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Three Representations for Set Partitions

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ABSTRACT The Set Partitioning Problem (SPP) aims to obtain non-empty disjoint subsets of objects such that their union equals the whole set of objects, and the partition meets some prespecified criteria. The ubiquity of SPP is impressive, given that it has a lot of theoretical and practical motivations. In the theoretical side, the study of the SPP is closely related to Bell numbers, Stirling numbers of the second kind, integer partitions, Eulerian numbers, Restricted Growth Strings (RGS), factoradic number system, power calculations, etc. In the practical side, SPP is intimately related to classification problems, clustering problems, reduction of dimensionality problems, and so on. In this work, three representations for instances of SPP are presented, these representations use: Restricted Growth Strings (RGS), factoradic number system, and a number system with a fixed base. Two cases for these representations will be presented: where the number of subsets is unbounded (i.e. the number of subsets can be the number of objects); and where the number of subsets is less than the number of objects. Bidirectional mappings between these three representations will be introduced, also the mapping among these three representations and the power of a base is defined. Given, that these three representations can be used to solve instances of SPP using exact, greedy, and metaheuristic algorithms, that require to do small changes to one possible solution and/or recombination of two possible solutions, definitions of mutation and recombination operators for the three representations will be shown. In order to motivate the use of the three representations for the solution of particular instances of SPP, it was decided to present their application to solve an instance of a set partition of integers problem (SPIP) using a simple genetic algorithm.

INDEX TERMS Bell numbers, factoradic number system, restricted growth strings number system, stirling numbers of the second kind, Eulerian numbers.

I. INTRODUCTION

The set partitioning problem (SPP) aims to obtain non-empty disjoint subsets of objects such that their union equals the whole set of objects, and the partition meets some prespecified criteria. SPP is ubiquitous, it has a lot of theoretical and practical motivations. In the theoretical side, the study of SPP is closely related to: Bell numbers [1] (page 4), Stirling numbers of the second kind [2] (chapter 9), integer partitions [3], Eulerian numbers [2] (chapter 10), Restricted Growth Strings (RGS) [4] (page 81), factoradic number system [5], and power calculations [2] (chapters 9, 10, 15). In the practical side, SPP is intimately related to classification problems [6], clustering problems [7], reduction of dimensionality problems [8], self-organizing maps [9], text clustering for web mining [10], and so on.

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In this work, three representations for instances of SPP using: Restricted Growth Strings (RGS), factoradic number system, and a positional number system with a fixed base will be presented. Two cases for these representations will be presented, the first one permits that the number of partitions be equal to the number of objects, and the second one restricts the number of partitions to a certain value less-than the number of objects.

The three representations express each SPP possible solution with different redundancy degree (i.e., one solution in one representation can be equivalent to many solutions in another representation, and vice versa) to evidence this fact, bijective mappings between the three representations will be defined, also the three representations are bijectively mapped to an integer.

Relevant features of the three representations, like: redundancy degree for each representation; mutation of a solution (i.e. do a small change in one partition); recombination of two partitions (i.e. produce two child partitions using two parent

TABLE 1. A short table for integer partitions of k with v parts $p(k, v)$, the last column indicates the value of $p(k)$.

k \ v	1	2	3	4	5	6	7	8	$p(k)$
1	1								**1**
2	1	1							**2**
3	1	1	1						**3**
4	1	2	1	1					**5**
5	1	2	2	1	1				**7**
6	1	3	3	2	1	1			**11**
7	1	3	4	3	2	1	1		**15**
8	1	4	5	5	3	2	1	1	**22**

partitions); and repairing of a modified set partition, will be highlighted. All these features are relevant for deciding which representation to use, according the SPP to be solved and the algorithm that will be used to solve it. In order to motivate the use of the three representations, it is presented the solution of an instance of a set partition of integers problem(SPIP) using a simple genetic algorithm.

The remaining of this work is organized in additional five sections. Section II presents definitions of items that will be used for the three representations (i.e. Integer partitions, Set partitions, Bell numbers, Stirling numbers of the second kind, Eulerian numbers, positional number system with a fixed base, and sum of powers). In Section III the three representations based on Restricted Growth Strings, factoradic number system, and a positional number system with fixed base, are presented; the mappings between them and the redundancy degree of each representation, will be highlighted. Section IV shows how to implement mutation and recombination operators using the three representations and how to repair the resulting set partition (in case it is necessary). Section V was devoted to present the solution of one instances of a set partition of integers problem (SPIP) using a simple genetic algorithm. Finally Section VI shows some conclusions that can be extracted from the presented work.

II. DEFINITIONS NEEDED FOR THE THREE REPRESENTATIONS

This section presents definitions of concepts that are relevant to the three representations based on: Restricted Growth Strings, Factoradic Number System, and a Positional Number System with fixed base, that will be presented in Section III. The definitions in this section are: integer partitions, set partitions, Bell numbers, Stirling numbers of the second kind, Eulerian numbers, counting of number of functions, computation of an integer power, and summation of integer powers. A table with a summary of the formulas of this section is presented to end the section.

A. INTEGER PARTITIONS

Counting the ways in which a positive integer k can be expressed as sums of positive integers is known as the integer partition problem [11] and is denoted as $p(k)$. For instance there are five integer partitions of 4:

$$\{4,\ \ 3{+}1,\ \ 2{+}2,\ \ 2{+}1{+}1,\ \ 1{+}1{+}1{+}1\}$$

As is commonly used, the integer partitions will be expressed in vector notation in descending order, then the partitions of 4 are:

$$\{(4),\ \ (3, 1),\ \ (2, 2),\ \ (2, 1, 1),\ \ (1, 1, 1, 1)\}.$$

The number of partitions of the integer k in exactly v parts is expressed by $p(k, v)$, its definition is given in (1), a short table for $p(k, v)$ is given in Table 1.

$$p(k, v) = \begin{cases} 0 & \textbf{if } (v = 0) \vee (k = 0) \\ 1 & \textbf{if } (v = 1) \vee (v = k) \\ p(k\text{-}1, v\text{-}1) & \\ \ +p(k\text{-}v, v) & \textbf{otherwise} \end{cases} \tag{1}$$

The relation of $p(k)$ in terms of $p(k, v)$ is given in: (2)

$$p(k) = \sum_{v=1}^{k} p(k, v) \tag{2}$$

When it is needed to use the explicit partitions of an integer k or the explicit partitions of an integer k in v parts, these problems are referred by $\mathbb{P}(k)$ and $\mathbb{P}(k, v)$ respectively. Considering that $\mathbb{X} \in \mathbb{P}(k)$, then the number of elements (or size) of $\mathbb{X}$ is expressed by $|\mathbb{X}|$. In order to define how to compute explicit partitions of size k using explicit partitions of size k-1 the next definitions are needed: the symbol $\oplus$ indicates the concatenation of one element to a partition; the last element of a partition $\mathbb{X}$ is defined by $\lambda(\mathbb{X})$; and the last element of a partition $\mathbb{X}$ is removed by the expression $\mathbb{X}\backslash\lambda(\mathbb{X})$. The base case is $\mathbb{P}(1) = \{(1)\}$, the explicit partitions of an integer k are defined by (3).

$$\begin{aligned} & \mathbb{P}(k) = \emptyset; \\ & \forall \mathbb{X} \in \mathbb{P}(k\text{-}1)\{ \\ & \quad \mathbb{P}(k) = \mathbb{P}(k) \cup (\mathbb{X} \oplus 1); \\ & \quad \mathbb{Y} = \mathbb{X}\backslash\lambda(\mathbb{X}); \\ & \quad \textbf{if } (\mathbb{Y} = \emptyset)\ \mathbb{P}(k) = \mathbb{P}(k) \cup (\lambda(\mathbb{X}){+}1); \\ & \quad \textbf{elseif } (\lambda(\mathbb{Y}) \geq \lambda(\mathbb{X}){+}1) \\ & \qquad \mathbb{P}(k) = \mathbb{P}(k) \cup (\mathbb{Y} \oplus (\lambda(\mathbb{X}){+}1)); \\ & \} \end{aligned} \tag{3}$$

The integer partitions of k in exactly v parts in terms of $\mathbb{P}(k\text{-}1, v\text{-}1)$ and $\mathbb{P}(k\text{-}1, v)$ are defined in (4), please consider that $\mathbb{P}(1, 1) = \{(1)\}$, and $\mathbb{P}(i, 0) = \{\emptyset\}$. A table for $\mathbb{P}(k, v)$ for small values of k is given in Table 2, the last column of this table indicates the value of $p(k)$.

TABLE 2. A short table of $\mathbb{P}(k, v)$, the last column indicates the value of $p(k)$.

k \ v	1	2	3	4	5	6	7	8	$p(k)$
1	(1)								**1**
2	(2)	(1,1)							**2**
3	(3)	(2,1)	(1,1,1)						**3**
4	(4)	(3,1) (2,2)	(2,1,1)	(1,1,1,1)					**5**
5	(5)	(4,1) (3,2)	(3,1,1) (2,2,1)	(2,1,1,1)	(1,1,1,1,1)				**7**
6	(6)	(5,1) (4,2) (3,3)	(4,1,1) (3,2,1) (2,2,2)	(3,1,1,1) (2,2,1,1)	(2,1,1,1,1)	(1,1,1,1,1,1)			**11**
7	(7)	(6,1) (5,2) (4,3)	(5,1,1) (4,2,1) (3,3,1) (3,2,2)	(4,1,1,1) (3,2,1,1) (2,2,2,1)	(3,1,1,1,1) (2,2,1,1,1)	(2,1,1,1,1,1)	(1,1,1,1,1,1,1)		**15**
8	(8)	(7,1) (6,2) (5,3) (4,4)	(6,1,1) (5,2,1) (4,3,1) (4,2,2) (3,3,2)	(5,1,1,1) (4,2,1,1) (3,3,1,1) (3,2,2,1) (2,2,2,2)	(4,1,1,1,1) (3,2,1,1,1) (2,2,2,1,1)	(3,1,1,1,1,1) (2,2,1,1,1,1)	(2,1,1,1,1,1,1)	(1,1,1,1,1,1,1,1)	**22**

$\mathbb{P}(k, v) = \emptyset$;
$\forall \mathbb{X} \in \mathbb{P}(k\text{-}1, v\text{-}1)\ \mathbb{P}(k, v) = \mathbb{P}(k, v) \cup (\mathbb{X} \oplus 1)$;
$\forall \mathbb{X} \in \mathbb{P}(k\text{-}1, v)\{$
  $\mathbb{Y} = \mathbb{X} \backslash \lambda(\mathbb{X})$;
  if $(\mathbb{Y} = \emptyset)\ \mathbb{P}(k, v) = \mathbb{P}(k, v) \cup (\lambda(\mathbb{X})+1)$;
  elseif $(\lambda(\mathbb{Y}) \geq (\lambda(\mathbb{X})+1))$
    $\mathbb{P}(k, v) = \mathbb{P}(k, v) \cup (\mathbb{Y} \oplus (\lambda(\mathbb{X})+1))$;
$\}$ (4)

B. BELL NUMBERS

Respect to counting the number of set partitions, the Bell numbers $B(k)$ solves the issue [1] (page 4). A recursive definition for $B(k)$ is in (5). Using (5), $B(1) = 1$, $B(2) = 2$, $B(3) = 5$, $B(4) = 15$, $B(5) = 52$.

$$B(k) = \sum_{i=0}^{k-1} \binom{k\text{-}1}{i} B(i),\ B(0) = 1 \tag{5}$$

C. STIRLING NUMBERS OF THE SECOND KIND

When it is desired to count the number of set partitions of k elements in exactly v parts, the Stirling numbers of the second kind $S(k, v)$ [2] (chapter 9) can be used. A recursive definition for $S(k, v)$ is given in (6). Using this formula, for instance, $S(5, 3) = 25$. A well known identity that relates Bell numbers and Stirling numbers of the second kind [1] (page 5) is given in (7)

$$S(k, v) = \begin{cases} 0 & \textbf{if } (v = 0) \vee (v > k) \\ 1 & \textbf{if } (v = 1) \vee \\ & ((v = 0) \wedge (k = 0)) \\ S(k\text{-}1, v\text{-}1) & \\ \quad +v \cdot S(k\text{-}1, v) & \textbf{otherwise} \end{cases} \tag{6}$$

$$B(k) = \sum_{i=1}^{v} S(k, i) \tag{7}$$

A short table for $S(k, v)$ is given in Table 3, the last column indicates the summation of the row elements, i.e., the value of $B(k)$.

The numbers $S(k, v)$ can be used to compute the powers of an integer [2] (page 116), the expression to do that is shown in (8).

$$v^k = \sum_{i=0}^{k} \binom{v}{i} \cdot i! \cdot S(k, i) \tag{8}$$

Manipulating the expression in (8), it can be concluded that: $S(k, 2) = \frac{1}{2!}\left(2^k\text{-}\binom{2}{1}\right) = 2^{k-1}\text{-}1$, $S(k, 3) = \frac{1}{3!}\left(3^k\text{-}\binom{3}{1} \cdot 2^k + \binom{3}{2}\right)$, and generalizing this result [2] an expression for $S(k, v)$ as a summation of powers of integers with alternated signs is obtained, see (9).

$$S(k, v) = \frac{1}{v!} \sum_{i=0}^{v-1} (\text{-}1)^i \cdot \binom{v}{i} \cdot (v\text{-}i)^k \tag{9}$$

Connections of Stirling numbers of the second kind and integer partitions were treated in [12] and references herein. In particular, it is interesting to express the value of $S(k, v)$ in terms of $\mathbb{P}(k, v)$, this was done in Theorem 13.2 of [3] (page 215), the formula that describes this relation is presented in (10) as $S(k, v, \mathbb{X})$, where $\mathbb{X}$ is one of the integer partitions needed. The function $\eta(\mathbb{X})$ returns the elements of the partition $\mathbb{X}$ that are distinct, the function $\mu(l, \mathbb{X})$ returns the number of times that the element l appears in partition $\mathbb{X}$, and each component of a partition $\mathbb{X}$ is denoted by $\mathbb{X}_i$.

$$S(k, v, \mathbb{X}) = \frac{k!}{\prod_{i=1}^{v} \mathbb{X}_i! \prod_{l \in \eta(\mathbb{X})} \mu(l, \mathbb{X})!}, \quad \mathbb{X} \in \mathbb{P}(k, v) \tag{10}$$

TABLE 3. A short table of Stirling numbers of the second kind $S(k, v)$, the last column indicates the value of $B(k)$.

k \ v	1	2	3	4	5	6	7	8	9	$B(k)$
1	1									**1**
2	1	1								**2**
3	1	3	1							**5**
4	1	7	6	1						**15**
5	1	15	25	10	1					**52**
6	1	31	90	65	15	1				**203**
7	1	63	301	350	140	21	1			**877**
8	1	127	966	1,701	1,050	266	28	1		**4,140**
9	1	255	3,025	7,770	6,951	2,646	462	36	1	**21,147**

Henceforth, the notation $S(k, v, \mathbb{X})$ will be used to refer to the number of set partitions of k elements in v parts that follows the part-distribution specified by the integer partition $\mathbb{X} \in \mathbb{P}(k, v)$. Then, the expression in (11) is obtained. For instance $S(4, 2) = S(4, 2, (3, 1))+S(4, 2, (2, 2)) = 4+3 = 7$.

$$S(k, v) = \sum_{\forall \mathbb{X} \in \mathbb{P}(k,v)} S(k, v, \mathbb{X}) \quad (11)$$

Given that $S(k, v)$ is at least the number of elements present in $\mathbb{P}(k, v)$, $S(k, v) \geq p(k, v)$ and using the fact that $B(k)$ can be computed as the summation of $S(k, v)$ for $1 \leq v \leq k$, the computation of Bell numbers $B(k)$ in terms of the explicit integer partitions $\mathbb{P}(k)$, is expressed in (12). Then, it can be concluded that $B(k) \geq p(k)$.

$$B(k) = \sum_{v=1}^{k} \sum_{\forall \mathbb{X} \in \mathbb{P}(k,v)} \frac{k!}{\prod_{i=1}^{v} \mathbb{X}_i! \prod_{l \in \eta(\mathbb{X})} \mu(l, \mathbb{X})!} \quad (12)$$

The explicit partitions whose cardinality is defined by Stirling numbers of the second kind ($S(k, v)$, $S(k, v, \mathbb{X})$), and Bell numbers ($B(k)$), will be defined using Restricted Growth Strings (RGS) in Section III.

D. EULERIAN NUMBERS

The Eulerian numbers [2], denoted by $E(k, v)$, count how many permutations of size k have exactly v-1 *ascents*. An *ascent* occurs when an element in the permutation is greater than the previous elements.

Example 1: Assuming $k = 3$ the permutations are:

(1, 2, 3) has 2 ascents *countsfor* $E(3, 3)$
(1, 3, 2) has 1 ascents *countsfor* $E(3, 2)$
(2, 1, 3) has 1 ascents *countsfor* $E(3, 2)$
(2, 3, 1) has 1 ascents *countsfor* $E(3, 2)$
(3, 1, 2) has 1 ascents *countsfor* $E(3, 2)$
(3, 2, 1) has 0 ascents *countsfor* $E(3, 1)$

Then, $E(3, 1) = 1$; $E(3, 2) = 4$; $E(3, 3) = 1$. △

The Eulerian numbers $E(k, v)$ satisfies the relation given in (13),

$$\sum_{v=1}^{k} E(k, v) = k! \quad (13)$$

Eulerian numbers $E(k, v)$ can be computed according to [2], see (14), and also an interesting relation is that $E(k, v) = E(k, k\text{-}v+1)$

$$E(k, v) = \begin{cases} 0 & \textbf{if } (v = 0) \vee (v > k) \\ 1 & \textbf{if } (v = 1) \\ & \quad \vee ((v = 0) \wedge (k = 0)) \\ (k\text{-}(v\text{-}1)) \cdot E(k\text{-}1, v\text{-}1) & \\ \quad +v \cdot E(k\text{-}1, v) & \textbf{otherwise} \end{cases} \quad (14)$$

A short table of Eulerian numbers is presented in Table 4

It is well known [2] (page 139) that Eulerian numbers $E(k, v)$ can be used to compute powers of an integer, in particular in (15) an expression to fulfill this end is given.

$$v^k = \sum_{i=0}^{k} \binom{v+i\text{-}1}{k} \cdot E(k, i) \quad (15)$$

Manipulating the expression in (15), it can be concluded that: $E(k, 2) = 2^k\text{-}\binom{k+1}{1}$,$E(k, 3) = 3^k\text{-}\binom{k+1}{1}\cdot 2^k+\binom{k+1}{2}$, and generalizing this result [2] (page 140) an expression for $E(k, v)$ as a summation of powers of integers with alternated signs is obtained,see (16).

$$E(k, v) = \sum_{i=0}^{v} (\text{-}1)^i \cdot \binom{k+1}{i} \cdot (v\text{-}i)^k \quad (16)$$

The explicit elements counted by $E(k, v)$ can be listed using the factoradic notation that will be presented in detail in Section III.

E. NUMBERS TO COUNT CARDINALITY OF FUNCTIONS

In [13] (Section 5.2) a deep treatment of Catalan combinatorics is presented, in particular is of interest the definition of *Endomorphisms* given in page 168 and reported as sequence **A090657** in [14]. This numbers will be designated by $N(k)$

TABLE 4. Table of Eulerian Numbers, the last columns is the summation of the values of each row and is equal to $k!$.

k \ v	1	2	3	4	5	6	7	8	$\sum_{v=1}^{k} E(k,v)$
1	1								**1**
2	1	1							**2**
3	1	4	1						**6**
4	1	11	11	1					**24**
5	1	26	66	26	1				**120**
6	1	57	302	302	57	1			**720**
7	1	120	1,191	2,416	1,191	120	1		**5,040**
8	1	247	4,293	15,619	15,619	4,293	247	1	**40,320**

and refers to the total number of functions from $\{0,\ldots,k\text{-}1\}$ to $\{0,\ldots,k\text{-}1\}$. They are defined in (17).

$$N(k) = \sum_{v=0}^{k} v!\cdot S(k,v)\cdot\binom{k}{v} \tag{17}$$

The value of $N(k)$ is exactly k^k, and the calculation of this number with v parts is denoted with $N(k,v)$, it has the meaning of the cardinality of functions from $\{0,\ldots,k\text{-}1\}$ to $\{0,\ldots,k\text{-}1\}$ such that the image contains exactly v elements, see (18).

$$N(k,v) = v!\cdot\binom{k}{v}\cdot S(k,v) \tag{18}$$

And also, it can be stated the relationship shown in (19)

$$S(k,v) = \frac{N(k,v)}{v!\cdot\binom{k}{v}} \tag{19}$$

Example 2: $N(3,2)$ counts the numbers with 3 digits that have 2 different symbols. According (18) $N(3,2)) = 2!\cdot S(3,2)\cdot\binom{3}{2} = 2\cdot 3\cdot 3 = 18$, the counted numbers are:

$$(0,0,1)(0,0,2)(1,1,0)(1,1,2)(2,2,0)(2,2,1)$$
$$(0,1,0)(0,2,0)(1,0,1)(1,2,1)(2,0,2)(2,1,2)$$
$$(1,0,0)(2,0,0)(0,1,1)(2,1,1)(0,2,2)(1,2,2)$$

△

A connection between integer partitions and $N(k,v)$ is easily established, replacing (10) in (18), and it is obtained (20).

$$N(k,v) = v!\cdot\binom{k}{v}\cdot\sum_{\forall\mathbb{X}\in\mathbb{P}(k,v)} \frac{k!}{\prod_{i=1}^{v}\mathbb{X}_i!\ \prod_{l\in\eta(\mathbb{X})}\mu(l,\mathbb{X})!} \tag{20}$$

The $N(k,v)$ numbers can be used to compute integer powers as is stated in (21).

$$v^k = \sum_{i=0}^{v} \frac{\binom{v}{i}}{\binom{k}{i}}\cdot N(k,i) \tag{21}$$

The direct calculation of $N(k,v)$ in terms of sums of powers is given in (22)

$$N(k,v) = \binom{k}{v}\cdot\sum_{i=0}^{v}(\text{-}1)^i\cdot\binom{v}{i}\cdot(v\text{-}i)^k \tag{22}$$

A formula for $\mathbb{N}(k,v)$ involving $\mathbb{N}(k\text{-}1,v\text{-}1)$ and $\mathbb{N}(k\text{-}1,v)$, can be easily derived using (6), (18) and (19), this is done in (23).

from (18)

$$N(k,v) = v!\cdot\binom{k}{v}\cdot S(k,v)$$

using (6)

$$N(k,v) = v!\cdot\binom{k}{v}\cdot\big(S(k\text{-}1,v\text{-}1)+v\cdot S(k\text{-}1,v)\big)$$

using (19)

$$N(k,v) = v!\cdot\binom{k}{v}\cdot\Big(\frac{N(k\text{-}1,v\text{-}1)}{(v\text{-}1)!\cdot\binom{k\text{-}1}{v\text{-}1}}+v\cdot\frac{N(k\text{-}1,v)}{v!\cdot\binom{k\text{-}1}{v}}\Big)$$

simplifying

$$N(k,v) = k\cdot N(k\text{-}1,v\text{-}1)+\frac{k\cdot v}{k\text{-}v}\cdot N(k\text{-}1,v) \tag{23}$$

Then the recursive definition for $N(k,v)$ is taken from (23) and is presented in (24), this also corresponds to the sequence **A090657** of [14].

$$N(k,v)=\begin{cases} 0 & \textbf{if } (v=0)\vee(v>k)\\ 1 & \textbf{if } (v=1)\vee\\ & \quad((v=0)\wedge(k=0))\\ k\cdot N(k\text{-}1,v\text{-}1) & \\ +\frac{k\cdot v}{k\text{-}v}\cdot N(k\text{-}1,v) & \textbf{otherwise}\end{cases} \tag{24}$$

A short table for $N(k,v)$ is given in Table 5.

The explicit elements that are counted by $N(k,v)$ will be presented in Section III, using numbers expressed in a positional number system with a fixed base.

F. POWERS OF AN INTEGER

The powers of an integer are presented here for its direct relationship with $S(k,v)$, $E(k,v)$, and $N(k,v)$. Few values of v^k are shown in Table 6, the last column contains the result of $\sum_{v=1}^{k} v^k$. A definition for v^k in terms of v^{k-1} is simply: $v\cdot v^{k-1}$. $S(k,v)$,$E(k,v)$ and $N(k,v)$ can be used to compute values of v^k as is stated in (8), (15) and (21).

G. ADDITION OF INTEGER POWERS

The problem of computing the result of adding consecutive integers from $1,\ldots,k$ raised to a power r is expressed in (25).

$$A(k,r) = \sum_{v=1}^{k} v^r \tag{25}$$

TABLE 5. Table for $N(k, v)$, last column contains the summation of all the elements of each row, and is equal to k^k.

k \ v	1	2	3	4	5	6	7	$\sum_{v=1}^{k} N(k,v)$
1	1							**1**
2	2	2						**4**
3	3	18	6					**27**
4	4	84	144	24				**256**
5	5	300	1,500	1,200	120			**3,125**
6	6	930	10,800	23,400	10,800	720		**46,656**
7	7	2,646	63,210	294,000	352,800	105,840	5,040	**823,543**

TABLE 6. Table for v^k, the last column contains the summation of the row values.

k \ v	1	2	3	4	5	6	7	$\sum_{v=1}^{k} v^k$
1	1							**1**
2	1	4						**5**
3	1	8	27					**36**
4	1	16	81	256				**354**
5	1	32	243	1,024	3,125			**4,425**
6	1	64	729	4,096	15,625	46,656		**67,171**
7	1	128	2,187	16,384	78,125	279,936	823,543	**1,200,304**

The computation of $A(k, r)$ has attracted the attention of many important mathematicians like Jakob Bernoulli, instead of giving a general expression for $A(k, r)$ using Bernoulli numbers [2] (chapter 15), $S(k, v)$, $E(k, v)$ and $N(k, v)$ will be used.

In [2] (page 212) $A(k, r)$ is expressed using Stirling numbers of the second kind ($S(k, v)$), see (26).

$$A(k,r) = \sum_{i=1}^{r} \binom{k+1}{i+1} \cdot i! \cdot S(r,i) \tag{26}$$

In [15] the computation of $A(k, r)$ is given using Eulerian numbers, see (27)

$$A(k,r) = \sum_{i=1}^{r} \binom{k+i}{r+1} \cdot E(r,i) \tag{27}$$

Using (18) and (26), an expression for $A(k, r)$ using $N(k, v)$ is given in (28)

$$A(k,r) = \sum_{i=1}^{r} \frac{\binom{k+1}{i+1} \cdot i!}{\binom{k}{i} \cdot i!} \cdot N(r,i) = \sum_{i=1}^{r} \frac{k+1}{i+1} \cdot N(r,i) \tag{28}$$

H. TABLE OF NUMBERS DEFINED IN THIS SECTION

A little manipulation of the formulas given for $S(k, v)$, $E(k, v)$, $N(k, v)$, v^k, and $A(k, r)$ enables the creation of a table that summarizes the formulas given in this section, see Table 7. The last row of the table contains the summation of each of the labels of the columns.

III. THREE REPRESENTATIONS FOR SET PARTITIONS

In this section explicit objects that are counted by the numbers: $S(k, v)$, $E(k, v)$, and $N(k, v)$, are presented. These objects are based on Restricted Growth Strings (RGS), factoradic number system, and a positional number system with a fixed base. The names of the explicit objects can be viewed as sets, for this reason the names: $\mathbb{S}(k, v)$ for $S(k, v)$, $\mathbb{E}(k, v)$ for $E(k, v)$, and $\mathbb{N}(k, v)$ for $N(k, v)$, will be used. It is important to say that members of a set are represented enclosed between { and }, and can be processed in any order; and members of a sequence are represented enclosed between [and], and must be processed sequentially.

A. REPRESENTATION USING RESTRICTED GROWTH STRINGS

The Restricted Growth Strings (RGS) are a kind of number system in which the positional value and the valid symbols in each position depends on previous symbols, see [4] (page 81). This number system will be designated as $\mathbb{S}(k)$, then $\mathbb{X} \in \mathbb{S}(k)$ satisfies that:

$$(|\mathbb{X}| = k) \wedge (\mathbb{X}_0 = 0)$$
$$\mathbb{X}_i \in \{0, \ldots, 1 + \max_{j=0}^{i-1}(\mathbb{X}_j)\}$$

The numbers that belongs to $\mathbb{S}(k)$ can be used to represent without redundancy, all the possible solutions for the SPP; for instance, if $\mathbb{X} \in \mathbb{S}(k)$ the i-th object goes in partition $\mathbb{X}_i$. This way, $|\mathbb{S}(k)|$ is described by (29).

$$|\mathbb{S}(k)| = B(k) \tag{29}$$

Example 3: The members of $\mathbb{S}(k)$ for $k = 4$ are: {(0,0,0,0) (0,0,0,1) (0,0,1,0) (0,0,1,1) (0,0,1,2) (0,1,0,0) (0,1,0,1) (0,1,0,2) (0,1,1,0) (0,1,1,1) (0,1,1,2) (0,1,2,0) (0,1,2,1) (0,1,2,2) (0,1,2,3)}

The cardinality is 15 that is the same as $B(4) = 15$ △

The number of solutions for SPP with v parts is described exactly by $S(k, v)$ as stated in (30).

$$|\mathbb{S}(k,v)| = S(k,v) \tag{30}$$

Example 4: The members of $\mathbb{S}(k, v)$ for $k = 4, v = 2$ are: {(0,0,0,1) (0,0,1,0) (0,1,0,0) (0,1,1,1) (0,0,1,1) (0,1,0,1) (0,1,1,0)}

Seven members, that coincide with $S(4, 2) = 7$ △

TABLE 7. Summary of formulas presented in Section II, the last row of the table contains the summation of each of the labels of the columns.

	S(k,v)	*E(k,v)*	*N(k,v)*	v^k
S(k,v)=	$S(k\text{-}1,v\text{-}1)+v\cdot S(k\text{-}1,v)$	$\frac{1}{v!}\cdot\sum_{i=0}^{k}\binom{i\text{-}1}{k\text{-}v}\cdot E(k,i)$	$\frac{1}{v!\cdot\binom{k}{v}}\cdot N(k,v)$	$\frac{1}{v!}\cdot\sum_{i=0}^{v}(\text{-}1)^i\cdot\binom{v}{i}\cdot(v\text{-}i)^k$
E(k,v)=	$\sum_{i=0}^{k}(\text{-}1)^{v+i}\cdot\binom{k\text{-}i}{k\text{-}v}\cdot i!\cdot S(k,i)$	$(k\text{-}(v\text{-}1))\cdot E(k\text{-}1,v\text{-}1)+v\cdot E(k\text{-}1,v)$	$\sum_{i=0}^{k}\frac{(\text{-}1)^{v+i}}{\binom{k}{i}}\cdot\binom{k\text{-}i}{k\text{-}v}\cdot N(k,i)$	$\sum_{i=0}^{v}(\text{-}1)^i\cdot\binom{k+1}{i}\cdot(v\text{-}i)^k$
N(k,v)=	$v!\cdot S(k,v)\cdot\binom{k}{v}$	$\binom{k}{v}\cdot\sum_{i=0}^{k}\binom{i\text{-}1}{k\text{-}v}\cdot E(k,i)$	$k\cdot N(k\text{-}1,v\text{-}1)+\frac{k\cdot v}{k\text{-}v}\cdot N(k\text{-}1,v)$	$\binom{k}{v}\cdot\sum_{i=0}^{v}(\text{-}1)^i\cdot\binom{v}{i}\cdot(v\text{-}i)^k$
$v^k=$	$\sum_{i=0}^{k}\binom{v}{i}\cdot i!\cdot S(k,i)$	$\sum_{i=0}^{k}\binom{v+i-1}{k}\cdot E(k,i)$	$\sum_{i=0}^{k}\frac{\binom{v}{i}}{\binom{k}{i}}\cdot N(k,i)$	$v\cdot v^{k\text{-}1}$
$\sum_{\mathbf{v=1}}^{\mathbf{k}}\{\cdot\}=$	$\boldsymbol{B(k)}$	$\boldsymbol{k!}$	$\boldsymbol{k^k}$	$\sum_{\mathbf{i=1}}^{\mathbf{k}}\binom{\mathbf{k+1}}{\mathbf{i+1}}\cdot\mathbf{i!}\cdot\mathbf{S(k,i)}$ $\sum_{\mathbf{i=1}}^{\mathbf{k}}\binom{\mathbf{k+i}}{\mathbf{k+1}}\cdot\mathbf{E(k,i)}$ $\sum_{\mathbf{i=1}}^{\mathbf{k}}\frac{\mathbf{k+1}}{\mathbf{i+1}}\cdot\mathbf{N(k,i)}$

The cardinality of members of $\mathbb{S}(k, v)$ that have an specific pattern $\mathbb{X} \in \mathbb{P}(k, v)$ is described exactly by $S(k, v, \mathbb{X})$ see (10), then in (31) the cardinality of $\mathbb{S}(k, v, \mathbb{X})$ is obtained.

$$|\mathbb{S}(k, v, \mathbb{X})| = \frac{k!}{\prod_{i=1}^{v} \mathbb{X}_i! \prod_{l\in\eta(\mathbb{X})} \mu(l, \mathbb{X})!}, \quad \mathbb{X} \in \mathbb{P}(k, v) \quad (31)$$

The construction of $\mathbb{S}(k, v)$ using $\mathbb{S}(k\text{-}1, v\text{-}1)$ and $\mathbb{S}(k\text{-}1, v)$ is defined in (32), take into account that $\mathbb{S}(i, 1) = \{(0, 0, \ldots, 0)\}$ (i.e. i zeros), $\mathbb{S}(i, 0) = \{\emptyset\}$ and $k \geq v$.

$$\begin{aligned}
&\mathbb{S}(k, v) = \{\emptyset\}\\
&\forall \mathbb{X} \in \mathbb{S}(k\text{-}1, v\text{-}1)\ \mathbb{S}(k, v) = \mathbb{S}(k, v) \cup (\mathbb{X} \oplus (v\text{-}1));\\
&\forall \mathbb{X} \in \mathbb{S}(k\text{-}1, v)\{\\
&\quad \forall\ a \in\ \eta(\mathbb{X})\ \mathbb{S}(k, v) = \mathbb{S}(k, v) \cup (\mathbb{X} \oplus a);\\
&\}
\end{aligned} \quad (32)$$

Example 5: Assume we want to compute $\mathbb{S}(5, 3)$ we need $\mathbb{S}(4, 2)$ and $\mathbb{S}(4, 3)$. Then, we have that:

$\mathbb{S}(4, 2)$ ={(0,0,0,1) (0,0,1,0) (0,0,1,1) (0,1,0,0) (0,1,0,1) (0,1,1,0) (0,1,1,1)}

And, $\mathbb{S}(4, 3)$ ={(0,0,1,2)(0,1,0,2)(0,1,1,2) (0,1,2,0) (0,1,2,1)

(0,1,2,2)} Using $\mathbb{S}(4, 2)$ we have:
{((0,0,0,1)⊕2) ((0,0,1,0)⊕2) ((0,0,1,1)⊕2)
((0,1,0,0)⊕2) ((0,1,0,1)⊕2) ((0,1,1,0)⊕2)
((0,1,1,1)⊕2)}
And using $\mathbb{S}(4, 3)$, it is constructed:
{((0,0,1,2)⊕0) ((0,0,1,2)⊕1) ((0,0,1,2)⊕2)
((0,1,0,2)⊕0) ((0,1,0,2)⊕1) ((0,1,0,2)⊕2)
((0,1,1,2)⊕0) ((0,1,1,2)⊕1)
((0,1,1,2)⊕2)
((0,1,2,0)⊕0) ((0,1,2,0)⊕1) ((0,1,2,0)⊕2)
((0,1,2,1)⊕0) ((0,1,2,1)⊕1) ((0,1,2,1)⊕2)
((0,1,2,2)⊕0) ((0,1,2,2)⊕1) ((0,1,2,2)⊕2)}

Finally we have $\mathbb{S}(5, 3)$ ={(0,0,0,1,2) (0,0,1,0,2) (0,0,1,1,2) (0,1,0,0,2) (0,1,0,1,2) (0,1,1,0,2) (0,1,1,1,2) (0,0,1,2,0) (0,0,1,2,1) (0,0,1,2,2) (0,1,0,2,0) (0,1,0,2,1) (0,1,0,2,2) (0,1,1,2,0) (0,1,1,2,1) (0,1,1,2,2) (0,1,2,0,0) (0,1,2,0,1) (0,1,2,0,2) (0,1,2,1,0) (0,1,2,1,1) (0,1,2,1,2) (0,1,2,2,0) (0,1,2,2,1) (0,1,2,2,2)} △

The correctness of (32) is evident from two facts: for each element of $\mathbb{S}(k\text{-}1, v\text{-}1)$ one element to $\mathbb{S}(k, v)$ is added; and for each element of $\mathbb{S}(k\text{-}1, v)$ the addition of v elements to $\mathbb{S}(k, v)$ is done (a consequence that $\eta(\mathbb{S}(k\text{-}1, v)) = v$). Therefore,

$$|\mathbb{S}(k, v)| = |\mathbb{S}(k\text{-}1, v\text{-}1)| + v\cdot|\mathbb{S}(k\text{-}1, v)|$$

and this corresponds to the definition of the Stirling numbers of the second kind see (6). Then, it is reiterated that that $|\mathbb{S}(k, v)| = S(k, v)$

A short table for $\mathbb{S}(k, v)$ in terms of $\mathbb{S}(k, v, \mathbb{X})$ is given in Table 8, take note that each partition has as super index the cardinality of set partitions that satisfy the pattern given by the integer partition $\mathbb{X}$, the final result (after equal sign) express the value of $|\mathbb{S}(k, v)|$.

By the name *ranking* it is meant to assign a set of consecutive 0-based integers to all the objects that are managed, and by the name *unranking* it is meant to construct the object that corresponds to an integer given as input. The ranking and unranking [4] of elements of $\mathbb{S}(k)$ require the definition of a table of positional values that depend on the maximum number that has appeared in a partial RGS. The positional values will be denoted by $\mathbb{O}(k)$ and the access to its elements will be done by $\mathbb{O}(k)_{i,j}$, the size of this table is $(k+1)\times(k+1)$. The construction of the table is defined in (33).

$$\begin{aligned}
&\forall\ j \in [0, \ldots, k]\ \mathbb{O}(k)_{0,j} = 1;\\
&\forall\ i \in [1, \ldots, k]\ \{\\
&\quad \forall\ j \in [0, \ldots, k-1]\ \{\\
&\qquad \mathbb{O}(k)_{i,j} = j\cdot\mathbb{O}(k)_{i\text{-}1,j} + \mathbb{O}(k)_{i\text{-}1,j+1};
\end{aligned}$$

TABLE 8. Table for $\mathbb{S}(k) = \sum_{\forall \mathbb{X} \in \mathbb{P}(k,v)} \mathbb{S}(k, v, \mathbb{X})$, the super index notation computes the value of: $k!/(\prod_{i=1}^{v} \mathbb{X}_i! \prod_{l \in \eta(\mathbb{X})} \mu(l, \mathbb{X})!)$.

k \ v	1	2	3	4
1	$(1)^1 = 1$			
2	$(1)^1 = 1$	$(1,1)^1 = 1$		
3	$(1)^1 = 1$	$(2,1)^3 = 3$	$(1,1,1)^1 = 1$	
4	$(1)^1 = 1$	$(3,1)^4 + (2,2)^3 = 7$	$(2,1,1)^6 = 6$	$(1,1,1,1)^1$

```
    }
}                                                    (33)
```

The result for $\mathbb{O}(5)$ (i.e. the size is 6×6) is given in Table 9.

The ranking algorithm for $\mathbb{X} \in \mathbb{S}(k)$ is given in (34) the result is *rank*.

$$
\begin{aligned}
& rank = 0; j = 1; \\
& \forall\, i \in [1, \ldots, k\text{-}1]\ \{ \\
& \quad rank = rank + \mathbb{X}_i \cdot \mathbb{O}(k)_{k-i-1,j}; \\
& \quad j = \max(j, \mathbb{X}_i + 1); \\
& \}
\end{aligned}
\tag{34}
$$

The unranking algorithm for the integer *rank* with k digits is returned in $\mathbb{X} \in \mathbb{S}(k)$ the computation is given in (35)

$$
\begin{aligned}
& 0 \le rank \le B(k) - 1 \\
& \mathbb{X}_0 = 0;\ j = 1; \\
& \forall\, i \in [1, \ldots, k\text{-}1]\ \{ \\
& \quad \textbf{if } (j \cdot \mathbb{O}(k)_{k\text{-}i\text{-}1,j} \le rank) \textbf{ then } \{ \\
& \quad\quad \mathbb{X}_i = j; \\
& \quad\quad rank = rank\text{-}j \cdot \mathbb{O}(k)_{k\text{-}i\text{-}1,j}; \\
& \quad\quad j = j + 1; \\
& \quad \} \\
& \quad \textbf{else}\{ \\
& \quad\quad \mathbb{X}_i = \lfloor \frac{rank}{\mathbb{O}(k)_{k\text{-}i\text{-}1,j}} \rfloor; \\
& \quad\quad rank = rank \mod \mathbb{O}(k)_{k\text{-}i\text{-}1,j}; \\
& \quad \} \\
& \}
\end{aligned}
\tag{35}
$$

Next, some cases for rank and unrank algorithms are shown:

$$
\begin{aligned}
&(0, 1, 2, 3) \implies 14 \\
&(0, 1, 0, 2, 0, 3) \implies 75 \\
&(0, 0, 0, 0, 1) \implies 1 \\
&14\ with\ k = 4 \implies (0, 1, 2, 3) \\
&30\ with\ k = 5 \implies (0, 1, 1, 1, 2)
\end{aligned}
$$

B. REPRESENTATION USING FACTORADIC NUMBER SYSTEM

The factoradic number system (also known as factorial number system) [5] is a positional number system with variable base, and it will be used the notation $\mathbb{E}(k)$. Assuming that k is the number of digits, the cardinality of the distinct numbers is $k!$, i.e. $|\mathbb{E}(k)| = k!$.

TABLE 9. Table of $\mathbb{O}(5)$, positional values for ranking, unranking of RGS/decimal number.

i \ j	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	2	3	4	5	
2	2	5	10	17		
3	5	15	37			
4	15	52				
5	52					

By convenience, the members of $\mathbb{E}(k)$ are indexed 0-based from left to right, and a member $\mathbb{X} \in \mathbb{E}(k)$ consists always of k digits, and the i-th digit denoted by $\mathbb{X}_i$ $0 \le i \le k\text{-}1$ satisfies $0 \le \mathbb{X}_i \le i$.

Example 6: The members of $\mathbb{E}(3)$ are {(0,0,0) (0,0,1) (0,0,2) (0,1,0) (0,1,1) (0,1,2)}
△

The factoradic system can be used to represent set partitions in the next way, let $\mathbb{X} \in \mathbb{E}(k)$, $\mathbb{X}_i$, $0 \le i \le k\text{-}1$ indicates the partition to which belongs the i-th element of the set.

Example 7: Let $\mathbb{X} \in \mathbb{E}(6)$ be $\mathbb{X} = (0, 0, 2, 2, 1, 3)$, the partition of the set {0, 1, 2, 3, 4, 5} is {{0, 1}, {2, 3}, {4}, {5}}
△

The members of $\mathbb{E}(k)$ that have v different symbols is expressed by $\mathbb{E}(k, v)$.

Example 8: The member of $\mathbb{E}(1)$ is:

$$(0) \in \mathbb{E}(1, 1)$$

The members of $\mathbb{E}(2)$ are:

$$
\begin{aligned}
(0, 0) &\in \mathbb{E}(2, 1) \\
(0, 1) &\in \mathbb{E}(2, 2)
\end{aligned}
$$

The members of $\mathbb{E}(3)$ are:

$$
\begin{aligned}
(0, 0, 0) &\in \mathbb{E}(3, 1) \\
(0, 0, 1) &\in \mathbb{E}(3, 2) \\
(0, 0, 2) &\in \mathbb{E}(3, 2) \\
(0, 1, 0) &\in \mathbb{E}(3, 2) \\
(0, 1, 1) &\in \mathbb{E}(3, 2) \\
(0, 1, 2) &\in \mathbb{E}(3, 3)
\end{aligned}
$$

△

The construction of permutations in factoradic notation $\mathbb{E}(k, v)$ can be done using $\mathbb{E}(k\text{-}1, v)$ and $\mathbb{E}(k\text{-}1, v\text{-}1)$ as is expressed in (36). Take note that: $\eta(\mathbb{X})$ returns the symbols without repetition that are in permutation $\mathbb{X}$;

$\mathbb{X} \oplus a$ returns $\mathbb{X}$ concatenated with a, also consider that $\mathbb{E}(i, 1) = \{(0, 0, \ldots, 0)\}$ (i.e. i zeros), $\mathbb{E}(i, 0) = \{\emptyset\}$ and $k \geq v$.

$$\begin{aligned}
&\mathbb{E}(k, v) = \{\emptyset\}\\
&\forall \mathbb{X} \in \mathbb{E}(k\text{-}1, v\text{-}1)\{\\
&\ \forall a \in (\{0, \ldots, k\text{-}1\} \backslash \eta(\mathbb{X}))\ \mathbb{E}(k, v) = \mathbb{E}(k, v) \cup (\mathbb{X} \oplus a);\\
&\}\\
&\forall \mathbb{X} \in \mathbb{E}(k\text{-}1, v)\{\\
&\ \forall a \in \eta(\mathbb{X})\ \mathbb{E}(k, v) = \mathbb{E}(k, v) \cup (\mathbb{X} \oplus a);\\
&\}
\end{aligned} \tag{36}$$

It is easy to see in (36) that: the cardinality of $\{0, \ldots, k\text{-}1\} \backslash \eta(\mathbb{X})$ is exactly k-(v-1), then for each permutation of $\mathbb{E}(k\text{-}1, v\text{-}1)$ the addition of k-(v-1) permutations of size k to $\mathbb{E}(k, v)$ is performed; and it is obvious that $\eta(\mathbb{X})$ for $\mathbb{X} \in \mathbb{E}(k\text{-}1, v)$ has cardinality of v, then for each permutation of $\mathbb{E}(k\text{-}1, v)$ the addition of v permutations of size k to $\mathbb{E}(k, v)$ is done.

Surprisingly, the cardinality of $\mathbb{E}(k, v)$ (as it can be inferred yet) is the same as the Eulerian numbers $E(k, v)$, i.e. $|\mathbb{E}(k, v)| = E(k, v)$, this will be stated in Theorem 1.

Theorem 1: The cardinality of $\mathbb{E}(k, v)$ is described by $E(k, v)$.

Proof: The construction of $\mathbb{E}(k, v)$ is done (see (36)) adding v permutations for each element of $\mathbb{E}(k\text{-}1, v)$, and adding k-(v-1) permutations for each element of $\mathbb{E}(k\text{-}1, v\text{-}1)$. Then it is obvious that:

$$|\mathbb{E}(k, v)| = (k\text{-}(v\text{-}1)) \cdot |\mathbb{E}(k\text{-}1, v\text{-}1)| + v \cdot |\mathbb{E}(k\text{-}1, v)|$$

Comparing this expression with (14), an equivalent definition is obtained, then: $|\mathbb{E}(k, v)| = E(k, v)$ This finishes the proof. ■

Example 9: For the construction of $\mathbb{E}(5, 3)$ using $\mathbb{E}(4, 2)$ and $\mathbb{E}(4, 3)$ we have that:

$\mathbb{E}(4, 2)$ ={(0,0,0,1) (0,0,0,2) (0,0,0,3) (0,0,1,0) (0,0,1,1) (0,0,2,0) (0,0,2,2) (0,1,0,0) (0,1,0,1) (0,1,1,0) (0,1,1,1)}

$\mathbb{E}(4, 3)$ ={(0,0,1,2) (0,0,1,3) (0,0,2,1) (0,0,2,3) (0,1,0,2) (0,1,0,3) (0,1,1,2) (0,1,1,3) (0,1,2,0) (0,1,2,1) (0,1,2,2)}

Using $\mathbb{E}(4, 2)$ we have:

$$\begin{aligned}
&\{((0, 0, 0, 1)\oplus 2)((0, 0, 0, 1)\oplus 3)((0, 0, 0, 1)\oplus 4)\\
&((0, 0, 0, 2)\oplus 1)((0, 0, 0, 2)\oplus 3)((0, 0, 0, 2)\oplus 4)\\
&((0, 0, 0, 3)\oplus 1)((0, 0, 0, 3)\oplus 2)((0, 0, 0, 3)\oplus 4)\\
&((0, 0, 1, 0)\oplus 2)((0, 0, 1, 0)\oplus 3)((0, 0, 1, 0)\oplus 4)\\
&((0, 0, 1, 1)\oplus 2)((0, 0, 1, 1)\oplus 3)((0, 0, 1, 1)\oplus 4)\\
&((0, 0, 2, 0)\oplus 1)((0, 0, 2, 0)\oplus 3)((0, 0, 2, 0)\oplus 4)\\
&((0, 0, 2, 2)\oplus 1)((0, 0, 2, 2)\oplus 3)((0, 0, 2, 2)\oplus 4)\\
&((0, 1, 0, 0)\oplus 2)((0, 1, 0, 0)\oplus 3)((0, 1, 0, 0)\oplus 4)\\
&((0, 1, 0, 1)\oplus 2)((0, 1, 0, 1)\oplus 3)((0, 1, 0, 1)\oplus 4)\\
&((0, 1, 1, 0)\oplus 2)((0, 1, 1, 0)\oplus 3)((0, 1, 1, 0)\oplus 4)\\
&((0, 1, 1, 1)\oplus 2)((0, 1, 1, 1)\oplus 3)((0, 1, 1, 1)\oplus 4)\}
\end{aligned}$$

Using $\mathbb{E}(4, 3)$ it is obtained:

$$\begin{aligned}
&\{((0, 0, 1, 2)\oplus 0)((0, 0, 1, 2)\oplus 1)((0, 0, 1, 2)\oplus 2)\\
&((0, 0, 1, 3)\oplus 0)((0, 0, 1, 3)\oplus 1)((0, 0, 1, 3)\oplus 3)\\
&((0, 0, 2, 1)\oplus 0)((0, 0, 2, 1)\oplus 1)((0, 0, 2, 1)\oplus 2)\\
&((0, 0, 2, 3)\oplus 0)((0, 0, 2, 3)\oplus 2)((0, 0, 2, 3)\oplus 3)\\
&((0, 1, 0, 2)\oplus 0)((0, 1, 0, 2)\oplus 1)((0, 1, 0, 2)\oplus 2)\\
&((0, 1, 0, 3)\oplus 0)((0, 1, 0, 3)\oplus 1)((0, 1, 0, 3)\oplus 3)\\
&((0, 1, 1, 2)\oplus 0)((0, 1, 1, 2)\oplus 1)((0, 1, 1, 2)\oplus 2)\\
&((0, 1, 1, 3)\oplus 0)((0, 1, 1, 3)\oplus 1)((0, 1, 1, 3)\oplus 3)\\
&((0, 1, 2, 0)\oplus 0)((0, 1, 2, 0)\oplus 1)((0, 1, 2, 0)\oplus 2)\\
&((0, 1, 2, 1)\oplus 0)((0, 1, 2, 1)\oplus 1)((0, 1, 2, 1)\oplus 2)\\
&((0, 1, 2, 2)\oplus 0)((0, 1, 2, 2)\oplus 1)((0, 1, 2, 2)\oplus 2)\}
\end{aligned}$$

Finally $\mathbb{E}(5, 3)$ = {(0,0,0,1,2) (0,0,0,1,3) (0,0,0,1,4) (0,0,0,2,1) (0,0,0,2,3) (0,0,0,2,4) (0,0,0,3,1) (0,0,0,3,2) (0,0,0,3,4) (0,0,1,0,2) (0,0,1,0,3) (0,0,1,0,4) (0,0,1,1,2) (0,0,1,1,3) (0,0,1,1,4) (0,0,2,0,1) (0,0,2,0,3) (0,0,2,0,4) (0,0,2,2,1) (0,0,2,2,3) (0,0,2,2,4) (0,1,0,0,2) (0,1,0,0,3) (0,1,0,0,4) (0,1,0,1,2) (0,1,0,1,3) (0,1,0,1,4) (0,1,1,0,2) (0,1,1,0,3) (0,1,1,0,4) (0,1,1,1,2) (0,1,1,1,3) (0,1,1,1,4) (0,0,1,2,0) (0,0,1,2,1) (0,0,1,2,2) (0,0,1,3,0) (0,0,1,3,1) (0,0,1,3,3) (0,0,2,1,0) (0,0,2,1,1) (0,0,2,1,2) (0,0,2,3,0) (0,0,2,3,2) (0,0,2,3,3) (0,1,0,2,0) (0,1,0,2,1) (0,1,0,2,2) (0,1,0,3,0) (0,1,0,3,1) (0,1,0,3,3) (0,1,1,2,0) (0,1,1,2,1) (0,1,1,2,2) (0,1,1,3,0) (0,1,1,3,1) (0,1,1,3,3) (0,1,2,0,0) (0,1,2,0,1) (0,1,2,0,2) (0,1,2,1,0) (0,1,2,1,1) (0,1,2,1,2) (0,1,2,2,0) (0,1,2,2,1) (0,1,2,2,2)} △

It has been shown that $|\mathbb{E}(k)| = k!$, $|\mathbb{E}(k, v)| = E(k, v)$, an expression that relates the cardinality of factoradic number system respect to the integer partitions $\mathbb{P}(k, v)$, i.e., $|\mathbb{E}(k, v, \mathbb{X})|, \mathbb{X} \in \mathbb{P}(k, v)$ is shown in (37), take into account that: $\lambda(\mathbb{X})$ refers to the last element of the factoradic $\mathbb{X}$; $\mu(\mathbb{X}_i\text{-}1, \mathbb{Y})$ returns how many times $\mathbb{X}_i$-1 appears in the factoradic $\mathbb{Y}$; $\delta_{\lambda(\mathbb{X}),1}$ returns 1 if the last element of $\mathbb{X}$ is 1, otherwise it returns 0, it is the Kronecker δ; and $\rho(\mathbb{X}, \mathbb{X}_i, \mathbb{X}_i\text{-}1)$ returns the result of changing the $\mathbb{X}_i$ element with $\mathbb{X}_i$-1 (this does not modify $\mathbb{X}$).

$$\begin{aligned}
&|\mathbb{E}(k, v, \mathbb{X})|\\
&\quad = |\mathbb{E}(k\text{-}1, v\text{-}1, \mathbb{X} \backslash \lambda(\mathbb{X}))| \cdot (k\text{-}(v\text{-}1)) \cdot \delta_{\lambda(\mathbb{X}),1} +\\
&\qquad \sum_{\substack{\forall \mathbb{X}_i | (\mathbb{X}_i \neq 1) \wedge (\mathbb{X}_i - 1 \geq \mathbb{X}_{i+1}),\\ \mathbb{Y} = \rho(\mathbb{X}, \mathbb{X}_i, \mathbb{X}_i - 1)}} |\mathbb{E}(k\text{-}1, v, \mathbb{Y})| \cdot \mu(\mathbb{X}_i\text{-}1, \mathbb{Y})
\end{aligned} \tag{37}$$

It is shown in Table 10 the values of $|\mathbb{E}(k, v, \mathbb{X})|$ as super indices of the integer partition. In each cell it comes after the equal sign the value of $|\mathbb{E}(k, v)| = E(k, v)$.

The ranking of $\mathbb{X} \in \mathbb{E}(k)$ is given in (38) and is returned as the value *rank*.

$$\begin{aligned}
&\mathbb{X} \in \mathbb{E}(k);\ rank = 0;\\
&\forall\ i \in [1, \ldots, k\text{-}1]\ rank = rank \cdot (i+1) + \mathbb{X}_i;\\
&\textbf{return}\ rank
\end{aligned} \tag{38}$$

TABLE 10. Table for $|\mathbb{E}(k, v, \mathbb{X})|$ for $x \in \mathbb{P}(k, v)$, see Equation (37).

k \ v	1	2	3	4
1	$(1)^1 = 1$			
2	$(1)^1 = 1$	$(1,1)^1 = 1$		
3	$(1)^1 = 1$	$(2,1)^4 = 4$	$(1,1,1)^1 = 1$	
4	$(1)^1 = 1$	$(3,1)^7 + (2,2)^4 = 11$	$(2,1,1)^{11} = 11$	$(1,1,1,1)^1$

The unranking of the number *rank* is done in (39) and returned as $\mathbb{X}$ with k digits.

$$
\begin{aligned}
&0 \le rank \le k!\text{-}1;\\
&\mathbb{X}_0 = 0;\\
&\forall\, i \in [k\text{-}1, \ldots, 1]\{\\
&\quad \mathbb{X}_i = rank \mod (i{+}1);\\
&\quad rank = \frac{rank - \mathbb{X}_i}{i{+}1};\\
&\}\\
&\textbf{return } \mathbb{X}
\end{aligned} \tag{39}
$$

Example 10: Assuming $k = 5$, $\mathbb{X} \in \mathbb{E}(5)$ with $\mathbb{X} = (0, 1, 1, 2, 4)$, the ranking done by (38) is equivalent to:

$$((((0{\cdot}2{+}1){\cdot}3{+}1){\cdot}4{+}2){\cdot}5{+}4) = 94$$

The unranking for $k = 5$ and $rank = 119$ gives as result $\mathbb{X} = (0, 1, 2, 3, 4)$.

It can be verified that in general:

$$\sum_{i=0}^{k-1} i{\cdot}i! = k!\text{-}1$$

△

Even we have proved that Eulerian permutations (according the number of ascents) and factoradic representation are equivalent, it was not shown how to convert an Eulerian representation to a factoradic representation and vice versa. This gap will be filled next, let $\mathbb{F}$ represent a factoradic number whose size is k and it has v different digits; and let $\mathbb{R}$ be a permutation of size k that will have $v - 1$ ascents (i.e. the permutation is a member of the permutations counted by $E(k, v)$). First the mapping from $\mathbb{F}$ to $\mathbb{R}$ will be presented and later the mapping in the other direction.

In (40) the mapping from a factoradic representation $\mathbb{F}$ to an Eulerian permutation $\mathbb{R}$ is presented. The factoradic representation is an script that is traversed from position 0 to position k-1, to do right rotations using the permutation $\mathbb{R}$ that initially has zero ascents. In order to do the mapping two sets $\mathbb{C}$ and $\mathbb{Y}$ are used, the set $\mathbb{C}$ contains the candidate elements that are taken the first time a symbol appears in $\mathbb{F}$ and determines the number of ascents that must be traversed by a right rotation, once an element in $\mathbb{C}$ is processed it is inserted in the set $\mathbb{Y}$, the set $\mathbb{Y}$ is used to determine the number of non-ascents that must be traversed by a right rotation when it is processed an element in $\mathbb{F}$ that have been processed previously.

$$
\begin{aligned}
&\mathbb{F} = \textbf{Factoradic size } k \textbf{ with } v \textbf{ different digits};\\
&\mathbb{R} = \{k, \ldots, 1\};\ \mathbb{C} = \{1, .., k\};\ \mathbb{Y} = \emptyset;\\
&\forall i \in [1, \ldots, k]\{\\
&\quad \textbf{if } (\mathbb{F}_i \neq 0)\{\\
&\quad\quad \textbf{if } (\mathbb{F}_i \in \mathbb{C})\{\\
&\quad\quad\quad \forall j \in [1, \ldots, |\mathbb{C}|]\ \textbf{if } (\mathbb{F}_i = \mathbb{C}_j)\ \textbf{break};\\
&\quad\quad\quad l = 0;\ \forall c \in [i\text{-}1, \ldots, 1]\{\\
&\quad\quad\quad\quad \textbf{if } (c = 1) \vee (\mathbb{R}_{c-1} {>} \mathbb{R}_c)\ l = l{+}1;\\
&\quad\quad\quad\quad \textbf{if } (l = j)\ \textbf{break};\\
&\quad\quad\quad \}\\
&\quad\quad\quad \textbf{rotate right } \mathbb{R},\ \textbf{positions } c \textbf{ and } i;\\
&\quad\quad\quad \mathbb{C} = \mathbb{C} \setminus \mathbb{F}_i;\ \mathbb{Y} = \mathbb{Y} \cup \mathbb{F}_i;\ \textbf{sort} \mathbb{Y};\\
&\quad\quad \}\\
&\quad\quad \textbf{else}\{\\
&\quad\quad\quad \forall j \in [1, \ldots, |\mathbb{Y}|]\ \textbf{if } (\mathbb{F}_i = \mathbb{Y}_j)\ \textbf{break};\\
&\quad\quad\quad l = 0; \forall c \in [i\text{-}1, \ldots, 1]\{\\
&\quad\quad\quad\quad \textbf{if } (\mathbb{R}_{c-1} {<} \mathbb{R}_c)\ l = l{+}1;\\
&\quad\quad\quad\quad \textbf{if } (l = j)\ \textbf{break};\\
&\quad\quad\quad \}\\
&\quad\quad\quad \textbf{rotate right } \mathbb{R},\ \textbf{positions } c \textbf{ and } i;\\
&\quad\quad \}\\
&\quad \}\\
&\}
\end{aligned} \tag{40}
$$

In (41) the mapping from a permutation $\mathbb{R}$ to a factoradic representation $\mathbb{F}$ is presented. The construction of $\mathbb{F}$ undoes the step made in (40). In order to not destruct the permutation $\mathbb{R}$ a copy of it is used as $\mathbb{R}'$. The mapping from $\mathbb{R}'$ to $\mathbb{F}$ also uses the sets $\mathbb{C}$ and $\mathbb{Y}$. Essentially the process implies the coding of left rotations that must be made over the permutation $\mathbb{R}'$ in order that it has 0 ascents. The first part of the process (the one that uses the first $\forall i \in [1, \ldots, k]\{\ldots\}$) determines the positions to do left rotations, in $\mathbb{F}$ a zero value indicates that it not needed to do a left rotation; a positive number is recorded to indicate that set $\mathbb{C}$ must be used (i.e. this implies to traverse a certain number of ascents when the left rotation is done); and a negative number indicates that the set $\mathbb{Y}$ must be used (i.e. this implies to traverse certain number of non-ascents when the left rotation is done). The second part (the one that uses the second $\forall i \in [1, \ldots, k]\{\ldots\}$)transforms the previous coding made in $\mathbb{F}$ (that contains positive and negative numbers) to the valid factoradic representation using the sets (C) and $\mathbb{Y}$.

$$
\begin{array}{l}
\mathbb{R} = \textbf{Permutation size } k \textbf{ with } v\text{-}1 \textbf{ ascents};\\
\mathbb{R}' = \mathbb{R};\\
\forall i \in [1,\ldots,k]\{\\
\quad \textbf{if } (i = \mathbb{P}'_{k-i+1})\ \mathbb{F}_{k\text{-}i+1} = 0;\\
\quad \textbf{else}\{\\
\quad\quad \forall c \in [1,\ldots,k]\ \textbf{if } (i = \mathbb{R}'_c)\ \textbf{break};\\
\quad\quad \textbf{if } (c = 1) \vee (\mathbb{R}'_{c-1} > \mathbb{R}'_{c+1})\{\\
\quad\quad\quad l = 1;\\
\quad\quad\quad \forall j \in [c{+}1,\ldots,k\text{-}i]\ \textbf{if } (\mathbb{R}'_j > \mathbb{R}'_{j+1})\ l = l{+}1;\\
\quad\quad\quad \textbf{rotate left } \mathbb{R}' \textbf{ positions } c \textbf{ and } k\text{-}i{+}1;\\
\quad\quad\quad \mathbb{F}_{k\text{-}i+1} = l;\\
\quad\quad \}\\
\quad\quad \textbf{else}\{\\
\quad\quad\quad l = 1;\\
\quad\quad\quad \forall j \in [c{+}1,\ldots,k\text{-}i]\ \textbf{if } (\mathbb{R}'_j < \mathbb{R}'_{j+1})\ l = l{+}1;\\
\quad\quad\quad \textbf{rotate left } \mathbb{R}' \textbf{ positions } c \textbf{ and } k\text{-}i{+}1;\\
\quad\quad\quad \mathbb{F}_{k\text{-}i+1} = -l;\\
\quad\quad \}\\
\quad \}\\
\}\\
\mathbb{C} = \{1,\ldots,k\};\ \mathbb{Y} = \emptyset;\\
\forall i \in [1,\ldots,k]\{\\
\quad \textbf{if } (\mathbb{F}_i > 0)\{\\
\quad\quad \mathbb{F}_i = \mathbb{C}_{\mathbb{F}_i};\ \mathbb{C} = \mathbb{C} \setminus \mathbb{F}_i;\ \mathbb{Y} = \mathbb{Y} \cup \mathbb{F}_i;\ \textbf{sort } \mathbb{Y};\\
\quad \}\\
\quad \textbf{else if } (\mathbb{F}_i < 0)\ \mathbb{F}_i = \mathbb{Y}_{|\mathbb{F}_i|};\\
\}
\end{array} \tag{41}
$$

Example 11: Next some examples to transform from factoradic with v different symbols to permutations with v-1 ascents is given:

$$(0, 0, 0, 3, 3) \to (2, 1, 5, 4, 3)$$
$$(0, 0, 2, 0, 0, 5, 6, 0, 8, 6) \to (2, 4, 1, 5, 8, 10, 9, 7, 6, 3)$$

Examples to transform permutations to factoradic notation are given next:

$$(4, 2, 5, 3, 1) \to (0, 1, 0, 1, 0)$$
$$(8, 2, 3, 10, 9, 7, 1, 4, 6, 5) \to (0, 0, 2, 0, 0, 0, 3, 3, 7, 4)$$

△

C. REPRESENTATION USING A NUMBER SYSTEM WITH FIXED BASE

A number system base k to represent the possible solutions of SPP with k objects is referred as $\mathbb{N}(k)$. Each element $\mathbb{X} \in \mathbb{N}(k)$ has k digits with values in $\{0,\ldots,k\text{-}1\}$, and $\mathbb{X}_i$ indicates the number of partition to which belongs the i-th object (0-based).

Example 12: Assuming $k = 5$, $\mathbb{X} \in \mathbb{N}(5)$ the element $(4, 0, 0, 3, 2)$, represents the set partition of the objects $\{0, 1, 2, 3, 4\}$: $\{\{1, 2\}\{4\}\{3\}\{0\}\}$ △

The cardinality of $\mathbb{N}(k)$ is $|\mathbb{N}(k)| = k^k$, and the cardinality of the elements of $\mathbb{N}(k)$ that have v different symbols is represented by $\mathbb{N}(k, v)$. The construction of $\mathbb{N}(k, v)$ in terms of $\mathbb{N}(k\text{-}1, v\text{-}1)$ and $\mathbb{N}(k\text{-}1, v)$ is given in (42), take into account that $\mathbb{N}(i, 1) = \{(0, 0, \ldots, 0)\}$ (i.e. i zeros), $\mathbb{N}(i, 0) = \{\emptyset\}$ and $k \geq v$.

The construction of $\mathbb{N}(k, v)$ is shown in (42).

The first part of the construction uses $\mathbb{N}(k\text{-}1, v\text{-}1)$ and works in two steps: firstly, for each member in $\mathbb{N}(k\text{-}1, v\text{-}1)$ add k-(v-1) numbers resulting from concatenating at the end each of the k-(v-1) symbols not in the member; secondly, for each member add the result of replacing each distinct symbol (in total v-1) in it with the symbol k-1 and concatenating to the end the replaced symbol. This way, the net result is the addition of $k\text{-}(v\text{-}1)+(v\text{-}1) = k$ members for each element in $\mathbb{N}(k\text{-}1, v\text{-}1)$.

The second part is more elaborated, and operates in two main steps: firstly, for each member of $\mathbb{N}(k\text{-}1, v)$, concatenate each of the v symbols present in $\mathbb{N}(k\text{-}1, v)$, this result is stored as $\mathbb{T}_1$ and added to $\mathbb{N}(k, v)$; secondly, each of the members of $\mathbb{T}_1$ (in total $v \cdot |\mathbb{N}(k\text{-}1, v)|$) is subject to the next process: for each element of $\mathbb{T}_1$ replace one of the v symbols with the symbol k-1 generating v new elements (i.e. we generate $v^2 \cdot \mathbb{N}(k\text{-}1, v)$ elements), but many of the generated elements are identical, in order to see how many elements are identical consider that each solution has $(k\text{-}1)\text{-}(v\text{-}1) = k\text{-}v$ solutions that are almost identical (only are different in one symbol), then the net number of non repeated elements is exactly: $v^2 \cdot |\mathbb{N}(k\text{-}1, v)| \cdot \frac{1}{k\text{-}v}$, henceforth, we have added in the second part: $v \cdot |\mathbb{N}(k\text{-}1, v)| + \frac{v^2}{k\text{-}v} \cdot |\mathbb{N}(k\text{-}1, v)| = |\mathbb{N}(k\text{-}1, v)| \cdot (v + \frac{v^2}{k\text{-}v})$ simplifying we have that this second part contributes with: $|\mathbb{N}(k\text{-}1, v)| \cdot \frac{k \cdot v}{k\text{-}v}$ elements.

Example 13: In the second part of the previous analysis assuming $k = 5$, $v = 2$ and that a member of $\mathbb{T}_1$ is $(0, 0, 1, 1, 0, 0)$ then there are in $\mathbb{T}_1$ the elements $(2, 2, 1, 1, 2, 2)$ and $(3, 3, 1, 1, 3, 3)$ that are different only respect one symbol. This three elements generate an identical element: $(4, 4, 1, 1, 4, 4)$ i.e the number of identical elements are $k\text{-}v = 5\text{-}2 = 3$. △

Finally we can conclude that:

$$k \cdot |\mathbb{N}(k\text{-}1, v\text{-}1)| + \frac{kv}{k\text{-}v} \cdot |\mathbb{N}(k\text{-}1, v)|.$$

$$
\begin{array}{l}
\mathbb{N}(k, v) = \emptyset;\ \mathbb{T}_1 = \emptyset;\\
\forall \mathbb{X} \in \mathbb{N}(k\text{-}1, v\text{-}1)\{\\
\quad \forall a \in \{0,\ldots,k\text{-}1\} \backslash \eta(\mathbb{X})\ \mathbb{N}(k, v) = \mathbb{N}(k, v) \cup \mathbb{X} \oplus a;\\
\quad \forall a \in \eta(\mathbb{X})\{\\
\quad\quad \mathbb{Y} = \mathbb{X};\\
\quad\quad \forall i \in [0,\ldots,k\text{-}2]\ \textbf{if } (\mathbb{Y}_i = a)\ \mathbb{Y}_i = k\text{-}1;\\
\quad\quad \mathbb{N}(k, v) = \mathbb{N}(k, v) \cup (\mathbb{Y} \oplus a);
\end{array}
$$

$$
\begin{aligned}
&\quad\}\\
&\}\\
&\forall \mathbb{X} \in \mathbb{N}(k\text{-}1, v)\{\ \forall a \in \eta(\mathbb{X})\ \mathbb{T}_1 = \mathbb{T}_1 \cup (\mathbb{X} \oplus a);\ \}\\
&\mathbb{N}(k, v) = \mathbb{N}(k, v) \cup \mathbb{T}_1;\\
&\forall \mathbb{X} \in \mathbb{T}_1\{\forall a \in \eta(\mathbb{X})\{\\
&\quad \mathbb{Y} = \mathbb{X};\\
&\quad \forall i \in [0, \ldots, k\text{-}1]\ \textbf{if}\ (\mathbb{Y}_i = a)\ \mathbb{Y}_i = k\text{-}1;\\
&\quad \textbf{if}(\mathbb{Y} \notin \mathbb{N}(k, v))\ \mathbb{N}(k, v) = \mathbb{N}(k, v) \cup \mathbb{Y};\\
&\}\}
\end{aligned} \tag{42}
$$

The value of $|\mathbb{N}(k, v)|$ is shown in (43). In Theorem 2 it will be proved that $|\mathbb{N}(k, v)| = N(k, v)$

$$
|\mathbb{N}(k, v)| = \begin{cases} 0 & \textbf{if } (v > k) \wedge (v < 1) \\ 1 & \textbf{if } (k = 1) \\ k\cdot|\mathbb{N}(k\text{-}1, v\text{-}1)| + & \\ \quad \dfrac{kv}{k\text{-}v}\cdot|\mathbb{N}(k\text{-}1, v)| & \textbf{otherwise} \end{cases} \tag{43}
$$

Theorem 2: The cardinality of $\mathbb{N}(k, v)$ is equal to $N(k, v)$ i.e. $|\mathbb{N}(k, v)| = N(k, v)$.

Proof: Given that (43) is identical to (24) it is evident that $|\mathbb{N}(k, v)| = N(k, v)$. Another way, to prove this, is that $|\mathbb{N}(k, v)|$ counts the cardinality of numbers in base k with k digits and v distinct symbols, this is exactly the number of functions from $\{0, \ldots, k\text{-}1\}$ to $\{0, \ldots, k\text{-}1\}$ s.t. the image contains exactly v elements. This finishes the proof.

Example 14: Compute $\mathbb{N}(4, 2)$ using $\mathbb{N}(3, 1)$ and $\mathbb{N}(3, 2)$

$\mathbb{N}(3, 1)$ ={(0,0,0) (1,1,1) (2,2,2)}

$\mathbb{N}(3, 2)$ ={(0,0,1) (0,0,2) (0,1,0) (0,1,1) (0,2,0) (0,2,2) (1,0,0) (1,0,1) (1,1,0) (1,1,2) (1,2,1) (1,2,2) (2,0,0) (2,0,2) (2,1,1) (2,1,2) (2,2,0) (2,2,1)}

Using $\mathbb{N}(k\text{-}1, v\text{-}1)$ and processing $\forall a \in \{0, \ldots, k\text{-}1\} \setminus \eta(\mathbb{X})$ gives:

{((0,0,0)⊕0) ((0,0,0)⊕2) ((0,0,0)⊕3) ((1,1,1)⊕0) ((1,1,1)⊕2) ((1,1,1)⊕3) ((2,2,2)⊕0) ((2,2,2)⊕1) ((2,2,2)⊕3)

Using $\mathbb{N}(k\text{-}1, v\text{-}1)$ and processing $\forall a \in \eta(\mathbb{X})$ gives:

{((3,3,3)⊕0) ((3,3,3)⊕1) ((3,3,3)⊕2)}

In total we have 12 elements generated using $\mathbb{N}(k\text{-}1, v\text{-}1)$.

Now, using $\mathbb{N}(k\text{-}1, v)$ the elements stored in $\mathbb{T}_1$ are:

{((0,0,1)⊕0) ((0,0,1)⊕1) ((0,0,2)⊕0) ((0,0,2)⊕2) ((0,1,0)⊕0) ((0,1,0)⊕1) ((0,1,1)⊕0) ((0,1,1)⊕1) ((0,2,0)⊕0) ((0,2,0)⊕2) ((0,2,2)⊕0) ((0,2,2)⊕2) ((1,0,0)⊕0) ((1,0,0)⊕1) ((1,0,1)⊕0) ((1,0,1)⊕1) ((1,1,0)⊕0) ((1,1,0)⊕1) ((1,1,2)⊕1) ((1,1,2)⊕2) ((1,2,1)⊕1) ((1,2,1)⊕2) ((1,2,2)⊕1) ((1,2,2)⊕2) ((2,0,0)⊕0) ((2,0,0)⊕2) ((2,0,2)⊕0) ((2,0,2)⊕2) ((2,1,1)⊕1) ((2,1,1)⊕2) ((2,1,2)⊕1) ((2,1,2)⊕2) ((2,2,0)⊕0) ((2,2,0)⊕2) ((2,2,1)⊕1) ((2,2,1)⊕2)}

Now, using $\mathbb{N}(k\text{-}1, v)$, the part $\forall \mathbb{X} \in \mathbb{T}_1\{\forall a \in \eta(\mathbb{X})$ produces:

{(0,0,3,0) (0,0,3,3) (0,3,0,0) (0,3,0,3) (0,3,3,0) (0,3,3,3) (1,1,3,1) (1,1,3,3) (1,3,1,1) (1,3,1,3) (1,3,3,1) (1,3,3,3) (2,2,3,2) (2,2,3,3) (2,3,2,2) (2,3,2,3) (2,3,3,2) (2,3,3,3) (3,0,0,0) (3,0,0,3) (3,0,3,0) (3,0,3,3) (3,1,1,1) (3,1,1,3) (3,1,3,1) (3,1,3,3) (3,2,2,2) (3,2,2,3) (3,2,3,2) (3,2,3,3) (3,3,0,0) (3,3,0,3) (3,3,1,1) (3,3,1,3) (3,3,2,2) (3,3,2,3)}

In total for $\mathbb{N}(k\text{-}1, v)$ we have a total of 72 elements.

Then the final result for $\mathbb{N}(4, 2)$ has 84 elements, that are:

{(0,0,0,1) (0,0,0,2) (0,0,0,3) (0,0,1,0) (0,0,1,1) (0,0,2,0) (0,0,2,2) (0,0,3,0) (0,0,3,3) (0,1,0,0) (0,1,0,1) (0,1,1,0) (0,1,1,1) (0,2,0,0) (0,2,0,2) (0,2,2,0) (0,2,2,2) (0,3,0,0) (0,3,0,3) (0,3,3,0) (0,3,3,3) (1,0,0,0) (1,0,0,1) (1,0,1,0) (1,0,1,1) (1,1,0,0) (1,1,0,1) (1,1,1,0) (1,1,1,2) (1,1,1,3) (1,1,2,1) (1,1,2,2) (1,1,3,1) (1,1,3,3) (1,2,1,1) (1,2,1,2) (1,2,2,1) (1,2,2,2) (1,3,1,1) (1,3,1,3) (1,3,3,1) (1,3,3,3) (2,0,0,0) (2,0,0,2) (2,0,2,0) (2,0,2,2) (2,1,1,1) (2,1,1,2) (2,1,2,1) (2,1,2,2) (2,2,0,0) (2,2,0,2) (2,2,1,1) (2,2,1,2) (2,2,2,0) (2,2,2,1) (2,2,2,3) (2,2,3,2) (2,2,3,3) (2,3,2,2) (2,3,2,3) (2,3,3,2) (2,3,3,3) (3,0,0,0) (3,0,0,3) (3,0,3,0) (3,0,3,3) (3,1,1,1) (3,1,1,3) (3,1,3,1) (3,1,3,3) (3,2,2,2) (3,2,2,3) (3,2,3,2) (3,2,3,3) (3,3,0,0) (3,3,0,3) (3,3,1,1) (3,3,1,3) (3,3,2,2) (3,3,2,3) (3,3,3,0) (3,3,3,1) (3,3,3,2)}

△

It has been shown that $|\mathbb{N}(k)| = k^k$, $|\mathbb{N}(k, v)| = N(k, v)$, an expression that bounds the cardinality of factoradic number system respect to the integer partitions $\mathbb{P}(k, v)$, i.e., $|\mathbb{N}(k, v, \mathbb{X})|$, $\mathbb{X} \in \mathbb{P}(k, v)$ is shown in (44).

$$
|\mathbb{N}(k, v, \mathbb{X})| = v!\cdot\binom{k}{v}\cdot\frac{k!}{\prod_{i=1}^{v}\mathbb{X}_i!\ \prod_{l\in\eta(\mathbb{X})}\mu(l, \mathbb{X})!},\quad \mathbb{X} \in \mathbb{P}(k, v) \tag{44}
$$

Table 11 presents, the values of $|\mathbb{N}(k, v, \mathbb{X})|$ as super indices of the integer partition. In each cell after the equal sign the value of $|\mathbb{N}(k, v)| = N(k, v)$ is shown.

Interesting values for $|\mathbb{N}(k, v)|$ are $|\mathbb{N}(k, 1)| = k$, $|\mathbb{N}(k, k)| = k!$, and $\sum_{v=1}^{k} |\mathbb{N}(k, v)| = k^k$

The ranking of $\mathbb{X} \in \mathbb{N}(k)$ is very easy, transform to decimal a number with k digits in base k, see (45) the result is returned as *rank*.

$$
\begin{aligned}
&\mathbb{X} \in \mathbb{N}(k);\\
&rank = \mathbb{X}_0;\\
&\forall i \in [1, \ldots, k\text{-}1]\ rank = rank\cdot k + \mathbb{X}_i;\\
&\textbf{return}\ rank
\end{aligned} \tag{45}
$$

The unranking of the number *rank* is done in (46) and returned as $\mathbb{X}$ with k digits.

$$
\begin{aligned}
&0 \le rank \le k^k\text{-}1;\\
&\forall\ i \in [k\text{-}1, \ldots, 0]\{\\
&\quad \mathbb{X}_i = rank \mod k;\\
&\quad rank = \frac{rank - \mathbb{X}_i}{k};\\
&\}\\
&\textbf{return}\ \mathbb{X}
\end{aligned} \tag{46}
$$

D. COMMENTS ABOUT THE THREE REPRESENTATIONS

Respect the redundancy of the three representations, the $\mathbb{S}(k)$ representation has zero redundancy given that

TABLE 11. Table for $|\mathbb{N}(k, v, \mathbb{X})|$ for $\mathbb{X} \in \mathbb{P}(k, v)$, super indices have the cardinality and the value of $|\mathbb{N}(k, v)| = N(k, v)$ appears after equal sign.

k \ v	1	2	3	4
1	$(1)^1{=}1$			
2	$(1)^1{=}2$	$(1,1)^2{=}2$		
3	$(1)^1{=}3$	$(2,1)^{18}{=}18$	$(1,1,1)^6{=}6$	
4	$(1)^1{=}4$	$(3,1)^{48}+(2,2)^{36}{=}84$	$(2,1,1)^{144}{=}144$	$(1,1,1,1)^{24}{=}24$

$|\mathbb{S}(k)| = B(k)$, the $\mathbb{E}(k)$ representation has some redundancy given that $|\mathbb{E}(k)| = k!$, and $\mathbb{N}(k)$ has the greater redundancy, given that $|\mathbb{N}(k)| = k^k$. See Example 15 to get insight in the redundancy topic of the three representations.

Example 15:

$$\begin{aligned} &\textit{For } k \geq 3 \textit{ happens } |\mathbb{S}(k)|<|\mathbb{E}(k)|<|\mathbb{N}(k)| \\ &\textit{For } k = 2 \textit{ occurs } \quad |\mathbb{S}(k)| = |\mathbb{E}(k)|<|\mathbb{N}(k)| \\ &\textit{For } k = 1 \textit{ satisfies } |\mathbb{S}(k)| = |\mathbb{E}(k)| = |\mathbb{N}(k)| \end{aligned}$$

△

A comparison of $|\mathbb{S}(k, v, \mathbb{X})|$, $|\mathbb{E}(k, v, \mathbb{X})|$ and $|\mathbb{N}(k, v, \mathbb{X})|$ (see (44), (37) and (31)) leads to the conclusion that the greater redundancy is in $\mathbb{N}(k)$,followed with less redundancy $\mathbb{E}(k)$, and zero redundancy for $\mathbb{S}(k)$.

To see clearly this fact, assume that $\mathbb{X} \in \mathbb{S}(k)$ then the mapping of this element $\mathbb{X}$ to corresponding elements in $\mathbb{E}(k)$ and $\mathbb{N}(k)$ representations will be presented.

In (47) the process to transform $\mathbb{X} \in \mathbb{S}(k)$ to $\mathbb{N}(k)$ can be seen, take note that $\left\{\binom{k}{v}\right\}$ refers to the set of all v-wise combinations taken from k, and $\left\{v!\right\}$ refers to the set of permutations of size v. Therefore, one element in $\mathbb{S}(k)$ maps to $v!\cdot\binom{k}{v}$ members in $\mathbb{S}(k)$.

$$\begin{aligned} &\mathbb{X} \in \mathbb{S}(k); \\ &v = |\eta(\mathbb{X})|; \\ &\mathbb{R} = \emptyset \\ &\forall \mathbb{V} \in \left\{\binom{k}{v}\right\}\{\forall \mathbb{P} \in \left\{v!\right\}\{ \\ &\quad \forall i \in [0, \ldots, v\text{-}1]\ \mathbb{Z}_i = \mathbb{V}_{\mathbb{P}_i}; \\ &\quad \forall i \in [0, \ldots, k\text{-}1]\ \mathbb{Y}_i = \mathbb{Z}_{\mathbb{X}_i}; \\ &\quad \mathbb{R} = \mathbb{R} \cup \mathbb{Y}; \\ &\}\} \\ &\textbf{return } \mathbb{R}; \end{aligned} \tag{47}$$

Example 16: Let $\mathbb{X} \in \mathbb{S}(5, 3)$ be (0, 1, 1, 0, 2) the corresponding elements that belong to $\mathbb{N}(5, 3)$ are:

{(0,1,1,0,2) (0,1,1,0,3) (0,1,1,0,4) (0,2,2,0,1) (0,2,2,0,3) (0,2,2,0,4) (0,3,3,0,1) (0,3,3,0,2) (0,3,3,0,4) (0,4,4,0,1) (0,4,4,0,2) (0,4,4,0,3) (1,0,0,1,2) (1,0,0,1,3) (1,0,0,1,4) (1,2,2,1,0) (1,2,2,1,3) (1,2,2,1,4) (1,3,3,1,0) (1,3,3,1,2) (1,3,3,1,4) (1,4,4,1,0) (1,4,4,1,2) (1,4,4,1,3) (2,0,0,2,1) (2,0,0,2,3) (2,0,0,2,4) (2,1,1,2,0) (2,1,1,2,3) (2,1,1,2,4) (2,3,3,2,0) (2,3,3,2,1) (2,3,3,2,4) (2,4,4,2,0) (2,4,4,2,1) (2,4,4,2,3) (3,0,0,3,1) (3,0,0,3,2) (3,0,0,3,4) (3,1,1,3,0) (3,1,1,3,2) (3,1,1,3,4) (3,2,2,3,0) (3,2,2,3,1) (3,2,2,3,4) (3,4,4,3,0) (3,4,4,3,1) (3,4,4,3,2) (4,0,0,4,1) (4,0,0,4,2) (4,0,0,4,3) (4,1,1,4,0) (4,1,1,4,2) (4,1,1,4,3) (4,2,2,4,0) (4,2,2,4,1) (4,2,2,4,3) (4,3,3,4,0) (4,3,3,4,1) (4,3,3,4,2) } △

$$\begin{aligned} &\mathbb{X} \in \mathbb{S}(k); \\ &\mathbb{R} = \emptyset;\ \mathbb{F} = \emptyset;\ \mathbb{V} = \emptyset;\ \mathbb{I} = \emptyset;\ \mathbb{T} = \emptyset;\ \mathbb{Y} = \emptyset;\ \mathbb{Z} = \emptyset; \\ &\forall a \in \eta(\mathbb{X})\ \textbf{if}(\mathbb{X}_a = a)\ \mathbb{F} = \mathbb{F} \cup a; \\ &\forall i \in [0, \ldots, k\text{-}1]\{ \\ &\quad \textbf{if}\ (\mathbb{X}_i \notin \mathbb{F}) \wedge (\mathbb{X}_i \notin \mathbb{V})\{\ \mathbb{V} = \mathbb{V} \cup \mathbb{X}_i;\ \ \mathbb{I} = \mathbb{I} \cup i;\ \} \\ &\} \\ &\textbf{if}\ (\mathbb{V} = \emptyset)\ \mathbb{R} = \mathbb{X}; \\ &\textbf{else}\{ \\ &\quad \forall\ i \in \mathbb{I}\ \mathbb{T} = \mathbb{T} \times \{\mathbb{V}_0, \ldots, i\}; \\ &\quad \forall\ \mathbb{Y} \in \mathbb{T}\ \textbf{if}\ (|\mathbb{Y}| > |\eta(\mathbb{Y})|)\ \mathbb{T} = \mathbb{T} \setminus \mathbb{Y}; \\ &\quad \mathbb{Z} = \mathbb{X}; \\ &\quad \forall \mathbb{Y} \in \mathbb{T}\ \{ \\ &\quad\quad \forall i \in [0, \ldots, k\text{-}1]\{\forall j \in [0, \ldots, |\mathbb{V}|\text{-}1] \\ &\quad\quad\quad \{\ \textbf{if}\ (\mathbb{X}_i = \mathbb{V}_j)\ \mathbb{Z}_i = \mathbb{Y}_j;\ \}\} \\ &\quad\quad \mathbb{R} = \mathbb{R} \cup \mathbb{Z}; \\ &\quad \} \\ &\} \\ &\textbf{return } \mathbb{R}; \end{aligned} \tag{48}$$

In (48) the process to transform $\mathbb{X} \in \mathbb{S}(k)$ to $\mathbb{E}(k)$ is shown. The process to do this is more complex than the process to transform $\mathbb{X} \in \mathbb{S}(k)$ to $\mathbb{N}(k)$. At the beginning, the search of the first occurrence of symbol $a \in \mathbb{X}, 0 \leq a \leq k\text{-}1$ is done, if the index of the first occurrence of symbol a is a, henceforth such symbol is fixed (this symbol will belong to the set $\mathbb{F}$) and will not be subject to a process of being replaced, the other symbols will be subject to a process of being replaced (they will belong to the set $\mathbb{V}$ and its indices will be stored in the set $\mathbb{I}$), in case all the symbols of $\mathbb{X}$ are fixed, $\mathbb{R}$ will contain only $\mathbb{X}$. Take note that $\forall i \in \mathbb{I}\ \mathbb{T} = \mathbb{T} \times \{\mathbb{V}_0, \ldots, i\}$; computes the cartesian product of the sets formed from the first element that is not fixed through each index of the non-fixed elements, the elements that contain duplicated elements are removed from $\mathbb{T}$ with $\forall \mathbb{Y} \in \mathbb{T}$ **if**$(|\mathbb{Y}| < |\eta(\mathbb{Y})|\ \mathbb{T} = \mathbb{T} \cup \mathbb{Y}$;. A final comment about (48) is that it is needed to replace in $\mathbb{X}$, the elements that belongs to $\mathbb{V}$ using the values stored in $\mathbb{T}$.

Example 17: Let $\mathbb{X} \in \mathbb{S}(7, 5)$ be (0, 0, 1, 2, 2, 3, 4) the corresponding elements that belong to $\mathbb{E}(7, 5)$ are:

{(0,0,1,2,2,3,4)(0,0,1,2,2,3,5)(0,0,1,2,2,3,6) (0,0,1,2,2,4,3) (0,0,1,2,2,4,5) (0,0,1,2,2,4,6) (0,0,1,2,2,5,3) (0,0,1,2,2,5,4) (0,0,1,2,2,5,6) (0,0,1,3,3,2,4) (0,0,1,3,3,2,5) (0,0,1,3,3,2,6) (0,0,1,3,3,4,2) (0,0,1,3,3,4,5) (0,0,1,3,3,4,6) (0,0,1,3,3,5,2) (0,0,1,3,3,5,4) (0,0,1,3,3,5,6) (0,0,2,1,1,3,4) (0,0,2,1,1,3,5) (0,0,2,1,1,3,6) (0,0,2,1,1,4,3) (0,0,2,1,1,4,5) (0,0,2,1,1,4,6) (0,0,2,1,1,5,3) (0,0,2,1,1,5,4) (0,0,2,1,1,5,6) (0,0,2,3,3,1,4)

(0,0,2,3,3,1,5) (0,0,2,3,3,1,6) (0,0,2,3,3,4,1) (0,0,2,3,3,4,5) (0,0,2,3,3,4,6) (0,0,2,3,3,5,1) (0,0,2,3,3,5,4) (0,0,2,3,3,5,6)} △

The transformation of one element $\mathbb{X} \in \mathbb{N}(k)$ or $\mathbb{X} \in \mathbb{E}(k)$ to $\mathbb{S}(k)$ is very easy, only requires to do a relabeling of the symbols in $\mathbb{X}$ (see (49)), the first different element that appears in $\mathbb{X}$ is transformed to 0, the second different element that appears in $\mathbb{X}$ is mapped to 1, and so on. It is easy to see that in general, many elements in $\mathbb{N}(k)$ are transformed to the same element in $\mathbb{S}(k)$ (this is also true for an element that belongs to $\mathbb{E}(k)$). See (49).

$$
\begin{aligned}
&\mathbb{X} \textbf{ to be mapped to } \mathbb{S}(k);\\
&\mathbb{R} = \mathbb{X};\ \mathbb{V} = \emptyset;\\
&\forall i \in [0, \ldots, k\text{-}1]\{ \textbf{ if } (\mathbb{X}_i \notin \mathbb{V})\ \mathbb{V} = \mathbb{V} \cup \mathbb{X}_i;\ \}\\
&\forall i \in [0, \ldots, k\text{-}1]\{\\
&\quad \forall j \in [0, \ldots, |\mathbb{V} - 1|]\ \textbf{if } (\mathbb{X}_i = \mathbb{V}_j)\ \mathbb{R}_i = j;\\
&\textbf{return } \mathbb{R};
\end{aligned} \tag{49}
$$

Example 18: Let $\mathbb{X} \in \mathbb{N}(4)$ be (3, 1, 0, 0), $\mathbb{X}$ is mapped to $(0, 1, 2, 2) \in \mathbb{S}(4)$.

Let $\mathbb{X} \in \mathbb{E}(4)$ be (0, 0, 2, 3), $\mathbb{X}$ is mapped to $(0, 0, 1, 2) \in \mathbb{S}(4)$. △

The transformation of elements between $\mathbb{N}(k)$ and $\mathbb{E}(k)$ can be made using as intermediate the transformation from one representation to $\mathbb{S}(k)$, then transform $\mathbb{S}(k)$ to the other representation.

E. CASE WHEN THE NUMBER OF PARTITIONS IS BOUNDED

Until now if has been assumed that the number of partitions of a set is not bounded, i.e. for k objects the number of partitions can be from 1 to k. But, in some cases it is required to impose a bound in the maximum number of partitions, say r.

1) CARDINALITY OF r-BOUNDED $\{\mathbb{S}(k), \mathbb{E}(k), \mathbb{N}(k)\}$

The cardinality of r-bounded $\mathbb{S}(k)$ it is $\sum_{i=1}^{r} S(k, i)$ instead of $B(k)$. The cardinality of r-bounded $\mathbb{E}(k)$ it is $\sum_{i=1}^{r} E(k, i)$ instead of $k!$. The cardinality of r-bounded $N(k)$ is r^k instead of k^k.

2) RANKING AND UNRANKING ALGORITHMS FOR r-BOUNDED $\{\mathbb{S}(k), \mathbb{E}(k), \mathbb{N}(k)\}$

The ranking and unranking algorithms for r-bounded $\mathbb{S}(k)$ are controlled by the values of $\mathbb{O}(k)$, then only is required to change the construction of this table, see (50).

$$
\begin{aligned}
&\forall\, j \in [0, \ldots, r]\{\ \mathbb{O}(k)_{0,j} = 1;\ \}\\
&\forall\, i \in [1, \ldots, k]\{\\
&\quad \forall\, j \in [0, \ldots, \min(r, k\text{-}i)]\{\\
&\quad\quad \mathbb{O}(k)_{i,j} = j \cdot \mathbb{O}(k)_{i\text{-}1,j} + \mathbb{O}(k)_{i\text{-}1,j+1};\\
&\quad \}\\
&\}
\end{aligned} \tag{50}
$$

The result for 3-bounded $\mathbb{O}(5)$ is given Table 12.

TABLE 12. Table of 3-bounded $\mathbb{O}(5)$, positional values for ranking, unranking of RGS/decimal number.

i \ j	0	1	2	3
0	1	1	1	1
1	1	2	3	3
2	2	5	9	9
3	5	14	27	
4	14	41		
5	41			

The ranking and unranking for $\mathbb{X} \in r$-bounded$\mathbb{E}(k)$, must take into account that a set $\mathbb{A} = \bigcup_{i=1}^{r} \mathbb{E}(k, v)$ exists, and the i-th factoradic notation is done using $\mathbb{A}_i$ $0 \leq i \leq |\mathbb{A}| - 1$, then the unranking is very easy given that it is implicit in the construction of $\mathbb{A}$, for the ranking it is suggested to do a binary search using the set $\mathbb{A}$. See (51) and (52) for ranking and unranking procedures respectively.

$$
\begin{aligned}
&\mathbb{A} = \bigcup_{i=1}^{r} \mathbb{E}(k, v);\\
&\mathbb{X} \in r\text{-bounded } \mathbb{E}(k);\\
&d = 0;\ t = |\mathbb{A}|\text{-}1;\\
&\textbf{repeat}\{\\
&\quad rank = \lfloor \frac{d+t}{2} \rfloor;\\
&\quad \textbf{if } (\mathbb{X} = \mathbb{A}_{rank})\ \textbf{return } rank;\\
&\quad \textbf{if } (\mathbb{X} > \mathbb{A}_{rank})\ d = rank\text{+}1;\ \textbf{else } t = rank\text{-}1;\\
&\}
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
&\mathbb{A} = \bigcup_{i=1}^{r} \mathbb{E}(k, v);\\
&0 \leq rank \leq |\mathbb{A}|\text{-}1;\\
&\textbf{return } \mathbb{A}_{rank};
\end{aligned} \tag{52}
$$

The ranking and unranking algorithms for bounded $\mathbb{N}(k)$ is presented in (53), (54).

$$
\begin{aligned}
&\mathbb{X} \in \mathbb{N}(k);\ rank = \mathbb{X}_0\\
&\forall\, i \in [1, \ldots, k\text{-}1]\{\ rank = rank \cdot r + \mathbb{X}_i;\ \}\\
&\textbf{return } rank
\end{aligned} \tag{53}
$$

$$
\begin{aligned}
&0 \leq rank \leq r^k\text{-}1;\\
&\forall\, i \in [k\text{-}1, \ldots, 0]\{\\
&\quad \mathbb{X}_i = rank \mod r;\\
&\quad rank = \frac{(rank\text{-}\mathbb{X}_i)}{r};\\
&\}\\
&\textbf{return } \mathbb{X}
\end{aligned} \tag{54}
$$

3) Mapping BETWEEN r-BOUNDED $\mathbb{S}(k)$, $\mathbb{E}(k)$, $\mathbb{N}(k)$

The mapping for $\mathbb{X} \in r$-bounded $\mathbb{S}(k)$ to r-bounded $\mathbb{E}(k)$ is shown in (55), see that the only difference with (48) is the computation of the cartesian product.

$$
\begin{aligned}
&\mathbb{X} \in \mathbb{S}(k);\\
&\mathbb{R} = \emptyset;\ \mathbb{F} = \emptyset;\ \mathbb{V} = \emptyset;\ \mathbb{I} = \emptyset;\ \mathbb{T} = \emptyset;\ \mathbb{Y} = \emptyset;\ \mathbb{Z} = \emptyset;
\end{aligned}
$$

$\forall a \in \eta(\mathbb{X})$ **if**$(\mathbb{X}_a = a)$ $\mathbb{F} = \mathbb{F} \cup a$;
$\forall i \in [0, \ldots, k\text{-}1]\{$
  if $(\mathbb{X}_i \notin \mathbb{F}) \wedge (\mathbb{X}_i \notin \mathbb{V})\{$ $\mathbb{V} = \mathbb{V} \cup \mathbb{X}_i$; $\mathbb{I} = \mathbb{I} \cup i$; $\}$
$\}$
if $(\mathbb{V} = \emptyset)$ $\mathbb{R} = \mathbb{X}$;
else$\{$
 $\forall\, i \in \mathbb{I}$ $\mathbb{T} = \mathbb{T} \times \{\mathbb{V}_0, \ldots, \min(i, r\text{-}1)\}$;
 $\forall\, \mathbb{Y} \in \mathbb{T}$ **if** $(|\mathbb{Y}| > |\eta(\mathbb{Y})|)$ $\mathbb{T} = \mathbb{T} \setminus \mathbb{Y}$;
 $\mathbb{Z} = \mathbb{X}$;
 $\forall \mathbb{Y} \in \mathbb{T}\ \{$
  $\forall i \in [0, \ldots, k\text{-}1]\{\forall j \in [0, \ldots, |\mathbb{V}|\text{-}1]$
   $\{$ **if** $(\mathbb{X}_i = \mathbb{V}_j)$ $\mathbb{Z}_i = \mathbb{Y}_j$; $\}\}$
  $\mathbb{R} = \mathbb{R} \cup \mathbb{Z}$;
 $\}$
$\}$
return $\mathbb{R}$; (55)

Example 19: Let $\mathbb{X} \in \mathbb{S}(8)$ be $(0, 0, 0, 1, 0, 1, 2, 3)$ with $r = 5$, it is obtained:

$$
\begin{aligned}
\mathbb{E}(8, 5) = \{&(0, 0, 0, 1, 0, 1, 2, 3)(0, 0, 0, 1, 0, 1, 2, 4)\\
&(0, 0, 0, 1, 0, 1, 3, 2)(0, 0, 0, 1, 0, 1, 3, 4)\\
&(0, 0, 0, 1, 0, 1, 4, 2)(0, 0, 0, 1, 0, 1, 4, 3)\\
&(0, 0, 0, 2, 0, 2, 1, 3)(0, 0, 0, 2, 0, 2, 1, 4)\\
&(0, 0, 0, 2, 0, 2, 3, 1)(0, 0, 0, 2, 0, 2, 3, 4)\\
&(0, 0, 0, 2, 0, 2, 4, 1)(0, 0, 0, 2, 0, 2, 4, 3)\\
&(0, 0, 0, 3, 0, 3, 1, 2)(0, 0, 0, 3, 0, 3, 1, 4)\\
&(0, 0, 0, 3, 0, 3, 2, 1)(0, 0, 0, 3, 0, 3, 2, 4)\\
&(0, 0, 0, 3, 0, 3, 4, 1)(0, 0, 0, 3, 0, 3, 4, 2)\}
\end{aligned}
$$

△

The mapping for $\mathbb{X} \in r$-bounded $\mathbb{S}(k)$ to r-bounded $\mathbb{N}(k)$ is shown in (56). See that the only difference with (47) is the computation of $\left\{\binom{r}{v}\right\}$ instead of $\left\{\binom{k}{v}\right\}$.

$\mathbb{X} \in \mathbb{S}(k)$;
$v = |\eta(\mathbb{X})|$;
$\mathbb{R} = \emptyset$
$\forall \mathbb{V} \in \left\{\binom{r}{v}\right\}\{\forall \mathbb{P} \in \{v!\}\{$
 $\forall i \in [0, \ldots, v\text{-}1]$ $\mathbb{Z}_i = \mathbb{V}_{\mathbb{P}_i}$;
 $\forall i \in [0, \ldots, k\text{-}1]$ $\mathbb{Y}_i = \mathbb{Z}_{\mathbb{X}_i}$;
 $\mathbb{R} = \mathbb{R} \cup \mathbb{Y}$;
$\}\}$
return $\mathbb{R}$; (56)

Example 20: Let $\mathbb{X} \in \mathbb{S}(6)$ be $(0, 1, 0, 2, 2, 0)$ with $r = 4$, we have:

$$
\begin{aligned}
\mathbb{N}(6, 4) = \{&(0, 1, 0, 2, 2, 0)(0, 1, 0, 3, 3, 0)(0, 2, 0, 1, 1, 0)\\
&(0, 2, 0, 3, 3, 0)(0, 3, 0, 1, 1, 0)(0, 3, 0, 2, 2, 0)\\
&(1, 0, 1, 2, 2, 1)(1, 0, 1, 3, 3, 1)(1, 2, 1, 0, 0, 1)\\
&(1, 2, 1, 3, 3, 1)(1, 3, 1, 0, 0, 1)(1, 3, 1, 2, 2, 1)\\
&(2, 0, 2, 1, 1, 2)(2, 0, 2, 3, 3, 2)(2, 1, 2, 0, 0, 2)\\
&(2, 1, 2, 3, 3, 2)(2, 3, 2, 0, 0, 2)(2, 3, 2, 1, 1, 2)\\
&(3, 0, 3, 1, 1, 3)(3, 0, 3, 2, 2, 3)(3, 1, 3, 0, 0, 3)\\
&(3, 1, 3, 2, 2, 3)(3, 2, 3, 0, 0, 3)(3, 2, 3, 1, 1, 3)\\
\}&
\end{aligned}
$$

△

The procedure to map from r-bounded $\mathbb{E}(k)$ or r-bounded $\mathbb{N}(k)$ to r-bounded $\mathbb{S}(k)$ is the same as the unbounded case, and the mapping from r-bounded $\mathbb{E}(k)$ to r-bounded $\mathbb{N}(k)$ and vice versa, can be done using as bridge r-bounded $\mathbb{S}(k)$.

IV. OPERATORS FOR THE THREE REPRESENTATIONS

This section presents how to construct a random element for each representation, apply a mutation operator (a slight modification of a possible solution) to each of the representations, and recombine two elements of each representation. These three operators (random generation, mutation, and recombination) are key to implement optimization algorithms (greedy, exact, and metaheuristic) to solve specific SPP instances. In particular the mutation operators enables to do small changes to one specific potential solution (in this sense it enables to do exploration) and the recombination operator takes advantage of two potential solutions to produce two new solutions that inherits features from the parent solutions (this operation does in general more exploitation).

The random generation can be done in at least two ways: generate a random number in the valid rank and then use the corresponding unrank procedure, and the second way is to generate a valid element each position-value at-a-time.

The mutation of one member $\mathbb{X}$ that belongs to one of the three representations, is easy to implement, given that the modification of only one element of $\mathbb{X}$ according to its possible values is done, only for $\mathbb{S}(k)$ is needed to run over the result the relabel operation see (49). The recombination operator using two members $\mathbb{X}$ and $\mathbb{Y}$ belonging to one of the three representations, is direct for $\mathbb{E}(k)$ and $\mathbb{N}(k)$, but for $\mathbb{S}(k)$ a repair operation over the result of the recombination is needed(relabel operation see (49)).

A. RANDOM GENERATION

In case the first method of generation is used, a number x with a valid value is generated, and then corresponding unrank procedure is used: (35), (39), (46), (52), or (54). The valid ranks for x are shown in Table (13) for the unbounded and r-bounded cases.

The other case to generate a random element for each representation requires the generation one symbol at-a-time for a member of each representation.

The procedure to generate a random element for $\mathbb{S}(k)$ and r-bounded $\mathbb{S}(k)$ is given in (57), take note that for the unbounded case r must be replaced by k, also it is relevant to say that $\alpha = \min(r, 2 + \max_{j=0}^{i-1}(\mathbb{X}_j))$, computes the

TABLE 13. Table of valid ranks for the two cases (unbounded, and r-bounded) for the three representations $\mathbb{S}(k)$, $\mathbb{E}(k)$, and $\mathbb{N}(k)$.

Case / Repres.	unbounded	r-bounded
$\mathbb{S}(\boldsymbol{k})$	$0 \le x \le B(k)$-1	$0 \le x \le \left(\sum_{i=1}^{r} S(k,v)\right)$-1
$\mathbb{E}(\boldsymbol{k})$	$0 \le x \le E(k)$-1	$0 \le x \le \left(r! \cdot r^{k\text{-}r}\right) - 1$
$\mathbb{N}(\boldsymbol{k})$	$0 \le x \le k^k$-1	$0 \le x \le r^k$-1

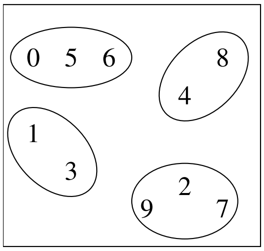


FIGURE 1. A graphical representation of.

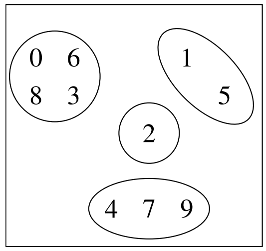


FIGURE 2. A graphical representation of $\mathbb{X} = (0, 1, 2, 0, 3, 1, 0, 3, 0, 3)$; $\mathbb{X} \in \mathbb{E}(10)$.

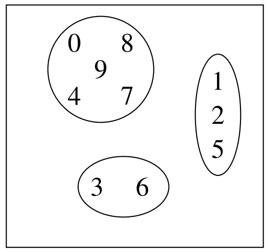


FIGURE 3. A graphical representation for $\mathbb{X} = (3, 2, 2, 1, 3, 2, 1, 3, 3, 3)$; $\mathbb{X} \in \mathbb{N}(10)$.

cardinality of the possible values that $\mathbb{X}_i$ could have.

$$\begin{aligned} &\mathbb{X} = \{\emptyset\}; \mathbb{X}_0 = 0; \\ &\forall i \in [1, \ldots, k-1] \, \{ \\ &\quad \alpha = \min(r, 2 + \max_{j=0}^{i-1}(\mathbb{X}_j)); \\ &\quad \mathbb{X}_i = \textbf{rand}() \bmod \alpha; \\ &\} \\ &\textbf{return } \mathbb{X}; \end{aligned} \tag{57}$$

Example 21: A random element for $\mathbb{S}(10, 4)$ is $\mathbb{X} = (0, 1, 2, 1, 3, 0, 0, 2, 3, 2)$, a graphic representation for it, is given in Figure 1. △

The procedure to generate a random element for $\mathbb{E}(k)$ and r-bounded $\mathbb{E}(k)$ is given in (58), take note that for the unbounded case r must be replaced by k, also note that $\alpha = \min(i+, r)$, computes the cardinality of the valid values that $\mathbb{X}_i$ could have.

$$\begin{aligned} &\mathbb{X} = \{\emptyset\}; \mathbb{X}_0 = 0; \\ &\forall \, i \in [1, \ldots, k\text{-}1]\{ \\ &\quad \alpha = \min(r, i{+}1); \\ &\quad \mathbb{X}_i = \textbf{rand}() \bmod \alpha; \\ &\} \\ &\textbf{return } \mathbb{X}; \end{aligned} \tag{58}$$

Example 22: Let $\mathbb{X} \in \mathbb{E}(10, 4)$ be $\mathbb{X} = (0, 1, 2, 0, 3, 1, 0, 3, 0, 3)$, its graphic representation in given in Figure 2. △

The procedure to generate a random element for $\mathbb{N}(k)$ and r-bounded $\mathbb{N}(k)$ is given in (59), take note that for the unbounded case r must be replaced by k, again the value of $\alpha = \min(k, r)$ indicates the cardinality of valid values of $\mathbb{X}_i$.

$$\begin{aligned} &\mathbb{X} = \{\emptyset\}; \\ &\alpha = \min(r, k); \\ &\forall \, i \in [0, \ldots, k\text{-}1]\{ \\ &\quad \mathbb{X}_i = \textbf{rand}() \bmod \alpha; \\ &\} \\ &\textbf{return } \mathbb{X}; \end{aligned} \tag{59}$$

Example 23: Let $\mathbb{X} \in \mathbb{N}(10, 4)$ be $\mathbb{X} = (3, 2, 2, 1, 3, 2, 1, 3, 3, 3)$, see its graphic representation in Figure 3. △

B. MUTATION OPERATORS

The mutation operator for $\mathbb{X} \in \mathbb{S}(k)$ is shown in (60), please note that $\alpha = \min(r, 2 + \max_{j=0}^{i-1}(\mathbb{X}_j))$, compute the cardinality of the possible values that could have $\mathbb{X}_i$, also note that: $(\mathbb{X}_i{+}1{+}\textbf{rand}() \bmod (\alpha\text{-}1)) \bmod \alpha$, computes a random value for $\mathbb{X}_i$ excluding its current value. It is possible that some $\mathbb{X}_j$ for $(j > i)$ need to be adjusted, for this reason, the relabel operator is called to get a valid RGS (see (49)) (for the unbounded case replace r with k. A final comment: given that always $\mathbb{X}_0 = 0$ we exclude this position to be mutated with $i = 1{+}\textbf{rand}() \bmod (k\text{-}1)$.

$$\begin{aligned} &\mathbb{X} \in \mathbb{S}(k) \text{ or } \mathbb{X} \in r\text{-bounded } \mathbb{S}(k); \\ &i = 1{+}\textbf{rand}() \bmod (k\text{-}1); \\ &\alpha = \min(r, 2 + \max_{j=0}^{i-1}(\mathbb{X}_j)); \\ &\mathbb{X}_i = (\mathbb{X}_i{+}1{+}\textbf{rand}() \bmod (\alpha\text{-}1)) \bmod \alpha; \\ &\mathbb{X} = \textbf{relabel}(\mathbb{X}); \\ &\textbf{return } \mathbb{X}; \end{aligned} \tag{60}$$

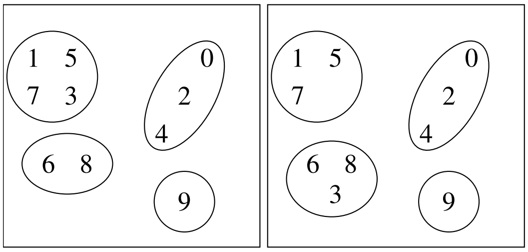


FIGURE 4. See in the left side $\mathbb{X} = (0, 1, 0, 1, 0, 1, 2, 1, 2, 3)$ and mutated $\mathbb{X} = (0, 1, 0, 2, 0, 1, 2, 1, 2, 3)$ at the right side, both belong to $\mathbb{S}(10, 4)$.

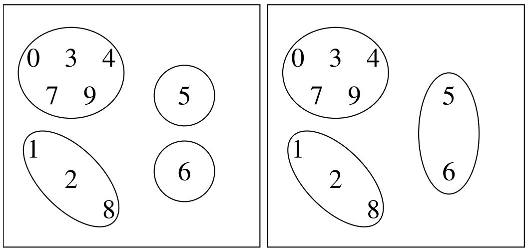


FIGURE 5. Figures of $\mathbb{X} = (0, 1, 1, 0, 0, 2, 3, 0, 1, 0)$ at the left and mutated $\mathbb{X} = (0, 1, 1, 0, 0, 3, 3, 0, 1, 0)$ at the right, both belong to $\mathbb{E}(10, 4)$.

Example 24: Let $\mathbb{X} \in \mathbb{S}(10, 4)$ be $\mathbb{X} = (0, 1, 0, 1, 0, 1, 2, 1, 2, 3)$ and mutated $\mathbb{X} = (0, 1, 0, 2, 0, 1, 2, 1, 2, 3)$, Figure 4 show in the left side a representation for $\mathbb{X}$ and in the right side its mutation.

△

The mutation operator for $\mathbb{X} \in \mathbb{E}(k)$ is shown in (61), replace the value of r with k for the unbounded case. Again the value of $\alpha = \min(r, i{+}1)$ indicates the cardinality of the values for $\mathbb{X}_i$; and $(\mathbb{X}_i{+}1{+}\textbf{rand}() \bmod (\alpha\text{-}1)) \bmod \alpha$ computes a valid value for $\mathbb{X}_i$ excluding its current value. A final comment: given that always $\mathbb{X}_0 = 0$ we exclude this position to be mutated with $i = 1{+}\textbf{rand}() \bmod (k\text{-}1)$.

$$\begin{aligned}
&\mathbb{X} \in \mathbb{E}(k) \text{ or } \mathbb{X} \in r\text{-bounded } \mathbb{E}(k);\\
&i = 1{+}\textbf{rand}() \bmod (k\text{-}1);\\
&\alpha = \min(r, i{+}1);\\
&\mathbb{X}_i = (\mathbb{X}_i{+}1{+}\textbf{rand}() \bmod (\alpha\text{-}1)) \bmod \alpha;\\
&\textbf{return } \mathbb{X};
\end{aligned} \tag{61}$$

Example 25: Let $\mathbb{X} \in \mathbb{E}(10, 4)$ be $\mathbb{X} = (0, 1, 1, 0, 0, 2, 3, 0, 1, 0)$ and mutated $\mathbb{X} = (0, 1, 1, 0, 0, 3, 3, 0, 1, 0)$, Figure 5 show in the left side a representation for $\mathbb{X}$ and in the right side its mutation.

△

The mutation operator for $\mathbb{X} \in \mathbb{N}(k)$ is shown in (62), the value of r is k for the unbounded case. The value of $\alpha = \min(k, r)$ indicates the cardinality of possible values, and $(\mathbb{X}_i{+}1 + \textbf{rand}() \bmod (\alpha\text{-}1)) \bmod \alpha$, computes a valid random value for $\mathbb{X}_i$ excluding its current value.

$$\begin{aligned}
&\mathbb{X} \in \mathbb{N}(k) \text{ or } \mathbb{X} \in r\text{-bounded } \mathbb{N}(k);\\
&i = \textbf{rand}() \bmod k;\\
&\alpha = \min(r, k);\\
&\mathbb{X}_i = (\mathbb{X}_i{+}1{+}\textbf{rand}() \bmod (\alpha\text{-}1)) \bmod \alpha;\\
&\textbf{return } \mathbb{X};
\end{aligned} \tag{62}$$

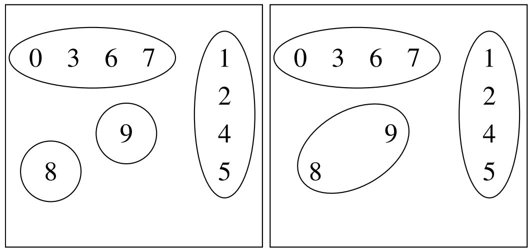


FIGURE 6. $\mathbb{X} \in \mathbb{N}(10, 4) = (3, 2, 2, 3, 2, 2, 3, 3, 0, 1)$ and its mutation $(3, 2, 2, 3, 2, 2, 3, 3, 1, 1)$ see their graphical representation from left to right.

Example 26: Let $\mathbb{X} \in \mathbb{N}(10, 4)$ be $\mathbb{X} = (3, 2, 2, 3, 2, 2, 3, 3, 0, 1)$ and its mutated version be $(3, 2, 2, 3, 2, 2, 3, 3, 1, 1)$, Figure 6 shows a graphical representation of $\mathbb{X}$ at its left and at its right the mutated version. △

The mutation operator that offers the greatest exploratory power is the mutation for $\mathbb{N}(k)$, given that the possible values for one random position are not constrained (only for the value of r). The mutation for $\mathbb{E}(k)$ has the constraint that the value in one position depends heavily on the position itself and the value for r, then the number of possible values in one position are constrained a lot. But, the mutation operator for $\mathbb{S}(k)$ is the most constrained, given that the set of possible values in one specific position, depends totally on all the previous elements to the position being modified. These facts suggest that the best representation is $\mathbb{N}(k)$ but remember that this representation has the biggest redundancy, and the zero redundancy representation is $\mathbb{S}(k)$.

C. RECOMBINATION OPERATORS

The recombination of two solutions $\mathbb{X}$ and $\mathbb{Y}$ that belongs to $\mathbb{S}(k)$ or r-bounded $\mathbb{S}(k)$ is presented only for one point of recombination, but the procedure shown in (63) can be easily extended for many points of recombination. After the recombination of two solutions is done, the solutions may be an incorrect RGS, then a repair is required, this is done using **relabel**($\mathbb{R}$) and **relabel**($\mathbb{T}$).

$$\begin{aligned}
&\mathbb{R} = \emptyset;\ \mathbb{T} = \emptyset;\\
&\mathbb{X} \in \mathbb{S}(k) \text{ or } \mathbb{X} \in r\text{-bounded } \mathbb{S}(k);\\
&\mathbb{Y} \in \mathbb{S}(k) \text{ or } \mathbb{Y} \in r\text{-bounded } \mathbb{S}(k);\\
&i = 1{+}\textbf{rand}() \bmod (k\text{-}1);\\
&\mathbb{R} = \mathbb{X}_{0,\ldots,i\text{-}1} \oplus \mathbb{Y}_{i,\ldots,k\text{-}1};\\
&\mathbb{T} = \mathbb{Y}_{0,\ldots,i\text{-}1} \oplus \mathbb{X}_{i,\ldots,k\text{-}1};\\
&\mathbb{R} = \textbf{relabel}(\mathbb{R});\ \mathbb{T} = \textbf{relabel}(\mathbb{T});\\
&\textbf{return } \mathbb{R} \text{ and } \mathbb{T};
\end{aligned} \tag{63}$$

Example 27: $\mathbb{X}, \mathbb{Y} \in \mathbb{S}(10, 4)$ are $\mathbb{X} = (0, 0, 0, 1, 2, 2, 3, 1, 3, 0)$, $\mathbb{Y} = (0, 0, 1, 1, 0, 2, 1, 3, 0, 3)$ and the recombination point exchanges the last three elements, then we have as result: $(0, 0, 0, 1, 2, 2, 3, 3, 0, 3)$ and $(0, 0, 1, 1, 0, 2, 1, 1, 3, 0)$.

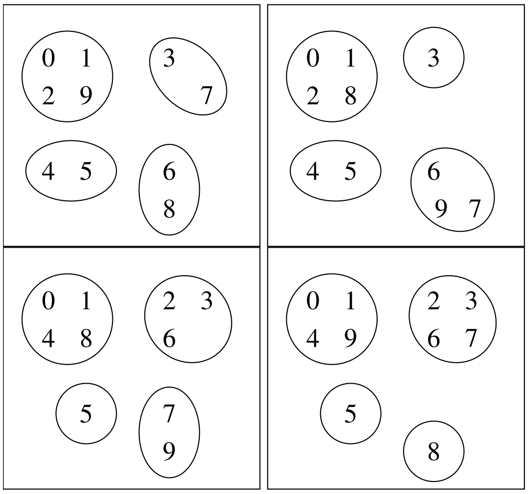


FIGURE 7. $\mathbb{X}, \mathbb{Y} \in \mathbb{S}(10, 4)$, $\mathbb{X} = (0, 0, 0, 1, 2, 2, 3, 1, 3, 0)$ and $\mathbb{Y} = (0, 0, 1, 1, 0, 2, 1, 3, 0, 3)$ are shown at the left, at the right the recombination exchanging the last three elements: $(0, 0, 0, 1, 2, 2, 3, 3, 0, 3)$ and $(0, 0, 1, 1, 0, 2, 1, 1, 3, 0)$.

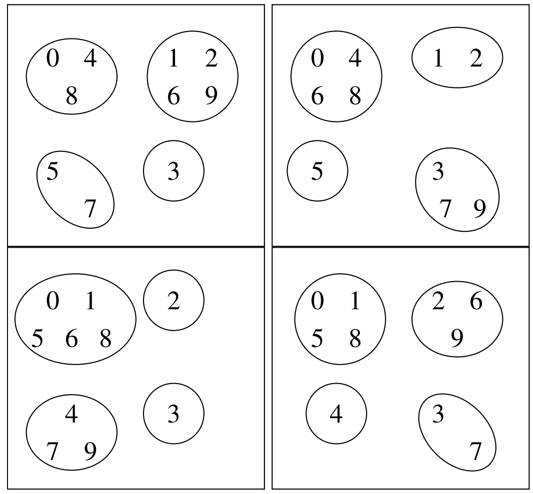


FIGURE 8. $\mathbb{X} = (0, 1, 1, 2, 0, 3, 1, 3, 0, 1)$, $\mathbb{Y} = (0, 0, 1, 3, 2, 0, 0, 2, 0, 2)$ $\in \mathbb{E}(10, 4)$ are shown at the left, and the result of recombination of the last 4 elements gives: $(0, 1, 1, 2, 0, 3, 0, 2, 0, 2)$, $(0, 0, 1, 3, 2, 0, 1, 3, 0, 1)$and are shown at the right.

See at the left of Figure 7, $\mathbb{X}$ and $\mathbb{Y}$ and the result of recombination at the right. △

The recombination of two solutions $\mathbb{X}$ and $\mathbb{Y}$ that belongs to $\mathbb{E}(k)$ or r-bounded $\mathbb{E}(k)$ is presented only using one point of recombination, but the procedure shown in (64) can be easily extended for many points of recombination.

$$\begin{aligned}
&\mathbb{R} = \emptyset; \mathbb{T} = \emptyset;\\
&\mathbb{X} \in \mathbb{E}(k) \text{ or } \mathbb{X} \in r\text{-bounded } \mathbb{E}(k);\\
&\mathbb{Y} \in \mathbb{E}(k) \text{ or } \mathbb{Y} \in r\text{-bounded } \mathbb{E}(k);\\
&i = 1+\textbf{rand}() \bmod (k\text{-}1);\\
&\mathbb{R} = \mathbb{X}_{0,\ldots,i\text{-}1} \oplus \mathbb{Y}_{i,\ldots,k\text{-}1};\\
&\mathbb{T} = \mathbb{Y}_{0,\ldots,i\text{-}1} \oplus \mathbb{X}_{i,\ldots,k\text{-}1};\\
&\textbf{return } \mathbb{R} \text{ and } \mathbb{T}; \qquad (64)
\end{aligned}$$

Example 28: Let $\mathbb{X}, \mathbb{Y} \in \mathbb{E}(10, 4)$ be $X = (0, 1, 1, 2, 0, 3, 1, 3, 0, 1)$ and $Y = (0, 0, 1, 3, 2, 0, 0, 2, 0, 2)$, assuming that the recombination point exchanges the last 4 elements, this results in: $(0, 1, 1, 2, 0, 3, 0, 2, 0, 2)$ and $(0, 0, 1, 3, 2, 0, 1, 3, 0, 1)$. See at the left of Figure 8 $\mathbb{X}$ and $\mathbb{Y}$; and at the left the result of doing the recombination. △

The recombination for the $\mathbb{N}(k)$ representation is almost identical to the one for $\mathbb{E}(k)$, see (65).

$$\begin{aligned}
&\mathbb{R} = \emptyset; \mathbb{T} = \emptyset;\\
&\mathbb{X} \in \mathbb{N}(k) \text{ or } \mathbb{X} \in r\text{-bounded } \mathbb{N}(k);\\
&\mathbb{Y} \in \mathbb{N}(k) \text{ or } \mathbb{Y} \in r\text{-bounded } \mathbb{N}(k);\\
&i = 1+\textbf{rand}() \bmod (k\text{-}1);\\
&\mathbb{R} = x_{0,\ldots,i\text{-}1} \oplus y_{i,\ldots,k-1};\\
&\mathbb{T} = y_{0,\ldots,i\text{-}1} \oplus x_{i,\ldots,k\text{-}1};\\
&\textbf{return } \mathbb{R} \text{ and } \mathbb{T}; \qquad (65)
\end{aligned}$$

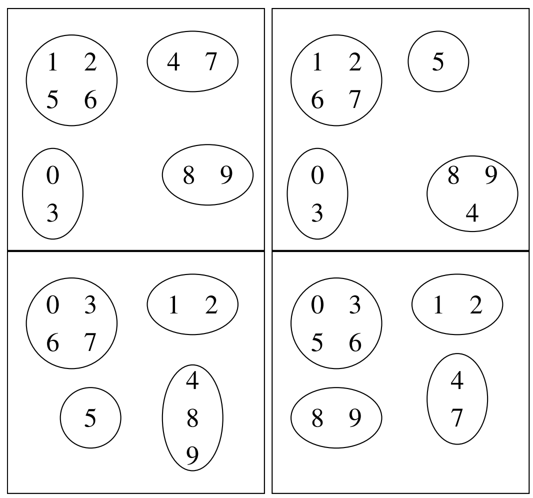


FIGURE 9. The recombination exchanges the six last elements of $\mathbb{X}, \mathbb{Y} \in \mathbb{N}(10, 4)$. $\mathbb{X} = (2, 0, 0, 2, 1, 0, 0, 1, 3, 3)$, $\mathbb{Y} = (0, 2, 2, 0, 3, 1, 0, 0, 3, 3)$ are shown at the left, and the results of recombination: $(2, 0, 0, 2, 3, 1, 0, 0, 3, 3)$ and $(0, 2, 2, 0, 1, 0, 0, 1, 3, 3)$are shown at right.

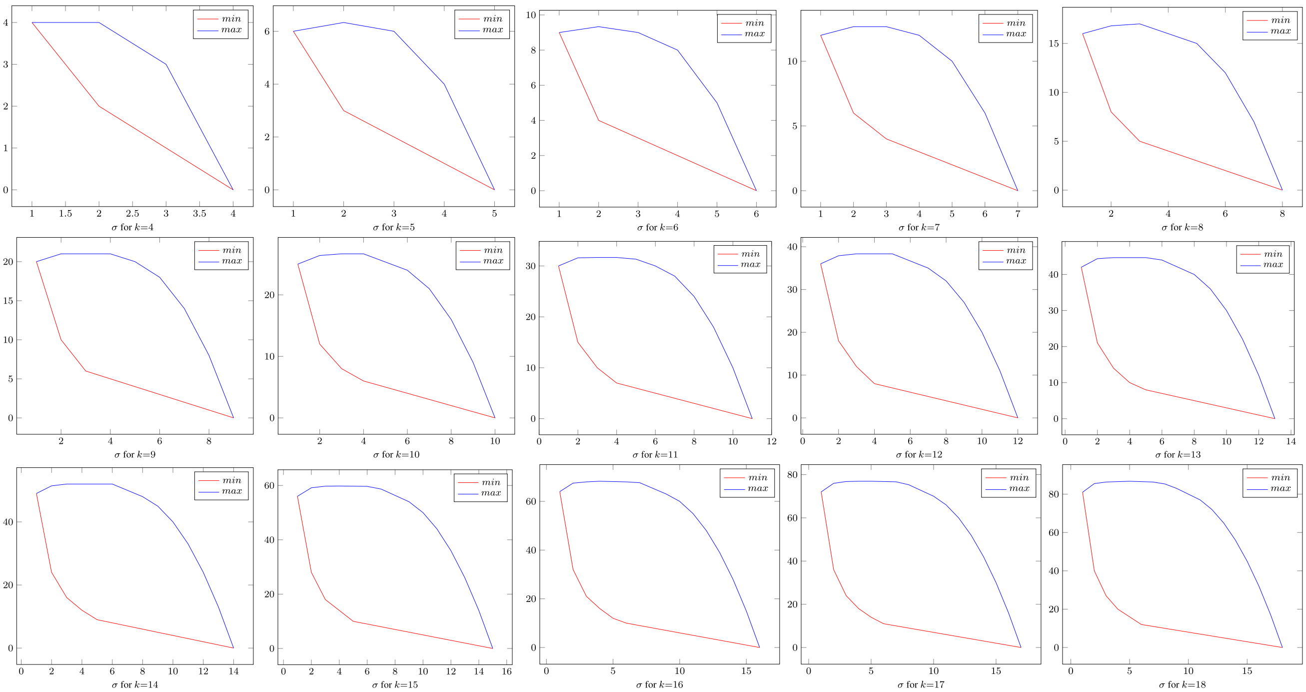
Example 29: Let $\mathbb{X}, \mathbb{Y} \in \mathbb{N}(10, 4)$ be $\mathbb{X} = (2, 0, 0, 2, 1, 0, 0, 1, 3, 3)$, $\mathbb{Y} = (0, 2, 2, 0, 3, 1, 0, 0, 3, 3)$, and the recombination exchanges the six last elements giving: $(2, 0, 0, 2, 3, 1, 0, 0, 3, 3)$ and $(0, 2, 2, 0, 1, 0, 0, 1, 3, 3)$. In Figure 9 at the left are shown $\mathbb{X}$ and $\mathbb{Y}$. △

The recombination for $\mathbb{E}(k)$ and $\mathbb{N}(k)$ does not require to repair the results, i.e., all the recombinations are valid. Even, the recombination for $\mathbb{S}(k)$ demands to repair (relabeling), this procedure is easy to implement.

V. USE OF THE THREE REPRESENTATIONS WITH ONE SPECIFIC PROBLEM

In order to get some insight about the performance of using the three representations ($\mathbb{N}(k)$, $\mathbb{S}(k)$, $\mathbb{E}(k)$) we will solve an instance of the set partition of integers problem (SPIP)

TABLE 14. The vertical axis indicates the minimum and maximum value of the function $\sum_{i=0}^{\sigma\text{-}1}\sum_{\forall a\in\mathbb{S}_i}|a\text{-}\zeta_i|$, $\zeta_i=\frac{1}{|S_i|}\cdot\sum_{\forall a\in\mathbb{S}_i}a$; the horizontal axis indicates the value of σ according the number of objects k; the problem instance is $\mathbb{P}=\{1,\ldots,k\}$.



using a simple genetic algorithm (one that uses only mutation of one element in each possible solution, and recombination/crossover with one point).

A. SPIP SET PARTITION OF INTEGERS PROBLEM

Two problems will be defined next, one in which near-optimal solution occurs with a large number of partitions and other problem in which near-optimal solution requires a small number of partitions, respectively we call the first one minimization problem and the second one maximization problem.

Minimization problem: Given a set $\mathbb{P}$ of positive integers, find σ subsets $\mathbb{S}_0,\ldots,\mathbb{S}_{\sigma\text{-}1}$, $1\le\sigma\le k$, $\mathbb{S}_i\cap\mathbb{S}_j=\emptyset$, $\bigcup_{i=0}^{\sigma\text{-}1}\mathbb{S}_i=\mathbb{P}$, such that (66) is minimized. Intuitively we minimize (66) using a large number of clusters, i.e. almost one cluster for each number to be partitioned.

Maximization problem: Given a set $\mathbb{P}$ of positive integers, find σ subsets $\mathbb{S}_0,\ldots,\mathbb{S}_{\sigma\text{-}1}$, $1\le\sigma\le k$, $\mathbb{S}_i\cap\mathbb{S}_j=\emptyset$, $\bigcup_{i=0}^{\sigma\text{-}1}\mathbb{S}_i=\mathbb{P}$, such that (66) is maximized. Intuitively we maximize (66) using a small number of clusters.

$$\sum_{i=0}^{\sigma\text{-}1}\sum_{\forall a\in\mathbb{S}_i}|a-\zeta_i|,\ \zeta_i=\frac{1}{|S_i|}\cdot\sum_{\forall a\in\mathbb{S}_i}a \tag{66}$$

Example 30: Let $\mathbb{P}=\{1,4,7,10,13,16\}$.

The value of the function $\sum_{i=0}^{\sigma\text{-}1}\sum_{\forall a\in\mathbb{S}_i}|a-\zeta_i|$ is:

0 with the set partition: $\mathbb{S}_0=\{1\}$,$\mathbb{S}_1=\{4\}$, $\mathbb{S}_2=\{7\}$, $\mathbb{S}_3=\{10\}$, $\mathbb{S}_4=\{13\}$, $\mathbb{S}_4=\{16\}$

9 with the set partition: $\mathbb{S}_0=\{1,4\}$,$\mathbb{S}_1=\{7,10\}$,$\mathbb{S}_2=\{13,16\}$

12 with the set partition: $\mathbb{S}_0=\{1,4,7\}$,$\mathbb{S}_1=\{10,13,16\}$ $\triangle$

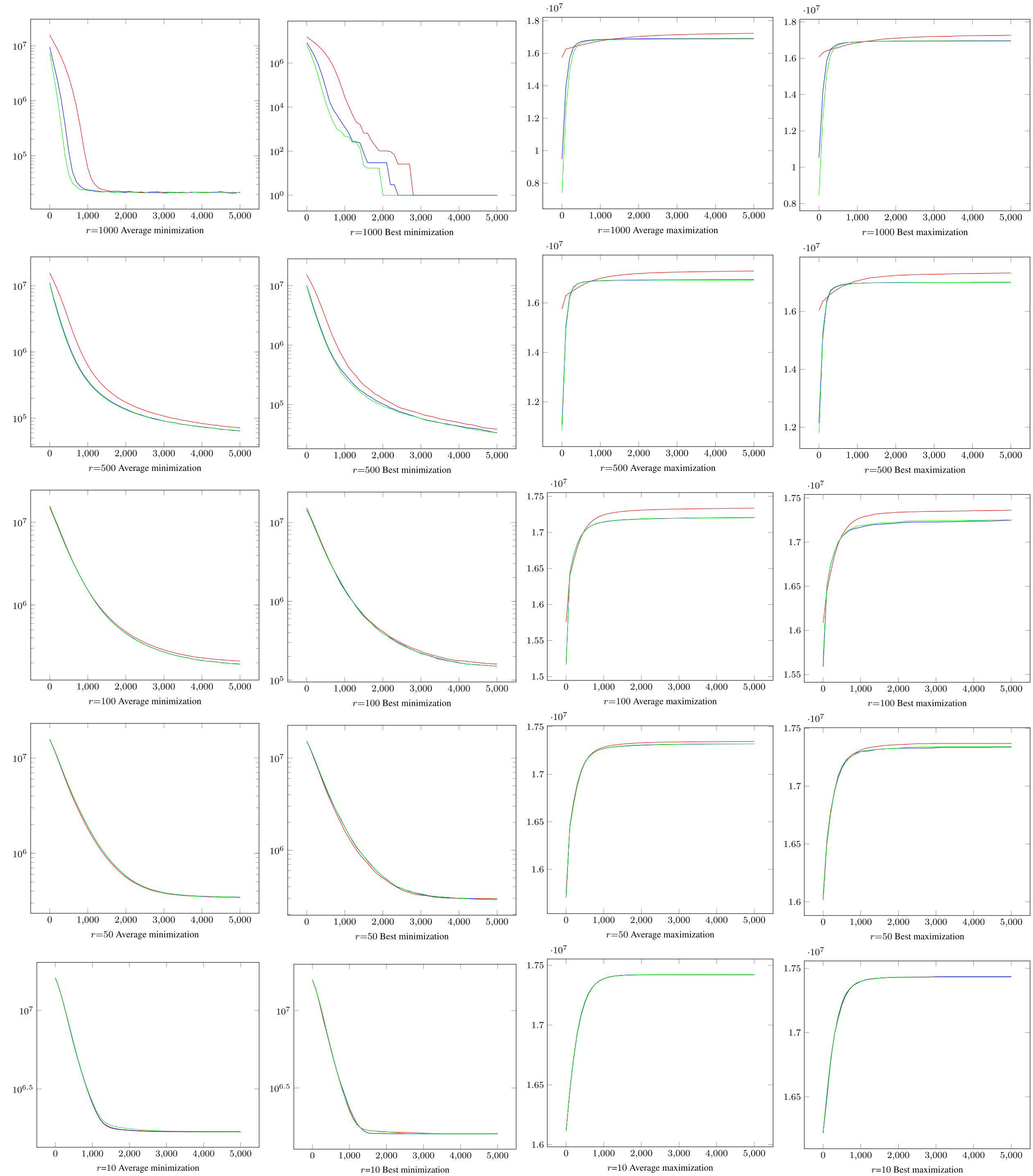
TABLE 15. Maximum values of function in equation (66), the value of k, and the value(s) of number of partitions (σ).

max,k,{σ}	max,k,{σ}	max,k,{σ}
4.000, 4, {1,2}	6.333, 5, {2}	9.333, 6, {2}
12.667, 7, {2,3}	17.000, 8, {3}	21.000, 9, {2,3,4}
26.667, 10, {3,4}	31.667, 11, {3,4}	38.333, 12, {3,4,5}
44.667, 13, {3,4,5}	52.000, 14, {3,4,5}	59.800, 15, {4}
68.267, 16, {4}	76.933, 17, {4,5}	86.733, 18, {5}

With this small example, it can be created the intuition that in order to minimize the value of the function: $\sum_{i=0}^{\sigma\text{-}1}\sum_{\forall a\in\mathbb{S}_i}|a-\zeta_i|$ the number of required subsets (σ) will be large, and to maximize its value, the number of required subsets will be small.

In order to verify the intuition related to the number of partitions needed for the maximization and minimization versions of SPIP, we have defined instances with $4\le k\le 18$ and $\mathbb{P}=\{1,\ldots,k\}$, for the 15 instances all the $B(k)$ possible solutions were constructed and determined the maximum and minimum value of (66). The results are shown graphically in Table 14 and in Table 15 are depicted the maximum value of the function including the corresponding σ values (the minimum is not shown in Table 15 given that always is zero and occurs with a value of $\sigma=k$). It can be verified the initial intuition that solutions with minimum values of the function occurs with the maximum number of partitions (i.e. k) with a zero value, and that solutions with maximum values of the function occurs near the minimum number of partitions(i.e. one partition) but in general the resulting σ values seems to be greater than 1.

TABLE 16. Results of population average and best results for minimizing and maximizing function in equation (66), each row contains the results for a particular value of $r \in \{10, 50, 100, 500, 1000\}$. Lines in red, blue and green refers respectively to $\mathbb{S}$, $\mathbb{E}$ and $\mathbb{N}$ representations.



B. A SIMPLE GENETIC ALGORITHM

We explore the maximization and minimization of the function defined in (66) with a simple genetic algorithm in order to test the capabilities of the three representations to approximate good solutions requiring a small number of partitions and a large number of partitions.

The characteristics defined for the simple genetic algorithm are:

- A problem instance $\mathbb{P}$ with size $k = 1000$, each member of the problem instance is a random number in the rank $\{0, \ldots, 65535\}$
- The values for r-bounded representations are: $r \in \{10, 50, 100, 500, 1000\}$.

- Function to be maximized or minimized: $\sum_{i=0}^{\sigma\text{-}1}\sum_{\forall a \in \mathbb{S}_i}|a-\zeta_i|$, $\zeta_i = \frac{1}{|S_i|}\cdot\sum_{\forall a\in\mathbb{S}_i} a$.
- Population Size 100 ($POPULATION = 100$).
- Number of generations.
- Tournament selection of size 2, the elements for a recombination are selected according the function (66) and if the problem is the maximization or minimization version (see variable *case*). Then for each generation *POPULATION* tournament selections are done.
- *POPULATION*/2 recombinations are done (each recombination produces two elements).
- For both elements resulting from a recombination a mutation of one position is done.
- The best element of the current generation replaces the worst element in the new generation (best and worst changes its meaning considering if the problem is the maximization or the minimization version).

Once the instance $\mathbb{P}$ is created, the operation of the simple genetic algorithm is depicted in (67).

$$
\begin{array}{l}
\forall r \in \{10, 50, 100, 500, 1000\}\{\\
\quad \forall i \in [0, \ldots, POPULATION\text{-}1]\{\\
\quad\quad \mathbb{G}_{0,i} = \textbf{random}\text{-}\{\mathbb{S}, \mathbb{E}, \textit{or } \mathbb{N}\};\\
\quad \}\\
\quad \textbf{evaluate } \mathbb{G}_0;\\
\quad case = 0 \textbf{ for minimization},\\
\quad\quad\quad\quad 1 \textbf{ for maximization};\\
\quad \forall i \in [0, GENERATIONS\text{-}1]\{\\
\quad\quad \forall j \in [0, \dfrac{POPULATION}{2}\text{-}1]\{\\
\quad\quad\quad \textbf{select } \{a, b\}\ a \neq b\\
\quad\quad\quad\quad \textbf{using tournament}(\mathbb{G}_{i-1}, case);\\
\quad\quad\quad \{a', b'\} =\\
\quad\quad\quad r\text{-}bounded\{\mathbb{S}, \mathbb{E}, \textit{ or } \mathbb{N}\}\ \textbf{recombination}(a, b);\\
\quad\quad\quad a'' = r\text{-}bounded\{\mathbb{S}, \mathbb{E}, \textit{ or } \mathbb{N}\}\ \textbf{mutation}(a);\\
\quad\quad\quad b'' = r\text{-}bounded\{\mathbb{S}, \mathbb{E}, \textit{ or } \mathbb{N}\}\ \textbf{mutation}(b);\\
\quad\quad\quad \mathbb{G}_{i,2\cdot j} = a'';\ \mathbb{G}_{i,2\cdot j+1} = b'';\\
\quad\quad \}\\
\quad\quad \textbf{evaluate } \mathbb{G}_i;\\
\quad\quad a = \textbf{best}(\mathbb{G}_{i-1}, case);\\
\quad\quad \textbf{replace worst}(\mathbb{G}_i, case) \textbf{ with } a;\\
\quad \}\\
\}
\end{array} \tag{67}
$$

C. RESULTS OF THE PSIP USING A SIMPLE GENETIC ALGORITHM

Each experiment expressed in (67) were ran 31 times, and the results are depicted graphically in Table (16) a summary of the results is presented in Table (17).

According to the results obtained some preliminary conclusions can be extracted:

TABLE 17. Best results for minimizing and maximizing function (66) using a simple genetic algorithm, according the *r*-bounded representation; the best value of the function and the best σ value for maximization and minimization is indicated; best results indicated with bold font.

r	Representation	Minimum value	σ	Maximum value	σ
1000	$\mathbb{S}$	**0.00**	**998**	**17265648.20**	**31**
	$\mathbb{E}$	**0.00**	**1000**	16965788.90	355
	$\mathbb{N}$	**0.00**	**998**	16946294.93	392
500	$\mathbb{S}$	38678.81	500	**17326033.70**	**35**
	$\mathbb{E}$	33886.32	500	17007307.47	323
	$\mathbb{N}$	**33464.25**	**500**	17003183.37	351
100	$\mathbb{S}$	160175.81	100	**17363482.00**	**40**
	$\mathbb{E}$	150936.23	100	17250068.76	100
	$\mathbb{N}$	**150513.55**	**100**	17252402.89	100
50	$\mathbb{S}$	296296.74	50	**17369615.13**	**35**
	$\mathbb{E}$	293406.85	50	17341156.33	50
	$\mathbb{N}$	**291995.78**	**50**	17336955.09	50
10	$\mathbb{S}$	**1598576.16**	**10**	**17435999.13**	**10**
	$\mathbb{E}$	**1598576.16**	**10**	17434561.47	10
	$\mathbb{N}$	**1598576.16**	**10**	17432426.74	10

- the representation $\mathbb{N}$ is the best when the solution searched has a large number of partitions;
- the representation $\mathbb{S}$ is the best when the solution searched has a small number of partitions;
- the representation $\mathbb{E}$ has an average behavior in both cases (maximization and minimization).

D. ADVICE ABOUT THE CONCLUSIONS OF PSIP RESULTS

It is important to say that the performance of the three representations may change for another kind of SPP instances, i.e. that the best representation for the PSIP instance may not be the best representation for another kind of SPP instances.

VI. CONCLUSION

Important concepts related to the integer partition problem and SPP were presented, the focus was in three possible representations of SPP solutions. Remarkably, the three representations presented: $\mathbb{S}(k)$,$\mathbb{E}(k)$, and $\mathbb{N}(k)$, are based respectively on: Restricted Growth Strings, factoradic number system, and a positional number system with a fixed base. Also, the three representations have a strong mathematical foundation based on the Stirling numbers of the second kind ($S(k, v)$), Eulerian numbers ($E(k, v)$), and the cardinality of functions ($N(k, v)$). Two theorems related to the cardinalities of $\mathbb{E}(k)$ and $\mathbb{N}(k)$ were proved, for completeness ranking and unranking functions for the three representations and mapping procedures between the representations were provided.

To help practitioners of optimization algorithms to use and test the three representations: a) the case of r-bounded in the number of set partitions was included for the three representations; b) procedures to generate a random element for each representation were detailed; c) mechanisms to mutate (do a small change) to one member for each representation were provided; and d) recombination procedures for the three representations were presented.

The use of the three representations combined with a simple genetic algorithm was presented to solve an instance of PSIP. The main conclusions of this exercise are that: a) the representation $\mathbb{N}$ is the best when the solution searched has a large number of partitions; b) the representation $\mathbb{S}$ is the best when the solution searched has a small number of partitions; c) the representation $\mathbb{E}$ has an average behavior in both cases (when the solution searched has a large or small number of partitions).

Is is expected that many researchers will be benefited from the possibility of using the three representations to solve particular instances of SPP. Finally, and reiterating, we have to say that the representation with zero redundancy is $\mathbb{S}(k, v)$ and the one with more redundancy is $\mathbb{N}(k)$. But, it seems that the exploratory power of $\mathbb{N}(k)$ and $\mathbb{E}(k)$ is better than the exploratory power of $\mathbb{S}(k)$. This fact suggests that it will be a good alternative to use the representation based on factoradic number system ($\mathbb{E}(k)$), given that it seems to provide a balance between redundancy and exploratory power.

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