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Some Geometric Aggregation Operators Under q-Rung Orthopair Fuzzy Soft Information With Their Applications in Multi-Criteria Decision Making

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ABSTRACT The pioneer paradigm of soft set (S_{ft}S) was investigated by Molodtsov in 1999 by affixing parameterization tools in ordinary sets. S_{ft}S theory is free from inherit complexity and a nice mathematical tool for handle uncertainties and vagueness. The aim of this paper is to initiate the combine study of S_{ft}S and q-rung orthopair fuzzy set (q-ROFS) to get the new notion called q-rung orthopair fuzzy soft set (q-ROFS_{ft}S). The notion of q-ROFS_{ft}S is free from those complexities which suffering the contemporary theories because parameterization tool is the most significant character of q-ROFS_{ft}S. In this manuscript our main contribution to originate the concept of q-ROF soft weighted geometric (q-ROFS_{ft}WG), q-ROF soft ordered weighted geometric (q-ROFS_{ft}OWG) and q-ROF soft hybrid geometric (q-ROFS_{ft}HG) operators in q-ROFS_{ft}S environment. Moreover, some dominant properties of these developed operators are studied in detail. Based on these proposed approaches, a model is build up for multi-criteria decision making (MCDM) and their step wise algorithm is being presented. Finally, utilizing the developed approach an illustrative example is solved under q-ROFS_{ft} environment. Further a comparative analysis of the investigated models with some existing methods are presented in detail which shows the superiority, competence and ability of the developed model.

INDEX TERMS Pythagorean fuzzy sets, S_{ft}S, PFS_{ft}S, q-ROFS, q-ROFS_{ft}S, q-ROFS_{ft}WG operator, q-ROFS_{ft}OWG operator, q-ROFS_{ft}HG operator, MCDM.

I. INTRODUCTION

Decision Making (DM) is a pre-plan technique of identifying and selecting the best choice out of many alternatives. DM is a hard process because it can vary so obviously from one scenario to the next. Therefore, it is very important to judge the characteristics and limitations of each alternative. Also DM is a batter approach to increase the chance of selecting most appropriate alternative of the choice. It is essential to know that how much truly background information is required for decision maker and the best effective strategy is to keep an eye and focus on your goal. To handle these situation Zadeh [1] originated the preeminent concept of fuzzy set. Fuzzy set is characterized to assign membership grade (MG) to each alternative from [0, 1]. In many real phenomena decision maker needs nonmembership grade (NMG) for the same alternative. So, to handle this issue Atanassov [2] developed the theory of intuitionistic fuzzy set (IFS), by adding NMG with the ordinary fuzzy set. IFS is manly characterized by MG and NMG, and their sum belongs to [0, 1]. Since the inception of this prominent concept researchers studied their hybrid structures in various directions with their applications in DM. Many scholars studied different aggregation operators for IFS. Xu [3] investigated the weighted averaging (WA)

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operators. Similarly, Yager [4] proposed the idea of ordered weighted averaging (OWA) operator by giving weight to an object against their ranking position. Xu and Yager [5] initiated the idea of weighted geometric (WG) aggregation operators to aggregate different intuitionistic values into a single one. By using Einstein operation Wang and Liu [6] originated the notion of IF Einstein WA and Einstein OWA operators. He *et al.* [7] initiated the idea of IF interactive aggregation operators and Garg [8], [9] proposed the generalized concept of IF interactive operators and present novel IF operational laws. Chinram *et al.* [10] initiated the concept of some IF rough aggregation operators such as IF rough weighted, ordered weighted, hybrid averaging operators and IF rough weighted geometric, ordered weighed geometric, hybrid geometric aggregation operators by using EDAS method for MCGDM. Khan *et al.* [11] presented the novel model towards generalized IF soft sets. Ye [12] investigated hybrid arithmetic and geometric operators and initiated their applications in DM by using IF environment. Zhang and Yang [13] developed the tracking control problem for a family of strict-feedback systems in the presence of unknown nonlinearities and immeasurable system states, for detailed see [14], [15], [16]. From the inception and appearance of the dominant concept of IFS, a lot of research were done by different scholars in several directions. However, there exist some deficiency in this prominent notion due to which it fail to handle the situation and restrict the experts to the boundary range that sum of MG and NMG must not exceed 1.

To cope on this deficiency Yager [17] presented the resilience paradigm of Pythagorean fuzzy set (PFS). The notion of PFS provides additional space in boundary range for the experts to select the optimum decision. PFS is characterized by two mappings called MG and NMG, and their square sum belongs to [0,1]. Furthermore, Yager [18], [19] originated several aggregation operators such as PF weighted averaging (PFWA), PF WG, PF weighted power averaging and geometric operators and presented their applications in DM. The idea of PF division and subtraction operators and point aggregation operator are initiated by Peng and Yang [20], [21]. Garg [22], [23] presented the notions of generalized PFWA and generalized PFWG aggregation operators via Einstein operation. The idea of symmetric PFWA and PFWG (SPFWA/G) operators were initiated by Ma and Xu [24]. Hussain *et al.* [25] originated the notion of rough PF ideals by using algebraic structure of semigroups. Wang *et al.* [26] presented the concept of some PF interactive Hamacher power averaging and geometric aggregation operators. Wang and Li [27] proposed the notion of PF interaction power Bonferroni mean aggregation operator. Ashraf *et al.* [28] presented the concept of sine trigonometric by using PF information. Khan *et al.* [29] developed the notion of Dombi aggregation operators based on PF environment. However, PFSs also have some shortcoming because if the expert assigns MG 0.85 and NMG0.65, then $0.85^2 + 0.65^2 > 1$. So, in this case PFS cannot handle the situation. Therefore, scholars intensively need a new concept to cope on these situations.

Recently Yager [30] investigated the prominent generalization of PFS and called it q-ROFS which is also characterized two mappings called MG and NMG. These two mapping satisfying the condition that q^{th} power of MG and q^{th} power of NMG must belongs to [0,1]. From the analysis of this concept it is clear that the boundary range of q-ROFS is more capable and extensive than IFS and PFS. Ali [31] presented another view of q-ROFS by using the concept of orbits. Liu and Wang [32] generalized the existing concept to q-ROF weighted averaging geometric (q-ROFWA) and q-ROF weighted geometric (q-ROFWG) operators and proved their fundamental properties. Hussain *et al.* [33] initiated the generalized and group generalized averaging aggregation operators by using q-ROF information. Wang *et al.* [34] initiated the concept of some q-ROF soft rough aggregation operators such as q-ROF soft rough weighted, ordered weighted, hybrid averaging operators and q-ROF soft rough weighted geometric, ordered weighed geometric, hybrid geometric aggregation operators. The combine study of q-ROF numbers with Bonferroni mean operators are presented by Liu and Liu [35]. Xing *et al.* [36] initiated the concept of point weighted aggregation operators with new score function using q-ROF information. Hussain *et al.* [37] investigated the concept of roughness in q-ROFS and presented their application in DM by using TOPSIS method. Liu [38] presented entropy-based GLDS method for social capital selection of a PPP project by applying q-ROF information. Hussain *et al.* [39] initiated the concept of hesitant q-ROF weighted averaging and geometric aggregation operators. Ye *et al.* [40] originated the study of q-ROF continuous single variable information. Wang *et al.* [41]–[44] presented different aggregation operators by applying different decision making methods based on q-ROF information. Yin *et al.* [45] defined some product operations on q-ROF graphs and proved some theorems on the same concepts. In literature various traditional concepts are exist such as fuzzy set, IFS, PFS and rough set [46] which are generally utilized for handling the uncertain, complex and vague data but collectively it is observed that all these concepts have a deficiency of parameters information. Hence, these notions cannot remarkably use in real situations.

The prominent theory of S_{ft}S was initiated by Molodtsov [47], in which parameters are used to handle the complex and uncertain information. Ali *et al.* [48] improved some operations in the existing literature and defined some new operational laws. Maji [49], [50] presented the hybrid study of S_{ft}S, fuzzy set and IFS to get fuzzy S_{ft}S and IF soft set (IFS_{ft}S) which plays a significant role among these theories. Arora and Garg [51] gave the concept of aggregation operators in IFS_{ft}S. Feng *et al.* [52] improved some operations of generalized IFS_{ft}S and based on these new defined operation, they presented their application in DM. Based on [52] Hayat *et al.* [53] presented another look of group generalized IFS_{ft}S. Hussain *et al.* [54] investigated the combine study of S_{ft}S and q-ROFS to get the new notion of q-ROFS_{ft}S and studied different average aggregation operators and their basic properties on the same concept. The hybrid intelligent concept of soft sets and q-ROFS that is q-ROFS_{ft}S is a powerful mathematical tool to handle with inconsistent, indeterminate and incomplete information, which attract the attention of scholars. From the analysis of existing literature, it is clear that the aggregation operators great significance in decision making to aggregate the collective evaluated information of several experts into a single value. According to the best of our knowledge up till now, no application of the geometric aggregation operators with the hybridization of q-ROFS with soft set is reported in q-ROF environment. Therefore, this motivates the current research to investigate novel concept of geometric aggregation operators by applying q-ROFS_{ft} information because there have been enough space for new research on q-ROFS_{ft}S. The remaining of the paper is organized as.

The arrangement of the paper is as: Section **II**, is devoted for a brief study of basic literature which will be helpful in onward sections. Section **III**, consists of the hybrid study of q-OFS, S_{ft}S that is q-ROFS_{ft}S and their basic operations and relations. Further Section **VI**, consists of some geometric aggregation operators such as q-ROFS_{ft} weighted geometric (q-ROFS_{ft}WG), q-ROFS_{ft} ordered weighted geometric (q-ROFS_{ft}OWG) and q-ROFS_{ft} hybrid geometric (q-ROFS_{ft}HG) operators. The basic characteristics of these aggregation operators such as Idempotency, Boundedness, Monotonicity, Shift invariance and Homogeneity are presented in detail. In Section **V**, a technique for MCDM and its algorithm are presented. In Section **VI**, an illustrative example of medical diagnosis is initiated by using the developed model. The subsection **D**, contains the comparative analysis of proposed model with some existing methods and it is observed that the method originated in this manuscript is more capable and superior than existing methods.

II. PRELIMINARIES

In this section, a brief study of S_{ft}S, IFS, PFS and q-ROFS are presented which will assess in coming sections.

Definition 1: [2] Suppose a universal S and an IFS T in S is a set denoted and defined as:

$$T = \{< a, \mu_T(a), \eta_T(a) > | a \in S\}$$

where the mappings $\mu_T : S \to [0, 1]$, and $\eta_T : S \to [0, 1]$, represents the MG and NMG of an alternative $a \in S$ to the set T. It must satisfied the restriction that $0 \leq \mu_T(a) + \eta_T(a) \leq 1$.

Definition 2: [17] Consider S be initial universal set. A PFS T in S is an object denoted and defined by

$$T = \{< a, \mu_T(a), \eta_T(a) > | a \in S\}$$

where the mappings $\mu_T : S \to [0, 1]$, and $\eta_T : S \to [0, 1]$, denotes the MG and NMG of an alternative $a \in S$ to the set T. The PFS must satisfied that $0 \leq {\mu_T}^2(a) + \eta_T^2(a) \leq 1$.

Definition 3: [30]. Suppose S be a universal set. A q-ROFS T in S is denoted and defined as

$$T = \left\{< a, \mu_T(a), \eta_T(a) >_q | a \in S \text{ and } q \geq 1\right\}$$

where the mappings $\mu_T : S \to [0, 1]$, and $\eta_T : S \to [0, 1]$, denotes the MG and NMG of an element $a \in S$ to the set T. It must satisfied the conditions that $0 \leq {\mu_T}^q(a) + \eta_T^q(a) \leq 1$ for $q \geq 1$. Moreover, $\pi = \sqrt[q]{1 - \left(\mu_T^q(a) + \eta_T^q(a)\right)}$ denotes the hesitancy degree for each alternative $a \in S$.

In 1999, the prominent theory of S_{ft}S was initiated by Molodtsov [47], in which parameters are used to handle the complex and uncertain information. Then various researcher presented the hybrid study of S_{ft}S, fuzzy set and IFS to get fuzzy S_{ft}S and IF soft set (IFS_{ft}S) which plays a role of bridge among these theories. In literature various traditional concepts are exist such as fuzzy set, IFS, PFS and rough set which are generally utilized for handling the uncertain, complex and vague data but it is observed that all these concepts have a deficiency of parameters information. Therefore, these notions cannot remarkably utilized in real situations. Therefore, to cope on these shortcoming Molodtsov originated the powerful notion of S_{ft}S which is defined as:

Definition 4: [47]. Suppose a universal set S. Consider a set of parameters E and $\mathcal{A} \subseteq E$. The pair $(\mathrm{F}, \mathcal{A})$ is known to be a S_{ft}S over S, where is a mapping denoted by $\mathrm{F} : \mathcal{A} \to P(S)$. $P(S)$ denotes the power set of S.

Definition 5: [49] Suppose $(S, \mathcal{E})$ be a S_{ft}S and $A \subseteq \mathcal{E}$. A pair $(F, \mathcal{A})$ is said to be a FSS over S, where is a mapping denoted by $F : \mathcal{A} \to P^*(S)$; $P^*(S)$ denotes the collections of all fuzzy subsets of S, and is defined as

$$F(\hbar_j) = \{\prec a_i, \mu_j(a_i) \succ | a_i \in S \text{ and } \hbar_j \in \mathcal{A}\}$$

A FSS reduced to S_{ft}S, if $\hbar_j$ is a crisp subsets of S.

Definition 6: [54] Suppose a soft set $(S, \mathcal{E})$ and $\mathcal{A} \subseteq \mathcal{E}$. A pair $(\mathcal{P}, \mathcal{A})$ is said to be a Pythagorean fuzzy S_{ft}S (PFS_{ft}S) over$S$, where $\mathcal{P}$ is a mapping represented by $\mathcal{P} : \mathcal{A} \to PFS^{(S)}$, which is given as

$$\mathcal{P}_{\hbar_j}(x_i) = \{\prec x_i, \lambda_j(x_i), \xi_j(x_i) \succ | x_i \in S, \hbar_j \in \mathcal{A}\}$$

where $PFS^{(S)}$ represents the family of all PFSs of S. Here $\lambda_j(x_i), \xi_j(x_i)$ denotes the MG and NMG of an alternative $x_i \in S$ to a set $\mathcal{P}_{\hbar_j}$ respectively, and satisfying $0 \leq (\lambda_j(x_i))^2 + (\xi_j(x_i))^2 \leq 1$. For simplicity $\mathcal{P}_{\hbar_j}(x_i) = \prec x_i, \lambda_j(x_i), \xi_j(x_i) \succ$ is denoted by $\mathfrak{I}_{\hbar_{ij}} = (\lambda_{ij}, \xi_{ij})$ is called Pythagorean fuzzy number (PFN).

Let $\mathfrak{I}_{\hbar_{1j}} = (\lambda_{1j}, \xi_{1j})$ $(j = 1, 2)$, and $\mathfrak{I} = (\lambda, \xi)$ be any three PFNs and $\alpha, \alpha_1, \alpha_2 > 0$. Then the following operations are given as:

(i) $\mathfrak{I}_{\hbar_{11}} \cup \mathfrak{I}_{\hbar_{12}} = (max(\lambda_{11}, \lambda_{12}), min(\xi_{11}, \xi_{12}))$;
(ii) $\mathfrak{I}_{\hbar_{11}} \cap \mathfrak{I}_{\hbar_{12}} = (min(\lambda_{11}, \lambda_{12}), max(\xi_{11}, \xi_{12}))$;
(iii) $\mathfrak{I}^c = (\xi, \lambda)$, where $\mathfrak{I}^c$ denotes the complement of $\mathfrak{I}$;
(iv) $\mathfrak{I}_{\hbar_{11}} \preccurlyeq \mathfrak{I}_{\hbar_{12}}$ *if* $\lambda_{11} \geq \lambda_{12}, \xi_{11} \leq \xi_{12}$;
(v) $\mathfrak{I}_{\hbar_{11}} \oplus \mathfrak{I}_{\hbar_{12}} = \left(\sqrt{(\lambda_{11})^2 + (\lambda_{12})^2 - (\lambda_{11})^2(\lambda_{12})^2}, \xi_{11}\xi_{12}\right)$;
(vi) $\mathfrak{I}_{\hbar_{11}} \otimes \mathfrak{I}_{\hbar_{12}} = \left(\lambda_{11}\lambda_{12}, \sqrt{(\xi_{11})^2 + (\xi_{12})^2 - (\xi_{11})^2(\xi_{12})^2}\right)$;
(vii) $\alpha\mathfrak{I} = \left(\sqrt{1 - [1 - \lambda^2]^\alpha}, \xi^\alpha\right)$;
(viii) $\mathfrak{I}^\alpha = \left(\lambda^\alpha, \sqrt{1 - [1 - \xi^2]^\alpha}\right)$.

III. q-RUNG ORTHOPAIR FUZZY SOFT SET

The prominent notion of IFS was originated by Atanassov [2] in 1986, which is characterized by two function, that is MG and NMG and their sum belongs to [0,1]. After that Yager [17] presented the notion of PFS which gives more freedom and relaxation in the boundary range for the professional that is the square sum of MG and NMG must not exceed the unit interval [0,1], which attract the researchers of recent area. In 2016 Yager [30], investigated the prominent generalization of PFS and called it q-ROFS, in which the q^{th} power of MG and NMG belongs to [0,1]. Hussain *et al.* [54] initiated the combine study of q-ROFS and S_{ft}S and is called as q-ROFS_{ft}S, and is given as:

Definition 7: [54]. Suppose a soft universe $(S, \mathcal{E})$ and $\mathcal{A} \subseteq \mathcal{E}$. A pair $(T, \mathcal{A})$ is said to be a q-ROFS_{ft}S over S, where T is a mapping denoted by $\mathrm{T} : \mathcal{A} \to q - ROFS^{(S)}$, which is given as

$$\mathrm{T}_{\hbar_j}(a_i) = \left\{ \prec a_i, \mu_j(a_i), \eta_j(a_i) \succ_q | a_i \in S, \hbar_j \in \mathcal{A} \text{ and } q \geq 1 \right\}$$

where $\mathrm{q-ROFS}^{(S)}$ denotes all q-ROFSs of S. Here $\mu_j(a_i), \eta_j(a_i)$ denotes the MG and NMG of an alternative $a_i \in S$ to a set $\mathrm{T}_{\hbar_j}$ respectively, and hold the restriction that $0 \leq (\mu_j(a_i))^q + (\eta_j(a_i))^q \leq 1$ and $q \geq 1$. For simplicity $\mathrm{T}_{\hbar_j}(a_i) = \prec a_i, \mu_j(a_i), \eta_j(a_i) \succ_q$ is denoted by $\aleph_{\hbar_{ij}} = (\mu_{ij}, \eta_{ij})$ represents a q-ROFS_{ft} number (q-ROFS_{ft}N). Further, the hesitancy degree for q-ROFS_{ft}N is given as$\pi_{\aleph_{\hbar_{ij}}} = \sqrt[q]{1 - \left((\mu_{ij})^q + (\eta_{ij})^q\right)}$. A set of all q-ROF$S_{ft}$S is denoted by $q - ROFS_{ft}S^{(S)}$.

Let $\aleph_{\hbar_{1j}} = (\mu_{1j}, \eta_{1j})$ (j=1, 2), and $\aleph = (\mu, \eta)$ be any three q-ROFS_{ft}Ns and $\alpha > 0$. The following operations are defined as follows:

i $\aleph_{\hbar_{11}} \cup \aleph_{\hbar_{12}} = (max(\mu_{11}, \mu_{12}), min(\eta_{11}, \eta_{12}))$;

ii $\aleph_{\hbar_{11}} \cap \aleph_{\hbar_{12}} = (min(\mu_{11}, \mu_{12}), max(\eta_{11}, \eta_{12}))$;

iii $\aleph^c = (\eta, \mu)$, where $\aleph^c$ denotes the complement of $\aleph$;

iv $\aleph_{\hbar_{11}} \preccurlyeq \aleph_{\hbar_{12}} if\, \mu_{11} \leq \mu_{12}, \eta_{11} \geq \eta_{12}$;

v $\aleph_{\hbar_{11}} \oplus \aleph_{\hbar_{12}} = \left(\sqrt[q]{(\mu_{11})^q + (\mu_{12})^q - (\mu_{11})^q (\mu_{12})^q}, \eta_{11}\eta_{12}\right)$;

vi $\aleph_{\hbar_{11}} \otimes \aleph_{\hbar_{12}} = \left(\mu_{11}\mu_{12}, \sqrt[q]{(\eta_{11})^q + (\eta_{12})^q - (\eta_{11})^q (\eta_{12})^q}\right)$;

vii $\alpha\aleph = \left(\sqrt[q]{1 - [1 - \mu^q]^\alpha}, \eta^\alpha\right)$;

viii $\aleph^\alpha = \left(\mu^\lambda, \sqrt[q]{1 - [1 - \eta^q]^\alpha}\right)$.

Example 1: Suppose a decision maker purchase a cellphone form a set having of five objects that is $S = \{a_1, a_2, a_3, a_4, a_5\}$. Let $E = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4\}$ be the corresponding set of parameters where $\hbar_1$ = high quality audio, video and voice call, $\hbar_2$ = impressive design with high resolution camera, $\hbar_3$ = high battery timing, $\hbar_4$ = reasonable price. From the above attribute a decision maker gives their assessment for each alternatives in the form of q-ROFS_{ft} Ns as given in Table 1;

Theorem 1: Let $\aleph_{\hbar_{1j}} = (\mu_{1j}, \eta_{1j})$ (j = 1, 2), and $\aleph = (\mu, \eta)$ be any three q-ROFS_{ft}Ns and $\alpha, \alpha_1, \alpha_2 > 0$. Then the following are holds:

(i) $\aleph_{\hbar_{11}} \oplus \aleph_{\hbar_{12}} = \aleph_{\hbar_{12}} \oplus \aleph_{\hbar_{11}}$;

(ii) $\aleph_{\hbar_{11}} \otimes \aleph_{\hbar_{12}} = \aleph_{\hbar_{12}} \otimes \aleph_{\hbar_{11}}$;

(iii) $\alpha\left(\aleph_{\hbar_{11}} \oplus \aleph_{\hbar_{12}}\right) = \alpha\aleph_{\hbar_{11}} \oplus \alpha\aleph_{\hbar_{12}}$;

(iv) $(\alpha_1 + \alpha_2)\aleph = \alpha_1\aleph \oplus \alpha_2\aleph$;

(v) $\aleph^{(\alpha_1+\alpha_2)} = \aleph^{\alpha_1} \otimes \aleph^{\alpha_2}$;

(vi) $\aleph^\alpha_{\hbar_{11}} \otimes \aleph^\alpha_{\hbar_{12}} = \left(\aleph_{\hbar_{11}} \otimes \aleph_{\hbar_{11}}\right)^\alpha$.

Proof Proofs are straightforward.

Definition 8: [55] Consider a curve function $h(a) = \frac{e^a}{1+e^a}$; $a \in (-\infty, \infty)$, having the following properties:

(i) $h(a)$ is strictly monotonically increasing for $a \in (-\infty, \infty)$;

(ii) $h(0) = 0.5$ *and* $h(a) + h(-a) = 1$;

(iii) $0 < h(a) < 1$.

Definition 9: Let $\aleph_{\hbar_{ij}} = (\mu_{ij}, \eta_{ij})$ be a q-ROFS_{ft}N. Then score function for $\aleph_{\hbar_{ij}}$ is given as

$$S\left(\aleph_{\hbar_{ij}}\right) = \mu^q_{ij} - \eta^q_{ij} + \left(\frac{e^{\mu^q_{ij}-\eta^q_{ij}}}{e^{\mu^q_{ij}-\eta^q_{ij}}+1} - \frac{1}{2}\right)\pi^q_{\aleph_{\hbar_{ij}}} \text{ for } q \geq 1$$

and $S\left(\aleph_{\hbar_{ij}}\right) \in [-1, 1]$

Let $\aleph_{\hbar_{11}} = (\mu_{11}, \eta_{11})$ *and* $\aleph_{\hbar_{12}} = (\mu_{12}, \eta_{12})$ be two q-ROFS_{ft}Ns. Then

1) If $S\left(\aleph_{\hbar_{11}}\right) > S\left(\aleph_{\hbar_{12}}\right)$, then $\aleph_{\hbar_{11}} \succcurlyeq \aleph_{\hbar_{12}}$;
2) If $S\left(\aleph_{\hbar_{11}}\right) < S\left(\aleph_{\hbar_{12}}\right)$, then $\aleph_{\hbar_{11}} \preccurlyeq \aleph_{\hbar_{12}}$;
3) If $S\left(\aleph_{\hbar_{11}}\right) = S\left(\aleph_{\hbar_{12}}\right)$, then
   (a) If $\pi_{\aleph_{\hbar_{11}}} > \pi_{\aleph_{\hbar_{12}}}$ then $\aleph_{\hbar_{11}} \prec \aleph_{\hbar_{12}}$;
   (b) If $\pi_{\aleph_{\hbar_{11}}} = \pi_{\aleph_{\hbar_{12}}}$ then$\aleph_{\hbar_{11}} = \aleph_{\hbar_{12}}$.

IV. q-RUNG ORTHOPAIR FUZZY SOFT GEOMETRIC AGGREGATION OPERATOR

This section, is allotted to the detail study of q-ROFS_{ft} weighted geometric (q-ROFS_{ft}WG), q-ROFS_{ft} ordered weighted geometric (q-ROFS_{ft}OWG) and q-ROFS_{ft} hybrid geometric (q-ROFS_{ft}HG) operators and also proved their basic properties in detail.

A. q-RUNF ORTHOPAIR FUZZY SOFT WEIGHTED GEOMETRIC OPERATOR

This subsection, consists of the detail study of q-ROFS_{ft}WG operator and discuss their fundamental properties.

Definition 10: Let$\aleph_{\hbar_{ij}} = (\mu_{ij}, \eta_{ij})$ *for* $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, m$ be the collection of q-ROFS_{ft}Ns, and suppose the weight vectors $u = \{u_1, u_2, \ldots, u_n\}$for the decision makers a_i and $z = \{z_1, z_2, \ldots, z_m\}$ and for the parameters $\hbar'_j$ respectively; and satisfying the restrictions that $u_i, z_j \in [0, 1]$ with$\sum_{i=1}^n u_i = 1$ and $\sum_{j=1}^m z_j = 1$. Then q-ROFS_{ft}WG operator is a mapping denoted and defined as: $\mathrm{q-ROF}S_{ft}\mathrm{WG} : \mathcal{H}^n \to \mathcal{H}$, (where $\mathcal{H}$ contains the collection of q-ROFS_{ft}Ns)

$$\mathrm{q-ROF}S_{ft}\mathrm{WG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) = \otimes_{j=1}^m \left(\otimes_{i=1}^n \aleph^{u_i}_{\hbar_{ij}}\right)^{z_j}$$

The following Theorem 2, describe the aggregation result for q-ROFS_{ft} WG operator.

Theorem 2: Suppose the collections $\aleph_{\hbar_{ij}} = (\mu_{ij}, \eta_{ij})$ for $i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, m$, of q-ROFS_{ft}Ns. Then

TABLE 1. Tabular notation of q-ROFS_{ft}S $(\mathsf{T}, \mathcal{A})$; for $q \geq 3$.

S	$\hbar_1$	$\hbar_2$	$\hbar_3$	$\hbar_4$
a_1	(0.92, 0.45)	(0.82, 0.45)	(0.66, 0.35)	(0.9, 0.32)
a_2	(0.83, 0.32)	(0.55, 0.15)	(0.78, 0.45)	(0.87, 0.33)
a_3	(0.9, 0.23)	(0.77, 0.35)	(0.65, 0.4)	(0.7, 0.45)
a_4	(0.76, 0.4)	(0.84, 0.62)	(0.9, 0.2)	(0.75, 0.25)

the aggregation result for q-ROFS_{ft}WG operator is defined as:

$$q-\text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) = \otimes_{j=1}^{m}\left(\otimes_{i=1}^{n}\aleph_{\hbar_{ij}}^{u_i}\right)^{z_j} = \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{ij}^{u_i}\right)^{z_j}, \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\eta_{ij}^{q})^{u_i}\right)^{z_j}}\right) \quad (1)$$

where $u = \{u_1, u_2, \ldots, u_n\}$ and be weight vector for decision makers $z = \{z_1, z_2, \ldots, z_m\}$ be the for the parameters h_j respectively; which satisfying that $u_i, z_j \in [0, 1]$ with $\sum_{i=1}^{n} u_i = 1$ *and* $\sum_{j=1}^{m} z_j = 1$.

Proof : By utilizing mathematical induction to prove the aggregation result of Eq. 1.

Consider the operation laws of q-ROFS_{ft}S, that is

$$\aleph_{\hbar_{11}} \otimes \aleph_{\hbar_{12}} = \left(\mu_{11}\mu_{12}, \sqrt[q]{(\eta_{11})^q + (\eta_{12})^q - (\eta_{11})^q(\eta_{12})^q}\right)$$
$$and\ \aleph^{\alpha} = \left(, \mu^{\alpha}, \sqrt[q]{1-[1-\eta^q]^{\alpha}}\right) for\alpha \geq 1$$

First we will show that the Eq. 1, is true for $n = 2$ *and* $m = 2$, so we have, shown at the bottom of next page.

$$= \left(\prod_{j=1}^{2}\left(\prod_{i=1}^{2}\mu_{ij}^{u_i}\right)^{z_j}, \sqrt[q]{1-\prod_{j=1}^{2}\left(\prod_{i=1}^{2}\left(1-\eta_{ij}^{q}\right)^{u_i}\right)^{z_j}}\right)$$

Hence the result is true for $n = 2$ *and* $m = 2$, Next suppose that Eq. 1, is true for $n = k_1$ *and* $m = k_2$

$$q-\text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{h_{k_1k_2}}\right) = \otimes_{j=1}^{k_2}\left(\otimes_{i=1}^{k_1}\aleph_{\hbar_{ij}}^{u_i}\right)^{z_j} = \left(\prod_{j=1}^{k_2}\left(\prod_{i=1}^{k_1}\mu_{ij}^{u_i}\right)^{z_j}, \sqrt[q]{1-\prod_{j=1}^{k_2}\left(\prod_{i=1}^{k_1}\left(1-\eta_{ij}^{q}\right)^{u_i}\right)^{z_j}}\right)$$

Further suppose that Eq. 1, is true for $n = k_1 + 1$ *and* $m = k_2 + 1$

$$q-\text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{k_1k_2}}, \aleph_{h_{(k_1+1)(k_2+1)}}\right)$$
$$= q-\text{ROF}S_{ft}\text{WG}\left(\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{k_1k_2}}\right), \times \aleph_{\hbar_{(k_1+1)(k_2+1)}}\right)$$
$$= \left\{\otimes_{j=1}^{k_2}\left(\otimes_{i=1}^{k_1}\aleph_{\hbar_{ij}}^{u_i}\right)^{z_j}\right\} \otimes \left(\aleph_{\hbar_{(k_1+1)(k_2+1)}}^{u_{(k_2+1)}}\right)^{z_{(k_1+1)}}$$
$$= \left(\prod_{j=1}^{k_2}\left(\prod_{i=1}^{k_1}\mu_{ij}^{u_i}\right)^{z_j}, \sqrt[q]{1-\prod_{j=1}^{k_2}\left(\prod_{i=1}^{k_1}\left(1-\eta_{ij}^{q}\right)^{u_i}\right)^{z_j}}\right) \otimes \left(\aleph_{h_{(k_1+1)(k_2+1)}}^{u_{(k_2+1)}}\right)^{z_{(k_1+1)}}$$
$$= \left(\frac{\prod_{j=1}^{(k_2+1)}\left(\prod_{i=1}^{(k_1+1)}\mu_{ij}^{u_i}\right)^{z_j},}{\sqrt[q]{1-\prod_{j=1}^{(k_2+1)}\left(\prod_{i=1}^{(k_1+1)}\left(1-\eta_{ij}^{q}\right)^{u_i}\right)^{z_j}}}\right)$$

Hence Eq. 1, is true for $n = k_1 + 1$ *and* $m = k_2 + 1$. Therefore, by induction process the *Eq.*1 is true for all values of $m, n \geq 1$.

Moreover, to prove the aggregated result achieved from q-ROFS_{ft}WG operator is again a q-ROFS_{ft}N. Now for any $\aleph_{\hbar_{ij}} = (\mu_{ij}, \eta_{ij}), (i = 1, 2, \ldots, n)$ and $j = 1, 2, \ldots, m$, where $0 \leq \mu_{ij}, \eta_{ij} \leq 1$, satisfying that $0 \leq \mu_{ij}^q + \eta_{ij}^q \leq 1$, with weight vectors $u = \{u_1, u_2, \ldots, u_n\}$ and $z = \{z_1, z_2, \ldots, z_m\}$ for the decision maker a_i and for the parameters $\hbar_j$ respectively; which satisfying that $u_i, z_j \in [0, 1]$ with $\sum_{i=1}^{n} u_i = 1$ *and* $\sum_{j=1}^{m} z_j = 1$.

As,

$$0 \leq \mu_{ij} \leq 1 \Rightarrow 0 \leq \prod_{i=1}^{n}\mu_{ij}^{u_i} \leq 1$$
$$\Rightarrow 0 \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{ij}^{u_i}\right)^{z_j} \leq 1$$

Similarly, $0 \leq \eta_{ij} \leq 1 \Rightarrow 0 \leq 1 - \eta_{ij} \leq 1 \Rightarrow 0 \leq \left(1-\eta_{ij}^{q}\right)^{u_i} \leq 1$

$$\Rightarrow 0 \leq \prod_{i=1}^{n}\left(1-\eta_{ij}^{q}\right)^{u_i} \leq 1$$
$$\Rightarrow 0 \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\eta_{ij}^{q}\right)^{u_i}\right)^{z_j} \leq 1$$

$$\Rightarrow 0 \leq \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\eta_{ij}^{q}\right)^{u_i}\right)^{z_j}} \leq 1$$

As

$$\mu_{ij}^{q}+\eta_{ij}^{q} \leq 1 \Rightarrow \mu_{ij}^{q} \leq 1-\eta_{ij}^{q}$$
$$\Rightarrow \prod_{i=1}^{n}\left(\mu_{ij}^{q}\right)^{u_i} \leq \left(\prod_{i=1}^{n}\left(1-\eta_{ij}^{q}\right)^{u_i}\right)$$
$$\Rightarrow \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{ij}^{q}\right)^{u_i}\right)^{z_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\eta_{ij}^{q}\right)^{u_i}\right)^{z_j}$$
$$\Rightarrow \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{ij}^{u_i}\right)^{z_j}\right)^{q} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\eta_{ij}^{q}\right)^{u_i}\right)^{z_j} \quad (2)$$

Now we have

$$0 \leq \left\{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{ij}^{u_i}\right)^{z_j}\right\}^{q} + \left\{\sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\eta_{ij}^{q}\right)^{u_i}\right)^{z_j}}\right\}^{q}$$

by $Eq.2$, wehave $\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\eta_{ij}^{q}\right)^{u_i}\right)^{z_j} + 1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\eta_{ij}^{q}\right)^{u_i}\right)^{z_j} = 1$

Therefore,

$$0 \leq \left\{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{ij}^{u_i}\right)^{z_j}\right\}^{q} + \left\{\sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\eta_{ij}^{q}\right)^{u_i}\right)^{z_j}}\right\}^{q} \leq 1$$

Therefore, from the above analysis we observed that the aggregation result obtained from q-ROFS_{ft}WG operator is again a q-ROFS_{ft}N.

Remark 1: (a) When rung $q = 1$, so in this case the developed q-ROFS_{ft}WG operator degenerate into IF S_{ft}WG operator.

(b) When rung $q = 2$, so in this case the investigated q-ROFS_{ft}WG operator degenerate into PFS_{ft}WG operator.

(c) If the parameter set contain just one element, i.e.$\hbar_1$(mean $m = 1$), so in this case the developed q-ROFS_{ft}WG operator degenerate to q-ROFWG operator.

It is clear from Remark 1, that IFWG, IFS_{ft}WG, PFS_{ft}WG and q-ROFWG operators are the special cases of the developed operator.

Example 2: Consider a decision maker Mr. Z purchase a house in the domain set $S = \{a_1, a_2, a_3, a_4, a_5\}$ and let $\mathcal{E} = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4\}$ be the criterion (parameters) set, i.e. $\hbar_i(i = 1, 2, 3, 4)$ stands for $\hbar_1$ = beautiful, $\hbar_2$ = in green surrounding, $\hbar_3$ = expenxive, $\hbar_4$ =safety respectively. Suppose $u = \{0.26, 0.12, 0.23, 0.2, 0.19\}$ be the weight vectors for expert a_i and $z = \{0.26, 0.21, 0.29, 0.24\}$ be the weight vector for parameters $\hbar_j$ respectively. The decision maker gives their assessment for each alternative to against their parameters in the form of q-ROFS_{ft}Ns, which is given in Table 2.

By using $Eq.1$, we have, shown at the bottom of next page.

From the analysis of Theorem 2, the q-ROFS_{ft}WG operator fulfill the following properties for the collection q-ROFS_{ft}Ns $\aleph_{\hbar_{ij}} = \left(\mu_{ij}, \eta_{ij}\right)$, is being presented.

Theorem 3: Let $\aleph_{\hbar_{ij}} = \left(\mu_{ij}, \eta_{ij}\right), (i = 1, 2, \ldots, n)$ and $(j = 1, 2, \ldots, m)$, the collections of q-ROFSNs with weight vectors $u = (u_1, u_2, \ldots, u_n)^T$ and $z = (z_1, z_2, \ldots, z_m)^T$ for the decision makers a_i and for the parameters h_j respectively, such that $u_i, z_j \in [0, 1]$ with $\sum_{i=1}^{n} u_i = 1$ *and* $\sum_{j=1}^{m} z_j = 1$. Then the q-ROFS_{ft}WG operator satisfying the following properties;

i : (**Idempotency**) : If $\aleph_{\hbar_{ij}} = \mathcal{L}_{\hbar}$, where $\mathcal{L}_{\hbar} = (p, r)$ then

$$q-\text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) = \mathcal{L}_{\hbar}.$$

$$q-\text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar_{ij}}, \aleph_{\hbar_{ij}}\right) = \otimes_{j=1}^{2}\left(\otimes_{i=1}^{2}\aleph_{\hbar_{ij}}^{u_i}\right)^{z_j} = \left(\otimes_{i=1}^{2}\aleph_{\hbar_{i1}}^{u_i}\right)^{z_1} \otimes \left(\otimes_{i=1}^{2}\aleph_{\hbar_{i2}}^{u_i}\right)^{z_2}$$
$$= \left(\aleph_{\hbar_{11}}^{u_1} \otimes \aleph_{h_{21}}^{u_2}\right)^{z_1} \otimes \left(\aleph_{\hbar_{12}}^{u_1} \otimes \aleph_{\hbar_{22}}^{u_2}\right)^{z_2}$$
$$= \left\{\left(\mu_{11}^{u_1}, \sqrt[q]{1-(1-\eta_{11}^{q})^{u_1}}\right) \otimes \left(\mu_{21}^{u_2}, \sqrt[q]{1-(1-\eta_{21}^{q})^{u_2}}\right)\right\}^{z_1} \otimes \left\{\begin{array}{l}\left(\mu_{12}^{u_1}, \sqrt[q]{1-(1-\eta_{12}^{q})^{u_1}}\right) \otimes \\ \left(\mu_{22}^{u_2}, \sqrt[q]{1-(1-\eta_{22}^{q})^{u_2}}\right)\end{array}\right.$$
$$= \left\{\left(\prod_{i=1}^{2}\mu_{i1}^{u_i}, \sqrt[q]{1-\prod_{i=1}^{2}(1-\eta_{i1}^{q})^{u_i}}\right)\right\}^{z_1} \otimes \left\{\left(\prod_{i=1}^{2}\mu_{i2}^{u_i}, \sqrt[q]{1-\prod_{i=1}^{2}(1-\eta_{i2}^{q})^{u_i}}\right)\right\}^{z_2}$$
$$= \left(\left(\prod_{i=1}^{2}\mu_{i1}^{u_i}\right)^{z_1}, \sqrt[q]{1-\left(\prod_{i=1}^{2}(1-\eta_{i1}^{q})^{u_i}\right)^{z_1}}\right) \otimes \left(\left(\prod_{i=1}^{2}\mu_{i1}^{u_i}\right)^{z_2}, \sqrt[q]{1-\left(\prod_{i=1}^{2}(1-\eta_{i1}^{q})^{u_i}\right)^{z_2}}\right)$$

TABLE 2. Tabular notation of q-ROFS_{ft}S $(\mathcal{T}, \mathcal{A})$ for q ≥ 3.

S	$\hbar_1$	$\hbar_2$	$\hbar_3$	$\hbar_4$
a_1	(0.78, 0.34)	(0.86, 0.42)	(0.72, 0.26)	(0.93, 0.4)
a_2	(0.93, 0.25)	(0.76, 0.36)	(0.87, 0.41)	(0.87, 0.5)
a_3	(0.91, 0.24)	(0.92, 0.35)	(0.86, 0.42)	(0.77, 0.25)
a_4	(0.75, 0.26)	(0.85, 0.34)	(0.93, 0.25)	(0.94, 0.28)
a_5	(0.85, 0.35)	(0.94, 0.35)	(0.78, 0.3)	(0.92, 0.46)

ii : (**Boundedness**) :
If $\aleph^{-}_{\hbar_{ij}} = \left(\min_j \min_i \{\mu_{ij}\}, \max_j \max_i \{\eta_{ij}\}\right)$, and $\aleph^{+}_{\hbar_{ij}} = \left(\min_j \min_i \{\mu_{ij}\}, \max_j \max_i \{\eta_{ij}\}\right)$, then

$$\aleph^{-}_{\hbar_{ij}} \leq \text{q}-\text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) \leq \aleph^{+}_{\hbar_{ij}}.$$

iii : (**Monotonicity**) : If $\mathcal{L}_{\hbar_{ij}} = \left(\mathscr{p}_{ij}, \mathscr{r}_{ij}\right)$, ($i$= 1, 2, ..., n) and (j= 1, 2, ..., m), be the another collection of q-ROFS_{ft}Ns such that $\mu_{ij} \leq \mathscr{p}_{ij}$ *and* $\eta_{ij} \geq \mathscr{r}_{ij}$, then

$$\begin{aligned}&\text{q}-\text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right)\\&\qquad \leq \text{q}-\text{ROF}S_{ft}\text{WG}\left(\mathcal{L}_{\hbar_{11}}, \mathcal{L}_{\hbar_{12}}, \ldots, \mathcal{L}_{\hbar_{nm}}\right).\end{aligned}$$

iv : (**ShiftInvariance**) : If $\tilde{\mathcal{L}}_{\hbar} = (\mathscr{p}, \mathscr{r})$, is another q-ROF$S_{ft}$N, then

$$\begin{aligned}&\text{q}-\text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar_{11}} \otimes \mathcal{L}_{\hbar}, \aleph_{\hbar_{12}} \otimes \mathcal{L}_{\hbar}, \ldots, \aleph_{\hbar_{nm}} \otimes \mathcal{L}_{\hbar}\right)\\&\qquad = \text{q}-\text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) \otimes \mathcal{L}_{\hbar}.\end{aligned}$$

iv : (**Homogeneity**) : For a real number $\lambda > 0$, then

$$\begin{aligned}&\text{q}-\text{ROF}S_{ft}\text{WG}\left(\lambda\aleph_{\hbar_{11}}, \lambda\aleph_{\hbar_{12}}, \ldots, \lambda\aleph_{\hbar_{nm}}\right)\\&\qquad = \lambda\text{q}-\text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right).\end{aligned}$$

Proof **i** : (**Idempotency**) As it is given that if for all $\aleph_{\hbar_{ij}} = \mathcal{L}_{\hbar} = (\mathscr{p}, \mathscr{r})$ ($\forall i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, m$), then from Theorem 1, we have

$$\begin{aligned}&\text{q}-\text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right)\\
&= \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{ij}^{u_i}\right)^{z_j}, \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\eta_{ij}^{q}\right)^{\hbar_i}\right)^{z_j}}\right)\\
&= \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mathscr{p}^{\hbar_i}\right)^{z_j}, \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{r}^{q}\right)^{\hbar_i}\right)^{z_j}}\right)\\
&= \left(\left(\mathscr{p}^{\sum_{i=1}^{n}u_i}\right)^{\sum_{j=1}^{m}z_j}, \sqrt[q]{1-\left(\left(1-\mathscr{r}^{q}\right)^{\sum_{i=1}^{n}u_i}\right)^{\sum_{j=1}^{m}z_j}}\right)\\
&= \left(\mathscr{p}, \sqrt[q]{1-\left(1-\mathscr{r}^{q}\right)}\right) = (\mathscr{p}, \mathscr{r}) = \tilde{\mathcal{L}}_{\hbar}\end{aligned}$$

Therefore, $\text{q}-\text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) = \mathcal{L}_{\hbar}$.

ii : (**Boundedness**) :
As $\aleph^{-}_{\hbar_{ij}} = \left(\min_j \min_i \{\mu_{ij}\}, \max_j \max_i\{\eta_{ij}\}\right)$ and $\aleph^{+}_{\hbar_{ij}} = \left(\max_j \max_i \{\mu_{ij}\}, \min_j \min_i\{\eta_{ij}\}\right)$. To prove that $\aleph^{-}_{\hbar_{ij}} \leq \text{q}-\text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) \leq \aleph^{+}_{\hbar_{ij}}$, Now for every $i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, m$, we have

$$\begin{aligned}&\min_j \min_i \{\mu_{ij}\} \leq \mu_{ij} \leq \max_j \max_i \{\mu_{ij}\}\\
&\Leftrightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\min_j \min_i \{\mu_{ij}\}\right)^{u_i}\right)^{z_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mu_{ij}\right)^{u_i}\right)^{z_j}\end{aligned}$$

$$\begin{aligned}&\text{q}-\text{ROF}S_{ft}\text{WA}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{h_{54}}\right) = \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{ij}^{u_i}\right)^{z_j}, \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\eta_{ij}^{q}\right)^{u_i}\right)^{z_j}}\right)\\
&= \left(\begin{array}{l}\left\{\left(0.78^{0.26}\right)\left(0.93^{0.12}\right)\left(0.91^{0.23}\right)\left(0.75^{0.2}\right)\left(0.85^{0.19}\right)\right\}^{0.26}\left\{\left(0.86^{0.26}\right)\left(0.76^{0.12}\right)\left(0.92^{0.23}\right)\left(0.85^{0.2}\right)\left(0.94^{0.19}\right)\right\}^{0.21}\\ \left\{\left(0.72^{0.26}\right)\left(0.87^{0.12}\right)\left(0.86^{0.23}\right)\left(0.93^{0.2}\right)\left(0.78^{0.19}\right)\right\}^{0.29}\left\{\left(0.93^{0.26}\right)\left(0.87^{0.12}\right)\left(0.77^{0.23}\right)\left(0.94^{0.2}\right)\left(0.92^{0.19}\right)\right\}^{0.24}\end{array},\right.\\
&\left.\sqrt[3]{\begin{array}{l}1-\left\{\left(1-0.34^{3}\right)^{0.26}\left(1-0.25^{3}\right)^{0.12}\left(1-0.24^{3}\right)^{0.23}\left(1-0.26^{3}\right)^{0.2}\left(1-0.35^{3}\right)^{0.19}\right\}^{0.26}\\ \left\{\left(1-0.42^{3}\right)^{0.26}\left(1-0.36^{3}\right)^{0.12}\left(1-0.35^{3}\right)^{0.23}\left(1-0.34^{3}\right)^{0.2}\left(1-0.35^{3}\right)^{0.19}\right\}^{0.21}\\ \left\{\left(1-0.26^{3}\right)^{0.26}\left(1-0.41^{3}\right)^{0.12}\left(1-0.42^{3}\right)^{0.23}\left(1-0.25^{3}\right)^{0.2}\left(1-0.3^{3}\right)^{0.19}\right\}^{0.29}\\ \left\{\left(1-0.4^{3}\right)^{0.26}\left(1-0.5^{3}\right)^{0.12}\left(1-0.25^{3}\right)^{0.23}\left(1-0.28^{3}\right)^{0.2}\left(1-0.46^{3}\right)^{0.19}\right\}^{0.24}\end{array}}\right) = (0.849189, 0.350549).\end{aligned}$$

$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\max_j \max_i \{\mu_{ij}\}\right)^{\varpi_i}\right)^{z_j}$$

$$\Leftrightarrow \left(\left(\min_j \min_i \{\mu_{ij}\}\right)^{\sum_{i=1}^{n}\varpi_i}\right)^{\sum_{j=1}^{m} z_j}$$

$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mu_{ij})^{\varpi_i}\right)^{z_j} \leq \left(\left(\max_j \max_i \{\mu_{ij}\}\right)^{\sum_{i=1}^{n}\varpi_i}\right)^{\sum_{j=1}^{m} z_j}$$

this implies that

$$\min_j \min_i \{\mu_{ij}\} \leq \{\mu_{ij}\} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mu_{ij})^{\varpi_i}\right)^{z_j} \leq \max_j \max_i \{\mu_{ij}\} \tag{3}$$

Next for each $i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, m$, we have

$$\min_j \min_i \left\{\eta_{ij}^q\right\} \leq \eta_{ij}^q \leq \max_j \max_i \{\eta_{ij}\} \Leftrightarrow$$

$$1 - \max_j \max_i \left\{\eta_{ij}^q\right\} \leq 1 - \eta_{ij}^q \leq 1 - \min_j \min_i \left\{\eta_{ij}^q\right\}$$

$$\Leftrightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \max_j \max_i \left\{\eta_{ij}^q\right\}\right)^{\varpi_i}\right)^{z_j}$$

$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \eta_{ij}^q\right)^{\varpi_i}\right)^{z_j}$$

$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \min_j \min_i \left\{\eta_{ij}^q\right\}\right)^{\varpi_i}\right)^{z_j}$$

$$\Leftrightarrow \left(\left(1 - \max_j \max_i \left\{\eta_{ij}^q\right\}\right)^{\sum_{i=1}^{n}\varpi_i}\right)^{\sum_{j=1}^{m} z_j}$$

$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \eta_{ij}^q\right)^{\varpi_i}\right)^{z_j}$$

$$\leq \left(\left(1 - \min_j \min_i \left\{\eta_{ij}^q\right\}\right)^{\sum_{i=1}^{n}\varpi_i}\right)^{\sum_{j=1}^{m} z_j}$$

$$\Leftrightarrow \left(1 - \max_j \max_i \left\{\eta_{ij}^q\right\}\right) \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \eta_{ij}^q\right)^{\varpi_i}\right)^{z_j}$$

$$\leq \left(1 - \min_j \min_i \left\{\eta_{ij}^q\right\}\right)$$

$$\Leftrightarrow 1 - \left(1 - \min_j \min_i \left\{\eta_{ij}^q\right\}\right) \leq 1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \eta_{ij}^q\right)^{\varpi_i}\right)^{z_j}$$

$$\leq 1 - \left(1 - \left\{\eta_{ij}^q\right\}\right)$$

$$\min_j \min_i \left\{\eta_{ij}^q\right\} \leq 1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \eta_{ij}^q\right)^{\varpi_i}\right)^{z_j} \leq \max_j \max_i \left\{\eta_{ij}^q\right\}$$

and

$$\min_j \min_i \{\eta_{ij}\} \leq \sqrt[q]{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \eta_{ij}^q\right)^{\varpi_i}\right)^{z_j}} \leq \max_j \max_i \{\eta_{ij}\}$$

Hence

$$\min_j \min_i \{\eta_{ij}\} \leq \sqrt[q]{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \eta_{ij}^q\right)^{\varpi_i}\right)^{z_j}} \max_j \max_i \leq \{\eta_{ij}\} \tag{4}$$

Therefore, from *Eqs*.(3) *and* (4), we have

Let $\delta = \text{q} - \text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) = (\mu_\delta, \eta_\delta)$, then by using score function, we have

$$\begin{aligned} S(\delta) &= \mu_\delta^q - \eta_\delta^q + \left(\frac{e^{\mu_\delta^q - \eta_\delta^q}}{e^{\mu_\delta^q - \eta_\delta^q} + 1} - \frac{1}{2}\right)\pi_\delta^q \\ &\leq \left(\max_j \max_i \{\mu_{ij}\}\right)^q - \left(\min_j \min_i \{\eta_{ij}\}\right)^q \\ &\quad + \left(\frac{e^{\left(\max_j \max_i \{\mu_{ij}\}\right)^q - \left(\min_j \min_i \{\eta_{ij}\}\right)^q}}{e^{\left(\max_j \max_i \{\mu_{ij}\}\right)^q - \left(\min_j \min_i \{\eta_{ij}\}\right)^q} + 1} - \frac{1}{2}\right)\pi_{\aleph_{\hbar_{ij}}^+}^q \\ &= S\left(\aleph_{\hbar_{ij}}^+\right) \Rightarrow S(\delta) \leq S\left(\aleph_{\hbar_{ij}}^+\right) \end{aligned}$$

and

$$\begin{aligned} S(\delta) &= \mu_\delta^q - \eta_\delta^q + \left(\frac{e^{\mu_\delta^q - \eta_\delta^q}}{e^{\mu_\delta^q - \eta_\delta^q} + 1} - \frac{1}{2}\right)\pi_\delta^q \geq \\ &\quad \left(\min_j \min_i \{\mu_{ij}\}\right)^q - \left(\max_j \max_i \{\eta_{ij}\}\right)^q \\ &\quad + \left(\frac{e^{\left(\min_j \min_i \{\mu_{ij}\}\right)^q - \left(\max_j \max_i \{\eta_{ij}\}\right)^q}}{e^{\left(\min_j \min_i \{\mu_{ij}\}\right)^q - \left(\max_j \max_i \{\eta_{ij}\}\right)^q} + 1} - \frac{1}{2}\right)\pi_{\aleph_{\hbar_{ij}}^-}^q \\ &= S\left(\aleph_{\hbar_{ij}}^-\right) \Rightarrow S(\delta) \geq S\left(\aleph_{\hbar_{ij}}^-\right). \end{aligned}$$

From the above study, the following cases arises,

Case i: If $S(\delta) < S\left(\aleph_{\hbar_{ij}}^+\right)$ *and* $S(\delta) > S\left(\aleph_{\hbar_{ij}}^-\right)$, by comparing two q-ROF$S_{ft}$Ns, we get $\aleph_{\hbar_{ij}}^- < \text{q} - \text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) < \aleph_{\hbar_{ij}}^+$.

Case ii : If $S(\delta) = S\left(\aleph_{\hbar_{ij}}^+\right)$, that is

$$\mu_\delta^q - \eta_\delta^q + \left(\frac{e^{\mu_\delta^q - \eta_\delta^q}}{e^{\mu_\delta^q - \eta_\delta^q} + 1} - \frac{1}{2}\right)\pi_\delta^q$$

$$= \left(\max_j \max_i \{\mu_{ij}\}\right)^q - \left(\min_j \min_i \{\eta_{ij}\}\right)^q +$$

$$+\left(\frac{e^{\left(\max_j \max_i\{\mu_{ij}\}\right)^q-\left(\min_j \min_i\{\eta_{ij}\}\right)^q}}{e^{\left(\max_j \max_i\{\mu_{ij}\}\right)^q-\left(\min_j \min_i\{\eta_{ij}\}\right)^q}+1}-\frac{1}{2}\right)\pi^q_{\aleph^+_{\hbar ij}},$$

then by using the above inequalities, we get $\mu_\delta = \max_j \max_i \{\mu_{ij}\}$ *and* $\eta_\delta = \min_j \min_i \{\eta_{ij}\}$. Thus $\pi^q_\delta = \pi^q_{\aleph^+_{\hbar ij}}$,
Thus from the comparison of two q-ROFS_{ft}Ns, we have

$$\text{q}-\text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar 11},\aleph_{\hbar 12},\ldots,\aleph_{\hbar nm}\right)=\aleph^+_{\hbar ij}.$$

Caseiii : If $S(\delta) = S\left(\aleph^-_{\hbar ij}\right)$, that is

$$\mu^q_\delta-\eta^q_\delta+\left(\frac{e^{\mu^q_\delta-\eta^q_\delta}}{e^{\mu^q_\delta-\eta^q_\delta}+1}-\frac{1}{2}\right)\pi^q_\delta$$
$$=\left(\min_j \min_i\{\mu_{ij}\}\right)^q-\left(\max_j \max_i\{\eta_{ij}\}\right)^q$$
$$+\left(\frac{e^{\left(\min_j \min_i\{\mu_{ij}\}\right)^q-\left(\max_j \max_i\{\eta_{ij}\}\right)^q}}{e^{\left(\min_j \min_i\{\mu_{ij}\}\right)^q-\left(\max_j \max_i\{\eta_{ij}\}\right)^q}+1}-\frac{1}{2}\right)\pi^q_{\aleph^-_{\hbar ij}},$$

then by using the above inequalities, we get $\mu_\delta = \{\mu_{ij}\}$ *and* $\eta_\delta = \{\eta_{ij}\}$. Thus $\pi^q_\delta = \pi^q_{\aleph^-_{\hbar ij}}$, this implies $\text{q}-\text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar 11},\aleph_{\hbar 12},\ldots,\aleph_{\hbar nm}\right)=\aleph^-_{\hbar ij}$.
Therefore, it is proved that

$$\aleph^-_{\hbar ij}\le q-ROFS_{ft}\text{WG}\left(\aleph_{\hbar 11},\aleph_{\hbar 12},\ldots,\aleph_{\hbar nm}\right)\le\aleph^+_{\hbar ij}.$$

iii : (**Monotonicity**) : Since $\mu_{ij} \le p_{ij}$ *and* $\eta_{ij} \ge r_{ij}$, (i= 1, 2, ..., n) *and* (j= 1, 2, ..., m), then this implies that

$$\mu_{ij}\le p_{ij}\Rightarrow\left(\prod_{i=1}^n(\mu_{ij})^{u_i}\right)\le\prod_{i=1}^n(p_{ij})^{u_i}$$
$$\Rightarrow\prod_{j=1}^m\left(\prod_{i=1}^n(\mu_{ij})^{u_i}\right)^{z_j}\le\prod_{j=1}^m\left(\prod_{i=1}^n(p_{ij})^{u_i}\right)^{z_j}\quad(5)$$

Furthermore,

$$\eta_{ij}\ge r_{ij}\Rightarrow 1-r_{ij}\ge 1-\eta_{ij}\Rightarrow 1-r^q_{ij}\ge 1-\eta^q_{ij}$$
$$\Rightarrow\prod_{j=1}^m\left(\prod_{i=1}^n\left(1-r^q_{ij}\right)^{u_i}\right)^{z_j}\ge\prod_{j=1}^m\left(\prod_{i=1}^n\left(1-\eta^q_{ij}\right)^{u_i}\right)^{z_j}$$
$$\Rightarrow 1-\prod_{j=1}^m\left(\prod_{i=1}^n\left(1-\eta^q_{ij}\right)^{u_i}\right)^{z_j}$$
$$\ge 1-\prod_{j=1}^m\left(\prod_{i=1}^n\left(1-r^q_{ij}\right)^{u_i}\right)^{z_j}$$
$$\Rightarrow\sqrt[q]{1-\prod_{j=1}^m\left(\prod_{i=1}^n\left(1-\eta^q_{ij}\right)^{u_i}\right)^{z_j}}$$
$$\ge\sqrt[q]{1-\prod_{j=1}^m\left(\prod_{i=1}^n\left(1-r^q_{ij}\right)^{u_i}\right)^{z_j}}\quad(6)$$

Let $\delta_\aleph = \text{q}-\text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar 11},\aleph_{\hbar 12},\ldots,\aleph_{\hbar nm}\right) = (\mu_{\delta_\aleph}, \eta_{\delta_\aleph})$ and $\delta_\mathcal{L} = \text{q}-\text{ROF}S_{ft}\text{WG}\left(\mathcal{L}_{\hbar 11},\mathcal{L}_{\hbar 12},\ldots,\mathcal{L}_{\hbar nm}\right) = (p_{\delta_\mathcal{L}}, r_{\delta_\mathcal{L}})$ From *Eqs.*(5) *and* (6), we have

$$\mu_{\delta_\aleph}\le p_{\delta_\mathcal{L}}\ \textit{and}\ \eta_{\delta_\aleph}\ge r_{\delta_\mathcal{L}}$$

then from a score function, we have

$$S(\delta_\aleph)\le S(\delta_\mathcal{L})$$

In view of above condition, the following cases arises,
Case i : If $S(\delta_\aleph) < S(\delta_L)$, by comparing two q-ROF$S_{ft}$Ns, we get

$$\text{q}-\text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar 11},\aleph_{\hbar 12},\ldots,\aleph_{\hbar nm}\right)$$
$$<\text{q}-\text{ROF}S_{ft}\text{WG}\left(\mathcal{L}_{\hbar 11},\mathcal{L}_{\hbar 12},\ldots,\mathcal{L}_{\hbar nm}\right).$$

Caseii : If $S(\delta_\aleph) = S(\delta_L)$, that is

$$S(\delta_\aleph)=\mu^q_{\delta_\aleph}-\eta^q_{\delta_\aleph}+\left(\frac{e^{\mu^q_{\delta_\aleph}-\eta^q_{\delta_\aleph}}}{e^{\mu^q_{\delta_\aleph}-\eta^q_{\delta_\aleph}}+1}-\frac{1}{2}\right)\pi^q_{\delta_\aleph}$$
$$=\mu^q_{\delta_L}-\eta^q_{\delta_L}+\left(\frac{e^{\mu^q_{\delta_L}-\eta^q_{\delta_L}}}{e^{\mu^q_{\delta_L}-\eta^q_{\delta_L}}+1}-\frac{1}{2}\right)\pi^q_{\delta_L}=S(\delta_L),$$

then, by above inequality, we have

$$\mu_{\delta_\aleph}=p_{\delta_\mathcal{L}}\ \textit{and}\ \eta_{\delta_\aleph}=r_{\delta_\mathcal{L}}$$

Hence $\pi^q_{\delta_\aleph}=\pi^q_{\delta_\mathcal{L}}\Longrightarrow(\mu_{\delta_\aleph},\eta_{\delta_\aleph})=(p_{\delta_\mathcal{L}},r_{\delta_\mathcal{L}})$
Therefore, it is proved that

$$\text{q}-\text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar 11},\aleph_{\hbar 12},\ldots,\aleph_{\hbar nm}\right)$$
$$\le\text{q}-\text{ROF}S_{ft}\text{WG}\left(\mathcal{L}_{\hbar 11},\mathcal{L}_{\hbar 12},\ldots,\mathcal{L}_{\hbar nm}\right).$$

iv : (**Shift Invariance**) Since $\mathcal{L}_\hbar = (p, r)$ and $\aleph_{\hbar ij} = \left(\mu_{\hbar ij}, \eta_{\hbar ij}\right)$ are the q-ROFS_{ft}Ns, so

$$\aleph_{\hbar 11}\otimes\mathcal{L}_\hbar=\left(\mu_{11}p,\sqrt[q]{1-(1-\eta^q_{11})(1-r^q)}\right)$$

Therefore,

$$\text{q}-\text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar 11}\otimes\mathcal{L}_\hbar,\aleph_{\hbar 12}\otimes\mathcal{L}_\hbar,\ldots,\aleph_{\hbar nm}\otimes\mathcal{L}_\hbar\right)$$
$$=\otimes^m_{j=1}\left(\otimes^n_{i=1}\left(\aleph_{\hbar nm}\otimes\mathcal{L}_\hbar\right)^{u_i}\right)^{z_j}$$
$$=\left(\begin{array}{c}\prod_{j=1}^m\left(\prod_{i=1}^n\mu^{u_i}_{ij}p^{u_i}\right)^{z_j},\\ \sqrt[q]{1-\prod_{j=1}^m\left(\prod_{i=1}^n\left(1-\eta^q_{ij}\right)^{u_i}(1-r^q)^{u_i}\right)^{z_j}}\end{array}\right)$$
$$=\left(\begin{array}{c}p\prod_{j=1}^m\left(\prod_{i=1}^n\mu^{u_i}_{ij}\right)^{z_j},\\ \sqrt[q]{1-(1-r^q)\prod_{j=1}^m\left(\prod_{i=1}^n\left(1-\eta^q_{ij}\right)^{u_i}\right)^{z_j}}\end{array}\right)$$
$$=\left(\begin{array}{c}\prod_{j=1}^m\left(\prod_{i=1}^n\mu^{u_i}_{ij}\right)^{z_j},\\ \sqrt[q]{1-\prod_{j=1}^m\left(\prod_{i=1}^n\left(1-\eta^q_{ij}\right)^{u_i}\right)^{z_j}}\end{array}\right)\otimes(p,r)$$

$$= \text{q} - \text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) \otimes \mathcal{L}_{\hbar}$$

Thus we get required proof.

iv : (**Homogeneity**) Consider for real number $\lambda > 0$ and $\aleph_{\hbar_{ij}} = \left(\mu_{ij}, \eta_{ij}\right)$ be a q-ROFS_{ft}N, then

$$\aleph^{\lambda}_{\hbar_{ij}} = \left(\mu^{\lambda}_{ij}, \sqrt[q]{1 - \left(1 - \eta^{q}_{ij}\right)^{\lambda}}\right)$$

Now

$$\text{q} - \text{ROF}S_{ft}\text{WG}\left(\lambda\aleph_{\hbar_{11}}, \lambda\aleph_{\hbar_{12}}, \ldots, \lambda\aleph_{\hbar_{nm}}\right)$$

$$= \left(\begin{array}{c}\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{ij}^{\lambda u_i}\right)^{z_j}, \\ \sqrt[q]{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \eta^{q}_{ij}\right)^{\lambda u_i}\right)^{z_j}}\end{array}\right)$$

$$= \left(\begin{array}{c}\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{ij}^{u_i}\right)^{z_j}\right)^{\lambda}, \\ \sqrt[q]{1 - \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \eta^{q}_{ij}\right)^{u_i}\right)^{z_j}\right)^{\lambda}}\end{array}\right)$$

$$= \lambda\text{q} - \text{ROF}S_{ft}\text{WG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right)$$

Hence, the proof is completed.

B. q-RUNF ORTHOPAIR FUZZY SOFT ORDERED WEIGHTED GEOMETRIC OPERATOR

From the analysis of q-ROFS_{ft}WG operator, it is observed that q-ROFS_{ft}WG operator only weight the values of q-ROFS_{ft}N, while q-ROFS_{ft}OWG operator weight the ordered positions of q-ROFS_{ft}N through scoring instead of weighting the q-ROFS_{ft} values itself. So, in this subsection we will investigate the detailed study of q-ROFS_{ft}OWG operator and their related properties.

Definition 11: Let $\aleph_{\hbar_{ij}} = \left(\mu_{ij}, \eta_{ij}\right)$ (*for* $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$), be the collections of q-ROFS_{ft}Ns, and suppose the weight vectors $u = \{u_1, u_2, \ldots, u_n\}$ for the decision makers a_i and $z = \{z_1, z_2, \ldots, z_m\}$ for the parameters $\hbar_j$ respectively, and satisfying the restrictions that $u_i, z_j \in [0, 1]$ with $\sum_{i=1}^{n} u_i = 1$ and $\sum_{j=1}^{m} z_j = 1$. Then q-ROFS_{ft}OWG operator is a mapping denoted and defined as:$\text{q} - \text{ROF}S_{ft}\text{OWG} : \mathcal{H}^n \to \mathcal{H}$, (where $\mathcal{H}$ contains the collections of q-ROFS_{ft}Ns)

$$\text{q} - \text{ROF}S_{ft}\text{OWG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) = \otimes_{j=1}^{m}\left(\otimes_{i=1}^{n}\aleph^{u_i}_{\sigma h_{ij}}\right)^{z_j}.$$

The following Theorem 4, described the aggregation result for q-ROFS_{ft}OWA operator.

Theorem 4: Consider the collections $\aleph_{\hbar_{ij}} = \left(\mu_{ij}, \eta_{ij}\right)$ (*for* $i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, m$), of q-ROFS_{ft}Ns. Then the aggregation result using q-ROFS_{ft}OWG operator is defined as:

$$\begin{aligned}&\text{q} - \text{ROF}S_{ft}\text{OWG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) \\ &= \otimes_{j=1}^{m}\left(\otimes_{i=1}^{n}\aleph^{u_i}_{\sigma\hbar_{ij}}\right)^{z_j} \\ &= \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{\sigma ij}^{u_i}\right)^{z_j}, \sqrt[q]{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1 - \eta^{q}_{\sigma ij})^{u_i}\right)^{z_j}}\right) \quad (7)\end{aligned}$$

where $\aleph_{\sigma\hbar_{ij}} = \left(\mu_{\sigma ij}, \eta_{\sigma ij}\right)$, denotes the permutations of i^{th} *and* j^{th} largest value of an alternative of the collections of *ith row and jth column* of q-ROFS_{ft}Ns $\aleph_{\hbar_{ij}} = \left(\mu_{ij}, \eta_{ij}\right)$.

Proof. Proof is easy and directly follows from Theorem 2.

Remark 2:

(a) When rung $q = 1$, so in this case the investigated q-ROFS_{ft}OWG operator degenerate into IF S_{ft}OWG operator.

(b) When rung $q = 2$, so in this case the investigated q-ROFS_{ft}OWG operator degenerate into PFS_{ft}OWG operator.

(c) If the parameter set contains just one parameter that is $\hbar_1$ (means $m = 1$), then the developed q-ROFS_{ft}OWG operator in this manuscript reduces to q-ROFOWG operator.

Thus from the analysis of Remark 2, we observed that IFS_{ft}OWG, PFS_{ft}OWG and q-ROFOWG operators are the specially derived from the developed q-ROFS_{ft}OWG operator.

Example 3: Suppose that $\aleph_{\hbar_{ij}} = \left(\mu_{ij}, \eta_{ij}\right)$ be the collection q-ROFS_{ft}Ns. Take the values of q-ROFS_{ft}Ns from Table 2 of Example 2, then by utilizing score function, the tabular notations of $\aleph_{\sigma h_{ij}} = \left(\mu_{\sigma ij}, \eta_{\sigma ij}\right)$ is given in Table 3. Now by *Eq.* (7), we have, shown at the bottom of the page.

From the analysis of Theorem 4, the q-ROFS_{ft}OWG operator fulfill the following properties for the collection q-ROFS_{ft}Ns$\aleph_{\hbar_{ij}} = \left(\mu_{ij}, \eta_{ij}\right)$.

Theorem 5: Let $\aleph_{\hbar_{ij}} = \left(\mu_{ij}, \eta_{ij}\right)$, ($i = 1, 2, \ldots, n$) and ($j = 1, 2, \ldots, m$) be the collection of q-ROFS_{ft}Ns with weight vector $u = (u_1, u_2, \ldots, u_n)^T$ for experts a_i and $z = (z_1, z_2, \ldots, z_m)^T$ be weight vector the parameters h_j respectively, such that $u_i, z_j \in [0, 1]$ with $\sum_{i=1}^{n} u_i = 1$ *and* $\sum_{j=1}^{m} z_j = 1$. Then the q-ROFS_{ft}OWG operator satisfied the following:

i : (**Idempotency**) : If $\aleph_{\hbar_{ij}} = \mathcal{L}_{\hbar}$, where $\mathcal{L}_{\hbar} = (p, r)$, then

$$\text{q} - \text{ROF}S_{ft}\text{OWG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) = \mathcal{L}_{\hbar}.$$

ii : (**Boundedness**) :

If $\aleph^{-}_{\hbar_{ij}} = \left(\min_j \min_i \left\{\mu_{ij}\right\}, \max_j \max_i \left\{\eta_{ij}\right\}\right)$, and $\aleph^{+}_{\hbar_{ij}} = \left(\min_j \min_i \left\{\mu_{ij}\right\}, \max_j \max_i \left\{\eta_{ij}\right\}\right)$, then

$$\aleph^{-}_{\hbar_{ij}} \leq \text{q} - \text{ROF}S_{ft}\text{OWG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) \leq \aleph^{+}_{\hbar_{ij}}.$$

iii : (**Monotonicity**) : If $\mathcal{L}_{\hbar_{ij}} = \left(p_{ij}, r_{ij}\right)$, ($i = 1, 2, \ldots, n$) *and* ($j = 1, 2, \ldots, m$), be the another collection of q-ROFS_{ft}Ns such that $\mu_{ij} \leq p_{ij}$ *and* $\eta_{ij} \geq r_{ij}$, then

$$\begin{aligned}&\text{q} - \text{ROF}S_{ft}\text{OWG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) \\ &\quad \leq \text{q} - \text{ROF}S_{ft}\text{OWG}\left(\mathcal{L}_{\hbar_{11}}, \mathcal{L}_{\hbar_{12}}, \ldots, \mathcal{L}_{\hbar_{nm}}\right).\end{aligned}$$

TABLE 3. Tabular notation of q-ROFS_{ft}S $\aleph_{\sigma\hbar_{ij}} = (\mu_{\sigma ij}, \eta_{\sigma ij})$ for $q \geq 3$.

S	$\hbar_1$	$\hbar_2$	$\hbar_3$	$\hbar_4$
a_1	(0.93, 0.25)	(0.94, 0.35)	(0.93, 0.25)	(0.94, 0.28)
a_2	(0.91, 0.24)	(0.92, 0.35)	(0.87, 0.41)	(0.93, 0.4)
a_3	(0.85, 0.35)	(0.85, 0.34)	(0.86, 0.42)	(0.92, 0.46)
a_4	(0.78, 0.34)	(0.86, 0.42)	(0.78, 0.3)	(0.87, 0.5)
a_5	(0.75, 0.26)	(0.76, 0.36)	(0.72, 0.26)	(0.77, 0.25)

TABLE 4. The score values of q-ROFS_{ft}Ns $\tilde{\aleph}_{\hbar_{ij}} = n u_i r_j \aleph_{\hbar_{ij}}$ for $q \geq 3$.

	$\hbar_1$	$\hbar_2$	$\hbar_3$	$\hbar_4$
a_1	-0.22765	-0.08604	-0.26668	0.109494
a_2	-0.01185	-0.25119	-0.36466	-0.23585
a_3	-0.48968	-0.47161	-0.69634	-0.50436
a_4	-0.14662	0.008862	0.03341	0.295625
a_5	-0.45766	-0.21373	-0.52886	-0.33226

iv : (**ShiftInvariance**) : If $\mathcal{L}_{\hbar} = (\wp, \nu)$, is another q-ROF$S_{ft}$N, then

$$\mathrm{q-ROF}S_{ft}\mathrm{OWG}\left(\aleph_{\hbar_{11}} \otimes \mathcal{L}_{\hbar}, \aleph_{\hbar_{12}} \otimes \mathcal{L}_{\hbar}, \ldots, \aleph_{\hbar_{nm}} \otimes \mathcal{L}_{\hbar}\right) = \mathrm{q-ROF}S_{ft}\mathrm{OWG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) \otimes \mathcal{L}_{\hbar}.$$

iv : (**Homogeneity**) : If any $\lambda > 0$, then

$$\mathrm{q-ROF}S_{ft}\mathrm{OWG}\left(\lambda\aleph_{\hbar_{11}}, \lambda\aleph_{\hbar_{12}}, \ldots, \lambda\aleph_{\hbar_{nm}}\right) = \lambda \mathrm{q-ROF}S_{ft}\mathrm{OWG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right).$$

Proof. Proofs are straightforward.

C. q-RUNG ORTHOPAIR FUZZY SOFT HYBRID GEOMETRIC OPERATOR

In this subsection, we will initiate the detail study of q-ROFS_{ft}HG operator and it is observe that q-ROFS_{ft}HG operator weight q-ROFS_{ft}Ns and its order position as well. Here we will discuss their fundamental properties of q-ROFS_{ft}HG operators such as Idempotency, Boundedness, Monotonicity, etc. with detail.

Definition 12: Let $\aleph_{\hbar_{ij}} = (\mu_{ij}, \eta_{ij})$ $(fori = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m)$, be the collections of q-ROFS_{ft}Ns, and consider the weight vectors $u = \{u_1, u_2, \ldots, u_n\}$

$$\mathrm{q-ROF}S_{ft}\mathrm{OWG}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) = \otimes_{j=1}^{m}\left(\otimes_{i=1}^{n}\aleph_{\sigma h_{ij}}^{u_i}\right)^{z_j}$$
$$= \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{\sigma ij}^{u_i}\right)^{z_j}, \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\eta_{\sigma ij}^{q}\right)^{u_i}\right)^{z_j}}\right)$$
$$= \left(\begin{array}{c} \left\{(0.93^{0.26})(0.91^{0.12})(0.85^{0.23})(0.78^{0.2})(0.75^{0.19})\right\}^{0.26}\left\{\begin{array}{c}(0.94^{0.26})(0.92^{0.12})(0.85^{0.23})\\(0.86^{0.2})(0.76^{0.19})\end{array}\right\}^{0.21} \\ \left\{(0.93^{0.26})(0.87^{0.12})(0.86^{0.23})(0.78^{0.2})(0.72^{0.19})\right\}^{0.29}\left\{\begin{array}{c}(0.94^{0.26})(0.93^{0.12})(0.92^{0.23})\\(0.87^{0.2})(0.77^{0.19})\end{array}\right\}^{0.24}, \\ \sqrt[3]{\begin{array}{l} 1-\left\{(1-0.25^3)^{0.26}(1-0.24^3)^{0.12}(1-0.35^3)^{0.23}(1-0.34^3)^{0.2}(1-0.26^3)^{0.19}\right\}^{0.26} \\ \left\{(1-0.35^3)^{0.26}(1-0.35^3)^{0.12}(1-0.34^3)^{0.23}(1-0.42^3)^{0.2}(1-0.36^3)^{0.19}\right\}^{0.21} \\ \left\{(1-0.25^3)^{0.26}(1-0.41^3)^{0.12}(1-0.42^3)^{0.23}(1-0.3^3)^{0.2}(1-0.26^3)^{0.19}\right\}^{0.29} \\ \left\{(1-0.28^3)^{0.26}(1-0.4^3)^{0.12}(1-0.46^3)^{0.23}(1-0.5^3)^{0.2}(1-0.25^3)^{0.19}\right\}^{0.24} \end{array}} \end{array}\right)$$
$$= (0.854398, 0.353285).$$

and $z = \{z_1, z_2, \ldots, z_m\}$ for the professional experts a_i and for the parameters $h_j's$ respectively; and satisfying that $u_i, z_j \in [0, 1]$ with $\sum_{i=1}^{n} u_i = 1$ and $\sum_{j=1}^{m} z_j = 1$. Then q-ROFS_{ft}HA operator is a mapping denoted and defined as; q − ROFS_{ft}HA :$\mathcal{H}^n \to \mathcal{H}$, (where $\mathcal{H}$ contains the collections of all q-ROFS_{ft}Ns)

$$q-\text{ROF}S_{ft}\text{HA}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) = \otimes_{j=1}^{m}\left(\otimes_{i=1}^{n}(\tilde{\aleph}_{\hbar_{ij}}^{u_i})^{z_j}\right).$$

Based on Definition 12, the following Theorem 6, described the aggregation result for q-ROFS_{ft}HA operator.

Theorem 6: Suppose the collection $\aleph_{\hbar_{ij}} = (\mu_{ij}, \eta_{ij})$ (for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$) of q-ROFNs, with $u = (u_1, u_2, \ldots, u_n)^T$ and $r = (r_1, r_2, \ldots, r_m)^T$ are the weight vectors of $\aleph_{\hbar_{ij}} = (\mu_{ij}, \eta_{ij})$, such that $u_i, r_j \in [0, 1]$ with $\sum_{i=1}^{n} u_i = 1$ and $\sum_{j=1}^{m} r_j = 1$ and n denotes the number of elements and is called the balancing coefficient in i^{th}*rowand*j^{th}*column* with aggregation associated vectors $u = (u_1, u_2, \ldots, u_n)^T$ and $z = (z_1, z_2, \ldots, z_m)^T$ for the decision makers a_i and for the parameters $h_j's$ respectively, with $u_i, z_j \in [0, 1]$ such that $\sum_{i=1}^{n} u_i = 1$ and $\sum_{j=1}^{m} z_j = 1$.Then the aggregated result for q-ROFS_{ft}HA operator is given as:

$$q-\text{ROF}S_{ft}\text{HA}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) = \otimes_{j=1}^{m}\left(\otimes_{i=1}^{n}(\tilde{\aleph}_{\hbar_{ij}}^{u_i})^{z_j}\right). = \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\tilde{\mu}_{ij}^{u_i}\right)^{z_j}, \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\tilde{\eta}_{ij}^{q})^{u_i}\right)^{z_j}}\right) \quad (8)$$

where $\tilde{\aleph}_{\hbar_{ij}} = nu_i r_j \aleph_{\hbar_{ij}}$, represents the largest alternative of permutation of i^{th}*and*j^{th} of the collections of $i \times j$ q-ROFS_{ft}Ns $\tilde{\aleph}_{\hbar_{ij}} = (\tilde{\mu}_{ij}, \tilde{\eta}_{ij})$.

Proof: Proof is straightforward to Theorem1.

Remark 3:

(a) When $q = 1$, so in this case the investigated q-ROFS_{ft}HA operator degenerates into IFS_{ft}HA operator.

(b) When $q = 2$, so in this case the investigated q-ROFS_{ft}HA operator degenerates into PFS_{ft}HA operator.

(c) When the parameter set contains just one alternative that is $\hbar_1$ (means $m = 1$), then the investigated q-ROFS_{ft}HA operator degenerates to q-ROFHA operator.

(d) When $ur = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, so in this case the investigated q-ROFS_{ft}HA operator degenerates into q-ROFS_{ft}WA operator.

(e) When $uz = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, so in this case the investigated q-ROFS_{ft}HA operator degenerates into q-ROFS_{ft}OWA operator.

Thus from the analysis of Remark 3, we analyzed that IFS_{ft}HA, PFS_{ft}HA, q-ROFHA, q-ROFS_{ft}WA and q-ROFS_{ft}OWA operators are the special derived cases of the developed q-ROFS_{ft}HA operator.

Example 4: Suppose that $\aleph_{\hbar_{ij}} = (\mu_{ij}, \eta_{ij})$ be the collection of q-ROFS_{ft}Ns as described in Table 2, of Example 2, with $u = (0.26, 0.22, 0.1, 0.27, 0.15)^T$ be the weight vectors of experts $r = (0.23, 0.28, 0.2, 0.29)^T$ be the weight vector for parameter. Let the associated aggregate vectors $u = (0.27, 0.18, 0.1, 0.18, 0.27)^T$ and $z = (0.26, 0.24, 0.24, 0.26)^T$.By applying *Eq*. (9) and their score values are express in Table 4. The permutation of largest values of the collection q-ROFS_{ft}Ns$\tilde{\aleph}_{\hbar_{ij}} = nu_i r_j \aleph_{\hbar_{ij}}$, of i^{th} row and j^{th} column are express in Table 5. Since

$$\tilde{\aleph}_{\hbar_{ij}} = nu_i r_j \aleph_{\hbar_{ij}} = \left(\sqrt[3]{1-\left(1-\mu_{ij}^{3}\right)^{nu_i r_j}}, \eta^{nu_i r_j}\right) \quad (9)$$

Now by using *Eq*.8, of *Theorem* 6,

$$q-\text{ROF}S_{ft}\text{HA}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) = \otimes_{j=1}^{m} z_j\left(\otimes_{i=1}^{n} u_i \tilde{\aleph}_{\hbar_{ij}}\right) = \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\tilde{\mu}_{ij}^{u_i}\right)^{z_j}, \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\tilde{\eta}_{ij}^{q})^{u_i}\right)^{z_j}}\right) = (0.595792, 0.630295)$$

Based on Theorem 6, the investigated q-ROFS_{ft}HA operator satisfying some basic properties.

Theorem 7: Suppose $\aleph_{\hbar_{ij}} = (\mu_{ij}, \eta_{ij})$, ($i = 1, 2, \ldots, n$) and ($j = 1, 2, \ldots, m$), be the collection of q-ROFS_{ft}Ns with $u = (u_1, u_2, \ldots, u_n)^T$ be the weight vectors of a_i and $r = (r_1, r_2, \ldots, r_m)^T$ be the weight vectors of $\hbar_j$, with $u_i, r_j \in [0, 1]$ such that $\sum_{i=1}^{n} u_i = 1$ and $\sum_{j=1}^{m} r_j = 1$. Here n represent the number of alternatives in i^{th}*row* and j^{th}*column* and is called balancing coefficient. Let $u = (u_1, u_2, \ldots, u_n)^T$ and $z = (z_1, z_2, \ldots, z_m)^T$ be the aggregate associated weight vectors for the experts a_i and for the parameters $\hbar_j's$ respectively, with $u_i, z_j \in [0, 1]$ such that $\sum_{i=1}^{n} u_i = 1$and $\sum_{j=1}^{m} z_j = 1$. Then the following properties hold for q-ROFS_{ft}HA operator:

i : (**Idempotency**) : If $\aleph_{\hbar_{ij}} = \mathcal{L}_{\hbar}$, where $\mathcal{L}_{\hbar} = (p, r)$, then

$$q-\text{ROF}S_{ft}\text{HA}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) = \mathcal{L}_{\hbar}.$$

ii : (**Boundedness**) :

If $\aleph_{\hbar_{ij}}^{-} = \left(\min_j \min_i \{\mu_{ij}\}, \max_j \max_i \{\eta_{ij}\}\right)$ and $\aleph_{\hbar_{ij}}^{+} = \left(\max_j \max_i \{\mu_{ij}\}, \min_j \min_i \{\eta_{ij}\}\right)$, then

$$\aleph_{\hbar_{ij}}^{-} \le q-\text{ROF}S_{ft}\text{HA}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) \le \aleph_{\hbar_{ij}}^{+}.$$

iii : (**Monotonicity**) : If $\mathcal{L}_{\hbar_{ij}} = (p_{ij}, r_{ij})$, ($i = 1, 2, \ldots, n$) and ($j = 1, 2, \ldots, m$), be the another collection of q-ROFS_{ft}Ns such that $\mu_{ij} \le p_{ij}$ and $\eta_{ij} \ge r_{ij}$, then

$$q-\text{ROF}S_{ft}\text{HA}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) \le q-\text{ROF}S_{ft}\text{HA}\left(\mathcal{L}_{\hbar_{11}}, \mathcal{L}_{\hbar_{12}}, \ldots, \mathcal{L}_{\hbar_{nm}}\right).$$

iv : (**ShiftInvariance**) : If $\mathcal{L}_{\hbar} = (p, r)$, is another q-ROF$S_{ft}$N, then

$$q-\text{ROF}S_{ft}\text{HA}\left(\aleph_{\hbar_{11}} \otimes \mathcal{L}_{\hbar}, \aleph_{\hbar_{12}} \otimes \mathcal{L}_{\hbar}, \ldots, \aleph_{\hbar_{nm}} \otimes \mathcal{L}_{\hbar}\right) = q-\text{ROF}S_{ft}\text{HA}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right) \otimes \mathcal{L}_{\hbar}.$$

TABLE 5. Tabular description of q-ROFS_{ft}Ns$\tilde{\aleph}_{\hbar_{ij}} = n\mathcal{u}_i r_j \aleph_{\hbar_{ij}}$ for q ≥ 3.

	$\hbar_1$	$\hbar_2$	$\hbar_3$	$\hbar_4$
a_1	(0.6967,0.7042)	(0.6711, 0.6651)	(0.7089, 0.6878)	(0.7942, 0.6075)
a_2	(0.5389, 0.6582)	(0.6752, 0.7292)	(0.4854, 0.7045)	(0.7716, 0.7079)
a_3	(0.5594, 0.7243)	(0.6777, 0.8022)	(0.5949, 0.8219)	(0.6621, 0.8016)
a_4	(0.5331, 0.8344)	(0.5463, 0.7301)	(0.4515, 0.8348)	(0.654, 0.8446)
a_5	(0.5299, 0.8486)	(0.5752, 0.8633)	(0.4581, 0.9167)	(0.439, 0.8179)

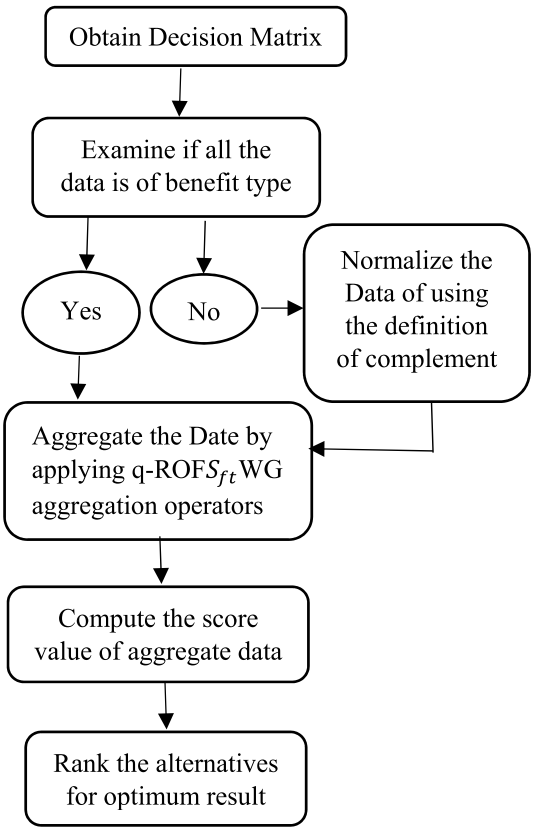

FIGURE 1. Flow chart for the proposed model un MCDM.

iv : (**Homogeneity**) : For any $\lambda > 0$, then

$$\mathrm{q-ROF}S_{ft}\mathrm{HA}\left(\lambda\aleph_{\hbar_{11}}, \lambda\aleph_{\hbar_{12}}, \ldots, \lambda\aleph_{\hbar_{nm}}\right) = \lambda\mathrm{q-ROF}S_{ft}\mathrm{HA}\left(\aleph_{\hbar_{11}}, \aleph_{\hbar_{12}}, \ldots, \aleph_{\hbar_{nm}}\right).$$

Proof. Proofs are easy.

V. AN APPROACH TO MCDM UNDER Q-ROF SOFT INFORMATION

In section is allotted for the DM process for the developed aggregation operators. In DM aggregation operators plays an important role because it aggregates the several evaluation values of experts into a single value. DM is a pre-plan process of identifying and selecting the best choice out of many alternatives. DM is a hard process because it can vary so obviously from one scenario to the next. Therefore, it is very important to judge the characteristics and limitations of alternative. Also DM is a batter approach to increase the chance of selecting most appropriate alternative of the choice. It is essential to know that how much truly background information is required for decision maker and the best effective strategy in DM is to keep an eye and focus on your goal.

Suppose a discrete set $S = \{x_1, x_2, \ldots, x_l\}$ of different objects and consider $\mathfrak{E} = \{\hbar_1, \hbar_2, \ldots, \hbar_n\}$ be set of parameter against alternatives $x_s(s = 1, 2, \ldots, l)$. The team of m professional experts $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_m$ are going to evaluate each object x_s against their given parameter $\hbar_j$. The group of professional experts described their evaluation in the form of $\aleph_{\hbar_{ij}} = \left(\mu_{ij}, \eta_{ij}\right)$ with weight vector $u = (\mathcal{u}_1, \mathcal{u}_2, \ldots, u_m)^T$ for senior experts D_i and let $z = (z_1, z_2, \ldots, z_n)^T$ be the weight vector for the parameters $\hbar_j$ with $\mathcal{u}_i, z_j \in [0, 1]$ such that $\sum_{i=1}^{m} \mathcal{u}_i = 1$ *and* $\sum_{j=1}^{n} z_j = 1$. The collective evaluation of professional experts are described in a decision matrix $\mathbb{M} = [\aleph_{\hbar_{ij}}]_{m\times n}$. By applying the developed model on evaluated decision matrix $\mathbb{M} = [\aleph_{\hbar_{ij}}]_{m\times n}$ we will get an aggregated q-ROFS_{ft}N$\xi_s = (\mu_s, \eta_s)$ for every object to against parameters. Finally by applying the score function on each aggregated q-ROFS_{ft}N$\xi_s = (\mu_s, \eta_s)$ for each object x_s and rank them in a specific ordered to the most desirable option out of total.

Based on above analysis, an algorithm is developed for the proposed model for solving MCDM applications.

A. ALGORITGM

Step 1: Collect the evaluation information of professional experts for every object to corresponding parameters and then established the decision matrix $M = [\aleph_{\hbar_{ij}}]_{m\times n}$ as:

$$\mathbb{M} = \begin{bmatrix} (\mu_{11}, \eta_{11}) & (\mu_{12}, \eta_{12}) & \cdots & (\mu_{1n}, \eta_{1n}) \\ (\mu_{21}, \eta_{21}) & (\mu_{22}, \eta_{22}) & \cdots & (\mu_{2n}, \eta_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{m1}, \eta_{m1}) & (\mu_{m2}, \eta_{m2}) & \cdots & (\mu_{mn}, \eta_{mn}) \end{bmatrix}$$

Step 2: Normalize the decision matrix by interchanging the cost and benefit parameters if there is any by applying the formula from [56] that is,

$$p_{ij} = \begin{cases} \aleph^c_{\hbar_{ij}}; & \textit{for cost type parameter} \\ \aleph_{\hbar_{ij}}; & \textit{for benefit type parameter} \end{cases}$$

TABLE 6. q-ROFS_{ft} matrix for patient $\wp_1$.

	$\hbar_1$ = Chest pain	$\hbar_2$ = Fever	$\hbar_3$ = Cough	$\hbar_4$ =Fatigue	$\hbar_5$ = Vomit
$\mathcal{D}_1$	(0.7,0.25)	(0.7,0.22)	(0.88,0.1)	(0.9,0.1)	(0.73,0.2)
$\mathcal{D}_2$	(0.6, 0.1)	(0.6,0.13)	(0.85,0.12)	(0.65,0.25)	(0.81,0.18)
$\mathcal{D}_3$	(0.54,0.15)	(0.7,0.2)	(0.75,0.24)	(0.68,0.25)	(0.6,0.26)
$\mathcal{D}_4$	(0.65,0.2)	(0.8,0.18)	(0.85,0.13)	(0.8,0.15)	(0.7,0.28)
$\mathcal{D}_5$	(0.6,0.3)	(0.75,0.18)	(0.67,0.25)	(0.6,0.3)	(0.45,0.15)

where $\aleph^c_{\hbar_{ij}} = (\eta_{ij}, \mu_{ij})$ represents the complement of $\aleph_{\hbar_{ij}} = (\mu_{ij}, \eta_{ij})$.

Step 3: By applying the developed model on evaluated decision matrix $\mathbb{M} = [\aleph_{\hbar_{ij}}]_{m\times n}$ we will get an aggregated q-ROFS_{ft}N$\xi_s = (\mu_s, \eta_s)$ for each alternative $x_s (s = 1, 2, \ldots, l)$ to their corresponding parameters.

Step 4: Determine the score value on each aggregated q-ROFS_{ft}N$\xi_s = (\mu_s, \eta_s)$ for each alternativex_s.

Step 5: Finally rank the score value in a specific ordered to get best choice out of total.

The flow chart of the algorithm for proposed model is given in Fig. 1.

VI. AN ILLUSTRATIVE EXAMPLE FOR THE PROPOSED MODEL TO MCDM

In this subsection through illustrative example we will present the medical diagnose problem by applying the developed model to determine the applicability and superiority of the developed methods based on q-ROF soft information adopted from [54].

Suppose a team of five professional Doctors $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ *and* $\mathcal{D}_5$ are going to describe their assessment report for four different under medical treatment patients $\wp_1, \wp_2, \wp_3$ *and* $\wp_4$ having weight vector $u = (0.18, 0.24, 0.21, 0.15, 0.22)^T$. Let $\mathfrak{E} = \{h_1 = chest\ \ pain, \hbar_2 = fever, \hbar_3 = cough, \hbar_4 = fatigue, \hbar_5 = vomit\}$ be the set of parameters having weight vector $z = (0.26, 0.22, 0.1, 0.27, 0.15)^T$. The experts means professional Doctors present their assessment report for each under medical treatment patient against their symptom in the form of q-ROFS_{ft} decision matrix. Based on above analysis, to diagnose the most illness patient via the algorithm for the proposed model is given below.

A. BY USING Q-ROF S_{ft} WG OPERATOR

Step 1: The collective evaluation information of professional experts for each patient to against parameters (symptoms) and their established the decision matrix $\mathbb{M} = [\aleph_{\hbar_{ij}}]_{m\times n}$ are given in Tables $6 - 9$ respectively:

Step 2: All the parameter are the same type so no need to normalize the assessment information in decision matrix.

Step 3: By applying the developed model on each evaluated decision matrix $\mathbb{M} = [\aleph_{\hbar_{ij}}]_{m\times n}$ for each patient p_i by using

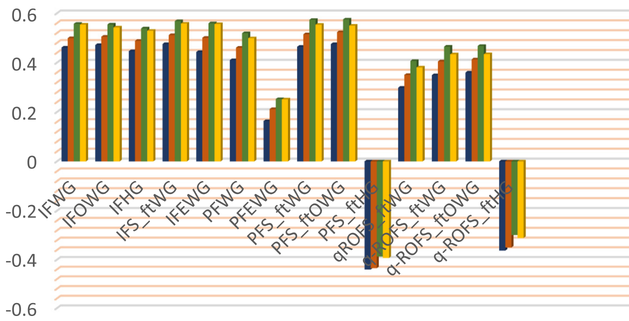

FIGURE 2. Comparative study of different geometric operators of Table 11.

$Eq.1,\ for\ q = 3$, and the aggregated result is given below:

$$\xi_1 = (0.676098, 0.217227),$$
$$\xi_2 = (0.711392, 0.213948),$$
$$\xi_3 = (0.745244, 0.192632),$$
$$\xi_4 = (0.726185, 0.183019)$$

Step 4: Determine the score value on each aggregated q-ROFN $\xi_s = (\mu_s, \eta_s)$ for each alternative x_s in **Step 3**, that is

$$S(\xi_1) = 0.349273, S(\xi_2) = 0.404847,$$
$$S(\xi_3) = 0.464826, S(\xi_4) = 0.4337$$

Step 5: In final step rank the score value in a specific ordered to get best choice out of total.

$$S(\xi_3) > S(\xi_4) > S(\xi_2) > S(\xi_1)$$

Hence, form the analysis of above calculation it is clear that under medical treatment patient p_3 has diagnose more illness in list.

B. BY USING Q-ROF S_{ft} OWG OPERATOR

Step 1: Similar as above.

*Step 2:*Similar as above.

Step 3: By applying the developed model on each evaluated decision matrix $\mathbb{M} = [\aleph_{\hbar_{ij}}]_{m\times n}$ for each patient p_i by using $Eq.7,\ for\ q = 3$, and the aggregated result is given below:

$$\xi_1 = (0.682695, 0.212683),$$
$$\xi_2 = (0.716155, 0.210147),$$

TABLE 7. q-ROFS$_{ft}$ matrix for patient p_2.

	$\hbar_1$ =Chest pain	$\hbar_2$ =Fever	$\hbar_3$ =Cough	$\hbar_4$ =Fatigue	$\hbar_5$ =Vomit
$\mathcal{D}_1$	(0.8,0.15)	(0.75,0.22)	(0.76,0.1)	(0.8,0.19)	(0.7,0.25)
$\mathcal{D}_2$	(0.75, 0.18)	(0.8,0.15)	(0.8,0.18)	(0.5,0.25)	(0.8,0.16)
$\mathcal{D}_3$	(0.78,0.13)	(0.7,0.2)	(0.7,0.25)	(0.76,0.21)	(0.76,0.23)
$\mathcal{D}_4$	(0.9,0.1)	(0.65,0.33)	(0.76,0.15)	(0.87,0.12)	(0.65,0.18)
$\mathcal{D}_5$	(0.65,0.3)	(0.55,0.2)	(0.6,0.3)	(0.7,0.23)	(0.55,0.15)

TABLE 8. q-ROFS$_{ft}$ matrix for patient p_3.

	$\hbar_1$ =Chest pain	$\hbar_2$ =Fever	$\hbar_3$ =Cough	$\hbar_4$ =Fatigue	$\hbar_5$ =Vomit
$\mathcal{D}_1$	(0.71,0.25)	(0.78,0.1)	(0.88,0.11)	(0.81,0.18)	(0.78,0.2)
$\mathcal{D}_2$	(0.8,0.15)	(0.85,0.12)	(0.9,0.1)	(0.65,0.25)	(0.74,0.23)
$\mathcal{D}_3$	(0.76,0.1)	(0.88,0.11)	(0.84,0.12)	(0.86,0.1)	(0.79,0.2)
$\mathcal{D}_4$	(0.78,0.22)	(0.75,0.25)	(0.74,0.2)	(0.75,0.25)	(0.65,0.16)
$\mathcal{D}_5$	(0.6,0.25)	(0.8,0.19)	(0.75,0.16)	(0.6,0.2)	(0.5,0.1)

TABLE 9. q-ROFS$_{ft}$ matrix for patient p_4.

	$\hbar_1$ =Chest pain	$\hbar_2$ =Fever	$\hbar_3$ =Cough	$\hbar_4$ =Fatigue	$\hbar_5$ =Vomit
$\mathcal{D}_1$	(0.76,0.22)	(0.75,0.22)	(0.85,0.14)	(0.78,0.2)	(0.65,0.26)
$\mathcal{D}_2$	(0.72,0.12)	(0.79,0.18)	(0.6,0.12)	(0.73,0.15)	(0.8,0.14)
$\mathcal{D}_3$	(0.82,0.16)	(0.83,0.1)	(0.84,0.13)	(0.82,0.12)	(0.77,0.2)
$\mathcal{D}_4$	(0.6,0.27)	(0.6,0.3)	(0.7,0.2)	(0.83,0.13)	(0.6,0.25)
$\mathcal{D}_5$	(0.55,0.1)	(0.81,0.12)	(0.8,0.15)	(0.72,0.17)	(0.55,0.15)

TABLE 10. Aggregated values of q-ROFS$_{ft}$ matrix for patients.

	p_1	p_2	p_3	p_4
$\mathcal{D}_1$	(0.7713, 0.1999)	(0.7691,0.1960)	(0.7783,0.1932)	(0.7538,0.2168)
$\mathcal{D}_2$	(0.6641, 0.1820)	(0.6929,0.1974)	(0.7666,0.1951)	(0.7358,0.1478)
$\mathcal{D}_3$	(0.6388, 0.2267)	(0.7453,0.2019)	(0.8245,0.1301)	(0.8164,0.1470)
$\mathcal{D}_4$	(0.7474, 0.1984)	(0.7774,0.2151)	(0.7406,0.2280)	(0.6651,0.2457)
$\mathcal{D}_5$	(0.6103, 0.2607)	(0.6184,0.2483)	(0.6360,0.2015)	(0.6687,0.1411)

$$\xi_3 = (0.747071, 0.194893),$$
$$\xi_4 = (0.726911, 0.188214)$$

Step 4: Determine the score value on each aggregated q-ROFN $\xi_s = (\mu_s, \eta_s)$ for each alternative x_s in **Step 3**, that

$$S(\xi_1) = 0.360011, S(\xi_2) = 0.41323,$$
$$S(\xi_3) = 0.467677, S(\xi_4) = 0.434245$$

Step 5: In final step rank the score value in a specific ordered to get best choice out of total.

$$S(\xi_3) > S(\xi_4) > S(\xi_2) > S(\xi_1)$$

Hence, form the analysis of above calculation it is clear that under medical treatment patient p_3 has diagnose more illness in list.

C. BY USING Q-ROFS$_{ft}$ HG OPERATOR

Step 1: Similar as above.

TABLE 11. Comparative Studies of different methods.

Methods	*Score values of patients*				Ranking
	$\wp_1$	$\wp_2$	$\wp_3$	$\wp_4$	
IFWG [33]	0.46119,	0.49917,	0.5572,	0.55418	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
IFOWG [33]	0.47123,	0.50500,	0.55447,	0.54260	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
IFHG [33]	0.44665,	0.48839,	0.53872,	0.52907	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
IFS_{ft}WG [3]	0.47536,	0.51160,	0.56806,	0.55816	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
IFEWG [28]	0.44290,	0.50071,	0.55935,	0.55612	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
PFWG [15]	0.41024,	0.46069,	0.51919,	0.49886	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
PFEWG [15]	0.16317,	0.21196,	0.25247,	0.25114	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
PFS_{ft}WG (proposed)	0.46427,	0.51474,	0.57343,	0.55374	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
PFS_{ft}OWG (proposed)	0.47513,	0.52336,	0.57502,	0.54971	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
PFS_{ft}HG (proposed)	-0.44065,	-0.435,	-0.38704,	-0.39273	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
q-ROFWG [19]	0.29865,	0.35021,	0.40678,	0.38045	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
q-ROFS_{ft}WG (proposed)	0.349273,	0.404847,	0.464826,	0.4337	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
q-ROFS_{ft}OWG (proposed)	0.360011,	0.41323,	0.467677,	0.434245	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
q-ROFS_{ft}HG (proposed)	-0.3625,	-0.34969,	-0.2981,	-0.31024	$\xi_3 > \xi_4 > \xi_2 > \xi_1$

TABLE 12. Characteristic analysis of different models.

	Fuzzy information	*Aggragate parameter informatiom*
IFWG [5]	*Yes*	*No*
IFOWG [5]	*Yes*	*No*
IFHG [5]	*Yes*	*No*
IFEWG [57]	*Yes*	*No*
IFS_{ft}WG [31]	*Yes*	*Yes*
PFWG [23]	*Yes*	*No*
PFEWG [23]	*Yes*	*No*
q-ROFWG [32]	*Yes*	*No*
Proposed Operators	*Yes*	*Yes*

Step 2: Similar as above.

Step 3: By applying the developed model on each evaluated decision matrix $\mathbb{M} = [\aleph_{\hbar_{ij}}]_{m\times \boldsymbol{n}}$ for each patient $\wp_i$ by using $Eq.8$, $for\ q = 3$, with $u = (0.15, 0.2, 0.17, 0.3, 0.18)^T$ $and\ r = (0.16, 0.21, 0.13, 0.26, 0.24)^T$ be the weight vectors of $\aleph_{\hbar_{ij}} = \left(\mu_{ij}, \eta_{ij}\right)$, and n represent the number of alternatives ini^{th} *row and* j^{th}*column* and is called balancing coefficient. Let $u = (0.18, 0.24, 0.21, 0.15, 0.22)^T$ and $z = (0.26, 0.22, 0.1, 0.27, 0.15)^T$ be the aggregate associated weight vectors for professional Doctor $\mathfrak{D}_i$ and for the parameters $\hbar_j'{s}$ respectively, the aggregated result is given below:

$$\xi_1 = (0.418696, 0.732854),$$
$$\xi_2 = (0.444599, 0.735594),$$
$$\xi_3 = (0.469079, 0.715699),$$
$$\xi_4 = (0.450482, 0.71448)$$

Step 4: Determine the score value on each aggregated q-ROFN $\xi_s = (\mu_s, \eta_s)$ for each alternative x_s in **Step 3**, that

$$S(\xi_1) = -0.3625, S(\xi_2) = -0.34969,$$
$$S(\xi_3) = -0.2981, S(\xi_4) = -0.31024$$

Step 5: In final step rank the score value in a specific ordered to get best choice out of total:

$$S(\xi_3) > S(\xi_4) > S(\xi_2) > S(\xi_1)$$

Hence, form the analysis of above calculation it is clear that under medical treatment patient $\wp_3$ has diagnose more illness in list.

D. COMPARATIVE ANALYSIS

To present the applicability and superiority of the investigated aggregation models, a comparative study of the investigated model with some existing literatures (see [3, 15, 19, 28, 33]) have been presented. If a decision maker assign MG 0.95 and NMG 0.55, then their sum $0.95 + 0.55 > 1$, so the methods investigated in [4, 15, 28, 33] are failed to handle the decision makers prefer choice. Similarly, if we acknowledge Tables 6 to 9, then the methods initiated in [15, 19, 28, 33] are failed to handle the experts prefer evaluations and the methods investigated in this manuscript still handle all these scenarios. By applying proposed weighted geometric operators on Tables 6-9 to aggregate the different parameters of q-ROFS_{ft}Ns with weigh vector $z = (0.26, 0.22, 0.1, 0.27, 0.15)^T$ to achieve the decision matrix as summarized in Table 10 for different patients $\wp_i(i = 1, 2, 3, 4)$. Based on Table 10, a comparative study of the different existing models have been presented and their summarized results for each patient $\wp_i$ are given Table 11. The graphical representation of comparative study of proposed models with different existing geometric aggregation operations are given in Fig. 2. Hence, form the above calculation of Table 11, and Fig. 2, it is clear that under medical treatment patient $\wp_3$ has diagnose more illness in list. The Characteristic summery of proposed models with some existing literatures are presented in Table 12. Thus from the analysis of Table 12, it is observed that existing models give in [15, 19, 28, 33] having no information about parameterization tools. The main advantage of the investigated model is the capability and superiority to solve real life problems by utilizing parameterization properties. Therefore, the proposed approach is more capable and superior than existing methods under q-ROFS_{ft} environment.

VII. CONCLUSION

Decision making is a pre-plan process of identifying and choosing the logical choice out of several alternatives. DM is a hard process because it can vary so obviously from one scenario to the next. Therefore, it is very important to judge the characteristics and limitations of alternative. Also DM is a batter approach to increase the chance of selecting most appropriate alternative of the choice. It is essential to know that how much truly background information is required for decision maker and the best effective strategy in DM is to keep an eye and focus on your goal. The pioneer paradigm of S_{ft}S was investigated by Molodtsov by affixing parameterization tools in ordinary sets. S_{ft}S theory is free from inherit complexity and a nice mathematical tool to cope uncertainties in parametric manner. The aim of this manuscript is to initiate the combine study of S_{ft}S and q-ROFS to get the new notion called q-ROFS_{ft}S. The notion of q-ROFS_{ft}S is free from those complexities which suffering the ordinary theories because parameterization tool is the most significant character of q-ROFS_{ft}S. In this manuscript our main contribution to originate the concept of q-ROFS_{ft}WG, q-ROFS_{ft}OWG and q-ROFS_{ft}HG operators in q-ROFS_{ft}S environment. Moreover, some dominant properties of these developed operators are studied with detail. Based on these proposed approach, a model is build up for MCDM and their step wise algorithm is being presented. Finally, utilizing the developed approach an illustrative example is solved under q-ROFS_{ft} environment. Further a comparative analysis of the investigated models with existing methods are presented in detail which shows the competence and ability of the developed models. The main advantage of the investigated model is the capability to solve real problems by utilizing parameterization properties. Therefore, the proposed approach is more capable and superior than existing methods under q-ROFS_{ft} environment.

In future work, we shall extend our research in different directions, including different aggregation operators, applying similarities measure and entropy measure. We will also focus on the applications of the proposed method by using q-ROF information in different real life problems. Moreover, we will extend the developed method to other generalization of fuzzy sets as well and apply it to other fields, such as medical diagnosis.

DATA AVAILABILITY

The data used in this article are original, artificial and hypothetical, and anyone can use these data before prior permission by just citing this article.

CONFLICTS OF INTEREST

The authors declare that they have no conflicts of interest.

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