IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received January 15, 2021, accepted February 18, 2021, date of publication February 26, 2021, date of current version March 9, 2021.

Digital Object Identifier 10.1109/ACCESS.2021.3061600

On the Path Cover Number of Connected Quasi-Claw-Free Graphs

HUIQING LIU, JIAN LU, SHUNZHE ZHANG, AND XIUYU ZHONG
Hubei Key Laboratory of Applied Mathematics, Faculty of Mathematics and Statistics, Hubei University, Wuhan 430062, China

Corresponding author: Shunzhe Zhang (shunzhezhang@hubu.edu.cn)

This work was supported in part by the National Natural Science Foundation of China under Grant 11971158 and Grant 61373019.

ABSTRACT Detecting vertex disjoint paths is one of the central issues in designing and evaluating an interconnection network. It is naturally related to routing among nodes and fault tolerance of the network. A path cover of a graph G is a spanning subgraph of G consisting of vertex disjoint paths, and a path cover number of G denoted by $p(G) = \min\{|\mathcal{P}| : \mathcal{P}$ is a path cover of $G\}$. In this paper, we show that if the minimum degree sum of an independent set with $k+1$ vertices in a connected quasi-claw-free graph G of order n is no less than $n-k$, then $p(G) \leq k-1$, where $k \geq 2$. Examples illustrate that the degree sum condition in our result is sharp.

INDEX TERMS Path cover number, quasi-claw-free graph, vertex disjoint path.

I. INTRODUCTION

It is well known that a multiprocessor system plays a crucial role in parallel and distributed computing. Such a system has an underlying topology, which is frequently represented as a graph in which vertices and edges correspond to nodes and links, respectively. Since a lot of mutually conflicting requirements are inevitable in designing the topology of an interconnection network, it is almost impossible to design a network which is optimum from all aspects. In order to design a suitable network satisfying the requirements and its properties, it is necessary to study how well other networks can be embedded into this network.

Linear arrays (i.e., paths) is a fundamental network for parallel and distributed computing. Based on linear arrays, many efficient algorithms are originally designed to solve a variety of algebraic problems, graph problems and some parallel applications, especially those in image and signal processing [9]. Hence, it is important to detect an effective path embedding in a network. Finding parallel paths among nodes is one of the central issues concerned with efficient data transmission. Since full utilization of network nodes is important [14], parallel paths are usually studied in terms of vertex disjoint paths in graphs. It has drawn considerable research attention to its properties of vertex disjoint paths embedding, see [3], [8] and [10].

The associate editor coordinating the review of this manuscript and approving it for publication was Zhenliang Zhang.

Let $G = (V(G), E(G))$ be a simple, undirected and finite graph. For a vertex $v \in V(G)$, $N_G(v)$ is the neighbors of v and $d_G(v) = |N_G(v)|$. If $S \subseteq V(G)$, then $N(S)$ denotes the neighbors of S and $G[S]$ denotes the subgraph induced by S. For a subgraph H of G, let $G - H := G[V(G) \setminus V(H)]$. When $H = \{v\}$, we simplify $G - \{v\}$ to $G - v$. For two vertices u and v, the distance between u and v, denoted by $d_G(u, v)$, is the number of edges in a shortest path joining u and v in G. For $S, T \subseteq V(G)$, $E(S, T) = \{uv \in E(G) : u \in S, v \in T\}$.

For a graph G, let $\sigma_{k+1}(G) = \min\{\sum_{v \in S} d(v) : S$ is an independent set in a graph G with $|S| = k+1\}$ if $\alpha(G) \geq k+1$, otherwise $\sigma_{k+1}(G) = +\infty$, where $\alpha(G)$ is the independence number of G. A graph G is $K_{1,r}$*-free* if G contains no induced $K_{1,r}$ subgraphs, and $K_{1,3}$*-free* graphs is also called *claw-free* graphs. For $x, y \in V(G)$, let $J(x, y) = \{u : u \in N(x) \cap N(y), N(u) \subseteq N[x] \cup N[y]\}$. Ainouche [1] defined that a graph G is *quasi-claw-free* if $J(x, y) \neq \emptyset$ for $d(x, y) = 2$. Obviously, a claw-free graph is quasi-claw-free, but a quasi-claw-free graph may not be claw-free (see Figure 1, given in [1]).

A *path cover* of a graph G is a spanning subgraph of G consisting of vertex disjoint paths, and let $p(G) = \min\{|\mathcal{P}| : \mathcal{P}$ is a path cover of a graph $G\}$ denote the path cover number of G. A path cover $\mathcal{P}$ in G with $|\mathcal{P}| = p(G)$ is called a minimum path cover of G. The disjoint path cover problem finds applications in many areas such as software testing, database design, and code optimization, see [2] and [11].

Clearly, the research on path cover number of a graph is a generalization of determining whether a graph is

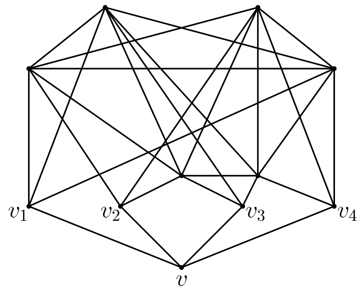


FIGURE 1. This graph is a quasi-claw-free graph, but it is not claw-free.

hamiltonian or traceable. It is well known that it is NP-hard to determine if a graph is hamiltonian or traceable. Thus there are lots of results about the sufficient conditions such as $K_{1,r}$-free graphs and quasi-claw-free graphs to be hamiltonian or traceable, see [1], [5], [6], [13] and [16]. Ore [12] showed that the path cover number is no more than $n - \sigma_2(G)$ for a graph G of order n. Hartman [7] proposed that for a graph G with connectivity k, if $\alpha(G) > k$, then $p(G) \leq \alpha(G) - k$. The path cover number of regular graphs [15] are also discussed. Recently, Chen *et al.* [4] gave the following result.

Theorem 1 (see [4]): Let k be a positive integer with $k \geq 2$. If G is a quasi-claw-free graph of order n and $\sigma_{k+1}(G) \geq n - k$, then $p(G) \leq k$.

The bound in Theorem 1 is not sharp for a connected graph. It is natural to consider the same sufficient condition to determine whether a connected graph has path cover number less than k. In this paper, we improves Theorem 1 when G is connected.

Theorem 2: Let k be a positive integer with $k \geq 2$. If G is a connected quasi-claw-free graph of order n and $\sigma_{k+1}(G) \geq n - k$, then $p(G) \leq k - 1$.

Remark 1: The bound in Theorem 2 is sharp. In Figure 2, G is a connected quasi-claw-free graph of order $n = a+b+c$ with $a, b, c \geq 2$. It is easy to show that $\sigma_3(G) = n - 3$. But $p(G) = 2$.

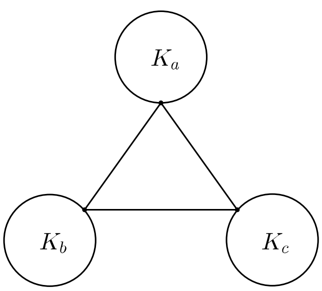


FIGURE 2. This graph is a connected quasi-claw-free graph of order $n = a + b + c$ with $a, b, c \geq 2$ and $\sigma_3(G) = n - 3$, but $p(G) = 2$.

II. PRELIMINARIES

Let C be a cycle/path with a given sense of traversal. For $u \in V(C)$, we use u^+ to denote the successor of u and u^- its predecessor. If $u, v \in V(C)$, we use $C[u, v]$ (or uCv) and $\overline{C}[v, u]$ (or $v\overline{C}u$) to denote the subpath $uu^+ \cdots v^- v$ of C and the same subpath in reverse order, respectively. Set $C(u, v] = C[u, v] \setminus \{u\}$ and $C[u, v) = C[u, v] \setminus \{v\}$, and consider them as both paths and vertex sets for convenience.

Suppose that G is a connected graph with $p(G) = t \geq 2$. Let $\mathcal{P} = \{P_1, P_2, \cdots, P_t\}$ be the minimum path cover of G with $P_i = u_{i1}u_{i2} \cdots u_{il_i}$, where $l_i = |V(P_i)|$, $1 \leq i \leq t$. By the minimality of $p(G)$, we can get the following two Lemmas.

Lemma 1: $xy \notin E(G)$ for $x \in \{u_{i1}, u_{il_i}\}$, $y \in \{u_{j1}, u_{jl_j}\}$ and $1 \leq i \neq j \leq t$.

Proof: Suppose, on the contrary, that $xy \in E(G)$ for $x \in \{u_{i1}, u_{il_i}\}$, $y \in \{u_{j1}, u_{jl_j}\}$. Then

$$P' := \begin{cases} u_{il_i}\overline{P_i}u_{i1}u_{j1}P_ju_{jl_j}, & \text{if } (x, y) = (u_{i1}, u_{j1}); \\ u_{i1}P_iu_{il_i}u_{j1}P_ju_{jl_j}, & \text{if } (x, y) = (u_{il_i}, u_{j1}); \\ u_{il_i}\overline{P_i}u_{i1}u_{jl_j}\overline{P_j}u_{j1}, & \text{if } (x, y) = (u_{i1}, u_{jl_j}); \\ u_{i1}P_iu_{il_i}u_{jl_j}\overline{P_j}u_{j1}, & \text{if } (x, y) = (u_{il_i}, u_{jl_j}). \end{cases}$$

Thus $\mathcal{P}' = (\mathcal{P} \setminus \{P_i, P_j\}) \cup \{P'\}$ is a path cover of G with $|\mathcal{P}'| < |\mathcal{P}|$, a contradiction to the minimum of $|\mathcal{P}|$. □

Lemma 2: There exists some i, $1 \leq i \leq t$ such that $u_{i1}u_{il_i} \notin E(G)$.

Proof: Suppose that $u_{i1}u_{il_i} \in E(G)$ for all $1 \leq i \leq t$. Then $C_i := u_{i1}P_iu_{il_i}u_{i1}$ is a cycle. Since G is connected, there exist two distinct i, j such that $E(P_i, P_j) \neq \emptyset$, where $1 \leq i \neq j \leq t$. Let $u \in V(P_i)$ and $v \in V(P_j)$ with $uv \in E(G)$. Then $P^* := u^+C_iuvC_jv^-$ is a path with $V(P^*) = V(P_i) \cup V(P_j)$. Thus $\mathcal{P}^* = (\mathcal{P} \setminus \{P_i, P_j\}) \cup \{P^*\}$ is a path cover of G with $|\mathcal{P}^*| < |\mathcal{P}|$, a contradiction. □

Zhang [16] proposed the definition of insertable vertex in order to research longest cycles problems of non-hamiltonian graphs. Motivated by this, Chen *et al.* [4] gave the following definition of insertable vertex in the background of a minimum path cover in G. For an edge $uw \in E(P_j)$ ($1 \leq j \leq t$), if there exists a vertex $v \in V(G) \setminus V(P_j)$ such that $uv, wv \in E(G)$, then v is called an insertable vertex, and a pair of vertices $\{u, w\}$ are called acceptor of v in P_j, otherwise, v is called a non-insertable vertex. By the definition of insertable vertex and the minimality of $p(G)$, Chen *et al.* [4] gave the following lemma.

Lemma 3 (see [4]): For each $P_i \in \mathcal{P}$, P_i contains a non-insertable vertex, where $1 \leq i \leq t$.

By Lemma 3, let w_i denote the first non-insertable vertex in each P_i for $1 \leq i \leq t$.

Lemma 4: (1) For any $u \in P_i[u_{i1}, w_i]$, we have $uu_{j1} \notin E(G)$ and $uu_{jl_j} \notin E(G)$ for $1 \leq i \neq j \leq t$;

(2) For any $u \in P_i[u_{i1}, w_i]$ and $v \in P_j[u_{j1}, w_j]$ with $1 \leq i \neq j \leq t$, we have $uv \notin E(G)$.

Proof: (1) If $uu_{j1} \in E(G)$, then let $P' := u_{il_i}\overline{P_i}uu_{j1}P_ju_{jl_j}$ and $\mathcal{P}' = (\mathcal{P} \setminus \{P_i, P_j\}) \cup \{P'\}$. By Lemma 1, $u \neq u_{i1}$ and thus, $P_i[u_{i1}, u^-]$ exists. Since w_i is the first non-insertable vertex in each P_i, all vertices in $P_i[u_{i1}, u^-]$ can be inserted into some paths in $\mathcal{P}'$, and $\mathcal{P}'$ is a path cover consisting of $t - 1$ paths, a contradiction. Thus $uu_{j1} \notin E(G)$. By a similar argument, $uu_{jl_j} \notin E(G)$.

(2) Suppose that $uv \in E(G)$. Choose $u = u_{ip} \in P_i[u_{i1}, w_i]$ and $v = u_{jq} \in P_j[u_{j1}, w_j]$ with the minimum subscripts sum $p+q$. By the choice of w_i, w_j, each vertex in $P_i[u_{i1}, w_i) \cup P_j[u_{j1}, w_j)$ is an insertable vertex.

Let $P^* = u_{il_i}\overline{P_i}uvP_ju_{jl_j}$ and $\mathcal{P}^* = (\mathcal{P} \setminus \{P_i, P_j\}) \cup \{P^*\}$. By Lemma 4(1), $u \neq u_{i1}$ and $v \neq u_{j1}$. This implies two paths $P_i[u_{i1}, u^-]$ and $P_j[u_{j1}, v^-]$ exist. By the choice of u and v, $E(P_i[u_{i1}, u^-], P_j[u_{j1}, v^-]) = \emptyset$. Thus there is no vertex in $P_i[u_{i1}, u^-]$ inserted between acceptor $\{v^-, v\}$ in P_j and there is no vertex in $P_j[u_{j1}, v^-]$ inserted between acceptor $\{u^-, u\}$ in P_i.

Next, we claim that, for any $u_{ia} \in P_i[u_{i1}, u^-]$ and $u_{jb} \in P_j[u_{j1}, v^-]$, u_{ia} and u_{jb} cannot be inserted between the same acceptor in any path from $\mathcal{P}\setminus\{P_i, P_j\}$. To the contrary, assume that u_{ia} and u_{jb} can be inserted between the same acceptor $\{z, w\}$ in some path $P_s \in \mathcal{P} \setminus \{P_i, P_j\}$ with the minimum subscripts sum $a+b$ (see Figure 3). Let $P_i^{**} := u_{il_i}\overline{P_i}u_{ia}wP_su_{sl_s}$, $P_j^{**} := u_{jl_j}\overline{P_j}u_{jb}z\overline{P_s}u_{s1}$ and $\mathcal{P}^{**} = (\mathcal{P} \setminus \{P_i, P_j, P_s\}) \cup \{P_i^{**}, P_j^{**}\}$. By the minimality of subscript sum $a+b$, each vertex in $P_i[u_{i1}, u_{ia}^-] \cup P_j[u_{j1}, u_{jb}^-]$ cannot be inserted between the same acceptor in any path from $\mathcal{P} \setminus \{P_i, P_j\}$. It follows that all vertices in $P_i[u_{i1}, u_{ia}^-] \cup P_j[u_{j1}, u_{jb}^-]$ can be inserted in some paths in $\mathcal{P}^{**}$ and we can get a path cover consisting of $t-1$ paths, a contradiction.

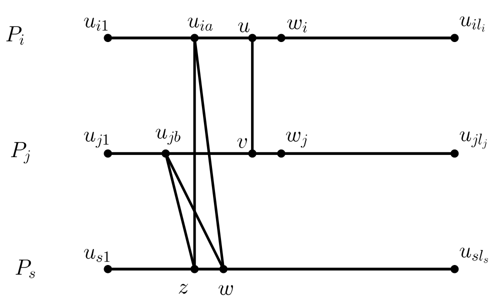


FIGURE 3. Illustration of Lemma 4(2).

Therefore, all vertices in $P_i[u_{i1}, u^-] \cup P_j[u_{j1}, v^-]$ can be inserted into some paths of $\mathcal{P}^*$, and $\mathcal{P}^*$ is a path cover consisting of $t-1$ paths, a contradiction. □

III. PROOF OF THEOREM 2

Suppose that G satisfies the conditions of Theorem 2 with $p(G) = t \geq k \geq 2$. Let $\mathcal{P} = \{P_1, P_2, \cdots, P_t\}$ is the minimum path cover of G with $P_i = u_{i1}u_{i2}\cdots u_{il_i}$, where $l_i = |V(P_i)|$, $1 \leq i \leq t$. By Lemma 2, we can choose $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ such that

$(T1)$ $u_{11}u_{1l_1} \notin E(G)$.

$(T2)$ $|V(P_1)| = l_1$ is as large as possible, subject to $(T1)$.

Note that $u_{11}u_{1l_1} \notin E(G)$. Denote $S^- = \{u_{1i} : u_{11}u_{1,i+1} \in E(G)$ and $1 \leq i \leq l_1 - 2\}$ and $S^+ = \{u_{1j} : u_{1,j-1}u_{1l_1} \in E(G)$ and $3 \leq j \leq l_1\}$. Obviously, $u_{11}u_{12}, u_{1,l_1-1}u_{1l_1} \in E(G)$. Then $u_{11} \in S^-$ and $u_{1l_1} \in S^+$.

Claim 1: (1) $G[P_1]$ contains no hamiltonian cycle.

(2) For any vertex $u_{1i} \in S^-$ with $1 \leq i \leq l_1 - 2$, $N(u_{1i}) \subseteq V(P_1)\setminus\{u_{1l_1}\}$. Similarly, $N(u_{1j}) \subseteq V(P_1)\setminus\{u_{11}\}$ for any vertex $u_{1j} \in S^+$ with $3 \leq j \leq l_1$. Furthermore, $N(u_{11}) \cup N(u_{1l_1}) \subseteq P_1(u_{11}, u_{1l_1})$.

Proof: (1) To the contrary, suppose that $G[P_1]$ contains a hamiltonian cycle C_1. Since G is connected, there exists some $2 \leq j \leq t$ such that $E(C_1, P_j) \neq \emptyset$. Let $x \in V(C_1)$ and $z \in V(P_j)$ with $xz \in E(C_1, P_j)$ for some $2 \leq j \leq t$. We claim that $z \neq u_{jl_j}$ and $x^+u_{j1} \notin E(G)$ (otherwise, let

$$P' := \begin{cases} x^+C_1xu_{jl_j}\overline{P_j}u_{j1}, & \text{if } z = u_{jl_j}; \\ u_{jl_j}\overline{P_j}u_{j1}x^+C_1x, & \text{if } x^+u_{j1} \in E(G); \end{cases}$$

then $\mathcal{P}' = (\mathcal{P} \setminus \{P_1, P_j\}) \cup \{P'\}$ is a path cover consisting of $t-1$ paths, a contradiction). Thus $P_j[z^+, u_{jl_j}]$ exists. Let $P_1'' := x^+C_1xz\overline{P_j}u_{j1}$ and $P_i'' := z^+P_ju_{jl_j}$. Then $\mathcal{P}'' = (\mathcal{P} \setminus \{P_1, P_j\}) \cup \{P_1'', P_i''\}$ is a path cover of G with $|\mathcal{P}''| = |\mathcal{P}|$ and $x^+u_{j1} \notin E(G)$. But $|V(P_1'')| > |V(P_1)|$, a contradiction to $(T2)$.

(2) Since $u_{1i} \in S^-$, $\underline{u_{11}}u_{1,i+1} \in E(G)$. If $u_{1i}u_{1l_1} \in E(G)$, then $u_{11}u_{1,i+1}P_1u_{1l_1}u_{1i}\overline{P_1}u_{11}$ is a hamiltonian cycle in $G[P_1]$, a contradiction to Claim 1(1). So $u_{1i}u_{1l_1} \notin E(G)$.

Now, it suffices to show that $N(u_{1i}) \setminus V(P_1) = \emptyset$. Suppose that there exists $z \in V(P_j)$ for some $2 \leq j \leq t$ such that $u_{1i}z \in E(G)$. We claim that $z \neq u_{jl_j}$ (otherwise, $P^* := u_{j1}P_ju_{jl_j}u_{1i}\overline{P_1}u_{11}u_{1,i+1}P_1u_{1l_1}$ is a path with $V(P^*) = V(P_1) \cup V(P_j)$ and we can get a path cover consisting of $t-1$ paths, a contradiction). Then $P_j[z^+, u_{jl_j}]$ exists. Let $P_1^{**} := u_{j1}P_jzu_{1i}\overline{P_1}u_{11}u_{1,i+1}P_1u_{1l_1}$ and $P_j^{**} := z^+P_ju_{jl_j}$. Then by Lemma 1, $\mathcal{P}^{**} = (\mathcal{P} \setminus \{P_1, P_j\}) \cup \{P_1^{**}, P_j^{**}\}$ is a path cover of G with $|\mathcal{P}^{**}| = |\mathcal{P}|$ and $u_{j1}u_{1l_1} \notin E(G)$. But $|V(P_1^{**})| > |V(P_1)|$, a contradiction to $(T2)$. Thus $N(u_{1i}) \subseteq V(P_1) \setminus \{u_{1l_1}\}$. □

By Lemma 3, let w_i denote the first non-insertable vertex in each P_i for $2 \leq i \leq t$, and let $I(\mathcal{P}) = \{u_{11}, u_{1l_1}, w_2, \ldots, w_t\}$.

Claim 2: Let $x, y \in I(\mathcal{P})$ with $d(x, y) = 2$, then $\{x, y\} = \{u_{11}, u_{1l_1}\}$.

Proof: First, we show that $\{x, y\} \neq \{w_i, w_j\}$ for $2 \leq i \neq j \leq t$. Suppose $d(w_i, w_j) = 2$ for some $2 \leq i \neq j \leq t$. Since G is quasi-claw-free, $J(w_i, w_j) \neq \emptyset$. Let $v \in J(w_i, w_j)$. By Lemma 4, $v \notin P_s[u_{s1}, w_s] \cup \{u_{sl_s}\}$ for any $2 \leq s \leq t$. Without loss of generality, assume $v \in P_s(w_s, u_{sl_s})$ for some $2 \leq s \leq t$ and $s \neq j$. If $s \neq i$, then $N(v^+) \cap \{w_i, w_j\} = \emptyset$ or $N(v^-) \cap \{w_i, w_j\} = \emptyset$ as w_i, w_j are non-insertable, a contradiction to $v \in J(w_i, w_j)$. Thus $v \in P_i(w_i, u_{il_i})$ and $w_iv^+ \in E(G)$ as $v^+w_j \notin E(G)$ (see Figure 4(a)). Let $P'' := u_{il_i}\overline{P_i}v^+w_iP_ivw_jP_ju_{jl_j}$ and $\mathcal{P}'' = (\mathcal{P} \setminus \{P_i, P_j\}) \cup \{P''\}$. By a similar argument to the proof of Lemma 4(2), we can insert all vertices in $P_i[u_{i1}, w_i) \cup P_j[u_{j1}, w_j)$ into some paths in $\mathcal{P}''$, and $\mathcal{P}''$ is a path cover consisting of $t-1$ paths, a contradiction.

Next, we show that $\{x, y\} \neq \{u_{11}, w_j\}$ and $\{x, y\} \neq \{u_{1l_1}, w_j\}$. To the contrary, suppose that $d(u_{11}, w_j) = 2$ for some $2 \leq j \leq t$. Let $v' \in J(u_{11}, w_j)$. Then by Claim 1(2), $v' \in V(P_1)$ and $v' \neq u_{1l_1}$. Denote $v' := u_{1i}$. Then $w_ju_{1i} \in E(G)$ (see Figure 4(b)). Since w_j is a non-insertable vertex, $w_ju_{1,i+1} \notin E(G)$. Thus $u_{11}u_{1,i+1} \in E(G)$ as $u_{1i} \in J(u_{11}, w_j)$, which implies $u_{1i} \in S^-$. By Claim 1(2), $N(u_{1i}) \subseteq V(P_1) \setminus \{u_{1i}\}$, a contradiction to $w_ju_{1i} \in E(G)$.

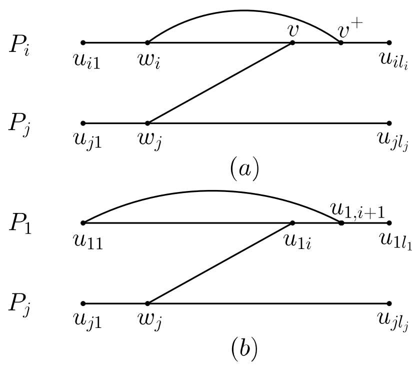


FIGURE 4. **Illustration of Claim 2.**

Note that $u_{11}u_{1l_1} \notin E(G)$. So $\{x, y\} = \{u_{11}, u_{1l_1}\}$. □

Recall that $I(\mathcal{P}) = \{u_{11}, u_{1l_1}, w_2, \ldots, w_t\}$. By Lemma 4, Claims 1 and 2, we have the following:

Claim 3: (1) $I(\mathcal{P})$ is an independent set.

(2) For $u \in P_i[u_{i1}, u_{il_i}]$ with $2 \le i \le t$, $d_{I(\mathcal{P})}(u) \le 1$.

(3) For $u \in P_1(u_{11}, u_{1l_1})$, either $N(u) \cap I(\mathcal{P}) \subseteq \{u_{11}, u_{1l_1}\}$ or $N(u) \cap I(\mathcal{P}) \subseteq \{w_i\}$ for some $2 \le i \le t$.

For $u \in P_1(u_{11}, u_{1l_1})$, we write:

- X for the vertex set $\{u : |N(u) \cap \{u_{11}, u_{1l_1}\}| = 2\}$;
- Y for the vertex set $\{u : |N(u) \cap \{u_{11}, u_{1l_1}\}| = 1\}$;
- Z for the vertex set $\{u : |N(u) \cap \{w_2, \ldots, w_t\}| = 1\}$;
- U for the vertex set $\{u : N(u) \cap I(\mathcal{P}) = \emptyset\}$.

By Claim 3(3), $\{X, Y, Z, U\}$ is a partition of all vertices in $P_1(u_{11}, u_{1l_1})$. Note that X is the vertex set of common neighbors of u_{11} and u_{1l_1} on P_1. If $X \neq \emptyset$, let $X = \{u_{1i_1}, u_{1i_2}, \ldots, u_{1i_r}\}$, where $2 \le i_1 < i_2 < \cdots < i_r \le l_1 - 1$ and $r \ge 1$. Let $U_0 = U \cap P_1(u_{11}, u_{1i_1})$ and $U_r = U \cap P_1(u_{1i_r}, u_{1l_1})$. Recall that $S^- = \{u_{1i} : u_{11}u_{1,i+1} \in E(G)$ and $1 \le i \le l_1 - 2\}$ and $S^+ = \{u_{1j} : u_{1,j-1}u_{1l_1} \in E(G)$ and $3 \le j \le l_1\}$.

Claim 4: If $U_0 = \emptyset$, then $P_1[u_{11}, u_{1i_1}) \subseteq S^-$. Similarly, if $U_r = \emptyset$, then $P_1(u_{1i_r}, u_{1l_1}] \subseteq S^+$.

Proof: To the contrary, suppose that $P_1[u_{11}, u_{1i_1}) \nsubseteq S^-$. Then $P_1[u_{12}, u_{1i_1}] \nsubseteq N(u_{11})$. Note that $\{u_{12}, u_{1i_1}\} \subseteq N(u_{11})$. Thus there exists some j ($3 \le j \le i_1 - 1$) such that $u_{11}u_{1j} \notin E(G)$ with the maximum j. By the maximality of j, $P_1[u_{1,j+1}, u_{1i_1}] \subseteq N(u_{11})$. Thus $P_1[u_{1j}, u_{1i_1}) \subseteq S^-$, and then $u_{1j} \in S^-$. By Claim 1(2), $N(u_{1j}) \subseteq V(P_1) \setminus \{u_{1l_1}\}$. This together with $u_{11}u_{1j} \notin E(G)$, we have $N(u_{1j}) \cap I(\mathcal{P}) = \emptyset$. It follows that $u_{1j} \in U_0$, a contradiction to the assumption $U_0 = \emptyset$. □

Claim 5: $|U| \ge |X|$.

Proof: Obviously, the conclusion holds when $X = \emptyset$. Now, assume that $X \neq \emptyset$. By way of contradiction, $|U| \le |X| - 1 = r - 1$. Since G is connected, there exists $j \in [2, t]$ such that $E(P_1, P_j) \neq \emptyset$. Let $x \in V(P_1)$ and $z \in V(P_j)$ with $xz \in E(P_1, P_j)$.

If $r = 1$, then $U = \emptyset$. By Claim 4, $P_1[u_{11}, u_{1i_1}) \subseteq S^-$ and $P_1(u_{1i_1}, u_{1l_1}] \subseteq S^+$, which implies $P_1 - u_{1i_1} \subseteq S^- \cup S^+$. This together with $xz \in E(P_1, P_j)$ and Claim 1(2), $x = u_{1i_1}$ and $u_{11}z \notin E(G)$. Then $u_{11}u_{1i_1}, zu_{1i_1} \in E(G)$. Thus $d(u_{11}, z) = 2$. Since G is quasi-claw-free, $J(u_{11}, z) \neq \emptyset$. Let $v \in J(u_{11}, z)$. Then $u_{11}v, zv \in E(G)$. Note that $P_1 - u_{1i_1} \subseteq S^- \cup S^+$. This together with $zv \in E(G)$ and Claim 1(2), $v = u_{1i_1}$ and hence, $u_{1i_1} \in J(u_{11}, z)$. By Claim 4, $P_1(u_{1i_1}, u_{1l_1}] \subseteq S^+$, and then $u_{1,i_1+1} \in S^+$. By Claim 1(2), $N(u_{1,i_1+1}) \subseteq V(P_1) \setminus \{u_{11}\}$, which implies $u_{1,i_1+1}u_{11} \notin E(G)$ and $u_{1,i_1+1}z \notin E(G)$, a contradiction to $u_{1i_1} \in J(u_{11}, z)$.

In the following, we always assume $r \ge 2$. Recall that $X = \{u_{1i_1}, u_{1i_2}, \ldots, u_{1i_r}\}$ and X is the vertex set of common neighbors of u_{11} and u_{1l_1} on P_1. By Claim 1(1), $G[P_1]$ contains no hamiltonian cycle. Then $u_{1i_s}u_{1i_{s+1}} \notin E(P_1)$ for $1 \le s \le r-1$. Thus $P_1(u_{1i_s}, u_{1i_{s+1}}) \neq \emptyset$. Let $U_s = U \cap P_1(u_{1i_s}, u_{1i_{s+1}})$ for $1 \le s \le r - 1$.

Fact 1: $|U_s| = 1$ for $1 \le s \le r - 1$ and $U_0 = U_r = \emptyset$.

Proof: Note that $u_{1l_1}u_{1i_s} \in E(G)$ for each $1 \le s \le r - 1$. Then, by Claim 1(1), $u_{11}u_{1,i_s+1} \notin E(G)$. This implies $P_1[u_{1,i_s+1}, u_{1i_{s+1}}] \nsubseteq N(u_{11})$. Since $u_{1i_{s+1}} \in X$, we have $u_{1i_{s+1}} \in N(u_{11})$. Thus there exists some j ($i_s + 1 \le j \le i_{s+1} - 1$) such that $u_{11}u_{1j} \notin E(G)$ with the maximum j. By the maximality of j, $P_1[u_{1,j+1}, u_{1i_{s+1}}] \subseteq N(u_{11})$. Thus $P_1[u_{1j}, u_{1i_{s+1}}) \subseteq S^-$, and then $u_{1j} \in S^-$. By Claim 1(2), $N(u_{1j}) \subseteq V(P_1) \setminus \{u_{1l_1}\}$. This together with $u_{11}u_{1j} \notin E(G)$, we have $N(u_{1j}) \cap I(\mathcal{P}) = \emptyset$. It follows that $u_{1j} \in U_s$, and then $|U_s| \ge 1$ for each $1 \le s \le r - 1$. This implies $|U| = \sum_{s=0}^{r} |U_s| \ge \sum_{s=1}^{r-1} |U_s| \ge r - 1$. By assumption, $|U| \le r - 1$. Thus $|U_s| = 1$ for $1 \le s \le r - 1$ and $U_0 = U_r = \emptyset$. □

By Fact 1, assume that $U_s = \{u_{1q_s}\}$ for $1 \le s \le r - 1$.

Fact 2: $P_1[u_{11}, u_{1i_1}) \cup P_1[u_{1q_s}, u_{1i_{s+1}}) \subseteq S^-$ for $1 \le s \le r - 1$. Similarly, $P_1(u_{1i_r}, u_{1l_1}] \cup P_1(u_{1i_s}, u_{1q_s}] \subseteq S^+$ for $1 \le s \le r - 1$. Furthermore, $P_1 - X \subseteq S^- \cup S^+$.

Proof: By Fact 1 and Claim 4, $P_1[u_{11}, u_{1i_1}) \subseteq S^-$. Next, it suffices to show that $P_1[u_{1q_s}, u_{1i_{s+1}}) \subseteq S^-$ for $1 \le s \le r - 1$. Clearly, $P_1[u_{1q_s}, u_{1i_{s+1}}) \subseteq S^-$, if $P_1[u_{1q_s}, u_{1i_{s+1}}) = \emptyset$. Now, assume $P_1[u_{1q_s}, u_{1i_{s+1}}) \neq \emptyset$. To the contrary, suppose $P_1[u_{1q_s}, u_{1i_{s+1}}) \nsubseteq S^-$ for some $1 \le s \le r - 1$. Then $P_1[u_{1,q_s+1}, u_{1i_{s+1}}] \nsubseteq N(u_{11})$. Note that $u_{11}u_{1i_{s+1}} \in E(G)$. Thus there exists some j ($q_s + 1 \le j \le i_{s+1} - 1$) such that $u_{11}u_{1j} \notin E(G)$ with the maximum j. By the maximality of j, $P_1[u_{1,j+1}, u_{1i_{s+1}}] \subseteq N(u_{11})$. Thus $P_1[u_{1j}, u_{1i_{s+1}}) \subseteq S^-$, and then $u_{1j} \in S^-$. By Claim 1(2), $N(u_{1j}) \subseteq V(P_1) \setminus \{u_{1l_1}\}$. This together with $u_{11}u_{1j} \notin E(G)$, we have $N(u_{1j}) \cap I(\mathcal{P}) = \emptyset$. It follows that $u_{1j} \in U_s$. Since $j \ge q_s + 1$, we have $u_{1j} \neq u_{1q_s}$. Thus $|U_s| \ge 2$, a contradiction to Fact 1. □

Recall that $xz \in E(P_1, P_j)$. By Fact 2, $P_1 - X \subseteq S^- \cup S^+$. This together with $xz \in E(P_1, P_j)$ and Claim 1(2), $x \in X$ and $u_{11}z \notin E(G)$. Note that $X \subseteq N(u_{11})$ and $xz \in E(G)$. Thus $d(u_{11}, z) = 2$. Since G is quasi-claw-free, $J(u_{11}, z) \neq \emptyset$. Let $v' \in J(u_{11}, z)$. Then $u_{11}v', zv' \in E(G)$. Note that $P_1 - X \subseteq S^- \cup S^+$. This together with $v'z \in E(G)$ and Claim 1(2), $v' \in X$. Denote $v' := u_{1i_s}$ for some $1 \le s \le r$, and thus $u_{1i_s} \in J(u_{11}, z)$. By Fact 2, $P_1(u_{1i_s}, u_{1q_s}] \subseteq S^+$, and then $u_{1,i_s+1} \in S^+$. By Claim 1(2), $N(u_{1,i_s+1}) \subseteq V(P_1) \setminus \{u_{11}\}$, which implies $u_{1,i_s+1}u_{11} \notin E(G)$ and $u_{1,i_s+1}z \notin E(G)$, a contradiction to $u_{1i_s} \in J(u_{11}, z)$. □

Claim 6: $d(u_{11}) + d(u_{1l_1}) \le l_1 - 2 - |Z|$.

Proof: By Claim 1(2), $N(u_{11}) \cup N(u_{1l_1}) \subseteq P_1(u_{11}, u_{1l_1})$. Recall that $\{X, Y, Z, U\}$ is a partition of all vertices in $P_1(u_{11}, u_{1l_1})$. Thus $|X|+|Y|+|Z|+|U| = l_1 - 2$. Combining this with Claim 5, $d(u_{11}) + d(u_{1l_1}) = 2|X| + |Y| \leq |X| + |Y| + |U| = l_1 - 2 - |Z|$. □

Now, we complete the proof of Theorem 2.

Clearly, $d(I(\mathcal{P})) = \sum_{u \in I(\mathcal{P})} d(u) = d(u_{11}) + d(u_{1l_1}) + \sum_{i=2}^{t} d(w_i)$. By Claims 3 and 6, $d(I(\mathcal{P})) \leq (l_1 - 2 - |Z|) + (\sum_{i=2}^{t}(l_i - 1) + |Z|) = n - (t+1)$. Recall that $I(\mathcal{P})$ is an independent set with $t+1$ vertices, and $d(I(\mathcal{P}))$ is the degree sum of $t+1$ vertices in $I(\mathcal{P})$. This together with the assumption $t \geq k$, we have $\sigma_{k+1} \leq \sigma_{t+1} \leq d(I(\mathcal{P})) \leq n - (t+1) < n - k$, a contradiction to the condition of Theorem 2.

REFERENCES

[1] A. Ainouche, "Quasi-claw-free graphs," *Discrete Math.*, vol. 179, nos. 1–3, pp. 13–26, Jan. 1998.

[2] K. Asdre and S. D. Nikolopoulos, "The 1-fixed-endpoint path cover problem is polynomial on interval graphs," *Algorithmica*, vol. 58, no. 3, pp. 679–710, Nov. 2010.

[3] J. A. Bondy and U. S. R. Murty, *Graph Theory*. New York, NY, USA: Springer, 2008.

[4] X. Chen, M. Liu, and H. Liu, "Minimum path cover in Quasi-claw-free graphs," *Bull. Iranian Math. Soc.*, vol. 46, no. 3, pp. 815–820, Jun. 2020.

[5] G. T. Chen and R. Schelp, "Hamiltonicity for $K_{1,r}$-free graphs," *J. Graph Theory*, vol. 20, no. 4, pp. 423–439, 1995.

[6] G. A. Dirac, "Some theorems on abstract graphs," *Proc. London Math. Soc.*, vol. 2, no. 1, pp. 69–81, 1952.

[7] I. B.-A. Hartman, "Variations on the Gallai-Milgram theorem," *Discrete Math.*, vol. 71, no. 2, pp. 95–105, 1988.

[8] L. H. Hsu and C. K. Lin, *Graph Theory and Interconnection Networks*. Boca Raton, FL, USA: CRC Press, 2008.

[9] F. T. Leighton, *Introduction to Parallel Algorithms and Architectures: Arrays, Trees, Hypercubes*. San Mateo, CA, USA: Morgan Kaufman, 1992.

[10] J. A. M. McHugh, *Algorithmic Graph Theory*. Upper Saddle River, NJ, USA: Prentice-Hall, 1990.

[11] S. C. Ntafos and S. L. Hakimi, "On path cover problems in digraphs and applications to program testing," *IEEE Trans. Softw. Eng.*, vol. SE-5, no. 5, pp. 520–529, Sep. 1979.

[12] O. Ore, "Arc coverings of graphs," *Annali di Matematica Pura ed Applicata*, vol. 55, no. 1, pp. 315–321, Dec. 1961.

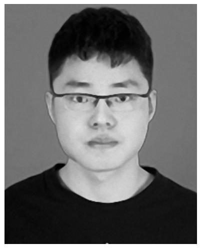
[13] O. Ore, "Note on Hamilton circuits," *Amer. Math. Monthly*, vol. 67, no. 1, p. 55, Jan. 1960.

[14] J.-H. Park, H.-C. Kim, and H.-S. Lim, "Many-to-many disjoint path covers in hypercube-like interconnection networks with faulty elements," *IEEE Trans. Parallel Distrib. Syst.*, vol. 17, no. 3, pp. 227–240, Mar. 2006.

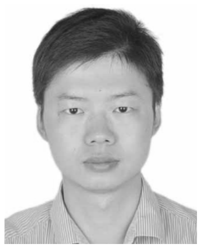
[15] G. Yu, "Covering 2-connected 3-regular graphs with disjoint paths," *J. Graph Theory*, vol. 88, no. 3, pp. 385–401, Jul. 2018.

[16] C. Q. Zhang, "Hamilton cycles in claw-free graphs," *J. Graph Theory*, vol. 12, no. 2, pp. 209–216, 1988.

HUIQING LIU received the M.S. degree from Central China Normal University, China, in 2001, and the Ph.D. degree from the Chinese Academy of Sciences, China, in 2014, both in mathematics. She is currently a Professor with the Faculty of Mathematics and Statistics, Hubei University, Hubei, China. Her research interests include operations research, and graph theory and its application.



JIAN LU received the B.S. degree from the Anhui University of Technology, China, in 2015, and the M.S. degree from Anhui University, China, in 2019, both in mathematics. He is currently pursuing the Ph.D. degree with the Faculty of Mathematics and Statistics, Hubei University, Hubei, China. His research interest includes graph theory.



SHUNZHE ZHANG received the M.S. degree from Hubei University, Hubei, China, in 2012, and the Ph.D. degree from Central China Normal University, China, in 2015, both in mathematics. He is currently a Lecturer with the Faculty of Mathematics and Statistics, Hubei University. His research interests include interconnection networks, and graph theory and its application.

XIUYU ZHONG received the B.S. degree in mathematics and applied mathematics from the Guangxi Normal University for Nationalities, China. She is currently pursuing the M.S. degree with the Faculty of Mathematics and Statistics, Hubei University, Hubei, China. Her research interest includes graph theory and its application.