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A Simple Sum of Products Formula to Compute the Reliability of the KooN System

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ABSTRACT Reliability block diagram (RBD) is a well-known, high-level abstract modeling method for calculating systems reliability. Increasing redundancy is the most important way for increasing Fault-tolerance and reliability of dependable systems. K-out-of-N (KooN) is one of the known redundancy models. The redundancy causes repeated events and increases the complexity of the computing system's reliability, and researchers use techniques like factorization to overcome it. Current methods lead to the cumbersome formula that needs a lot of simplification to change in the form of Sum of the Products (SoP) in terms of reliabilities of its constituting components. In This paper, a technique for extracting simple formula for calculating the KooN system's reliability in SoP form using the Venn diagram is presented. Then, the shortcoming of using the Venn diagram that is masking some joints events in the case of a large number of independent components is explained. We proposed the replacement of Lattice instead of Venn diagrams to overcome this weakness. Then, the Lattice of reliabilities that is dual of power set Lattice of components is introduced. Using the basic properties of Lattice of reliabilities and their inclusion relationships, we propose an algorithm for driving a general formula of the KooN system's reliability in SoP form. The proposed algorithm gives the SoP formula coefficients by computing elements of the main diagonal and elements below it in a squared matrix. The computational and space complexity of the proposed algorithm is $\theta((n-k)^2/2)$ that n is the number of different components and k denotes the number of functioning components. A lemma and a theorem are defined and proved as a basis of the proposed general formula for computing coefficients of the SoP formula of the KooN system. Computational and space complexity of computing all of the coefficients of reliability formula of KooN system using this formula reduced to $\theta(n-k)$. The proposed formula is simple and is in the form of SoP, and its computation is less error-prone.

INDEX TERMS Reliability analysis, reliability block diagram, K-out-of-N system, Lattice, probabilistic formula, Venn diagram.

I. INTRODUCTION

Computerized systems accomplish critical tasks like control of nuclear power plants, railway systems, aeroplanes, factories, and petroleums. The dependability of critical systems is a crucial issue for systems designers. Reliability and fault tolerance of the system are two main issues in dependability. A most common method for increasing reliability and fault tolerance is increasing the redundancy of components. Fault tree and reliability block diagram (RBD) are the two most commonly used methods for calculating the system's reliability, especially in redundant components. Each of these methods is appropriate for the specific type of systems.

RBD is one of the well-known and straightforward means for reliability modelling and reliability analysis of the system. It represents a high-level abstract model of functional dependencies of the underlying systems' components. We can categorize RBD models into four significant groups named; 1) series, 2) parallel, 3) combination of series and parallel, and 4) systems that do not fall in any of the mentioned groups [1]. For computing reliability of parallel and series-systems, simple probabilistic formulas based on the reliability of constituting components have been presented in textbooks [1], [2]. We can use these formulas for computing the reliability of type 3 systems too. Calculating the reliability of kind four systems and also systems with repeated events is a bit cumbersome. The factorization technique is a well-known method for converting a type four system into multiple type-three systems. After that, we compute their reliabilities and finally combine them [1], [3].

RBD is used to evaluate various failure-related characteristics, such as reliability [1], [4], [5], availability [6], and

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maintainability [7] in a wide range of systems. One of the most important and widely used structures is K-out-of-N (KooN). KooN system operates if at least K elements from N elements functions. Exceptional KooN structure cases are 1) NooN, also known as series structure, and 2) 1ooN that represents full parallel structure [2].

Lu *et al.* [8] proposed a moving extremum surrogate modeling strategy (MESMS) by considering multi-physics coupling with various dynamics and uncertainties to improve complex systems' dynamic reliability analysis. They adopted extremum thought for handling the dynamic process of input parameters and output response. To extract efficient samples and enhance dynamic reliability estimation efficiency, they used the importance sampling method. They used the moving least square (MLS) method to establish a more precise surrogate model by selecting good training samples. They validated MESMS by dynamic reliability analysis of turbine blisk radial deformation as a case study.

Increasing the reliability and performance of complex dynamic systems is a critical issue. Assembly relationships during the operation of dynamic systems require analysis of many components and their multi-discipline interactions. Distributed collaborative improved support-vector regression (DCISR) method was proposed by Fei *et al.* [9] to improve assembly relationship design. They developed a multilevel nested model to effectively perform the assembly relationship's reliability-based design optimization (RBDO). They developed improved support-vector regression as the basis function of the DCISR for reliability analysis. To find the optimal model parameters, they adopted the multi-population genetic algorithm (MPGA). They considered a multilevel nested model of the assembly relationship in the DCISR method as an optimization model. As a case study, they applied their proposed method for RBDO of turbine blade-tip running clearance.

Dynamic probabilistic analysis of the multi-component structures requires substantial simulations that are time-consuming. Using surrogate-based methods is an approach to overcome this simulation's overhead. Kriging is one of the well-known Surrogate Modeling Techniques. Lu *et al.* [10] proposed an improved decomposed-coordinated Kriging modeling strategy (IDCKMS) for improving the modeling efficiency and simulations' performance. They integrated decomposed coordinated (DC) strategy, extremum response surface method (ERSM), and genetic algorithm (GA) in the Kriging surrogate model. They used The GA to resolve the maximum-likelihood equation for computing the optimal values of the Kriging hyperparameter. ERSM is used for resolving the response process of outputs in the Kriging surrogate model. They sued DC strategy to coordinate the output responses of analytical objectives. They accomplished probabilistic analysis of an aeroengine high-pressure turbine blisk with blade and disk as a case study.

To improve simulation efficiency and the modeling precision and in the flexible mechanism reliability evaluation, Fei *et al.* [11] proposed an enhanced network learning method (ENLM). They introduced a multi-population genetic algorithm (MPGA) and generalized regression neural network (GRNN) into the extremum response surface method (ERSM). They adopted ERSM for handling transient issues in motion reliability analysis. They applied GRNN to overcome high-nonlinearity in surrogate modeling and used MPGA to find the optimal model parameters. As a case study, they evaluated the motion reliability of a two-link flexible robot manipulator.

Also, RBD is used in computer science to analyze the reliability of software and hardware systems. Researchers have been used the RBD method in a lot of works. Evaluating the smart grid substation's reliability analysis while considering its dynamic behavior is accomplished in [12]. Bein *et al.* [13] have modelled a fault-tolerant WSN system using the series-parallel RBD configuration. Shaikh *et al.* [14] have presented the wireless sensor network's data transmission process and modelled it using the parallel-series configuration of RBD. A communication protocol is presented and modelled using a series-parallel setting RBD [15]. A stochastic Petri Nets based RBD analysis of enterprise network is performed for reliability evaluation [16]. The reliability and availability analysis of the enterprise network is implemented using the RBD technique, where the RBD models are solved through Petri nets for analysis [17], [18].

One of the application areas of RBD is evaluating the reliability of a Redundant Array of Independent Disks (RAID) system. For RAID systems with 12 disks arranged in the 9-out-of-12 structure, third-degree fault tolerance is claimed in [19]. They proved their claim using RBD. Bistouni and Jahanshahi [20] presented a methodology. They used it to model the combination of series and parallel components RBDs instead of using estimated upper and lower bound RBDs. These are just a few of the thousands of works that accomplished using RBD. RBD is a commonly used tool for engineers and scientists. In this paper, we propose a fast and straightforward formula to calculate the KooN systems' reliability.

II. BASIC METHODS

A. BASICS OF RBD

RBD consists of blocks and connector lines. The blocks generally represent the system components, and the connector lines represent the functional relationships between them. One endpoint (usually drawn at the left) represents the start point of the process, and the other endpoint represents the end of the process. The system works if at least one path of properly functioning components from start to endpoint exists; otherwise, the system is failed [18]. Yang *et al.* [21] initially have used RBD-based models for reliability analysis of a hardware system that presents how the system's components are reliability-wise connected. Besides RBD, the Fault Tree (FT) is another widely used known method for computing a system's reliability. Each of RBD and FT is appropriate for calculating the reliability of the specific type of systems. Computing reliability of the system in case of

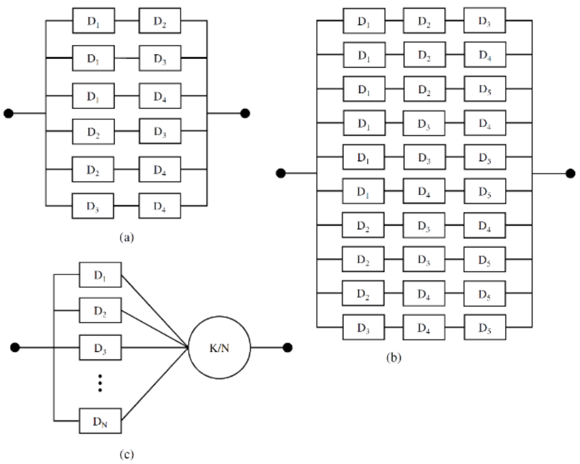

FIGURE 1. (a) RBD of 2oo4 system, (b) RBD of 3oo5 system, (c) General diagram of the KooN system.

repeated events is slightly different from the condition that no repeated events exist in a system. Except for the 1ooN and NooN redundant systems, all KooN systems have repeated events. Figure 1 illustrates some of the KooN structures. Previously known methods for calculating the reliability of RBD system with repeated events (duplicate nodes) are as follows:

1) Factorizing algorithm [1]
2) Sum of disjoint products (SDP) [1], [3]
3) Binary Decision Diagrams (BDD) algorithm [1]
4) Using (1) and simplifying it [1], [22]

$$R_{(k\,|\,n)} = 1 - \sum\nolimits_{|i|>k} \left(\prod\nolimits_{i\in I}(1-R_i)\right)\left(\prod\nolimits_{i\notin I} R_i\right) \quad (1)$$

where R_i denotes the reliability of the i^{th} component of a system that consists of $I = 1, \ldots, n$ components. The reliability of an element is the probability of the correct functioning. Index i ranges on all choices $i_1 < i_2 < \ldots < i_m$ such that $k \le m \le n$.

5) Using (2) and simplifying it [18]

$$R_{k|n} = \sum\nolimits_{i=k}^{n} \left(\binom{n}{k} R^i (1-R)^{n-1}\right) \quad (2)$$

6) Using (3) and simplifying it [2]

$$R_{k|n} = 1 - \prod\nolimits_{i=1}^{n} (1-\rho_i(x)) \quad (3)$$

7) Using Boolean Truth Table (BTT) [1]

Each of the mentioned methods has some limitations that are as follows:

Reliability formulas are probability formulas in algebraic form. A more widely used format of algebraic formulas is Sum of the Products (SoP). All of the known methods require a lot of tedious simplification of initial formulas to convert them to the form of SoP. Methods 1 up to 3 do not give a final simplified formula in the form of SoP. They represent approaches that need a lot of simplification to be performed by the person how wants to gain an algebraic formula in the form of SoP.

TABLE 1. The boolean truth table of 2oo4 and 3oo4 systems.

R_1	R_2	R_3	R_4	2oo4	3004
0	0	0	0	0	0
0	0	0	1	0	0
0	0	1	0	0	0
0	0	1	1	1	0
0	1	0	0	0	0
0	1	0	1	1	0
0	1	1	0	1	0
0	1	1	1	1	1
1	0	0	0	0	0
1	0	0	1	1	0
1	0	1	0	1	0
1	0	1	1	1	1
1	1	0	0	1	0
1	1	0	1	1	1
1	1	1	0	1	1
1	1	1	1	1	1

Methods 4, 6, and 7 are more straightforward than methods 1 to 3 in this regard. Method 5 is very simplified and can be used only for the particular case that all system components' reliability is the same.

Table 1 shows BTT for computing the Reliability of 2oo4 and 3004 system. Computing reliability of 2oo4 system using BTT is presented respectively in (4) and (5).

$$\begin{aligned}
R_{(2OO4)} &= R_1R_2R_3R_4 + R_1R_2R_3(1-R_4) \\
&\quad + R_1R_2R_4(1-R_3) + R_1R_3R_4(1-R_2) \\
&\quad + R_2R_3R_4(1-R_1) + R_1R_2(1-R_3)(1-R_4) \\
&\quad + R_1R_3(1-R_2)(1-R_4) \\
&\quad + R_1R_4(1-R_2)(1-R_3) \\
&\quad + R_2R_3(1-R_1)(1-R_4) \\
&\quad + R_2R_4(1-R_1)(1-R_3) \\
&\quad + R_3R_4(1-R_1)(1-R_2) \qquad (4) \\
&= R_1R_2 + R_1R_3 + R_1R_4 + R_2R_3 + R_2R_4 + R_3R_4 \\
&\quad - 2(R_1R_2R_3) - 2(R_1R_2R_4) \\
&\quad - 2(R_1R_3R_4) - 2(R_2R_3R_4) + 3(R_1R_2R_3R_4) \qquad (5)
\end{aligned}$$

Using BTT requires a lot of simplifications to convert its formula in the form of SoP. It is appropriate when only numerically computing the reliability of a system is needed.

B. THE VENN DIAGRAM APPROACH

Venn diagram is a well-known scheme that helps computing probability formulas, especially the KooN systems' reliability. Figure 2 shows a 3-set Venn diagram that we used for calculating the reliability of 2oo3 system. The system works if at least two different components of it operate correctly.

The probability of the coloured area in Fig. 2 represents the reliability of the 2oo3 system. Area R_1R_2 represents the case that both components 1 and 2 are functioning correctly. But this probability contains the probability of area $R_1R_2R_3$.

By considering all paired joint probability of three components in Fig. 2, we have considered three times the probability of area $R_1R_2R_3$ that we should consider only once.

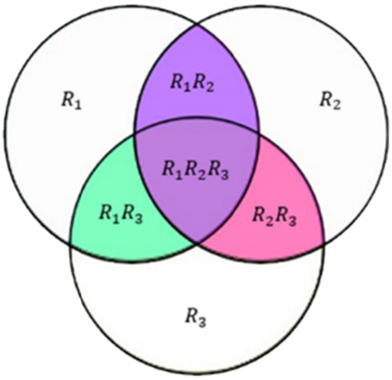

FIGURE 2. Venn diagram for computing reliability of the 2oo3 system.

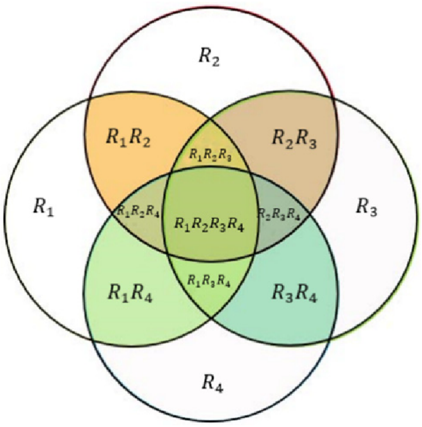

FIGURE 3. Venn diagram for computing reliability of the 2oo4 system.

Therefore the coefficient of the term $R_1R_2R_3$ in the formula must be -2 as is denoted in (6). We used this approach in this paper.

$$R_{(2oo3)} = R_1R_2 + R_1R_3 + R_2R_3 - 2\,(R_1R_2R_3) \tag{6}$$

C. A SHORTCOMING OF THE VENN DIAGRAM

Humans rarely face the case of considering upper than three-dimensional spaces in real life. Thinking and demonstrating higher than three-dimensional spaces in a single picture is difficult for some persons and may cause mistakes. Let assume that we want to compute the reliability of the 2oo4 system using the Venn diagram. Figure 3 tries to demonstrate a 4-set Venn diagram of all combinations of functioning systems components.

As is shown in Fig. 3, the areas of R_2R_4 and R_1R_3 are not visible. Because drawing a Venn diagram for extracting the reliability formula of 2oo4 system must be accomplished in 4-dimensional space. Some beginner students mistakenly drive incorrect formulas because the Venn diagram is a two-dimensional diagram that masks some of the joint probabilities if the system contains many components. The condition holds when we want to create the formula for the KooN system's reliability with larger values of N.

Equation (5) displays the formula of computing reliability of the 2oo4 system. We consider the probabilities of all colored parts of Fig. 3 and all other areas that are not visible in the Venn diagram in this formula. We regard once all 2-tire combinations of reliability of 4 components. All 3-tired combinations of reliability of 4 components appear three times in the all 2-tire combination of reliability of 4 elements. We consider precisely once the coloured area of the exact intersection of three components.

Therefore we set their coefficients as -2. Similarly, we conclude that 4-tired combinations of 4 elements must appear in (5) with coefficient +3.

Venn diagram suffers from the mentioned shortcoming. One solution is using the concept of Lattice for computing reliability of the KooN systems instead of the Venn diagram, but still following the same approach for deriving reliability formula. The next section presents the general form of the reliability formula of the KooN systems in the form of SoP before introducing the Lattice.

III. THE GENERAL FORM OF THE PROPOSED FORMULA

In the previous section, we calculated the reliability of the KooN systems of 2oo3 and 2oo4. So, we introduce a simple formula that is valid for computing the reliability of all KooN structures. We will obtain the final straightforward formula in SoP form only by substituting K, and N in this formula. In this paper, capital K and capital N are used interchangeably with small k and small n, respectively. Let assume notation for the sum of all K-elements (k-tiers) combinations of reliabilities of N components, as is shown in (7).

$$\begin{aligned}\bigwedge\nolimits_{i=1}^{n} R_i^k &= R_1R_2\ldots R_{k-1}R_k + R_1R_2\ldots R_{k-1}R_{k+1}\\ &\quad + \cdots + R_1R_2\ldots R_{k-1}R_n + \cdots\\ &\quad + R_{n-k+1}R_{n-k+1}\ldots R_{n-1}R_n\end{aligned} \tag{7}$$

Our proposed general formula for the reliability of *KooN* system using the mentioned notation is as follows:

$$\begin{aligned}R_{KooN} &= a_k^k \bigwedge\nolimits_{i=1}^{n} R_i^k\\ &\quad + a_{k+1}^{k+1}\left(\bigwedge\nolimits_{i=1}^{n} R_i^{k+1}\right)\\ &\quad + a_{k+2}^{k+2}\left(\bigwedge\nolimits_{i=1}^{n} R_i^{k+2}\right)\\ &\quad + \ldots + a_n^n\left(\bigwedge\nolimits_{i=1}^{n} R_i^n\right)\end{aligned} \tag{8}$$

The index $i \in \{1, \ldots, N\}$ varies over N components, and the k denotes the combination degree.

IV. BASICS OF PROPOSED APPROACH

A. LATTICE

A binary relation $\leq$ defined on a set A is a partial order if the following conditions hold on set A:

1) Reflexivity: $a \leq a$
2) Anti-symmetry: $a \leq b$ and $b \leq a$ imply $a = b$
3) Transitivity: $a \leq b$ and $b \leq c$ imply $a \leq c$

If besides, for every a, b in A

1) $a \leq b$ or $b \leq a$

then we say relation $\leq$ is a total order on A. A nonempty set with a partial order relation is called a partially ordered set, or more briefly, a Poset. If a relation on the set is a total

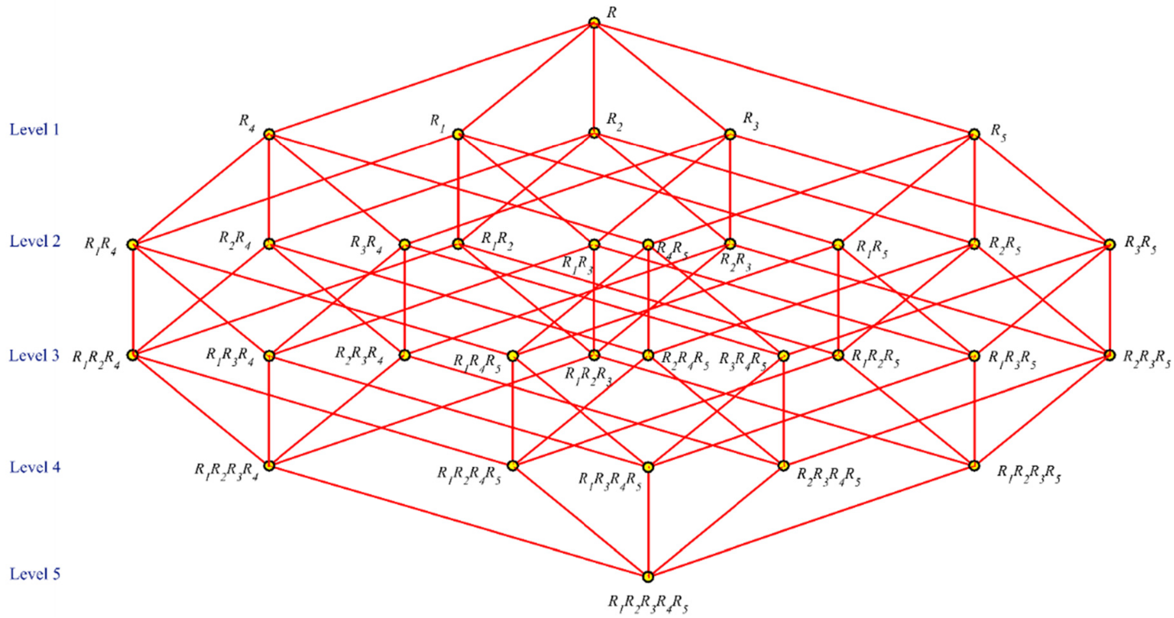

FIGURE 4. Reliabilities Lattice of a system with five components.

order, we name the set as a totally ordered, linearly ordered set or only a chain.

Let $\psi(A)$ denote the power set of A, (set of all subsets of A), then $\subseteq$ is a partial order on $\psi(A)$ [23]. In power set A, we use the expression $a \subset b$ to mean $a \subseteq b$ but $a \neq b$. Lattice is a Poset that each pair of elements of the set have a unique least upper bound (join) and a unique greatest lower bound (meet) [24], [25].

B. HASSE DIAGRAM

In order theory, Hasse is a type of mathematical diagram used to represent a finite Poset by drawing its transitive reduction. In the Hasse diagram of a Poset $(S, \leq)$ a vertex in the plane shows each element of S. A line segment or curve goes upward from x to y whenever y covers x (that is, whenever $x < y$ and there is no z such that $x < z < y$) [26]. For any set A, the power set of A can be ordered via subset inclusion to obtain a lattice. Set intersection and union can be interpreted as meet and join, respectively.

C. LATTICE OF RELIABILITY

The power set of a system's components with relation $\subseteq$ is a Lattice, and let name it as components Lattice of a system. Let assume dual of components Lattice as reliabilities Lattice. Let us replace each vertex of reliabilities Lattice with the multiplication of reliabilities of elements which are appeared in the corresponding dual node in the components Lattice. An edge of reliability Lattice represents the relation $\leq$ based on the characteristic of joint probabilities.

Figure 4 illustrates the reliability Lattice of a system with five elements $\{1, \cdots, 5\}$ which R_i denotes the reliability of each component i. Level i contains all subsets with norm i, and we use it for driving all formulas of the Koo5 structure. Hasse diagram of reliabilities Lattice of a system consisting of N components is a suitable replacement of the Venn diagram to drive the reliability formula. Lattice contains all required concepts for driving the reliability formula of the KooN system.

V. ALGORITHM OF COMPUTING COEFFICIENTS OF THE PROPOSED FORMULA

For example, in level 3 of Fig. 4, we see that each 3-tire combination of the N components' reliability is included in precisely three 2-tired combinations of elements reliability in level 2 and contains precisely two 4-tired combinations of reliability of components. Based on the concept of Lattice, we conclude that each i-tired combination of reliabilities of N components in level i is included (contained) in $\binom{i}{j}$ number of j-tired combinations of components in upper-level j that j is smaller than i as is shown in Fig 4.

Based on this property of Lattice, and following the approach that we used with the Venn diagram, Table 2 shows the process of producing coefficients of terms in the equation of reliability of KooN system of general proposed formula (7). Notation r denotes the row number, and c indicates column number and a_c^r denotes the value of the element of rth row and cth column of the table. If we consider this table as a squared matrix, we aim to compute the main diagonal elements. Element a_{k+i}^{k+i} denotes coefficients of terms in the general proposed formula (7) that are the combination of $k+i$ elements of the system's components reliabilities.

The process of filling the table starts from the top row $r = k$ to the last row $r = n$ and for each row, from the left column $c = k$ up to the column $c = r$. We compute only the lower triangle of the table.

For computing reliability of the KooN system, we consider once all k-tire combination of reliabilities of components,

TABLE 2. Tabular representation for computing a_c^r.

		c							
		k	$k+1$	$k+2$	$\cdots$	$k+i$	$k+i+1$	$\cdots$	n
r	k	1							
	$k+1$	$a_k^k\binom{k+1}{k}$	$1-a_k^{k+1}$						
	$k+2$	$a_k^k\binom{k+2}{k}$	$a_{k+1}^{k+1}\binom{k+2}{k+1}$	$1-(a_{k+1}^{k+2}+a_k^{k+2})$					
	$\vdots$	$\vdots$	$\vdots$	$\vdots$					
	$k+i$	$a_k^k\binom{k+i}{k}$	$a_{k+1}^{k+1}\binom{k+i}{k+1}$	$a_{k+2}^{k+2}\binom{k+i}{k+2}$	$\cdots$	$1-\sum_{j=k}^{k+i-1}a_j^{k+i}$			
	$k+i+1$	$a_k^k\binom{k+i+1}{k}$	$a_{k+1}^{k+1}\binom{k+i+1}{k+1}$	$a_{k+2}^{k+2}\binom{k+i+1}{k+2}$	$\cdots$	$a_{k+i}^{k+i}\binom{k+i+1}{k+i}$	$1-\sum_{j=k}^{k+i}a_j^{k+i+1}$		
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$		
	n	$a_k^k\binom{n}{k}$	$a_{k+1}^{k+1}\binom{n}{k+1}$	$a_{k+2}^{k+2}\binom{n}{k+2}$	$\cdots$	$a_{k+i}^{k+i}\binom{n}{k+i}$	$a_{k+i+1}^{k+i+1}\binom{n}{k+i+1}$	$\cdots$	$1-\sum_{j=k}^{n-1}a_j^n$

so a_k^k always set to 1, as is shown in Table 1. Hence, the first column $c = k$ of the first-row $r = k$ always fills with 1. Following our mentioned approach, we consider only once the probability of each subarea in the Venn diagram. In the next row $r = k+1$, we know that each k-tired combinations of components reliabilities in upper-level k contains a k+1-tired combination $\binom{k+1}{k}$ times. Therefore, each of the k-tired combinations appears $a_k^k\binom{k+1}{k}$ times that we denote as a_k^{k+1}. To guarantee that each of the k-tired combinations only will be considered once in computing total system reliability, it must appear with coefficient $1 - a_k^k\binom{k+1}{k}$.

From the second-row $r = k + 1$ up to the last-row $c = n$, we fill columns of each row with the same approach. The first column $c = k$ up to the one column before diameter $c = k+i-1$ of row $r = k+i$ from left to right will be filled with the value $a_c^c\binom{r}{c}$. The last column of this row $c = k + i$ fills with value one minus sum of all previous row elements. The diameter in row $r = k + i$ is computed based on all earlier elements of the main diagonal of upper rows. Therefore we will have a recursive formula (in the recurrent form) in the main diagonal elements of table 2. Algorithms 1 represents steps of filling Table 2 in detail.

As the first example, Table 3 shows the calculation of the coefficients of the reliability formula of 2oo5 system based on algorithm 1. By substituting computed coefficients in the primary diameter of Table 3 in (8), we drive the formula of reliability of the 2oo5 system as the following formula in the form of SoP.

$$R_{2oo5} = \bigwedge_{i=1}^{5} R_i^2 - 2\bigwedge_{i=1}^{5} R_i^3 + 3\bigwedge_{i=1}^{5} 4 - 4\bigwedge_{i=1}^{5} R_i^5 \quad (9)$$

As another example, Table 4 shows the computation of the reliability formula's coefficients for the 3oo7 system using algorithm 1. Algorithm 1 shows the steps of computing items in table 2.

By substituting computed coefficients in the primary diameter of Table 4 in (8), we drive the reliability of the 3oo7 system as the following formula in the form of SoP.

Algorithm 1 Computing Coefficients of the Sum of Products for the KooN System

Input: Total number of system elements (n), Minimum number of required functioning elements of the system (k)
Output: all $\mathbf{a_c^r}$ of table 2
for r:= k **to** n **do**
  for c:= k **to** r **do**
    if r = k **then**
      $\mathbf{a_c^r \longleftarrow 1}$
    else
      if c < r **then**
        $\mathbf{a_c^r \longleftarrow a_c^c \binom{r}{c}}$
      else
        $\mathbf{a_c^r \longleftarrow 1 - \sum_{i=k}^{c-1} a_i^r}$
      end if
    end if
  end for
end for
end Algorithm

$$R_{3oo7} = \bigwedge_{i=1}^{7} R_i^3 - 3\bigwedge_{i=1}^{7} R_i^4 + 6\bigwedge_{i=1}^{7} R_i^5 - 10\bigwedge_{i=1}^{7} R_i^6 + 15\bigwedge_{i=1}^{7} R_i^7 \quad (10)$$

The computational and space complexity of algorithm 1 is:

$$(n-k+1)(n-k+2)/2 \quad (11)$$

that n is the number of different components (diversity of components) of the system, and k denotes the number of required functioning components.

VI. INDUCTIVE PROOF OF THE PROPOSED FORMULA

Computing coefficients of the formula of reliability of the KooN system by using algorithm 1 is a bit time-consuming. In this part of the paper, we drive a simple formula for computing coefficients of the reliability formula of the KooN system in the form of SoP.

TABLE 3. Coefficients of the formula of the 2oo5 system.

		c			
		2	3	4	5
r	2	+1			
	3	+3	-2		
	4	+6	-8	+3	
	5	+10	-20	+15	-4

TABLE 4. Coefficients of the formula of the 3oo7 system.

		c				
		3	4	5	6	7
r	3	+1				
	4	+4	-3			
	5	+10	-15	+6		
	6	+20	-45	+36	-10	
	7	+35	-105	+126	-70	+15

Lemma 1. The following formula holds:

$$\sum_{i=0}^{n} \frac{(-1)^i \binom{n}{i}}{k+i} = \frac{1}{k\binom{k+n}{k}} \tag{12}$$

Proof: Based on the binomial theorem, algebraic expansion of powers of a binomial expression is as follows:

$$(-1+x)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} x^i \tag{13}$$

Let multiply both sides of equation (13) by x^{k-1} as follows:

$$(x-1)^n x^{k-1} = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} x^{i+k-1} \tag{14}$$

Let integrate from both sides of (14) as follows:

$$\int_0^1 (x-1)^n x^{k-1} dx = \int_0^1 \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} x^{i+k-1} dx \tag{15}$$

Definition of Beta function (Euler integral of the first kind) is as follows:

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \frac{(m-1)!(n-1)!}{(m+n-1)!} \tag{16}$$

Based on the definition of the Beta function, equation (15) can be rewritten as follows [27]:

$$(-1)^n B(k,n+1) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \int_0^1 x^{i+k-1} dx \tag{17}$$

After computing the integral on the right-hand side of (17), we can rewrite it as follows:

$$(-1)^n B(k,n+1) = \sum_{i=0}^{n} \frac{(-1)^{n-i}\binom{n}{i}}{k+i} \tag{18}$$

After multiplying both sides of (18) with $(-1)^{-n}$ we will have the following formula:

$$B(k,n+1) = \sum_{i=0}^{n} \frac{(-1)^{-i}\binom{n}{i}}{k+i} \tag{19}$$

Based on the definition of Beta function in equation (16), we can rewrite the left-hand side of (18) as follows:

$$\frac{\Gamma(n+1)\Gamma(k)}{\Gamma(n+k+1)} = \sum_{i=0}^{n} \frac{(-1)^{-i}\binom{n}{i}}{k+i} \tag{20}$$

After simplification of the left side of (20), we can write it as follows:

$$\frac{1}{k\binom{k+n}{k}} = \sum_{i=0}^{n} \frac{(-1)^{i}\binom{n}{i}}{k+1} \tag{21}$$

Theorem 1. The answer of following recurrent relation

$$a_{k+i}^{k+i} = 1 - \left(\sum_{r=k}^{k+i-1} a_r^{k+i} \binom{k+i}{r}\right) \tag{22}$$

is as follows:

$$a_{k+i}^{k+i} = (-1)^i \binom{k+i-1}{i}, i \in [0, 1, \ldots, (n-k)] \tag{23}$$

Proof: We prove this relation by induction.

Case 1: By substituting value $i = 0$ in both (22) and (23), value of $a_k^k = 1$ yields.

Case 2: Let assume that for all $i \geq 0$, the answers of (22) are upon (23). Now we want to prove that based on the recurrent (22) value of coefficient with index $k+i+1$ based on (23) is as follows:

$$a_{k+i+1}^{k+i+1} = (-1)^{i+1} \binom{k+i}{i+1} \tag{24}$$

Recurrent (22) for the case $k+i+1$ is as follows:

$$a_{k+i+1}^{k+i+1} = 1 - \left(\sum_{r=k}^{k+i} a_r^{k+i+1} \binom{k+i+1}{r}\right) \tag{25}$$

Let assume expansion of the right side of (25) based on (23) as follows:

$$\begin{aligned} a_{k+i+1}^{k+i+1} = 1 - \Bigg(&\binom{k+i+1}{k} - k\binom{k+i+1}{k+1} + \frac{k(k+1)}{2!}\binom{k+i+1}{k+2} - \frac{(k+2)(k+1)k}{3!} \\ &+ \ldots + (-1)^j \frac{(k+j-1)(k+j-2)\ldots k}{j!} \times \binom{k+i+1}{k+j} + \ldots + (-1)^i \\ &\times \frac{(k+i-1)(k+i-2)\ldots k}{i!}\binom{k+i+1}{k+i}\Bigg) \end{aligned} \tag{26}$$

$$\begin{aligned} = 1 - \frac{(k+i+1)!}{k!}\Bigg(&\frac{1}{0!(i+1)!} - \frac{k}{1!i!(k+1)} + \frac{k(k+1)}{2!(i-1)!(k+2)(k+1)} + \ldots + (-1)^j \\ &\times \frac{(k+j-1)(k+j-2)\ldots k}{j!(i+1-j)!(k+j)\ldots(k+1)} + \ldots + (-1)^i \\ &\times \frac{(k+i-1)(k+i-2)\ldots k}{i!1!(k+i)(k+i-1)\ldots(k+1)}\Bigg) \end{aligned} \tag{27}$$

$$= 1 - \frac{(k+i+1)!}{k!}\left(\frac{k}{0!\,(i+1)!k} - \frac{k}{1!i!\,(k+1)} + \frac{k}{2!\,(i-1)!\,(k+2)} + \ldots + (-1)^j \frac{k}{j!\,(i+1-j)!\,(k+j)} + \ldots + (-1)^i \frac{k}{i!1!\,(k+i)}\right) \quad (28)$$

$$= 1 - \frac{(k+i+1)!}{(k-1)!}\left(\frac{1}{0!\,(i+1)!k} - \frac{1}{1!i!\,(k+1)} + \frac{1}{2!\,(i-1)!\,(k+2)} + \ldots + \frac{(-1)^j}{j!\,(i+1-j)!\,(k+j)} + \ldots + \frac{(-1)^i}{i!1!\,(k+i)}\right) \quad (29)$$

$$= 1 - \frac{(k+i+1)!}{(k-1)!}\left(\sum_{r=0}^{i} \frac{(-1)^r}{r!\,(i+1-r)!\,(k+r)}\right) \quad (30)$$

After simplification of the right hand of (30), the following relation holds:

$$a_{k+i+1}^{k+i+1} = 1 - \frac{(k+i+1)!}{(k-1)!\,(i+1)!}\left(\sum\nolimits_{r=0}^{i} \frac{(-1)^r \binom{i+1}{r}}{k+r}\right) \quad (31)$$

Based on Lemma 1, we have the following formula:

$$\sum\nolimits_{r=0}^{i+1} \frac{(-1)^r \binom{i+1}{r}}{k+r} = \frac{1}{k\binom{k+i+1}{r}} \quad (32)$$

Let extract the last sentence of sigma in the above relation as follows:

$$\sum\nolimits_{r=0}^{i} \frac{(-1)^r \binom{i+1}{r}}{k+r} + \frac{(-1)^{i+1}\binom{i+1}{i+1}}{k+i+1} = \frac{1}{k\binom{k+i+1}{r}} \quad (33)$$

We can rewrite (33) as follows:

$$\sum\nolimits_{r=0}^{i} \frac{(-1)^r \binom{i+1}{r}}{k+r} = \frac{1}{k\binom{k+i+1}{k}} + \frac{(-1)^i}{k+i+1} \quad (34)$$

By replacing the expression inside the rightmost parentheses of (31) based on (34), we can simplify it as follows:

$$a_{k+i+1}^{k+i+1} = 1 - \frac{(k+i+1)!}{(k-1)!\,(i+1)!}\left(\frac{1}{k\binom{k+i+1}{k}} + \frac{(-1)^i}{k+i+1}\right) \quad (35)$$

$$= 1 - \left(\frac{(k+i+1)!}{(k-1)!\,(i+1)!} \times \frac{k!\,(i+1)!}{k \times (k+i+1)!} + \frac{(k+i+1)!}{(k-1)!\,(i+1)!} \times \frac{(-1)^i}{k+i+1}\right) \quad (36)$$

$$= 1 - \left(1 + \frac{(k+i+1)!\,(-1)^i}{(k-1)!\,(i+1)!(k+i+1)}\right) \quad (37)$$

$$= \frac{(-1)^{i+1}\,(k+i)!}{(k-1)!\,(i+1)!} \quad (38)$$

$$= (-1)^{i+1}\binom{k+i}{i+1} \quad (39)$$

The result is the same as the right-hand side of (24), and the proof is complete.

Based on Theorem 1, we can summarize the general formula of reliability of the KooN that is presented in (8) as follows:

$$R_{KooN} = \sum\nolimits_{i=0}^{n-k} (-1)^i \binom{k+i-1}{i} \bigwedge\nolimits_{j=1}^{n} R_j^{k+i} \quad (40)$$

Computational and space complexity of computing all of the coefficients of reliability formula of KooN system using (40) reduces to $n-k+1$.

VII. CONCLUSION AND FUTURE WORK

Reliability evaluation is one of the critical requirements in the design of fault-tolerant and reliable systems. As one of the famous and abstract reliability modeling methods, RBD is widely used between engineers of different majors. One of the known and basic methods of increasing fault-tolerance and the systems' reliability is increasing the redundancy of basic functioning components. KooN is one of the widely used configurations of systems' redundancy in some of the dependable systems. The main topic of this article is simplifying the reliability evaluation of KooN redundant systems modeled by RDB. Many well-known methods and formulas for computing various systems' reliability with the different operational relationships between its components exist.

This paper presented a technique for extracting simple formula for calculating the KooN systems' reliability in SoP form using the Venn diagram. Some engineering students who do not have the necessary experience working in dimensional space higher than three are the source of some mistakes. A Venn diagram is a two-dimensional diagram that some of the relationships between independent random variables in higher dimensions will be masked. We clarified this shortcoming of the Venn diagram at first. This weakness caused that at the second step, we replaced the Venn diagram with Lattice to demonstrate all inclusion relationships between combinations of components reliabilities. In the third step, we introduced the Lattice of reliabilities that is dual of power set Lattice of components.

In the fourth step, we presented a method for driving a general formula of the KooN system's reliability using the basic properties of Lattice of reliabilities and their inclusion relationships. In the fifth step, we proposed an algorithm that presents a squared table structure that, by computing elements of the main diagonal and elements below it based on Lattice's properties, we can compute the SoP formula's coefficients. The computational and space complexity of the proposed algorithm is $\theta((\text{n-k})^2/2)$ that n is the number of different components and k denotes the number of required functioning components of the system. At last, we provide a lemma and theorem with their proof. Using this theorem, we presented a general formula for computing coefficients of the SoP formula of the KooN system without using the proposed algorithm and computing the table's required elements. Computational and space complexity of computing all of the coefficients of reliability formula of KooN system using this formula reduced to $\theta(n-k)$.

This paper gives a straightforward and concise SoP formula for the KooN system's reliability. The drawback of the proposed formula is that it is limited to compute only the KooN systems' reliability. The advantage of using the proposed method and formula in this paper are as follows: 1) The researcher generally considers the relationships between the reliabilities of the system's components using the Venn diagram. Using the Venn diagram is cumbersome and potentially is error-prone. 2) Our proposed method decreases the chance of mistakes in the computation of reliability of KooN systems. This formula eases the process of driving the formula for the reliability of the KooN systems. 3) Deriving SoP formula using our proposed method is effortless and short.4) In the proposed method, users can determine the system's reliability without considering and Venn or Lattice diagram and can accomplish it only by knowing K and N's value. 5) Using our proposed SoP formula of the reliability of KooN system simplifies the design of dedicated hardware for this purpose.

As future work, we design a novel rule-of-thumb method for driving the simplified SoP reliability formula of the KooN system. This method can be used to calculate the reliability of KooN and all RBD structures. We intend to use the result of this paper as the correctness-evidence of our future proposed method.

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