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Internally Positive Representation to Stability of Delayed Timescale-Type Differential-Difference Equation

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ABSTRACT This paper considers exponential stability for a class of timescale-type differential-difference equation with bounded time-varying delay. Based on time scale theory, internally positive representation technique, as well as the existing exponential results of positive differential-difference equation on time scale, criteria of exponential stability of the system under consideration are obtained and they are robust to time delay and time scale to some extent. The theoretical results are applied to study stability for a class of linear singular system, and they are validated via two numerical examples.

INDEX TERMS Internally positive representation, differential-difference equation, time scale, bounded delay, stability analysis.

I. INTRODUCTION

Differential-difference equation (DDE) with delays is a class of delayed system which has been paid long-lasting attention in the last past years. It has been involved in many practical applications as those related to lossless propagation problems in engineering science, singular models in chemical and nuclear reaction process [1]–[3]. It is also conducive to study dynamics of various systems like neutral system and delayed commensurate system [4], [5]. In [6], Shen *et al.* solved asymptotic stability for continuous-time DDE with an ingenious comparison method based on positivity constraint. Recently, the authors in [20] discussed exponential stability issue for timescale-type DDE which is also based on its positivity. It is worthy of noting that positivity of a system indeed contributes to analyzing its dynamics due to the nature positivity, for example, no sign issue raises in the analysis. Although positive filters design for the issue of positive state-space system realization allowing to tackle their state variable with only nonnegative quantities has been reported among a variety of literature [7], [8], it has also been demonstrated that the positivity constraint in the filter design may restrict the filter performance [9], [10]. To overcome such a limitation, a positivity-free filter for signal processing is necessary while it is realized based on combination of positive system [11]. It can be seen that the technique of combination of positive systems is an effective way to address systems without positivity constraint while maintain the advantages of positive systems. Recently, internally positive representation (IPR), as an extension of such technique, has been investigated in the analysis of continuous-time system [12], [13], discrete-time system [14], and differential-difference system [22].

Timescale system is a class of system that involves on a time scale, which is an extension of the real set and the integer set. Timescale theory was proposed by S. Hilger in the late 1980's, the original aim is to unify and extend the existing differential equation and difference equation theories [15]. However, timescale theory also have some practical applications. For example, stability analysis of coupled dynamical systems with communication disappearance or loss can be transformed into stability of a class of timescale system which evolves on a combination of non-overlapping time intervals. There are some nice results that time scales were taken into consideration [16]–[19].

This paper considers the exponential stability of a class of timescale-type differential-difference equation (TS-DDE) with bounded time-varying delays on the basis of some existing IPR techniques. At first, some basic concepts on IPR are given, the concept of Δ-Metzler positive representation is defined, and the definition of IPR for the TS-DDE under consideration is shown. Then some global exponential stability criteria for the TS-DDE under consideration are derived

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based on IPR technique. The obtained results show that they are robust against time scales to some extent provided that the supreme of graininess function of time scale is less than one threshold. The main results are also employed to study exponential stability of a class of timescale-type linear singular system.

II. INTERNALLY POSITIVE REPRESENTATION OF DDE

Notations: $\mathbb{PC}_{rd}(X, Y)$ denotes the set of all bounded and piecewise rd-continuous functions defined on X with its values in Y. $\mathcal{I}_n = \{1, 2, \cdots, n\}$. Given two vectors $\alpha, \beta \in \mathbb{R}^n$, $\alpha \succ (\prec)\beta \Leftrightarrow \alpha_i > (<)\beta_i(\forall i \in \mathcal{I}_n)$. For a matrix $M = (M_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, $\mathbb{D}(M) = \mathrm{diag}(M_{11}, M_{22}, \cdots, M_{nn})$, and M is said to be Δ-Metzler if it is Metzler and $1 + \mu(t)M_{ii} \geq 0$ for all $t \in \mathbb{T}$ and $i \in \mathcal{I}_n$. 0 denotes a zero vector or scalar in a concrete context, and $0_{n\times m}$ is an $n \times m$ matrix. I_n is $n \times n$ identity matrix. x_i and P_{ij} represent the i-th and (i, j)-th entry of vector x and matrix P respectively.

We consider the following TS-DDE with bounded time-varying delay

$$x^{\Delta}(t) = Ax(t) + By(t - \tau(t)),$$
$$y(t) = Cx(t) + Dy(t - \tau(t)), \qquad (1)$$

where $x(t) \in \mathbb{R}^p$, $y(t) \in \mathbb{R}^q$ denote the state of system (1), $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times q}$, $C \in \mathbb{R}^{q\times p}$, $D \in \mathbb{R}^{q\times q}$ are system matrices, $\tau(t) \in [0, +\infty)$ denotes the bounded time delay with $\tau_m = \sup_{t\in[t_0,+\infty)_{\mathbb{T}}}\{\tau(t)\}$, which is also assumed to be rd-continuous and $t - \tau(t) \in \mathbb{T}$, $t - \tau_m \in \mathbb{T}$, $\forall t \in [t_0, +\infty)_{\mathbb{T}}$. A set of initial state of (1) is given by

$$x(t_0) = \phi \in \mathbb{R}^p,$$
$$y(t) = \psi(t) \in \mathbb{PC}_{rd}([t_0 - \tau_m, t_0)_{\mathbb{T}}, \mathbb{R}^q). \qquad (2)$$

Here we assume that the existence and uniqueness of solution of system (1) is admitted. It should be noted that this assumption is widely used in various works [6], [20], [22].

For simplicity, we also denote system (1) as $\mathbb{S} = \{A, B, C, D, (p, q)\}$ with its solution

$$(x(t), y(t)) = \mathbb{S}(t, t_0, \phi, \psi), \quad \forall t \in [t_0, +\infty)_{\mathbb{T}}.$$

Now the positivity definition of a system in the form of (1) is given as follows.

Definition 1 [20]: A system in the form of TS-DDE (1) is (internally) positive if any $\phi \in \mathbb{R}^p_+$ and $\psi \in \mathbb{PC}_{rd}([t_0 - \tau_m, t_0)_{\mathbb{T}}, \mathbb{R}^q_+)$ implies $x(t) \in \mathbb{R}^p_+$ and $y(t) \in \mathbb{R}^q_+$, $\forall t \in [t_0, +\infty)_{\mathbb{T}}$.

Some conditions of positivity of system (1) have been given in the literature, which are described as the following results.

Lemma 1 [20]: TS-DDE (1) is (internally) positive provided that A is Δ-Metzler, D is Schur, and B, C, D are nonnegative matrices.

Lemma 2 [20]: Suppose that there exists a positive real τ_0 such that $\tau(t) \geq \tau_0$ for all $t \in [t_0, +\infty)_{\mathbb{T}}$, then TS-DDE (1) is (internally) positive if and only if A is Δ-Metzler, and B, C, D are nonnegative matrices.

Next we review some basic IPR-relevant concepts which have been developed well in the literature.

Given a real number a, $a^+ = (|a| - a)/2$, $a^- = (|a| - a)/2$. Given a real matrix $P = (p_{ij})_{r_1\times r_2}$, $P^+ = (p^+_{ij})_{r_1\times r_2}$, $P^- = (p^-_{ij})_{r_1\times r_2}$, and $|P| = P^+ - P^-$. Denote $\alpha_n = [I_n, I_n] \in \mathbb{R}^{n\times 2n}$.

Definition 2 [21]: A positive representation of a real scalar $x \in \mathbb{R}$ is an arbitrary vector $\tilde{x} \in \mathbb{R}^2_+$ such that

$$x = \alpha_1\tilde{x},$$

and its min-positive representation is denoted by $\pi(x)$ with

$$\pi(x) = \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \in \mathbb{R}^2_+.$$

Definition 3 [21]: A positive representation of a real vector $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$ is an arbitrary vector $\tilde{x} \in \mathbb{R}^{2n}_+$ such that

$$x = \alpha_n\tilde{x},$$

and its min-positive representation is denoted by $\pi(x) \in \mathbb{R}^{2n}$ where $((\pi(x))_i, (\pi(x))_{n+i})^T$ is the min-positive representation of x_i, $\forall i \in \mathcal{I}_n$.

Definition 4 [21]: A positive representation of a real matrix $X \in \mathbb{R}^{n\times m}$ is an arbitrary matrix

$$\tilde{X} = \begin{bmatrix} \tilde{X}_0 & \tilde{X}_1 \\ \tilde{X}_1 & \tilde{X}_0 \end{bmatrix} \in \mathbb{R}^{2n\times 2m}_+$$

such that $\forall i, j$,

$$((\tilde{X}_0)_{ij}, (\tilde{X}_1)_{ij})^T \in \mathbb{R}^2_+$$

is a positive representation of X_{ij}, and its min-positive representation is denoted by $\pi(X) \in \mathbb{R}^{2n\times 2m}_+$ where both

$$((\pi(X))_{ij}, (\pi(X))_{i(n+j)})^T$$

and

$$((\pi(X))_{(n+i)(n+j)}, (\pi(X))_{(n+i)j})^T$$

are the min-positive representation of X_{ij}, $\forall i, j \in \mathcal{I}_n$.

Next we introduce a Δ-Metzler positive representation of an arbitrary matrix where the diagonal elements form a Δ-Metzler matrix, which will be used in the stability analysis of TS-DDE.

Definition 5: A Δ-Metzler positive representation of a real square matrix $Y \in \mathbb{R}^{n\times n}$ which satisfies that $\mathbb{D}(Y)$ is a Δ-Metzler matrix is an arbitrary matrix

$$\tilde{Y}^M = \begin{bmatrix} \mathbb{D}(Y) + \tilde{Y}_0 & \tilde{Y}_1 \\ \tilde{Y}_1 & \mathbb{D}(Y) + \tilde{Y}_0 \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$$

such that

$$((\tilde{Y}_0)_{ij}, (\tilde{Y}_1)_{ij})^T \in \mathbb{R}^2_+$$

is a positive representation of $(Y - \mathbb{D}(Y))_{ij}$ for all $i, j \in \mathcal{I}_n$, $i \neq j$, and its Δ-Metzler min-positive representation is denoted by

$$\pi^M(Y) = I_2 \otimes \mathbb{D}(Y) + \pi(Y - \mathbb{D}(Y)) \in \mathbb{R}^{2n\times 2n}.$$

Lemma 3 [21]: Given a real matrix $X \in \mathbb{R}^{n\times m}$, a real vector $x \in \mathbb{R}^m$, and let $\tilde{X}$ and $\tilde{x}$ be the positive representation of X and x respectively. Then the product $\tilde{X}\tilde{x}$ is a positive representation of the product Xx.

Lemma 4: Given a real matrix $Y \in \mathbb{R}^{n\times n}$ with $\mathbb{D}(Y)$ being Δ-Metzler, a real vector $x \in \mathbb{R}^n$, and let $\tilde{Y}^M$ be the Δ-Meztler positive representation of Y, $\tilde{x}$ be the positive representation of x. Then $\tilde{Y}^M$ is a Δ-Metzler matrix and the product $\tilde{Y}^M\tilde{x}$ is a positive representation of the product Yx.

Proof: Since $\mathbb{D}(Y)$ is Δ-Metzler,

$$1+\mu(t)(\mathbb{D}(Y))_{ii} \geq 0,$$

for all $i \in \mathcal{I}_n$ and $t \in [t_0, +\infty)_{\mathbb{T}}$, then

$$\begin{aligned}1+\mu(t)(\mathbb{D}(Y)+\tilde{A}_0)_{ii} &= 1+\mu(t)(\mathbb{D}(Y))_{ii}\\ &\geq 0,\end{aligned}$$

for all $i \in \mathcal{I}_n$ and $t \in [t_0, +\infty)_{\mathbb{T}}$, which means $\tilde{Y}^M$ is Δ-Metzler. Based on the fact that $\tilde{Y}_0$ and $\tilde{Y}_1$ are nonnegative matrices, it can be readily observed that $\tilde{Y}^M$ is a Δ-Metzler matrix. Since $Y-\mathbb{D}(Y)=\tilde{Y}_0-\tilde{Y}_1$, by simple computation, one can obtain $Yx=\alpha_n\tilde{Y}^M\tilde{x}$. ■

Finally the definition of IPR of a TS-DDE is given as follows.

Definition 6: Given an arbitrary real system

$$\mathbb{S}=\{A, B, C, D, (p, q)\}$$

in the form of (1) with its initial value $\phi \in \mathbb{R}^p$ and $\psi \in \mathbb{PC}_{rd}([t_0-\tau_m, t_0)_{\mathbb{T}}, \mathbb{R}^q)$, its IPR is an internally positive system

$$\bar{\mathbb{S}}=\{\bar{A}, \bar{B}, \bar{C}, \bar{D}, (\bar{p}, \bar{q})\}$$

together with four transformations

$$\begin{aligned}\mathcal{T}_1:&\quad \mathbb{R}^p \to \mathbb{R}^{\bar{p}}_+,\\ \mathcal{T}_2:&\quad \mathbb{R}^{\bar{p}}_+ \to \mathbb{R}^p,\\ \mathcal{T}_3:&\quad \mathbb{R}^q \to \mathbb{R}^{\bar{q}}_+,\\ \mathcal{T}_4!:&\quad \mathbb{R}^{\bar{q}}_+ \to \mathbb{R}^q,\end{aligned}$$

such that

$$\begin{aligned}\bar{\phi} &= \mathcal{T}_1(\phi),\\ \bar{\psi} &= \mathcal{T}_3(\psi),\end{aligned}$$

implies

$$\begin{aligned}x(t) &= \mathcal{T}_2(\bar{x}(t)),\\ y(t) &= \mathcal{T}_4(\bar{y}(t)),\end{aligned}$$

where

$$\begin{aligned}(x(t), y(t)) &= \mathbb{S}(t, t_0, \phi, \psi), \quad \forall t \in [t_0, +\infty)_{\mathbb{T}},\\ (\bar{x}(t), \bar{y}(t)) &= \bar{\mathbb{S}}(t, t_0, \bar{\phi}, \bar{\psi}), \quad \forall t \in [t_0, +\infty)_{\mathbb{T}}.\end{aligned}$$

A schematic diagram of IPR is depicted in Fig. 1.

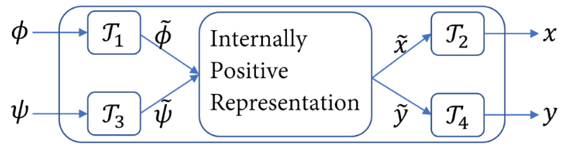

FIGURE 1. Schematic diagram of IPR.

III. MAIN RESULTS

This section is divided into two parts, the first part includes stability criteria of TS-DDE (1), and the second part contains a direct application of the stability results to a class of special timescale-type linear singular system.

A. STABILITY ANALYSIS OF TS-DDE VIA IPR

Theorem 1: For system (1), if $\mathbb{D}(A)$ is a Δ-Metzler matrix, then system

$$\tilde{\mathbb{S}}=\{\tilde{A}^M, \tilde{B}, \tilde{C}, \tilde{D}, (2p, 2q)\} \tag{3}$$

with the following four transformations

$$\begin{aligned}\mathcal{T}_1(x) &= \tilde{x},\\ \mathcal{T}_2(\tilde{x}) &= \alpha_p\tilde{x},\\ \mathcal{T}_3(y) &= \tilde{y},\\ \mathcal{T}_4(\tilde{y}) &= \alpha_q\tilde{y},\end{aligned} \tag{4}$$

is an IPR of system (1).

Proof: By Lemma 4, $\tilde{A}^M$ is a Δ-Metzler matrix, together Lemma 1 with the fact that $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ are nonnegative matrices, therefore $\tilde{\mathbb{S}}$ is a positive system. For any initial value of system (1) given by (2), denote

$$(\bar{x}_{new}(t), \bar{y}_{new}(t)) = \tilde{\mathbb{S}}(t, t_0, \tilde{\phi}, \tilde{\psi}), \quad \forall t \in [t_0, +\infty)_{\mathbb{T}},$$

namely, $\bar{x}_{new}(t)$ and $\bar{y}_{new}(t)$ solves

$$\begin{aligned}\bar{x}^{\Delta}_{new}(t) &= \tilde{A}^M\bar{x}_{new}(t)+\tilde{B}\bar{y}_{new}(t-\tau(t)),\\ \bar{y}_{new}(t) &= \tilde{C}\bar{x}_{new}(t)+\tilde{D}\bar{y}_{new}(t-\tau(t)),\end{aligned} \tag{5}$$

with

$$\begin{aligned}\bar{x}_{new}(t_0) &= \mathcal{T}_1(\phi) = \tilde{\phi},\\ \bar{y}_{new}(t) &= \mathcal{T}_3(\psi(t)) = \tilde{\psi}(t), \quad \forall t \in [t_0, +\infty)_{\mathbb{T}}.\end{aligned}$$

By (5), one gets

$$\begin{aligned}(\alpha_p\bar{x}_{new}(t))^{\Delta} &= \alpha_p\tilde{A}^M\bar{x}_{new}(t)+\alpha_p\tilde{B}\bar{y}_{new}(t-\tau(t)),\\ \alpha_q\bar{y}_{new}(t) &= \alpha_q\tilde{C}\bar{x}_{new}(t)+\alpha_q\tilde{D}\bar{y}_{new}(t-\tau(t)),\end{aligned}$$

By computation,

$$\begin{aligned}\alpha_p\tilde{A}^M &= A\alpha_p,\\ \alpha_p\tilde{B} &= B\alpha_q,\\ \alpha_q\tilde{C} &= C\alpha_p,\\ \alpha_q\tilde{D} &= D\alpha_q.\end{aligned}$$

It follows from (5) that

$$\begin{aligned}(\mathcal{T}_2(\bar{x}_{new}(t)))^{\Delta} &= A\mathcal{T}_2(\bar{x}_{new}(t))+B\mathcal{T}_4(\bar{y}_{new}(t-\tau(t))),\\ \mathcal{T}_4(\bar{y}_{new}(t)) &= C\mathcal{T}_2(\bar{x}_{new}(t))+D\mathcal{T}_4(\bar{y}_{new}(t-\tau(t))),\end{aligned}$$

Because of the existence and uniqueness of solution of system (1), one finally has

$$\begin{aligned}x(t) &= \mathcal{T}_2(\bar{x}_{new}(t)),\\ y(t) &= \mathcal{T}_4(\bar{y}_{new}(t)).\end{aligned}$$

By Definition 6, one can conclude that system $\tilde{\mathbb{S}}$ with the four transformations (4) is an IPR of system (1). ■

Definition 7: TS-DDE (1) is globally exponentially stable (GES) if trajectories $x(t)$ and $y(t)$ of (1) satisfy

$$\|x(t)\| \leq Q e_{\ominus\eta}(t, t_0)\|(\phi^T, \varphi^T)^T\|_\infty$$

and

$$\|y(t)\| \leq Q e_{\ominus\eta}(t, t_0)\|(\phi^T, \varphi^T)^T\|_\infty$$

for all $t \in [t_0, +\infty)_{\mathbb{T}}$, where $M \geq 1$, $\eta > 0$ are time-invariant.

Lemma 5 [20]: Assume that A is a Δ-Metzler matrix, B, C, D are nonnegative matrices, and further D is a Schur matrix, if there exist two positive compatible vectors γ_1 and γ_2 such that

$$A\gamma_1 + B\gamma_2 \prec 0, \tag{6}$$
$$C\gamma_1 + (D - I_q)\gamma_2 \prec 0, \tag{7}$$

then system $\mathbb{S} = \{A, B, C, D, (p, q)\}$ in the form of (1) is GES.

Lemma 6: For a given system in the form of (1), if it admits a global exponential stable IPR in the form of (3) and (4), then the system in the form of (1) is global exponential stable.

Proof: Since IPR (3) and (4) is GES, there exist scalars $\tilde{Q} \geq 1$, $\tilde{\eta} > 0$ such that

$$\|\bar{x}_{new}(t)\| \leq \tilde{Q} e_{\ominus\tilde{\eta}}(t, t_0)\|(\tilde{\phi}^T, \tilde{\varphi}^T)^T\|_\infty$$

and

$$\|\bar{y}_{new}(t)\| \leq \tilde{Q} e_{\ominus\tilde{\eta}}(t, t_0)\|(\tilde{\phi}^T, \tilde{\varphi}^T)^T\|_\infty$$

for all $t \in [t_0, +\infty)_{\mathbb{T}}$. It follows that $x(t)$ and $y(t)$, the solution of system (1), satisfy

$$\|x(t)\| \leq \tilde{Q} e_{\ominus\tilde{\eta}}(t, t_0)\|(\phi^T, \varphi^T)^T\|_\infty$$

and

$$\|y(t)\| \leq \tilde{Q} e_{\ominus\tilde{\eta}}(t, t_0)\|(\phi^T, \varphi^T)^T\|_\infty$$

for all $t \in [t_0, +\infty)_{\mathbb{T}}$. ■

Lemma 6 indicates that GES of an IPR of a TS-DDE implies that GES of the system itself. Based on the lemmas introduced above, a main result of GES of TS-DDE (1) is obtained.

Theorem 2: For system (1), if

$$\tilde{A}^M \text{ is } \Delta\text{-Metzler},$$
$$\tilde{D} \text{ is Schur}, \tag{8}$$

and there exist two positive vectors $\gamma_1 \in \mathbb{R}^{2p}$ and $\gamma_2 \in \mathbb{R}^{2q}$ such that

$$\tilde{A}^M\gamma_1 + \tilde{B}\gamma_2 \prec 0,$$
$$\tilde{C}\gamma_1 + (\tilde{D} - \mathbf{I})\gamma_2 \prec 0, \tag{9}$$

then system (1) is GES.

Proof: By (8), together with the definition of $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$, IPR of system (1) is positive. Combine (9) with Lemma 5, IPR of system (1) is GES. Based on Lemma 6, system (1) is GES. ■

Remark 1: Theorem 2 shows that GES of TS-DDE (1) is robust against any time-constant or bounded time-varying delay. In addition, the result is robust against time scale as long as the supreme of the graininess function of the time scale is identical. As such, if GES holds for a time scale $\bar{\mathbb{T}}$ with $\sup_{t\in[t_0,+\infty)_{\bar{\mathbb{T}}}} \mu(t) = \mu_0$, then the results hold for any time scale $\mathbb{T}$ if $\sup_{t\in[t_0,+\infty)_{\mathbb{T}}} \mu(t) \leq \mu_0$, and naturally hold for time scale $\mathbb{R}$.

Theorem 2 gives a GES result for TS-DDE (1) based on the existing positivity and GES result of [20]. However, one can see that the dimensions of all the parameters involved in Theorem 2 are doubled compared with those involved in system (1). So in the next step we aim to simplify it with lower dimensions. At first a useful lemma is given below.

Lemma 7 [22]: Let

$$M = \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix} \in \mathbb{R}^{2m\times 2m},$$

then

$$\rho(M) = \max\{\rho(M_1 + M_2), \rho(M_1 - M_2)\}.$$

Theorem 3: For system (1), if

$$\mathbb{D}(A) \text{ is } \Delta\text{-Metzler}, \tag{10}$$
$$|D| \text{ is Schur}, \tag{11}$$

and there exist four positive vectors $\gamma_1 \in \mathbb{R}^p$, $\gamma_2 \in \mathbb{R}^p$, $\zeta_1 \in \mathbb{R}^q$, and $\zeta_2 \in \mathbb{R}^q$ such that

$$\begin{aligned}
(\mathbb{D}(A) + A_0)\gamma_1 + A_1\gamma_2 + B_0\zeta_1 + B_1\zeta_2 &\prec 0,\\
(\mathbb{D}(A) + A_0)\gamma_2 + A_1\gamma_1 + B_0\zeta_2 + B_1\zeta_1 &\prec 0,\\
C_0\gamma_1 + C_1\gamma_2 + D_0\zeta_1 + D_1\zeta_2 &\prec \zeta_1,\\
C_0\gamma_2 + C_1\gamma_1 + D_0\zeta_2 + D_1\zeta_1 &\prec \zeta_2,
\end{aligned} \tag{12}$$

where

$$\begin{bmatrix} \mathbb{D}(A) + A_0 & A_1 \\ A_1 & \mathbb{D}(A) + A_0 \end{bmatrix} = \pi^M(A),$$
$$\begin{bmatrix} B_0 & B_1 \\ B_1 & B_0 \end{bmatrix} = \pi(B),$$
$$\begin{bmatrix} C_0 & C_1 \\ C_1 & C_0 \end{bmatrix} = \pi(C),$$
$$\begin{bmatrix} D_0 & D_1 \\ D_1 & D_0 \end{bmatrix} = \pi(D),$$

then system (1) is GES.

Proof: Condition (10) implies $\pi^M(A)$ is a Δ-Metzler matrix from Lemma 4. Based on Lemma 7,

$$\begin{aligned}
\rho(\pi(D)) &= \max\{\rho(D_0 + D_1), \rho(D_0 - D_1)\}\\
&= \max\{\rho(D), \rho(|D|)\}\\
&= \rho(|D|),
\end{aligned}$$

then (11) implies $\pi(D)$ is a Schur matrix.

Let $\gamma = (\gamma_1^T, \gamma_2^T)^T$ and $\zeta = (\zeta_1^T, \zeta_2^T)^T$, then $\gamma \succ 0$ and $\zeta \succ 0$. Moreover, (12) implies

$$\begin{aligned} \pi^M(A)\gamma + \pi(B)\zeta &\prec 0, \\ \pi(C)\gamma + (\pi(D) - \mathbf{I})\zeta &\prec 0, \end{aligned}$$

By a special case of Theorem 2, system (1) is GES. ■

Remark 2: It can be seen from Theorem 3 that the dimensions of all the parameters involved have been simplified. Another difference between Theorem 3 and Theorem 2 is that the IPR used in the former result is unique, while it is not unique in the latter result.

Since TS-DDE is a class of general system that may used to study some special systems like timescale-type singular or neutral systems. In the next subsection, the obtained main results are used to investigate GES of a class of timescale-type singular system.

B. STABILITY ANALYSIS OF TIMESCALE-TYPE LINEAR SINGULAR SYSTEM

Here we consider the following timescale-type linear neutral system with time-varying delay

$$Ex^{\Delta}(t) = Ax(t) + Bx(t - \tau(t)), \tag{13}$$

where $x(t) \in \mathbb{R}^p$, $E \in \mathbb{R}^{p\times p}$, $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times p}$ are system matrices, and the rank of matrix E is $r < p$ which corresponds to the singularity of system (13). For simplicity, we assume that these system matrices admit the following forms

$$E = \begin{bmatrix} I_r & 0_{r(p-r)} \\ 0_{(p-r)r} & 0_{(p-r)(p-r)} \end{bmatrix}, \quad A = \begin{bmatrix} A_f & 0_{r(p-r)} \\ A_s & -I_{p-r} \end{bmatrix}, \quad B = \begin{bmatrix} 0_{rr} & B_f \\ 0_{(p-r)r} & B_s \end{bmatrix},$$

then system (13) is recast into the following TS-DDE

$$\begin{aligned} x_1^{\Delta}(t) &= A_f x_1(t) + B_f x_2(t - \tau(t)), \\ x_2(t) &= A_s x_1(t) + B_s x_2(t - \tau(t)), \end{aligned} \tag{14}$$

where $x_1(t) \in \mathbb{R}^r$ and $x_2(t) \in \mathbb{R}^{p-r}$.

The following two results can be derived from Theorems 2 and 3 directly.

Corollary 1: For system (13), if

$$\begin{aligned} &\tilde{A_f}^M \text{ is } \Delta\text{-Metzler}, \\ &\tilde{B}_s \text{ is Schur}, \end{aligned} \tag{15}$$

and there exist two positive vectors $\gamma_1 \in \mathbb{R}^{2r}$ and $\gamma_2 \in \mathbb{R}^{2(p-r)}$ such that

$$\begin{aligned} \tilde{A_f}^M \gamma_1 + \tilde{B}_f \gamma_2 &\prec 0, \\ \tilde{A}_s \gamma_1 + (\tilde{B}_s - \mathbf{I})\gamma_2 &\prec 0, \end{aligned} \tag{16}$$

then system (13) is GES.

Corollary 2: For system (13), if

$$\mathbb{D}(A_f) \text{ is } \Delta\text{-Metzler}, \tag{17}$$

$$|B_s| \text{ is Schur}, \tag{18}$$

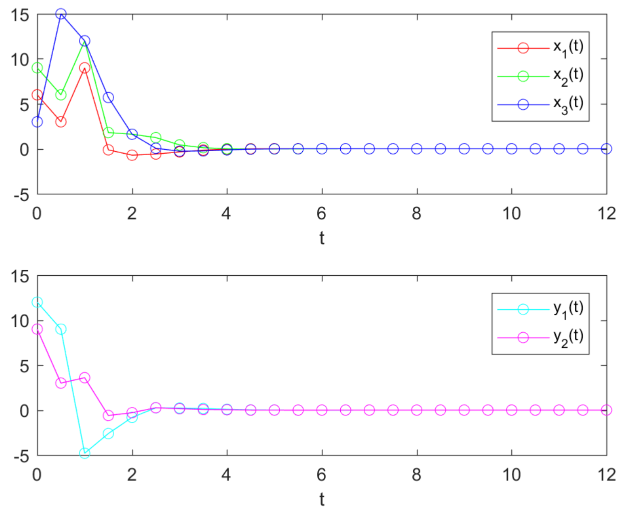

FIGURE 2. State evolution of TS-DDE (1) with parameters (20) on time scale $0.5\mathbb{Z}$.

and there exist four positive vectors $\gamma_1 \in \mathbb{R}^r$, $\gamma_2 \in \mathbb{R}^r$, $\zeta_1 \in \mathbb{R}^{p-r}$, and $\zeta_2 \in \mathbb{R}^{p-r}$ such that

$$\begin{aligned} (\mathbb{D}(A_f) + A_{f0})\gamma_1 + A_{f1}\gamma_2 + B_{f0}\zeta_1 + B_{f1}\zeta_2 &\prec 0, \\ (\mathbb{D}(A_f) + A_{f0})\gamma_2 + A_{f1}\gamma_1 + B_{f0}\zeta_2 + B_{f1}\zeta_1 &\prec 0, \\ A_{s0}\gamma_1 + A_{s1}\gamma_2 + B_{s0}\zeta_1 + B_{s1}\zeta_2 &\prec \zeta_1, \\ A_{s0}\gamma_2 + A_{s1}\gamma_1 + B_{s0}\zeta_2 + B_{s1}\zeta_1 &\prec \zeta_2, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \begin{bmatrix} \mathbb{D}(A_f) + A_{f0} & A_{f1} \\ A_{f1} & \mathbb{D}(A_f) + A_{f0} \end{bmatrix} &= \pi^M(A_f), \\ \begin{bmatrix} B_{f0} & B_{f1} \\ B_{f1} & B_{f0} \end{bmatrix} &= \pi(B_f), \\ \begin{bmatrix} A_{s0} & A_{s1} \\ A_{s1} & A_{s0} \end{bmatrix} &= \pi(A_s), \\ \begin{bmatrix} B_{s0} & B_{s1} \\ B_{s1} & B_{s0} \end{bmatrix} &= \pi(B_s), \end{aligned}$$

then system (13) is GES.

IV. NUMERICAL EXAMPLES

A. EXAMPLE 1

This example is designed to validate the correctness of Theorem 3 for TS-DDE (1) with parameters given by

$$\begin{aligned} A &= \begin{bmatrix} -1.5 & -0.3 & -0.2 \\ -0.3 & -1.4 & 0.2 \\ 0.5 & -0.2 & -1.3 \end{bmatrix}, \\ B &= \begin{bmatrix} 0.1 & -0.2 \\ -0.4 & 0.1 \\ 0.2 & -0.3 \end{bmatrix}, \\ C &= \begin{bmatrix} -0.3 & 0.1 & -0.4 \\ 0.1 & 0.2 & -0.1 \end{bmatrix}, \\ D &= \begin{bmatrix} 0.3 & -0.4 \\ 0.1 & 0.2 \end{bmatrix}, \end{aligned} \tag{20}$$

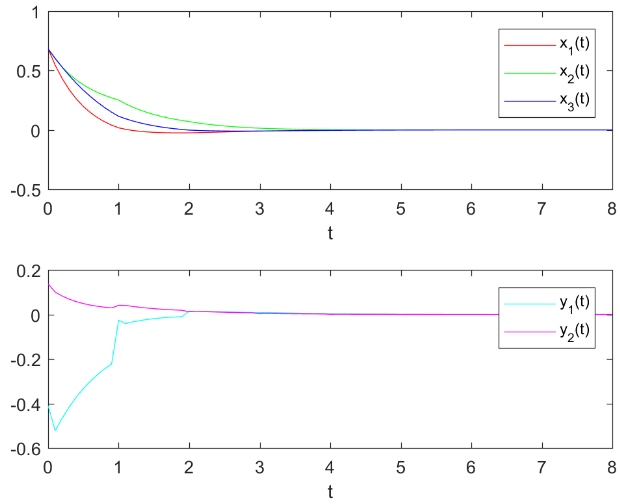

FIGURE 3. State evolution of TS-DDE (1) with parameters (20) on time scale $\mathbb{R}$.

and $\tau(t) = \text{mod}(t, 1)$. Then one can verify that $\mathbb{D}(A) = \text{diag}(-1.2, -1.4, -1.3)$ is a Δ-Metzler matrix (for time scale $\mathbb{T} = 0.5\mathbb{Z}$, and also hold for $\mathbb{T} = \mathbb{R}$) and $\rho(D) = 0.3162 < 1$, it follows that conditions (10)-(11) of Theorem 3 are satisfied. Moreover, by computation, one gets

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.2 \\ 0.5 & 0 & 0 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} 0 & 0.3 & 0.2 \\ 0.3 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix},$$
$$B_0 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \\ 0.2 & 0 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0 & 0.2 \\ 0.4 & 0 \\ 0 & 0.3 \end{bmatrix},$$
$$C_0 = \begin{bmatrix} 0 & 0.1 & 0 \\ 0.1 & 0.2 & 0 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0.3 & 0 & 0.4 \\ 0 & 0 & 0.1 \end{bmatrix},$$

and

$$D_0 = \begin{bmatrix} 0.3 & 0 \\ 0.1 & 0.2 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 0 & 0.4 \\ 0 & 0 \end{bmatrix}$$

Using Matlab software, one gets a set of feasible solution for (12)

$$\gamma_1 = (0.4024, 0.5742, 0.5364)^T \succ 0,$$
$$\gamma_2 = (0.3433, 0.5426, 0.5667)^T \succ 0,$$
$$\zeta_1 = (0.9956, 0.4942)^T \succ 0,$$

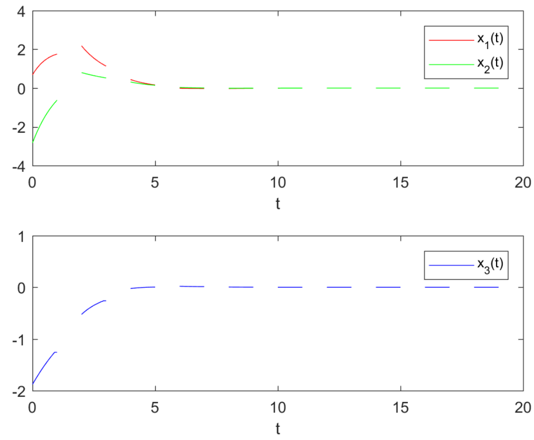

FIGURE 4. State evolution of TS-DDE (21) with parameters (22) on time scale $\mathbb{P}_{1,1}$.

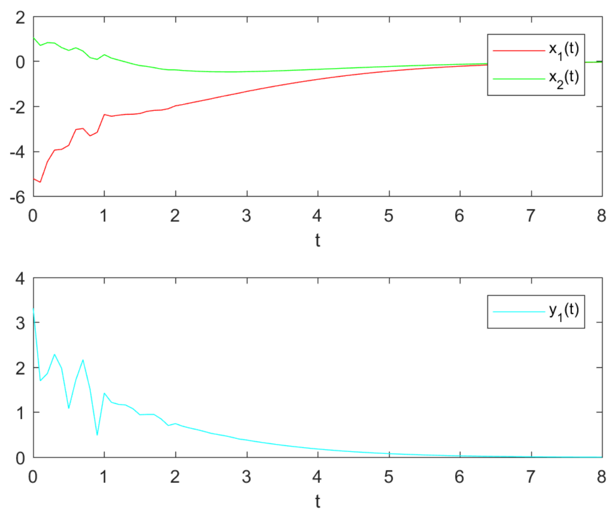

FIGURE 5. State evolution of TS-DDE (21) with parameters (22) on time scale $\mathbb{R}$.

and

$$\zeta_2 = (0.9743, 0.3996)^T \succ 0.$$

Based on Theorem 3, system (1) with parameters (20) on time scale $0.5\mathbb{Z}$ is GES. Figs. 2 and 3 depict the state evolution of TS-DDE (1) with parameters (20) on time scales $0.5\mathbb{Z}$ and $\mathbb{R}$ respectively with initial values randomly chosen.

B. EXAMPLE 2

Here we aim to verify Corollary 2 with a special timescale-type linear singular system, which is described as follows

$$Ex^{\Delta}(t) = Fx(t) + Gx(t - \tau(t)), \tag{21}$$

where $t \in \mathbb{P}_{1,1} = \cup_{k=0}^{\infty}[2k, 2k+1]$, $\tau(t) = t - 2k, \forall t \in [2k, 2k+1]$. The parameters of system (21) are given by

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} -0.7 & -0.4 & 0 \\ 0.1 & -0.9 & 0 \\ -0.4 & 0.5 & -1 \end{bmatrix},$$
$$G = \begin{bmatrix} 0 & 0 & -0.7 \\ 0 & 0 & -0.3 \\ 0 & 0 & 0.1 \end{bmatrix}, \tag{22}$$

and $x(t) = (x_1^T, x_2)^T$, where $x_1(t) \in \mathbb{R}^2$ and $x_2(t) \in \mathbb{R}$, then system (21) is transformed to

$$\begin{cases} x_1^{\Delta}(t) = F_1 x_1(t) + G_1 x_2(t-\tau(t)), \\ x_2(t) = F_2 x_1(t) + G_2 x_2(t-\tau(t)), \end{cases} \tag{23}$$

where

$$F_1 = \begin{bmatrix} -0.7 & -0.4 \\ 0.1 & -0.9 \end{bmatrix},$$
$$F_2 = \begin{bmatrix} -0.4 & 0.5 \end{bmatrix},$$
$$G_1 = \begin{bmatrix} -0.7 \\ -0.3 \end{bmatrix},$$
$$G_2 = 0.1.$$

By computation, one gets $\mathbb{D}(F_1) = \mathrm{diag}(-0.7, -0.9)$, and

$$F_{10} = \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \end{bmatrix},$$
$$F_{11} = \begin{bmatrix} 0 & 0.4 \\ 0 & 0 \end{bmatrix},$$
$$F_{20} = \begin{bmatrix} 0 & 0.5 \end{bmatrix},$$
$$F_{21} = \begin{bmatrix} 0.4 & 0 \end{bmatrix},$$
$$G_{10} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$G_{11} = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix},$$
$$G_{20} = 0.1,$$
$$G_{21} = 0,$$

Similarly, one gets a set of feasible solution of (19) as follows

$$\gamma_1 = (2.3416, 0.8999)^T \succ 0,$$
$$\gamma_2 = (2.9837, 1.0632)^T \succ 0,$$
$$\zeta_1 = 1.8788 > 0,$$
$$\zeta_2 = 1.7086 > 0.$$

Moreover, it can be verified that $\mathbb{D}(F_1)$ is Δ-Metzler, and $|G_2|$ is Schur. Therefore, according to Corollary 2, system (21) is GES on time scale $\mathbb{P}_{1,1}$. Figs. 4 and 5 depict the state evolution of TS-DDE (21) with parameters (22) on time scales $\mathbb{P}_{1,1}$ and $\mathbb{R}$ (in this case, we specify $\tau(t) = t - 1$, $\forall t \in \mathbb{T}$) with initial values randomly chosen.

V. CONCLUSION

Based on the existing exponential stability results of timescale-type differential-difference equations (TS-DDE) with bounded time-varying delays, together with the timescale-type internally positive representation proposed in this paper, we derived two global exponential stability criteria of TS-DDE with bounded time-varying delays. The results are robust to the time delay and time scale to a certain extent.

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