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Adaptive Predefined-Time Synchronization of Two Different Fractional-Order Chaotic Systems With Time-Delay

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ABSTRACT This paper devotes to the adaptive globally synchronization within predefined-time of two time-delayed fractional-order chaotic systems. Firstly, through fractional calculus, two novel different fractional-order systems with time-delay are proposed, whose convergence is guaranteed and phase trajectory is given. Secondly, by exploiting the non-negative Lyapunov function and inequality theorem, a novel global predefined-time stability theorem is proposed, which can ensure the settling time tunable. And the upper bound of the settling time estimation is more accurate compared with the classical results. With the help of novel predefined-time stability theorem, two active controllers are designed, namely the fixed-time synchronization controller and predefined-time synchronization controller, to achieve the fixed-time synchronization and the predefined-time synchronization of two different time-delayed fractional-order chaotic systems respectively. Finally, several numerical simulations are presented in order to show the effectiveness of the proposed methods.

INDEX TERMS Adaptive controller, fixed-time synchronization, nonlinear fractional-order systems, predefined-time synchronization.

I. INTRODUCTION

The system synchronization plays a significant role in control fields and industrial applications, especially in those situations where fractional calculus are demanded, such as secure communication [1], [2], complex neural networks [3], [4] and automatic control [5], [6]. Fractional calculus has a history of more than 300 years, which has been widely applied to physics, engineering and control system in recent years [7]–[10]. As a generalization of ordinary differentiation and integration to arbitrary order, fractional systems can be used to accurately describe system in many interdisciplinary fields [11]–[13]. At present, chaotic behavior of fractional-order nonlinear systems is a pervasive phenomenon. Meanwhile, numerous fractional-order systems display dynamical chaotic behavior obviously, which have been researched by many scholars [1], [14]–[17]. Fractional-order chaotic system have plenty of forms, including fractional-order Chen system, fractional-order Liu system, fractional-order Ameodo system, fractional-order Lorenz system, fractional-order Chua system, etc [7], [18], [19]. So far, various research is devoted to the properties of fractional-order chaotic system, and the characteristics of system synchronization have been found. After the pioneering research about chaotic system synchronization was published by Pecora and Carroll in [20], the synchronization of chaotic system gradually became the research focus [21]–[24]. In 1953, the definition of finite-time stability was proposed by Kamenkov [25]. According to the generalized Lyapunov function and the finite-time stability theorem, Ref. [26] designed an adaptive and state-feedback controller to investigate the finite-time stabilization problem of fractional-order chaotic system. The terminal sliding modes control function was designed in [1] to study the

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finite-time synchronization of fractional-order Chua system. Ref. [27] provided a feedback controller to analyse the finite-time synchronization of fractional-order time-delayed system.

Considering the disadvantage of finite-time stability that the convergence time is related to the initial state. Whilst, it's difficult to accurately obtain the initial values in practical systems. In 2012, fixed-time stability was put forward by Andrey [28] to solve this problem. Adopting to the fixed-time stability theorem in [14], [28], Ref. [29] studied the fixed-time synchronization of memristor neural networks, and the system could converge without considering the initial values. By using the properties of Lyapunov function and Weiner process, some fixed-time synchronization standards were obtained in [30]. Ref. [31] designed a fractional-order terminal sliding mode control function to achieve the fractional-order chaotic system stability within fixed-time.

Unfortunately, the main drawback of fixed-time stability is that the relationship between the convergence time and the parameters of drive-response systems are not explicit. Thus, it is desired that the maximum stabilization time, also called the settling time, of system synchronization can be determined. Then, Sanchez-Torres gave the definition of predefined-time stability in [32], whose upper bound of the settling time can be obtained by tuning parameters [10], [23]. Further, a series of dynamic chaotic systems with the settling time in predefined-time were proposed [17], [23], [33]. In Ref. [23], an active controller of predefined-time sliding mode synchronization was designed. For achieving predefined-time sliding mode synchronization, Ref. [33] proposed a novel formulation and designed a series of controllers. To enable the synchronization of drive-response systems within predefined-time, the modified function of hyperchaotic systems was investigated and an approaching sliding synchronization controller was designed in [34].

Additionally, it has been found that time-delay exists in many synchronization systems and practical applications in engineering. However, time-delay is an undesirable phenomenon in synchronization system, and effective controller is expected to suppress it. In this respect, the effects of time-delays in synchronization system have been examined in the literature [13], [35], [36]. By exploiting the finite integral of the past control values, Ref. [37] designed a delay compensation feedback term to suppress the time-delay. By designing three synchronization controllers, Ref. [4] achieved the synchronization of multilayer networks with time-delay.

Inspired by aforementioned analyses, this paper will concentrate on the adaptive predefined-time synchronization of two different fractional-order time-delayed systems. The innovative points of this paper summarize as follows.

(1) Novel fractional-order drive system and response system with time-delay are proposed. Meanwhile, the phase trajectories are given.
(2) A fixed-time synchronization controller is designed to enforce the drive-response system convergence within the upper bound of the settling time.
(3) The predefined-time synchronization controller is provided to ensure the settling time tunable.
(4) Under the circumstances of diverse initial values, the effectiveness of predefined-time stability theorem is proved by several numerical simulations.

The rest of this paper is organized as follows. Several common definitions and lemmas are presented in Section II. Novel fractional-order time-delayed chaotic systems and corresponding phase trajectories are provided in Section III. Section IV proposes a new predefined-time stability approach of fractional-order time-delayed chaotic system. Section V design the controllers of fixed-time synchronization and predefined-time synchronization. Section VI give the numerical simulation examples to testify the effectiveness of predefined-time stability theorems. Finally, several valuable conclusions are drawn in Section VII.

II. PRELIMINARIES

In this section, useful definitions and lemmas of fractional-order chaotic system are proposed, which are essential for the design of the synchronization controller.

Definition 1 ([7]): The fractional integral of $h(t)$ is defined as

$$I^q h(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} h(s) ds,$$

where $q > 0$ is the order of integral; t_0 is the initial time and $t > t_0$; $h(t) : (0, +\infty) \rightarrow R$ and $\Gamma(\cdot)$ is the generalized Euler's Gamma function

$$\Gamma(q) = \int_0^{\infty} t^{q-1} e^{-t} dt.$$

Definition 2 ([38], [39]): There are three differential operators of the fractional-order derivative: Caputo, Riemann-Liouville and Grunwald-Letnikov. The definition of Caputo derivative for $h(t)$ is given as following

$${}_{t_0}^{C}D_t^{\beta} h(t) = \begin{cases} \dfrac{1}{\Gamma(n-\beta)} \int_{t_0}^{t} \dfrac{h^{(n)}(s)}{(t-s)^{\beta-n+1}} ds, & n-1 < \beta < n, \\ \dfrac{d^n h(t)}{dt^n}, & \beta = n, \end{cases}$$

where $\beta > 0$ is the order of derivative, t_0 is the initial time and $t \geq t_0$; n is integer and $n-1 < \beta \leq n$.

The definition of βth-order Riemann-Liouville derivative is given below

$${}_{t_0}^{RL}D_t^{\beta} h(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dt}\right)^n \int_{t_0}^{t} \frac{h(s)}{(t-s)^{\beta-n+1}} ds.$$

The βth-order Grunwald-Letnikov definition is described by

$${}_{t_0}^{GL}D_t^{\beta} h(t) = \lim_{a \to 0} a^{-\beta} \sum_{i=0}^{[(t-t_0)/a]} \binom{-\beta}{i} h(t-ia).$$

In the rest of this paper, the Caputo fractional-order derivative is employed. For simplifying, the αth-order Caputo fractional-order derivative denote as D^{α}.

Due to the research on fractional calculus, some commonly properties are provided to analyse the fractional-order chaotic system.

Lemma 1 ([40]): For the Caputo fractional derivative and Riemann-Liouville fractional derivative, an equation holds that

$$ {}_{t_0}D_t^{\alpha}\left({}_{t_0}D_t^{m}f(t)\right) = {}_{t_0}D_t^{\alpha+m}f(t), $$

where $m \in N$ and $n-1 < \alpha < n \in N^{+}$.

Lemma 2 ([7]): If $f(t) \in C^{m}[0,\infty)$, $n-1 < q < n \in N^{+}$, then

$$ {}_{t_0}D_t^{q}\,{}_{t_0}D_t^{-q}f(t) = f(t). $$

Lemma 3 ([3]): Fractional derivative and integer derivative have similar linear characteristic, for any constant γ and δ, the linear characteristic of Caputo fractional derivatives describes as:

$$ {}_{t_0}D_t^{\alpha}[\gamma f(t) + \delta g(t)] = \gamma\,{}_{t_0}D_t^{\alpha}f(t) + \delta\,{}_{t_0}D_t^{\alpha}g(t). $$

Lemma 4 ([41]): Consider the continuous unbounded function $y(t): R^{n} \to R_{+}$ satisfying the differential inequality

$$ \dot{y}(t) = -(\alpha y^{q}(t) + b)^{k}, \quad y(0) = y_0 \tag{1} $$

where $\alpha > 0$, $b > 0$, $q > 0$, $k > 0$ and satisfying $qk > 1$, then $y \equiv 0$, $\forall t \geq T(x_0)$. Thus $y(t)$ is fixed-time stable and the $T(x_0)$ is the settling time upper bounded, satisfying:

$$ T(x_0) \leq T_{max}^{1} \triangleq \frac{1}{b^{k}}\left(\frac{b}{\alpha}\right)^{\frac{1}{q}}\left(1 + \frac{1}{qk-1}\right). \tag{2} $$

Lemma 5 ([5]): For any real numbers $\eta_j \in R^{+}(j = 1, 2, \ldots, n)$ and $\epsilon \in R^{+}$, satisfying the following inequalities:

$$ \begin{cases} \left(\sum_{j=1}^{N} |\eta_j|\right)^{\epsilon} \leq \sum_{j=1}^{N} |\eta_j|^{\epsilon}, \; 0 < \epsilon \leq 1 \\ n^{1-\epsilon}\left(\sum_{j=1}^{N} |\eta_j|\right)^{\epsilon} \leq \sum_{j=1}^{N} |\eta_j|^{\epsilon}, \; \epsilon > 1. \end{cases} $$

III. SYSTEMS DESCRIPTION

Considering the n-dimensional dynamic system with time-delay as drive system

$$ {}_{t_0}D_t^{\alpha}x_i(t) = -\sigma_i x_i(t) + f_i(X,t) + F_i(X,t-\tau), \tag{3} $$

where $0 < \alpha < 1$ is the order of drive system; $i = 1, 2, \ldots, n$, σ_i represents the self-inhibition of drive system and $\sigma_i > 0$; $x_i(t)$ represents the state variates of drive system; $\tau > 0$ represents the time-delay term; $X = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}} \in R^{n}$ is the state vector of drive system; $f_i(\cdot)$ stands for a known nonlinear function and $F_i(\cdot)$ denote the uncertain delayed function. When $f_i(\cdot)$ is a continuous and derivable function, meanwhile, $F_i(\cdot)$ is a continuous decreased function, the system (3) can show chaotic behavior. The response system is described by

$$ {}_{t_0}D_t^{\alpha}y_i(t) = -\sigma_i y_i(t) + f_i(Y,t) + F_i(Y,t-\tau) + u_i(t), \tag{4} $$

where $0 < \alpha < 1$, $\tau > 0$, $\sigma_i > 0$ is the self-inhibition of response system; $y_i(t)$ represents the state variates of response system; $Y = (y_1(t), y_2(t), \ldots, y_n(t))^{\mathrm{T}}$ is the state vector of response system; $f_i(\cdot)$ is a known continuous function; $F_i(\cdot)$ is a uncertain delayed function; $u_i(t)$ is the control function, which will be explained later.

The initial values of derive-response systems describe as $X(0) = (x_1(0), x_2(0), \ldots, x_n(0))^{\mathrm{T}}$, $Y(0) = (y_1(0), y_2(0), \ldots, y_n(0))^{\mathrm{T}}$, respectively. If $t \in [0, \tau]$, we can know that $x_i(t-\tau) = x_i(0)$, $y_i(t-\tau) = y_i(0)$, $F_i(X, t-\tau) = F_i(X(0))$, $F_i(Y, t-\tau) = F_i(Y(0))$.

Example 1: Considering the chaotic behavior of fractional-order time-delayed drive system and fractional-order time-delayed response system. Supposing the drive system is fourth-dimensional fractional-order chaotic system, that is $i = 1, 2, 3, 4$. Let the continuous nonlinear functions be $f_1(X,t) = a(x_2(t) - x_1(t))$, $f_2(X,t) = bx_1(t) - kx_1(t)x_3(t) + x_4(t)$, $f_3(X,t) = h(x_1(t))^2 - cx_3(t) + x_4(t)$, $f_4(X,t) = -rx_2(t)$, and the delayed function be $F_i(X, t-\tau) = 0.1\tanh(x_i(t-\tau))(i = 1, 2, 3, 4)$. Then, the fractional-order time-delayed drive system can describe as

$$ \begin{cases} D^{\alpha}x_1 = -\sigma_1 x_1(t) + a\,(x_2(t) - x_1(t)) \\ +0.1\tanh(x_1(t-\tau)) \\ D^{\alpha}x_2 = -\sigma_2 x_2(t) + bx_1(t) - kx_1(t)x_3(t) + x_4(t) \\ +0.1\tanh(x_2(t-\tau)) \\ D^{\alpha}x_3 = -\sigma_3 x_3(t) + h(x_1(t))^2 - cx_3(t) + x_4(t) \\ +0.1\tanh(x_3(t-\tau)) \\ D^{\alpha}x_4 = -\sigma_4 x_4(t) - rx_2(t) + 0.1\tanh(x_4(t-\tau)). \end{cases} \tag{5} $$

Supposing the response system is fourth-dimensional, and let $f_1(Y,t) = a_1\,(y_2(t) - y_1(t)) + y_4(t)$, $f_2(Y,t) = b_1 y_1(t) - y_2(t) - y_1(t)y_3(t)$, $f_3(Y,t) = y_1(t)y_2(t) - c_1 y_3(t)$, $f_4(Y,t) = -y_2(t)y_3(t) - r_1 y_4(t)$ and $F_i(Y, t-\tau) = 0.1\tanh(y_i(t-\tau))$ $(i = 1, 2, 3, 4)$. The corresponding response system describes as following

$$ \begin{cases} D^{\alpha}y_1 = -\sigma_1 y_1(t) + a_1\,(y_2(t) - y_1(t)) + y_4(t) \\ +0.1\tanh(y_1(t-\tau)) + u_1(t) \\ D^{\alpha}y_2 = -\sigma_2 y_2(t) + b_1 y_1(t) - y_2(t) - y_1(t)y_3(t) \\ +0.1\tanh(y_2(t-\tau)) + u_2(t) \\ D^{\alpha}y_3 = -\sigma_3 y_3(t) + y_1(t)y_2(t) - c_1 y_3(t) \\ +0.1\tanh(y_3(t-\tau)) + u_3(t) \\ D^{\alpha}y_4 = -\sigma_4 y_4(t) - y_2(t)y_3(t) - r_1 y_4(t) \\ +0.1\tanh(y_4(t-\tau)) + u_4(t). \end{cases} \tag{6} $$

The system order is $\alpha = 0.82$, and the initial values of drive system and response system are $X(0) = [0.6, 0.7, 0.3, 0.4]^{\mathrm{T}}$, $Y(0) = [3, -4, 2, 2]^{\mathrm{T}}$. The system parameters are selected as following

$$ \begin{aligned} &\sigma_i = 0.1(i = 1, 2, 3, 4), \quad \tau = 0.8, \\ &a = 10, \quad b = 40, \quad c = 2.5, \quad k = 10, \quad h = 4, \\ &r = 2.5, \quad a_1 = 10, \quad b_1 = 28, \quad c_1 = 8/3, \quad r_1 = 1. \end{aligned} $$

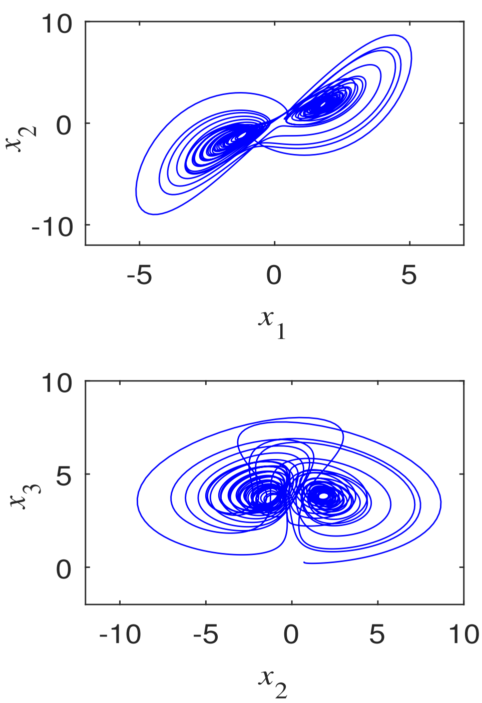

FIGURE 1. Phase portraits of drive system.

As revealed in Fig.1 and Fig.2, both the drive system (5) and response system (6) have obvious dynamic chaotic behavior. The phase portraits between different variables of fractional-order time-delayed system (5) are shown in Fig.1. The three-dimensional trajectory portrait of the response system (6) without control input $u_i(t)$ is shown in Fig.2.

IV. PREDEFINED-TIME STABILITY

In this section, a novel predefined-time stability theorem is established.

Theorem 1: *Supposing $V(\cdot) : R^n \to R_+ \cup 0$ is a continuous strictly monotonically decreased function and satisfies*

(1) $V(x) = 0 \Rightarrow x \in A$, *where $A \in R^n$ is a non-empty set and globally fixed-time attractive for system* (1)*;*

(2) $T_c \in \{r_1, \cdots, r_b\}$ *is a user defined parameter.*

(3) *For all $V(x) > 0$, there exist positive parameters α, k, b, q satisfying $q \in (0, 1]$, $qk \in (1, +\infty)$ such that*

$$\dot{V} \le -\frac{C_v}{T_c}(\alpha V^q + b)^k \tag{7}$$

where

$$C_v = \frac{1}{\alpha^{\frac{1}{q}}} \cdot \frac{1}{qk-1} \cdot b^{\frac{1-qk}{q}}.$$

Then, the system (1) *is globally predefined-time stable within predefined-time T_c.*

Proof: For all we known, when $V(x) > 0$,

$$\frac{dV}{dt} \le -\frac{C_v}{T_c} \cdot (\alpha V^q + b)^k$$
$$= -\frac{C_v}{T_c} \cdot ((\alpha^{\frac{1}{q}} V)^q + (b^{\frac{1}{q}})^q)^k.$$

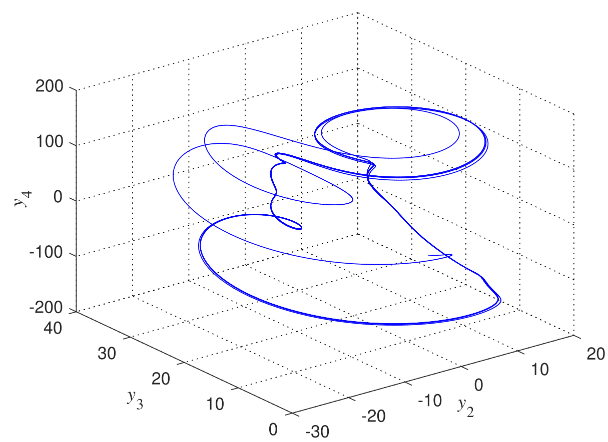

FIGURE 2. Trajectory portrait of response system without control inputs $u_i(t)$.

According to the supposing of Theorem 1, one can obtain that

$$T(x_0) = \int_0^{T(x_0)} dt$$
$$\le -\int_{V(x_0)}^0 \frac{T_c}{C_v} \cdot \frac{1}{((\alpha^{\frac{1}{q}} V)^q + (b^{\frac{1}{q}})^q)^k} dV.$$

By virtue of $q \in (0, 1)$, using Lemma 5, one has

$$(\alpha^{\frac{1}{q}} V)^q + (b^{\frac{1}{q}})^q \ge (\alpha^{\frac{1}{q}} V + b^{\frac{1}{q}})^q,$$

thus

$$T(x_0) = \int_0^{T(x_0)} dt \le \frac{T_c}{C_v}\int_0^{V(x_0)} \frac{1}{((\alpha^{\frac{1}{q}}V) + (b^{\frac{1}{q}}))^{qk}} dV$$
$$= \frac{T_c}{C_v}\frac{1}{\alpha^{\frac{1}{q}}(1\text{-}qk)}\cdot(\alpha^{\frac{1}{q}}V + b^{\frac{1}{q}})^{1\text{-}qk}\Big|_0^{V(x_0)}.$$

By virtue of $qk > 1$, $T(x_0)$ becomes

$$T(x_0) \le \frac{T_c}{C_v}\frac{1}{\alpha^{\frac{1}{p}}(1\text{-}qk)}\cdot\Big(\frac{1}{(\alpha^{\frac{1}{q}}V(x_0)+b^{\frac{1}{q}})^{qk-1}} - b^{\frac{1\text{-}qk}{q}}\Big)$$
$$= \frac{T_c}{C_v}\frac{1}{\alpha^{\frac{1}{q}}(qk-1)}\cdot\Big(b^{\frac{1\text{-}qk}{q}} - \frac{1}{(\alpha^{\frac{1}{q}}V(x_0)+b^{\frac{1}{q}})^{qk-1}}\Big).$$

If $V(x_0) = 0$, $T(x_0) = 0$ can be derived.
If $V(x_0) \to \infty$, one can derive that

$$T(x_0) \le \frac{T_b}{C_v}\frac{1}{\alpha^{\frac{1}{q}}(qk-1)}\Big(b^{\frac{1\text{-}qk}{q}} - \frac{1}{(\alpha^{\frac{1}{q}}V(x_0)+b^{\frac{1}{q}})^{qk-1}}\Big)$$
$$= \frac{T_c}{C_v}\frac{1}{\alpha^{\frac{1}{q}}(qk-1)}b^{\frac{1\text{-}qk}{q}} = T_c.$$

Therefore, T_c in Theorem 1 is a predefined-time for system (1). This proof is completed. □

Remark 1: In Theorem 1, the settling time of predefined-time stability is only related to the tuning parameter, namely the predefined-time T_c. After removing the parameters T_c and C_v, the predefined-time stability of Theorem 1 can be changed into the fixed-time stability. Then the fixed-time stability of system (1) can be achieved. And the upper bound of settling time can be indicated as $T^2_{max} = C_v$.

Theorem 2: If $p \in (0, 1]$, the settling time T^1_{max} in Lemma 4 is less precise than T^2_{max}.

Proof: Evidently, if $0 < q \le 1$

$$T^2_{max} - T^1_{max} = \frac{1}{\alpha^{\frac{1}{q}}}\cdot\frac{1}{qk-1}\cdot b^{\frac{1\text{-}qk}{q}} - \frac{1}{b^k}\Big(\frac{b}{\alpha}\Big)^{\frac{1}{q}}\Big(1+\frac{1}{qk-1}\Big)$$
$$= -\frac{1}{b^k}\Big(\frac{b}{\alpha}\Big)^{\frac{1}{q}} < 0.$$

then $T^2_{max} < T^1_{max}$. This proof is completed. □

Remark 2: Apparently, Theorem 1 provides a new proof process of predefined-time stability. Although Lyapunov function (7) is the same as that in Ref. [41], different settling time is obtained in Theorem 1. From Theorem 2, the settling time T^2_{max} obtained in Theorem 1 is more accurate than the settling time T^1_{max} in Lemma 4.

If $qk = 1$, by Lemma 5 and Theorem 1, one has

$$T(x_0)_1 \le \frac{T_c}{C_v}\frac{1}{\alpha^{\frac{1}{q}}}\cdot\ln\frac{(\alpha^{\frac{1}{q}}V(x_0)+b^{\frac{1}{q}})}{b^{\frac{1}{q}}}.$$

If $0 < qk < 1$, $T(x_0)$ satisfies

$$T(x_0)_2 \le \frac{T_c}{C_v}\frac{1}{\alpha^{\frac{1}{q}}(1-qk)}\cdot\big((\alpha^{\frac{1}{q}}V(x_0)+b^{\frac{1}{q}})^{1\text{-}qk} - b^{\frac{1\text{-}qk}{q}}\big).$$

which means that function $V(\cdot)$ converges to zero within finite-time and the convergence time is only dependent on the initial value $V(x_0)$. Therefore, the equilibrium solution of the system (1) can achieve finite-time stability when $0 < qk \le 1$. Meanwhile, it is easy to draw the following Corollary.

Corollary 1: Supposing $V(\cdot): R^n \to R_+ \cup 0$ is a continuous strictly monotonically decreased function and satisfies

(1) *$V(x) = 0 \Rightarrow x \in A$, where $A \in R^n$ is a non-empty set and globally fixed-time attractive for system (1);*
(2) *$T_c \in \{r_1, \cdots, r_b\}$ is a user defined parameter.*
(3) *For all $V(x) > 0$, there exist positive parameters α, b, q and satisfying $q \in (1, +\infty)$, such that*

$$\dot{V} \le -\frac{C_v}{T_c}(\alpha V^q + b) \tag{8}$$

where

$$C_v = \frac{1}{\alpha^{\frac{1}{q}}}\cdot\frac{2^{(q-1)}}{q-1}\cdot b^{\frac{1-q}{q}}.$$

The system (1) can achieve globally predefined-time stability within predefined-time T_c.

Proof: If $q \in (1, +\infty)$, by Lemma 5, one can get

$$(\alpha^{\frac{1}{q}}V)^q + (b^{\frac{1}{q}})^q \ge 2^{(1-q)}(\alpha^{\frac{1}{q}}V + b^{\frac{1}{q}})^q$$

thus

$$T(x_0) \le -\int_{V(x_0)}^0 \frac{T_c}{C_v}\cdot\frac{1}{(\alpha^{\frac{1}{q}}V)^q + (b^{\frac{1}{q}})^q}dV$$
$$\le \frac{T_c}{C_v}2^{(q-1)}\int_0^{V(x_0)}\frac{1}{(\alpha^{\frac{1}{q}}V + b^{\frac{1}{q}})^q}dV$$
$$= \frac{T_c}{C_v}\frac{2^{(q-1)}}{\alpha^{\frac{1}{q}}(1-q)}\cdot(\alpha^{\frac{1}{q}}V + c^{\frac{1}{q}})^{1-q}\Big|_0^{V(x_0)}$$
$$\le \frac{T_c}{C_v}\frac{2^{(q-1)}}{\alpha^{\frac{1}{q}}(1-q)}\cdot\Big(\frac{1}{(\alpha^{\frac{1}{q}}V(x_0)+b^{\frac{1}{q}})^{q-1}} - b^{\frac{1-q}{q}}\Big)$$
$$= \frac{T_c}{C_v}\frac{2^{(q-1)}}{\alpha^{\frac{1}{q}}(q-1)}\cdot\Big(b^{\frac{1-q}{q}} - \frac{1}{(\alpha^{\frac{1}{q}}V(x_0)+b^{\frac{1}{q}})^{q-1}}\Big).$$

If $V(x_0) = 0$, $T(x_0) = 0$ can be derived.
If $V(x_0) \to \infty$, one can drive that

$$T(x_0) \le \frac{T_c}{C_v}\frac{2^{(q-1)}}{\alpha^{\frac{1}{q}}(q-1)}\cdot\Big(b^{\frac{1-q}{q}} - \frac{1}{(\alpha^{\frac{1}{q}}V(x_0)+b^{\frac{1}{q}})^{q-1}}\Big)$$
$$= \frac{T_c}{C_v}\frac{2^{(q-1)}}{\alpha^{\frac{1}{q}}(q-1)}\cdot b^{\frac{1-q}{q}}$$
$$= T_c.$$

This proof is completed. □

Remark 3: Compared with Theorem 1, the Lyapunov function (8) in Corollary 1 has one less parameter k, which makes the Lyapunov function simpler. Theorem 1 and Corollary 1 can be selected and applied according to different situations.

Remark 4: From the hypothesis of Theorem 1, $qk > 1$, and hypothesis of Corollary 1, $q > 1$, one can see that *the exponents of the Lyapunov function (7) and (8) are not necessarily integer, but can also contain decimals. Therefore, Theorem 1 and Corollary 1 can not only be used for integer-order systems, but also for fractional-order chaotic systems with time-delay, such as Ref. [42], [43].*

V. FIXED-TIME SYNCHRONIZATION AND PREDEFINED-TIME SYNCHRONIZATION

In this section, the predefined-time synchronization of two different fractional-order chaotic systems with time-delay will be investigated. Further, the fixed-time and predefined-time synchronization controller for the response system will be designed. According to the definition of errors at drive system and response system, we have

$$e_i(t) = y_i(t) - x_i(t), i = 1, 2, \ldots, n. \tag{9}$$

Supposing that $E(t) = (e_1(t), e_2(t), \ldots, e_n(t))^{\mathrm{T}} \in R^n$ is the state vector of error system and the initial value is $E(0) = Y(0) - X(0)$. According to Lemma 2, the system (9) is described by

$${}_{t_0}D_t^{\alpha} e_i(t) = {}_{t_0}D_t^{\alpha} y_i(t) - {}_{t_0}D_t^{\alpha} x_i(t), \tag{10}$$

Therefore, formula (3) and (4) are substituted into (10) to obtian the following:

$$D^{\alpha} e_i(t) = -\sigma_i e_i(t) + h_i(e_i(t)) + H_i(e_i(t-\tau)) + u_i(t) \tag{11}$$

where $h_i(e_i(t)) = f_i(Y, t) - f_i(X, t)$, $H_i(e_i(t-\tau)) = F_i(Y, t-\tau) - F_i(X, t-\tau)$, and the control law $u_i(t)(i = 1, 2, \ldots, n)$ will be discussed later.

A. FIXED-TIME SYNCHRONIZATION

Definition 3 ([30]): Drive system and corresponding response system can achieve fixed-time synchronisation in all initial value $X(0)$, $Y(0)$, when there exists a positive constant $T(x_0)$ satisfying

$$\lim_{x \to T(x_0)} e_i(t) = \lim_{x \to T(x_0)} (y_i(t) - x_i(t)) = 0$$

and

$$e_i(t) \equiv 0, \forall t > T(x_0).$$

Assumption 1: For $\forall t, \tau > 0$, the delayed function $F_i(\cdot)$ is bounded. There exist positive constants $l_1 = \sup\{|F_i(X, t-\tau)|\}$, $l_2 = \sup\{|F_i(Y, t-\tau)|\}$. Then, choose a positive constant L, such that

$$\begin{aligned} |H_i(e_i(t-\tau))| &\le |F_i(Y, t-\tau)| + |F_i(X, t-\tau)| \\ &\le l_1 + l_2 = L, \quad \forall X(0), Y(0) \in R^n. \end{aligned}$$

Remark 5: In the practical system, it is difficult to acquire the accurate formula of the delayed function. However, it is easy to exactly estimate the upper bound of delayed function. The upper bound of the delayed function can be estimated by using adaptive techniques in [44], [45]. Therefore, Assumption 1 is reasonable.

Theorem 3: According to Definition 3 and Assumptions 1, for achieving fixed-time stability of fractional-order time-delayed system (3) and system (4), the controller will be designed as following

$$\begin{aligned} u_i(t) = \sigma_i e_i(t) - h_i(e_i(t)) - \operatorname{sign}(e_i(t))L - D^{\alpha-1}\Big(\frac{2^{k-1}}{N^{1-qk}} \\ \cdot \alpha_1 \operatorname{sign}(e_i(t))\, |e_i(t)|^{qk} + 2^{k-1}\lambda_1 \operatorname{sign}(e_i(t))\Big) \end{aligned} \tag{12}$$

where α_1, λ_1, k are positive constants and $qk > 1$.

Proof: Choose following continuous function as Lyapunov function

$$V_1(t) = \sum_{i=1}^{n} |e_i|\,. \tag{13}$$

By applying Lemma 1-3 and taking the derivative of $V_1(t)$, one can obtain that

$$\begin{aligned} \dot V_1(t) &= \sum_{i=1}^{N} \dot e_i(t) \operatorname{sign}(e_i(t)) \\ &= \sum_{i=1}^{N} \Big(D^{1-\alpha}\left(D^{\alpha} e_i(t)\right)\Big) \operatorname{sign}(e_i(t)) \\ &= \sum_{i=1}^{N} \operatorname{sign}(e_i(t)) \Big[D^{1-\alpha}\Big(-\sigma e_i(t) + h_i(e_i(t)) \\ &\quad + H_i(e_i(t-\tau)) + u_i(t)\Big)\Big] \\ &= \sum_{i=1}^{N} \operatorname{sign}(e_i(t)) \Big[D^{1-\alpha}\Big(H_i(e_i(t-\tau)) - \operatorname{sign}(e_i(t))L \\ &\quad - D^{\alpha-1}\Big(\frac{2^{k-1}}{N^{1-qk}} \cdot \alpha_1 \operatorname{sign}(e_i(t))\, |e_i(t)|^{qk} + 2^{k-1} \\ &\quad \cdot \lambda_1 \operatorname{sign}(e_i(t))\Big)\Big)\Big] \end{aligned}$$

By exploiting Assumption 1, one can derived that

$$\begin{aligned} &\operatorname{sign}(e_i(t))\Big(H_i(e_i(t-\tau)) - \operatorname{sign}(e_i(t))L\Big) \\ &\quad = \operatorname{sign}(e_i(t)) H_i(e_i(t-\tau)) - L \\ &\quad \le |H_i(e_i(t-\tau))| - L \le 0, \end{aligned}$$

thus

$$\begin{aligned} \dot V_1(t) &\le \sum_{i=1}^{N} \operatorname{sign}(e_i(t)) \Big[D^{1-\alpha}\Big(-D^{\alpha-1}\Big(\frac{2^{k-1}}{N^{1-qk}} \\ &\quad \cdot \alpha_1 \operatorname{sign}(e_i(t))\, |e_i(t)|^{qk} + 2^{k-1}\lambda_1 \operatorname{sign}(e_i(t))\Big)\Big)\Big] \\ &= -\sum_{i=1}^{N} \operatorname{sign}(e_i(t)) \cdot \Big[\frac{2^{k-1}}{N^{1-qk}} \alpha_1 \operatorname{sign}(e_i(t))\, |e_i(t)|^{qk} \\ &\quad + 2^{k-1}\lambda_1 \operatorname{sign}(e_i(t))\Big] \\ &= -\sum_{i=1}^{N} \left(\frac{2^{k-1}}{N^{1-qk}} \alpha_1 |e_i(t)|^{qk} + 2^{k-1}\lambda_1\right) \end{aligned}$$

Combined with Lemma 5, $\dot{V}_1(t)$ can be described as

$$\begin{aligned}\dot{V}_1(t) &\leq -N^{1-qk}\frac{2^{k-1}}{N^{1-qk}}\alpha_1\left(\sum_{i=1}^{N}|e_i(t)|\right)^{qk} - 2^{k-1}N\lambda_1\\ &= -2^{k-1}\alpha_1 V_1^{qk} - 2^{k-1}N\lambda_1\\ &\leq -(\alpha_1^{\frac{1}{k}}V_1^q + \frac{N}{N^{1-qk}}\lambda^{\frac{1}{k}})^k. \qquad (14)\end{aligned}$$

Simplifying (14), one has that

$$\dot{V}_1(t) = -(\alpha V_1^q + b)^k \qquad (15)$$

where $\alpha = \alpha_1^{\frac{1}{k}}$ and $b = \frac{N}{N^{1-qk}}\lambda^{\frac{1}{k}}$. According to Theorem 1 and Remark 1, the drive system (3) and response system (4) can achieve fixed-time synchronization. The upper bound of the settling time is

$$T_1 \leq \frac{1}{(\frac{N}{N^{1-qk}}\lambda^{\frac{1}{k}})^k}\left(\frac{(\frac{N}{N^{1-qk}}\lambda^{\frac{1}{k}})}{\alpha_1^{\frac{1}{k}}}\right)^{\frac{1}{q}}\left(\frac{1}{qk-1}\right). \qquad (16)$$

□

Remark 6: From the relationship between fixed time T_c and system parameters b, k, q, α above, it can be seen that in order to achieve stability or synchronization of the systems within a given time, it needs to be obtained through complex calculation. To overcome the above, predefined-time stability has been studied. The upper bound of the settling time appears explicitly in their tuning gains T_c, and can be conveniently adjusted according to the practical engineering application.

B. PREDEFINED-TIME SYNCHRONIZATION

Definition 4 ([46]): By adjusting the parameter T_c, the settling time $T(x_0)$ of fixed-time synchronization can be preset in advance. Then the drive system (3) and response system (4) can achieve globally predefined-time synchronization, i.e.,

$$T(x_0) \leq T_c, \forall x_0 \in R^n$$

Remark 7: Obviously, achieving predefined-time synchronization of the drive system (3) and response system (4) is equal to research the synchronization of dynamic error system (9). Hence, we will design an appropriate controller $u_i(t)$ to force the error system (9) convergence in the desired time.

Theorem 4: For achieving predefined-time synchronization, based on Assumptions 1, Definition 4 and Theorem 3, the control law will be designed as following:

$$\begin{aligned}u_i(t) = -h_i(e_i(t)) + \sigma_i e_i(t) - \operatorname{sign}(e_i(t))L - D^{\alpha-1}\Big[\frac{C_v}{T_c}\\ \cdot\Big(\beta_2\operatorname{sign}(e_i(t))\,|e_i(t)|^{qk} + \lambda_2\operatorname{sign}(e_i(t))\Big)\Big] \qquad (17)\end{aligned}$$

where T_c represents the tunable predefined-time; C_v is positive constant determined by other parameters; β_2, λ_2, q, k are positive constants.

Proof: Choose following continuous function as Lyapunov function

$$V_2(t) = \|e(t)\|_1 = \sum_{i=1}^{n}|e_i|.$$

Taking the derivative of $V_2(t)$, one can attain that

$$\begin{aligned}\dot{V}_2(t) &= \sum_{i=1}^{N}\dot{e}_i(t)\operatorname{sign}(e_i(t))\\ &= \sum_{i=1}^{N}\left(D^{1-\alpha}\left(D^{\alpha}e_i(t)\right)\right)\operatorname{sign}(e_i(t))\\ &= \sum_{i=1}^{N}\operatorname{sign}(e_i(t))\Big[D^{1-\alpha}\Big(-\sigma e_i(t) + h_i(e_i(t))\\ &\quad + H_i(e_i(t-\tau)) + u_i(t)\Big)\Big]\\ &= \sum_{i=1}^{N}\operatorname{sign}(e_i(t))\Big[D^{1-\alpha}\Big(H_i(e_i(t-\tau)) - L\\ &\quad \cdot\operatorname{sign}(e_i(t)) - D^{\alpha-1}\Big(\frac{C_v}{T_c}\Big(\beta_2\operatorname{sign}(e_i(t))\,|e_i(t)|^{qk}\\ &\quad + \lambda_2\operatorname{sign}(e_i(t))\Big)\Big)\Big)\Big]\\ &\leq -\frac{C_v}{T_c}\sum_{i=1}^{N}\operatorname{sign}(e_i(t))\Big[D^{1-\alpha}\Big(D^{\alpha-1}\Big(\beta_2\operatorname{sign}(e_i(t))\\ &\quad \cdot|e_i(t)|^{qk} + \lambda_2\operatorname{sign}(e_i(t))\Big)\Big)\Big]\\ &= -\frac{C_v}{T_c}\sum_{i=1}^{N}\operatorname{sign}(e_i(t))\Big(\beta_2\operatorname{sign}(e_i(t))\,|e_i(t)|^{qk}\\ &\quad + \lambda_2\operatorname{sign}(e_i(t))\Big)\\ &= -\frac{C_v}{T_c}\sum_{i=1}^{N}\left(\beta_2|e_i(t)|^{qk} + \lambda_2\right).\end{aligned}$$

Combined with Lemma 5, $\dot{V}_2(t)$ becomes:

$$\begin{aligned}\dot{V}_2(t) &\leq -\frac{C_v}{T_c}\left(N^{1-qk}\beta_2\left(\sum_{i=1}^{N}|e_i(t)|\right)^{qk} + N\lambda_2\right)\\ &= -\frac{C_v}{T_c}\left(N^{1-qk}\beta_2 V_2^{qk} + N\lambda_2\right)\\ &\leq -\frac{C_v}{T_c}(\beta_2^{\frac{1}{k}}V_2^q + \lambda_2^{\frac{1}{k}})^k\\ &= -\frac{C_v}{T_c}(\tilde{\alpha}V_2^q + \tilde{b})^k\end{aligned}$$

where $\tilde{\alpha} = \beta_2^{\frac{1}{k}}$, $\tilde{b} = \lambda_2^{\frac{1}{k}}$. According to Theorem 1 and Corollary 1, the C_v can be obtained as following:

$$C_v = \begin{cases}\dfrac{1}{\beta_2^{\frac{1}{qk}}}\cdot\dfrac{1}{qk-1}\cdot\lambda_2^{\frac{1-qk}{qk}} & \text{if} \quad 0 < q \leq 1,\\[2ex] \dfrac{1}{\beta_2^{\frac{1}{qk}}}\cdot\dfrac{2^{(q-1)}}{q-1}\cdot\lambda_2^{\frac{1-q}{qk}} & \text{if} \quad q > 1.\end{cases}$$

Then, for $\forall t > T_c$, we can obtain $V_2(t) = 0$. The dynamic error system (9) is synchronized within predefined-time T_c. □

VI. SIMULATION RESULTS

In this section, via two illustrative examples, we demonstrate the effectiveness of predefined-time theorem in Section IV. We present the numerical simulation results when the controllers $u_i(t)$ is designed based on Theorem 3 and Theorem 4 respectively. Besides, some analysis results are listed.

Example 2: We choose the fractional-order time-delayed system (5) as drive system and the system (6) as response system. Then, the following dynamic error system can be derived

$$D^{\alpha}e_i(t) = -\sigma_i e_i(t)+h_i(e_i(t))+H_i(e_i(t-\tau))+u_i(t). \quad (18)$$

According to Theorem 3, the fractional-order fixed-time synchronization controller $u_i(t)(i = 1, 2, 3, 4)$ is designed as

$$u_i(t) = -D^{\alpha-1}\Big[\frac{2^{k-1}}{N^{1-qk}}\alpha_1 \operatorname{sign}(e_i(t))\,|e_i(t)|^{qk} + 2^{k-1}\lambda_1 \cdot \operatorname{sign}(e_i(t))\Big] - h_i(e_i(t)) + \sigma_i e_i(t) - \operatorname{sign}(e_i(t))L \quad (19)$$

Combing formula (5), (6) and (18), the controllers are designed as following

$$\begin{cases} u_1(t) = \sigma_1 e_1(t) - a_1\,(y_2(t) - y_1(t)) - y_4(t) + a(x_2(t) \\ -x_1(t)) - D^{\alpha-1}\Big[\frac{2^{k-1}}{N^{1-qk}}\alpha_1 \operatorname{sign}(e_1(t)) \\ \cdot |e_1(t)|^{qk} + 2^{k-1}\lambda_1 \operatorname{sign}(e_1(t))\Big] - \operatorname{sign}(e_1(t))L \\ u_2(t) = \sigma_2 e_2(t) - b_1 y_1(t) + y_2(t) + y_1(t)y_3(t) + bx_1(t) \\ -kx_1(t)x_3(t) + x_4(t) - D^{\alpha-1}\Big[\frac{2^{k-1}}{N^{1-qk}}\alpha_1 \\ \cdot \operatorname{sign}(e_2(t))\,|e_2(t)|^{qk} + 2^{k-1}\lambda_1 \operatorname{sign}(e_2(t))\Big] \\ - \operatorname{sign}(e_2(t))L \\ u_3(t) = \sigma_3 e_3(t) - y_1(t)y_2(t) + c_1 y_3(t) + h(x_1(t))^2 \\ -cx_3(t) + x_4(t) - D^{\alpha-1}\Big[\frac{2^{k-1}}{N^{1-qk}}\alpha_1 \operatorname{sign}(e_3(t)) \\ \cdot |e_3(t)|^{qk} + 2^{k-1}\lambda_1 \operatorname{sign}(e_3(t))\Big] - \operatorname{sign}(e_3(t))L \\ u_4(t) = \sigma_4 e_4(t) + y_2(t)y_3(t) + r_1 y_4(t) - rx_2(t) \\ -D^{\alpha-1}\Big[\frac{2^{k-1}}{N^{1-qk}}\alpha_1 \operatorname{sign}(e_4(t))\,|e_4(t)|^{qk} + 2^{k-1} \\ \cdot\lambda_1 \operatorname{sign}(e_4(t))\Big] - \operatorname{sign}(e_4(t))L \end{cases} \quad (20)$$

where $\alpha = 0.82$, $N = 4$, $i = 1, 2, 3, 4$, $\sigma_i = 0.1$, $q = 0.5$, $k = 5.2$, $\alpha_1 = 0.764$. Because the function $tanh(\cdot)$ is bounded, then we can get $|F_i(\cdot)| \leq 0.1$. According to Assumption 1, we choose $L = 0.2 > |H_i(\cdot)|$.

From Theorem 3, Remark 1 and Definition 3, two different fraction-order time-delayed chaotic system (5) and (6) can achieve fixed-time synchronization. And different parameter λ_1 and diverse initial values $X(0), Y(0)$ are offered to illustrate the veracity of fixed-time stability theory. Fig. 3(a) shows the synchronization errors of the drive-response systems with the parameter $\lambda_1 = 0.598$, and the settling time estimation can be obtain that $T^2_{max} = 0.5294$. Fig.3(b) shows the synchronization errors of the drive-response systems with the parameter $\lambda_1 = 1.05$, and the settling time estimation can be obtain that $T^2_{max} = 0.593$. As can be seen from Fig.3, under different initial values, drive-response systems are able to achieve synchronization within the settling time T^2_{max}. According to the Lemma 4, when the parameter $\lambda_1 = 0.598$, one can obtain the $T^1_{max} = 1.3764 > T^2_{max} = 0.5294$. When the parameter $\lambda_1 = 1.05$, the $T^1_{max} = 1.5418 > T^2_{max} = 0.593$. Hence, the Theorem 2 is effectiveness. Compared with the convergence time in Fig.3 and the upper bound of the settling time estimation, it can be seen that the fixed-time synchronization controller (19) and the Theorem 3 are effectiveness and irrelevant of the initial values.

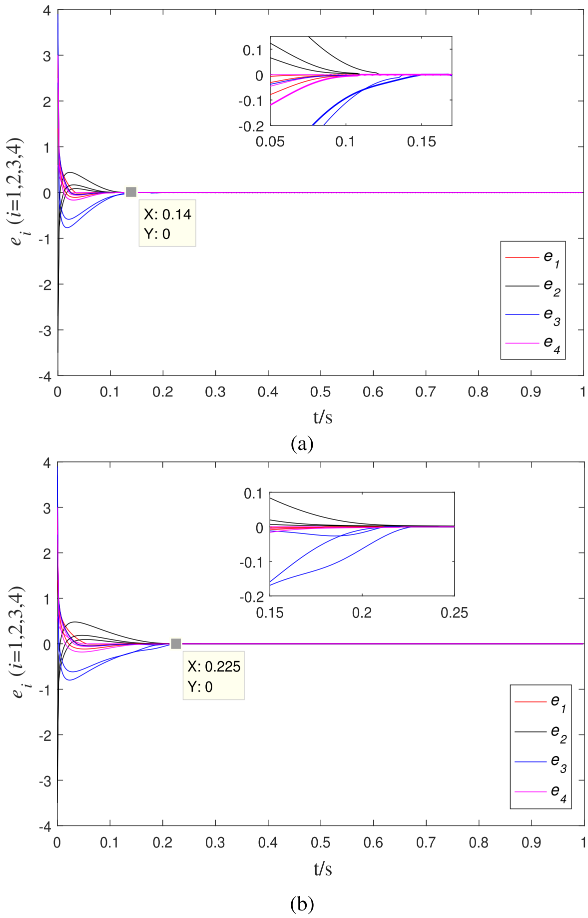

FIGURE 3. (a)When $\lambda_1 = 0.598$, the fixed-time synchronization error of drive-response systems with different initial values under the controller (19). (b)When $\lambda_1 = 1.05$, the fixed-time synchronization error of drive-response systems with different initial values under the controller (19).

Example 3: Considering predefined-time synchronization of drive system (5) and response system (6), according to Theorem 4, the controller $u_i(t)(i = 1, 2, 3, 4)$ is chosen as following

$$u_i(t) = \sigma_i e_i(t) - D^{\alpha-1}\Big[\frac{C_v}{T_c}\Big(\mu\,\mathrm{sign}(e_i(t))\,|e_i(t)|^{qk} + \omega \cdot \mathrm{sign}(e_i(t))\Big)\Big] - h_i(e_i(t)) - \mathrm{sign}(e_i(t))L \quad (21)$$

Based on system (5) and (6), we can obtain the predefined-time synchronization controller as follows

$$\begin{cases} u_1(t) = \sigma_1 e_1(t) - a_1\,(y_2(t) - y_1(t)) - y_4(t) + a(x_2(t) \\ -x_1(t)) - D^{\alpha-1}\Big[\frac{C_v}{T_c}\Big(\mu\,\mathrm{sign}(e_1(t))\,|e_1(t)|^{qk} \\ +\omega\,\mathrm{sign}(e_1(t))\Big)\Big] - \mathrm{sign}(e_1(t))L \\ u_2(t) = \sigma_2 e_2(t) - b_1 y_1(t) + y_2(t) + y_1(t)y_3(t) + bx_1(t) \\ -kx_1(t)x_3(t) + x_4(t) - D^{\alpha-1}\Big[\frac{C_v}{T_c}\Big(\mu\,\mathrm{sign}(e_2(t)) \\ \cdot\,|e_2(t)|^{qk} + \omega\,\mathrm{sign}(e_2(t))\Big)\Big] - \mathrm{sign}(e_2(t))L \\ u_3(t) = \sigma_3 e_3(t) - y_1(t)y_2(t) + c_1 y_3(t) + h(x_1(t))^2 \\ -cx_3(t) + x_4(t) - D^{\alpha-1}\Big[\frac{C_v}{T_c}\Big(\mu\,\mathrm{sign}(e_3(t)) \\ \cdot\,|e_3(t)|^{qk} + \omega\,\mathrm{sign}(e_3(t))\Big)\Big] - \mathrm{sign}(e_3(t))L \\ u_4(t) = \sigma_4 e_4(t) + y_2(t)y_3(t) + r_1 y_4(t) - rx_2(t) \\ -D^{\alpha-1}\Big[\frac{C_v}{T_c}\Big(\mu\,\mathrm{sign}(e_4(t))\,|e_4(t)|^{qk} \\ +\,\mathrm{sign}(e_4(t)) \cdot \omega\Big)\Big] - \mathrm{sign}(e_4(t))L \end{cases} \quad (22)$$

where $\alpha = 0.82$, $i = 1, 2, 3, 4$, $\sigma_i = 0.1$, $k = 5.2$, $\mu = 12.9$, $\omega = 11$. According to Assumption 1, we choose $L = 0.2$.

Case 1: When $0 < q \leq 1$, we can select $q = 0.5$. Through Theorem 1, we can get $C_v = 0.0534$.

From Theorem 4, the fractional-order time-delayed systems (5) and (6) can achieve predefined-time synchronization. To account for the effectiveness of the predefined-time synchronization theorem, we design the controller $u_i(t)$ ($i = 1, 2, 3, 4$) by using different T_c. By definition 4 and Theorem 1, it can be seen that the settling time of predefined-time stability is independent of the initial values $X(0)$ and $Y(0)$. By choosing different T_c, system (5) and (6) can synchronize within the given predefined-tiem T_c. The simulation results are shown in Fig.4(a) with $T_c = 1$ and Fig.5(b) with $T_c = 1.5$. Further, the time response of x_1, y_1 and x_3, y_3 under controller (22) shown in Fig.5. The Fig.5 demonstrates the excellent tracking performance between drive system (5) state variates x_i and response system (6) state variates y_i.

Case 2: When $q > 1$, to verify the validity of Corollary 1, we select $q = 2$ and the other parameters in (22) remain the same as Example 3. According to Corollary 1, we can obtain $C_v = 1.242$. Based on predefined-time synchronization controller (21), the simulation results are shown in Fig.6. The time response of drive system (5) and response system (6) shown in Fig.7. The Fig.7 exhibits the eminent synchronization between drive-response systems state vector x_i and y_i. The results in Fig.4-7 confirm that the predefined-time stability Theorem 3 and Theorem 4 are trustworthy and effectiveness.

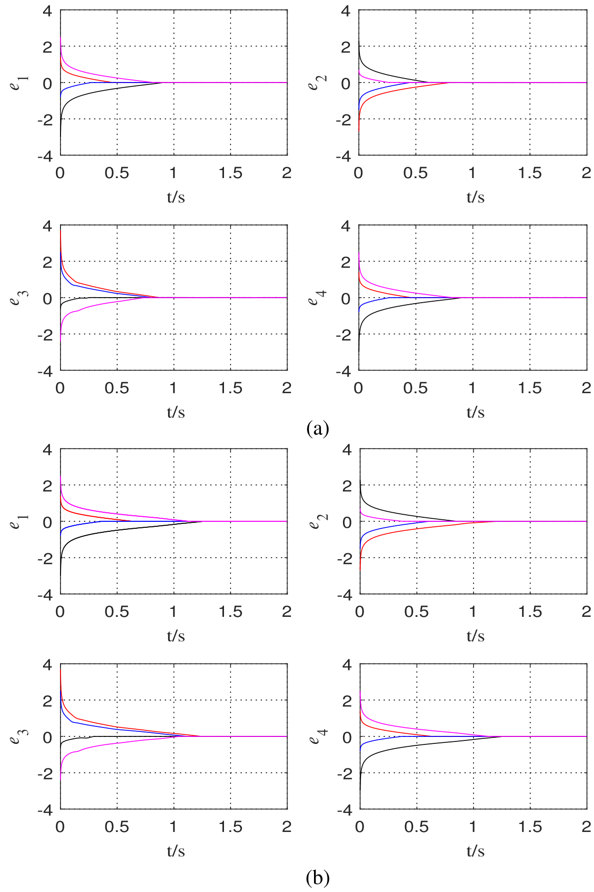

FIGURE 4. **(a)When $q = 0.5$, $T_c = 1$, the error system realizes predefined-time synchronization under controller (21) and diverse $X(0)$, $Y(0)$. (b)When $q = 0.5$, $T_c = 1.5$, the error system realizes predefined-time synchronization under controller (21) and diverse $X(0)$, $Y(0)$.**

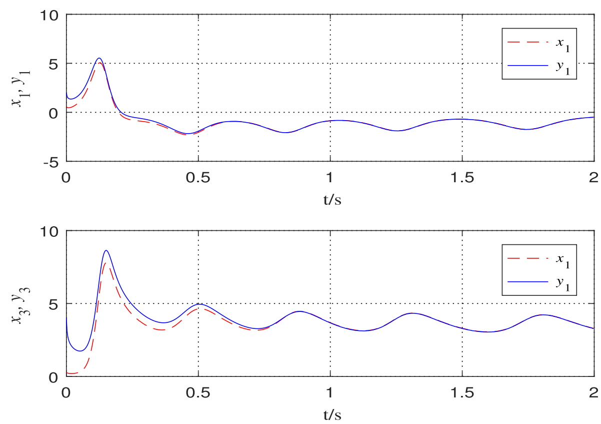

FIGURE 5. **Time response of x_1, y_1 and x_3, y_3 under controller (21) with $q = 0.5$, $T_c = 1$.**

Remark 8: According to Definition 4, the drive-response system can achieve synchronization within the predefined-time. Besides, Theorem 4 provides a tunable parameter T_c

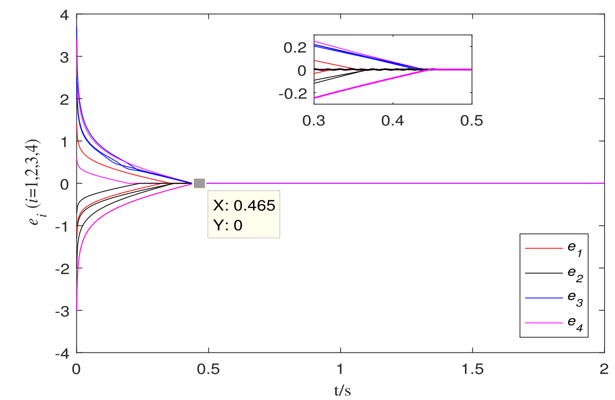

FIGURE 6. When $q = 2$ and $T_c = 1$, the synchronization error trajectories under controller (22).

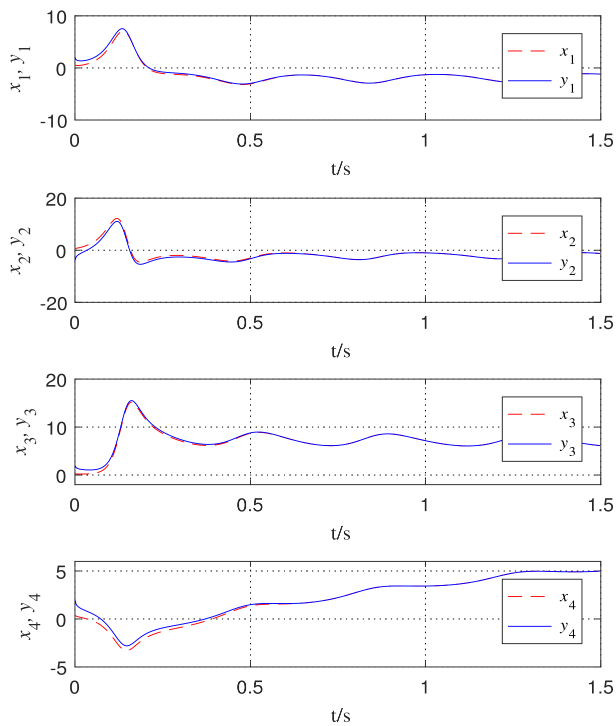

FIGURE 7. When $q = 2$ and $T_c = 1$, the time response of drive system and response system under controller (22).

in advance, which can tune the settling time according to the needs of practical application.

Remark 9: The method in Section V is feasible. The chattering phenomenon can be reduced by selecting appropriate parameters, but if the parameters are not selected rationally, more obvious chattering phenomenon may occur. In the future work, we will focus on the design of the controller to achieve higher precision predefined-time synchronization.

Example 4: Considering the situation that time-delay of fractional-order drive system and response system is non-negative time-varying variable. Supposing the time-dependent delayed function is denoted as

$$\tau(t) = \frac{e^t}{(1+e^t)}.$$

Then, the drive system is described by

$$\begin{cases} D^\alpha x_1 = -\sigma_1 x_1(t) + a\,(x_2(t) - x_1(t)) \\ +0.1\tanh(x_1(t-\tau(t))) \\ D^\alpha x_2 = -\sigma_2 x_2(t) + bx_1(t) - kx_1(t)x_3(t) + x_4(t) \\ +0.1\tanh(x_2(t-\tau(t))) \\ D^\alpha x_3 = -\sigma_3 x_3(t) + h(x_1(t))^2 - cx_3(t) + x_4(t) \\ +0.1\tanh(x_3(t-\tau(t))) \\ D^\alpha x_4 = -\sigma_4 x_4(t) - rx_2(t) + 0.1\tanh(x_4(t-\tau(t))) \end{cases} \tag{23}$$

Corresponding response system can be designed as

$$\begin{cases} D^\alpha y_1 = -\sigma_1 y_1(t) + a_1\,(y_2(t) - y_1(t)) + y_4(t) \\ +0.1\tanh(y_1(t-\tau(t))) + u_1(t) \\ D^\alpha y_2 = -\sigma_2 y_2(t) + b_1 y_1(t) - y_2(t) - y_1(t)y_3(t) \\ +0.1\tanh(y_2(t-\tau(t))) + u_2(t) \\ D^\alpha y_3 = -\sigma_3 y_3(t) + y_1(t)y_2(t) - c_1 y_3(t) \\ +0.1\tanh(y_3(t-\tau(t))) + u_3(t) \\ D^\alpha y_4 = -\sigma_4 y_4(t) - y_2(t)y_3(t) - r_1 y_4(t) \\ +0.1\tanh(y_4(t-\tau(t))) + u_4(t) \end{cases} \tag{24}$$

The other parameters in (23) and (24) stay the same as system (5) and (6). When time-delay $\tau(t)$ is variable, the drive system chaotic behavior and phase portraits between different variables are shown in Fig.8. The results of synchronization error are presented in Fig.9 under the controller (21) and the parameters $q = 0.5$, $T_c = 1.5$. The error system convergence time $T = 1.32 < T_c$, therefore, it can be obtained that Theorem 4 can still handle the case where the time-delay is variable.

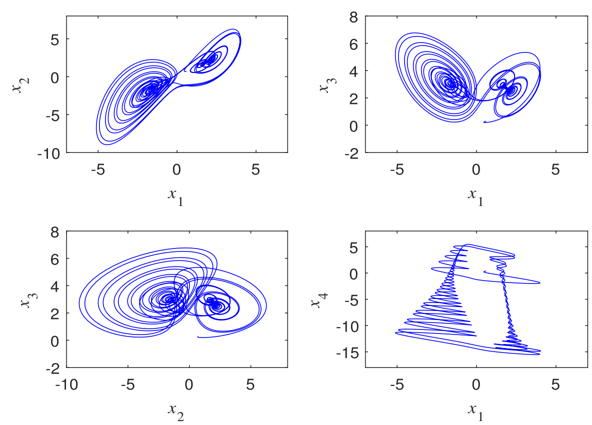

FIGURE 8. Phase portraits of drive system when time-varying is $\tau(t)$.

Remark 10: The time-varying input delay $\tau(t)$ is bounded such that $0 \leq \tau(t) \leq 1$. Besides, the time-delay is slow varied, and the rate of change of the delay satisfies $|\dot{\tau}(t)| < 1$. Although the time-delay is varied, the upper bound of $\tau(t)$ and delayed function $F_i(\cdot)$ still exist. That is, Assumption 1 holds, which resulting in the controller (21) appropriate in this case.

Remark 11: Theoretically, the time-delay in this paper can be set arbitrarily large. This paper mainly considers the case

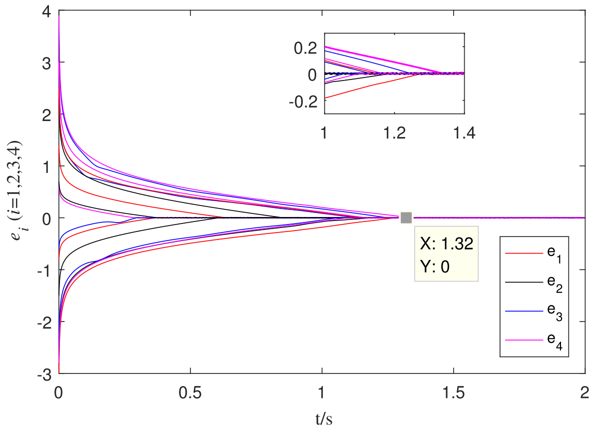

FIGURE 9. When $q = 0.5$, $T_c = 1.5$ and the time-varying $\tau(t)$, the error trajectories under controller (21).

that the time-delay is less than the predefined-time T_c, so that the influence of time-delay can be fully considered when choosing T_c. If the time-delay is greater than T_c, then the time-delay does not act before the drive-response systems synchronization, and when the time-delay acts, a small jitter may occur.

Remark 12: To show the effectiveness of the designed controller, numerical simulations at Example 2-4 have been offered by different values of p, T_c, $X(0)$ and $Y(0)$. Further, the effectiveness of fixed-time synchronization Theorem 3 and predefined-time synchronization Theorem 4 are verified. Evidently, the fractional-order drive system (3) and response system (4) can achieve predefined-time synchronization, by presupposing tunable parameter T_c. Then, the effectiveness of synchronization theorems have been verified.

VII. CONCLUSION

In this paper, two novel different fractional-order time-delayed chaotic systems have been investigated. On the basis of fixed-time synchronization theory and non-negative Lyapunov function, the globally predefined-time stability theorem of fractional-order chaotic system has been proposed. Even now, it has few been found that the references related to the adaptive predefined-time stability of fractional-order time-delayed system. For predefined-time stability, by setting the tunable parameter T_c in advance, the designed controller can achieve synchronization of drive system and corresponding response system within the upper bounded of settling time. And there are four advantages of predefined-time as follows:

(1) The convergence time is independent of the controller parameters and drive-response system parameters, which only affect by the certain tunable parameter T_c.
(2) The predefined-time stability has a direct relationship between the tunable parameter and the settling time, which appears explicitly in T_c.
(3) The upper bound of predefined-time is not a conservatively estimated value but a true minimum, which is more accurate than the settling time of Lemma 4.
(4) By selecting the value of the tunable parameter T_c, two different chaotic systems can synchronize at different times.

It can be discovered that the predefined-time stability is reliable under different conditions. The simulation results of two different fractional-order chaotic systems synchronization have been shown that the convergence time of Theorem 3 is more accurate than Lemma 4. The proposed predefined-time theory and the designed synchronization controller can be applied to other fractional-order time-delayed chaotic systems. In the future research work, we will continue to study the predefined-time synchronization under the influence of disturbances.

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