_0} s + \frac{[u_2^-, u_2^+] - \hat{\alpha}_0\, \overline{v}_2 - \hat{\alpha}_1\, \overline{v}_1}{\overline{v}_0} s^2 + \cdots \quad (68)$$

By matching (68) and (13), the TMs of interval model (10) can be written by equation (69)

$$\hat{\alpha}_m = \frac{u_m - \sum_{i=0}^{m-1} \hat{\alpha}_i\, \overline{v}_{m-i}}{\overline{v}_0}, \quad m = 1, 2, 3, \ldots \quad (69)$$

The same procedure is used to obtain MPs of interval model by expanding (10) around $s = \infty$ and comparing it with (14).

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