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Irregularity Strength of Circulant Graphs Using Algorithmic Approach

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ABSTRACT This paper deals with decomposition of complete graphs on n vertices into circulant graphs with reduced degree $r < n-1$. They are denoted as $C_n(a_1, a_2, \ldots, a_m)$, where a_1 to a_m are generators. Mathematical labeling for such bigger (higher order and huge size) and complex (strictly regular with so many triangles) graphs is very difficult. That is why after decomposition, an edge irregular k-labeling for these subgraphs is computed with the help of algorithmic approach. Results of k are computed by implementing this iterative algorithm in computer. Using the values of k, an upper bound for edge irregularity strength is suggested for $C_n(a_1, a_2, \ldots, a_m)$ that is $|E|/2 \log_2 |V|$.

INDEX TERMS Edge irregular labeling, circulant graph, graph algorithm, computational complexity, Sidon sequence.

I. INTRODUCTION AND PRELIMINARY RESULTS

Let G be a connected, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. In graph theory, degree of a graph is used as one of the most important graph invariant for comparison purpose. The *degree* of a vertex v is the number of edges incident to v, it is denoted as $\deg(v)$. The minimum and maximum degree of a graph G is denoted by $\delta(G)$ and $\Delta(G)$ respectively. If every vertex in a graph G has the same degree r, that is $\delta(G) = \Delta(G) = r$, then G is called an r-regular graph. A complete graph K_n is a circulant regular graph of order n and degree $r = n-1$, where every pair of distinct vertices is connected by a unique edge. That is why K_n is considered as super-circulant graph with biggest size that can be calculated using the property, $|E(K_n)| = n(n-1)/2$.

In graph theory new graphs are evolved from existing graphs by applying certain graph operations like by adding or deleting vertices or edges. Decomposition is one of the elementary graph operation in which a subgraph is extracted from a super-graph. Graph decomposition can be done by deleting 2-factors or Hamiltonian cycles from a supergraph that results as reduction of two-degree.

In 2001 graph decomposition was introduced by Knopfmacher and Mays [13], they used enumerations to develop formulae for few families of graphs like paths, cycles, trees etc. In 2012 Cichacz *et al.* [8] decomposed the complete graphs into (0, 2)-prisms. In 2015 Tichenor and Mays used decompositions for deleting edges from complete graphs and derived formulae for paths, cycles, star graphs and disjoint graphs using generating functions [21].

In computer science graphs play a vital role in computational linguistics, decision making software, coding theory and path determination in networks. In fifth-generation-computers, the interconnection network of the parallel processors are represented as complete graph K_n, by considering vertices as n processors and edges as link between them [17]. Circulant graphs also have wide applications in the field of cloud computing and network topologies like ring topology and fully connected networks [5]. Sipper and Ruppin in [18] using the idea of [6] worked on cellular programing based on circulant graphs. In [14] Lu worked on fast methods for designing circulant network topology by decomposing complete graphs to construct new circulant graphs.

In real world applications temporal graphs are used more than complete graphs. A *temporal graph* is a dynamic

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graph that changes with discrete time to develop connectivity between the nodes or vertices according to changed situation [16]. "Recent research shows that many graph properties and problems become radically different and substantially more difficult when an extra time dimension is added to them." This situation motivates to explore new graphs, their labeling and need of designing algorithms for more and more versatile types of graphs.

Chartrand *et al.* in [7] introduced an edge k-labeling of a graph G such that distinct vertices have distinct weights, where the weight of a vertex is the sum of labels of its incident edges. Such labeling is known as an *irregular assignment* whereas the *irregularity strength* $s(G)$ of a graph G is known as the minimum k for which G has an irregular assignment using labels at most k. Lot of research work has been carried out in the area of irregularity strength for different graph families [10]–[12], [15].

There are known many variations of irregular assignments. In 2014 Ahmad *et al.* [1] introduced a new type of a vertex labeling and established a lower bound for this type of labeling. A vertex k-labeling $V(G) \rightarrow \{1, 2, \ldots, k\}$ of a graph G is defined to be an *edge irregular k-labeling* if for every two different edges their weights are distinct. In this case the weight of an edge is defined as the sum of labels of its ends. The minimum k for which a graph G has an edge irregular k-labeling is called the *edge irregularity strength* of G denoted by $es(G)$. In recent years lot of work has been done on $es(G)$ for different families of graphs and trees [2], [3], [19], [20].

The following theorem given in [1], established a lower bound for the edge irregularity strength of a graph G.

Theorem 1: [1] Let $G = (V, E)$ be a simple graph with maximum degree $\Delta(G)$. Then

$$es(G) \geq \max\left\{\left\lceil \frac{|E(G)|+1}{2} \right\rceil, \Delta(G)\right\}.$$

An upper bound based on Fibonacci numbers is presented in the next theorem.

Theorem 2: [1] Let $G = (V, E)$ be a graph of order p. Let the sequence F_n of Fibonacci numbers be defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, $n \geq 3$, with seeds values $F_1 = 1$ and $F_2 = 2$. Then $es(G) \leq F_p$.

Primary objective of this article is to relate the $es(Kn)$ with the $es(Cn, r)$. Section-II, explains the circulant regular graphs as subgraphs of complete graphs. Hence the $es(Cn, r)$ must be less than $es(Kn)$.

II. DECOMPOSITION OF COMPLETE GRAPHS

In 2018 Asim *et al.* [4] computed an edge irregular k-labeling for complete graphs using algorithmic approach and suggested a better upper bound as $es(K_n) \leq n(n-1)/2 \log_2 n$ that is much better than previously known mathematical result using Fibonacci numbers F_n. Evidently, for complete graph K_n all vertex labels are unique and edge weights are also unique. These results are worth to compare them with a Sidon sequence to support the optimality of algorithmic results. A *Sidon sequence* $A = a_0, a_1, a_2, \ldots$ is a set of natural numbers in which all pairwise sums $a_i + a_j$, $i \leq j$ must be different. Sidon introduced this concept for Fourier series. In 2010 Cilleruelo *et al.* [9] solved the issue by giving a formula to calculate Sidon sequence terms. Table 1 shows the comparison between the algorithmic results of k computed in [4], Sidon sequence terms from [9] and Fibonacci numbers F_n as an upper bound from [1]. It can be seen clearly that algorithmic results are better than both sequences.

Successful results of $es(K_n)$ computed in [4] motivated us to extend the same approach for similar graph families, that is why edge irregular k-labeling of circulant graphs is computed in this article. Let n, m and $a_1, a_2, \ldots, a_m$ be positive integers, $1 \leq a_i \leq \lfloor n/2 \rfloor$ and $a_i \neq a_j$, for all $1 \leq i, j \leq m$. An undirected graph with the set of vertices $V = \{v_1, v_2, \ldots, v_n\}$ and the set of edges $E = \{v_i v_{i+a_j} : 1 \leq i \leq n, 1 \leq j \leq m\}$, the indices being taken modulo n, is called a *circulant graph* and denoted by $C_n(a_1, a_2, \ldots, a_m)$. The numbers $a_1, a_2, \ldots, a_m$ are called the generators and we say that the edge $v_i v_{i+a_j}$ is *of type* a_j. It is easy to see that the circulant graph $C_n(a_1, a_2, \ldots, a_m)$ is a regular graph of degree r, where

$$r = \begin{cases} 2m-1 & \text{if } \frac{n}{2} \in \{a_1, a_2, \ldots, a_m\} \\ 2m & \text{otherwise.} \end{cases}$$

Formation of circulant graphs leads us to the fact that degrees, generators and their decomposition depends on order of graph as even or odd. Both cases are explicitly explained as follows:

Case 1: $n \equiv 0 \pmod 2$

If the order is even, then degrees will be odd, starting from $r = n - 1$ for K_n up to $r = 3$ as maximum decomposed circulant graph. The value of $m = n/2$ and the graph can be represented as $C_n(1, 2, \ldots, n/2)$. The first decomposition can be done by deleting one 2-factor (outer cycle) that will result as $(n-3)$-regular graph and graph can be represented as $C_n(2, 3, \ldots, n/2) = C_{n,n-3}$. The second decomposition can be done by deleting the second 2-factor that means all edges of type 2 will be deleted and the graph representation will be like $C_n(3, 4, \ldots, n/2) = C_{n,n-5}$. This decomposition can continue until graph becomes 3-regular and can be represented as $C_n(n/2 + 1, n/2) = C_{n,3}$.

Case 2: $n \equiv 1 \pmod 2$

If the order is odd, then degrees will be even, starting from $r = n - 1$ for K_n up to $r = 4$ as maximum decomposed circulant graph. Circulant graph with degree $r = 2$ is innermost cycle C_n whose $es(C_n)$ has been proved already in [1]. The value of $m = (n-1)/2$ and the graph can be represented as $C_n(1, 2, \ldots, (n-1)/2)$. The first decomposition can be done by deleting a Hamiltonian cycle (outer cycle) that will result as $(n-3)$-regular and the graph can be represented as $C_n(2, 3, \ldots, (n-1)/2) = C_{n,n-3}$. The second decomposition can be done by deleting another 2-factor that means all edges of type 2 will be deleted and the graph representation will be like

TABLE 1. Comparison of upper bounds for $es(K_n)$ presented in [4] with sidon and fibonacci sequences.

n	lower bound	upper bounds		
	$\left\lceil \frac{n(n-1)+2}{4} \right\rceil$	$\left\lfloor \frac{n(n-1)}{2} \log_2 n \right\rfloor$	Sidon Sequence terms	Fibonacci number F_n
3	2	3	3	2
4	4	5	6	3
5	6	9	11	5
6	8	16	17	8
7	11	25	25	13
8	15	30	34	21
9	19	35	44	34
10	23	47	55	55
11	28	65	72	89

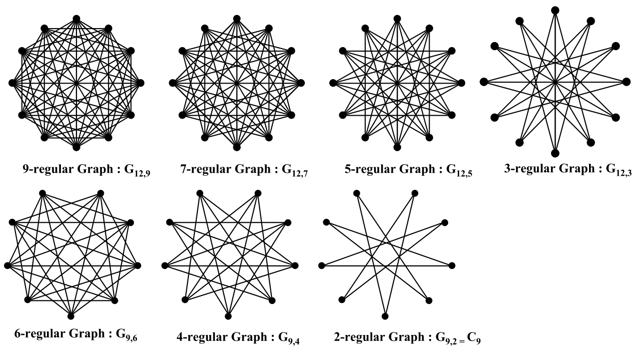


FIGURE 1. Decomposition of K_n from higher to lower degree.

$C_n(3, 4, \ldots, (n-1)/2) = C_{n,n-5}$. This decomposition can continue until graph becomes 4-regular and can be represented as $C_n((n-3)/2, (n-1)/2) = C_{n,4}$.

Phenomena of decomposition for both of the above cases is reflected in Figure 1.

III. ALGORITHMIC RESULTS

Using the decomposition process explained in Case 1 and 2, it is obvious that successive subgraphs have lesser size and according to handshaking lemma, size of each r-regular subgraph can be determined as $nr/2$. In [4] an upper bound for $es(K_n)$ is computed as $|E(K_n)| \log_2 |V(K_n)|$. The complete graph is super-circulant graph hence this result is considered as an upper bound, whereas a lower bound for any $es(G)$ is given ins Theorem 1. On the basis of these two references we established the following theorem and we proved it using algorithmic results.

Theorem 3: Let $C_{n,r}$ be an r-regular circulant graph of order n, $n \geq 5$ and $r \geq 3$. Then

$$\max\left\{\left\lceil \frac{nr+2}{4} \right\rceil, r\right\} \leq es(C_{n,r}) \leq \frac{nr}{2} \log_2 n.$$

Proof 1: The size of a circulant graph $C_{n,r}$ is $nr/2$, therefore using Theorem 1 we obtained the $(nr+2)/4$ as a lower bound for $es(C_{n,r})$. To prove the upper bound, an iterative algorithm is designed using back-tracking design strategy.

This algorithm computes an edge irregular k-labeling for a circulant graph $C_{n,r}$ with order $n \geq 5$ and any positive integer $4 \leq r \leq n-1$ as degree. Algorithm returns output in the form of an array containing label of vertices where n^{th} location vertex is k. Values of edge weights are computed, then compared their uniqueness and finally stored in a 2-D array.

Input: A positive integer $n \geq 5$ as the order of the graph and r as degree of the graph.

Output: Labels of vertices $V[n] \rightarrow \{1, 2, \ldots, k\}$

Algorithm 1 CR-Labeling(n, r)

```
V[n] ← {1, 2, 3, 4}
Diff ← n − r
dfact ← ⌊Diff/2⌋
for each edge w_φ(x, y) ← 1 where x ≠ y
t ← 5
m ← 4
repeat
    V[t] ← m + 1
    Edge-Calculate(G, t)
    if (Edge-Duplicate(G, t) ≠ TRUE)
        t ← t + 1
until t ≤ n
return V
```

Edge-Weights(G, t)

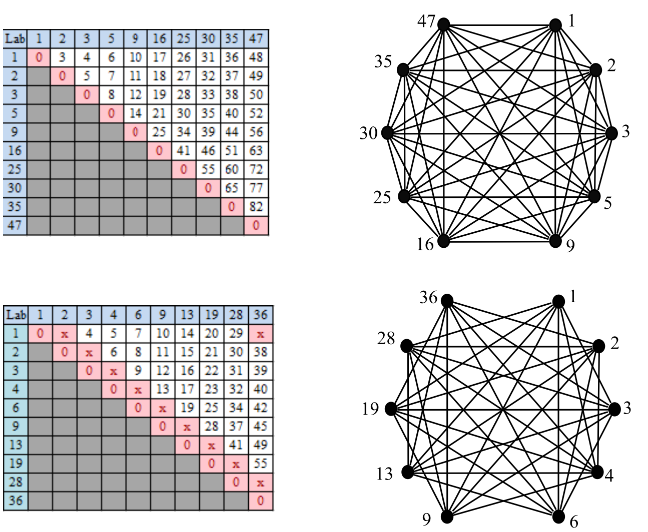
```
for i ← 1 to t − dfact − 1
    for j ← dfact + i + 1 to t

        if (i ≤ dfac AND j ≥ n − dfac + i)
            E[i][j] ← NULL
        else
            E[i][j] ← V[i] + V[j]
```

Description of the Algorithm: CR-Labeling(n, r) computes the label of vertices $\{1, 2, \ldots, k\}$ and store them in an array $V[n]$, where $\{1, 2, 3, 4\}$ are four initial labels used as seed values to initialize the back-tracking algorithm. Algorithm computes a variable "dfact" (difference factor), using the given inputs n and r that actually identi-

TABLE 2. Algorithmic results for circulant graphs $(n-3)$ to $(n-9)$ and suggested upper-bound.

n	$r=n-3$			$r=n-5$			$r=n-7$			$r=n-9$		
	$\|E\|$	k	bound	$\|E\|$	k	bound	$\|E\|$	k	bound	$\|E\|$	k	bound
7	14	13	20									
8	20	19	30	12	14	18						
9	27	28	43	18	19	29						
10	35	41	58	25	26	42	15	20	25			
11	44	55	76	33	36	57	22	26	38			
12	54	69	97	42	50	75	30	34	54	18	27	32
13	65	83	120	52	69	96	39	45	72	26	34	48
14	77	103	147	63	89	120	49	60	93	35	43	67
15	90	124	176	75	109	147	60	80	117	45	55	88
16	104	147	208	88	129	176	72	106	144	56	71	112
17	119	208	243	102	159	208	85	133	174	68	92	139
18	135	238	281	117	186	244	99	160	206	81	119	169
19	152	267	323	133	213	282	114	187	242	95	153	202
20	170	296	367	150	240	324	130	223	281	110	188	238
21	189	325	415	168	267	369	147	250	323	126	223	277
22	209	423	466	187	309	417	165	286	368	143	258	319
23	230	454	520	207	361	468	184	333	416	161	303	364
24	252	499	578	228	405	523	204	369	468	180	350	413
25	275	570	639	250	457	580	225	439	522	200	385	464
26	299	641	703	273	535	642	247	506	581	221	445	519
27	324	674	770	297	627	706	270	576	642	243	478	578
28	350	781	841	322	664	774	294	614	707	266	519	639
29	377	874	916	348	699	845	319	665	775	290	538	704
30	405	946	994	375	791	920	345	707	846	315	579	773

fies how many generators are deleted from K_n. This variable controls the starting and termination of loops. Algorithm is designed in a way that it can handle the even and odd value of n. Sub-procedure "Edge-Weights(G, t)" is based on two nested loops, that are executed in style of Gaussian arithmetic series to compute weights of all edges and then store in two dimensional array. Whereas "Weight-Duplicate(G, t)" ensures that all the edge weights are unique to verify the condition $w(u, v) \neq w(u', v')$ for all edges (u, v) and (u', v'). CR-Labeling(n, r) works in a fashion of back-tracking because any of the i^{th} label for $C_i(a_1, a_2, \ldots, a_m)$ is computed using the labels of $C_{i-1}(a_1, a_2, \ldots, a_m)$.



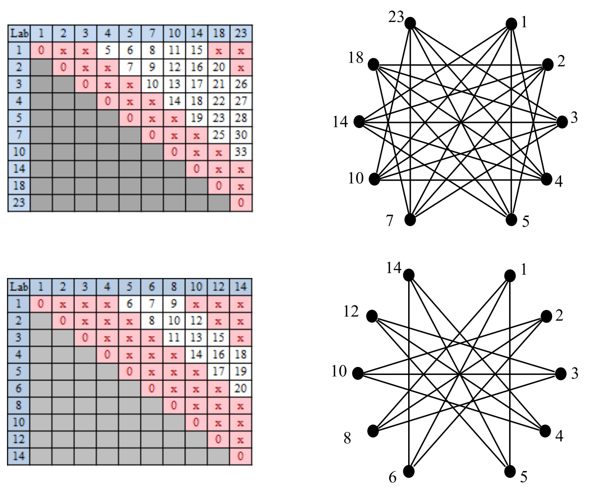


FIGURE 2. Labeling of circulant graph C_{10} from $(n-1)$ to $(n-7)$ regular.

Weight-Duplicate(G, t)

```
for i ← 2 to t − dfac − 2
    for j ← dfac + i + 1 to t − 1
        for l ← 1 to i − 1
            for w ← j + 1 to t
                if E[i][j] = E[l][w] AND E[l][w] ≠ NULL
                    return TRUE
                    break
return FALSE
```

Outcomes of the algorithm is shown pictorially in Figure 2 as labeled graphs and their representation in 2-D matrices.

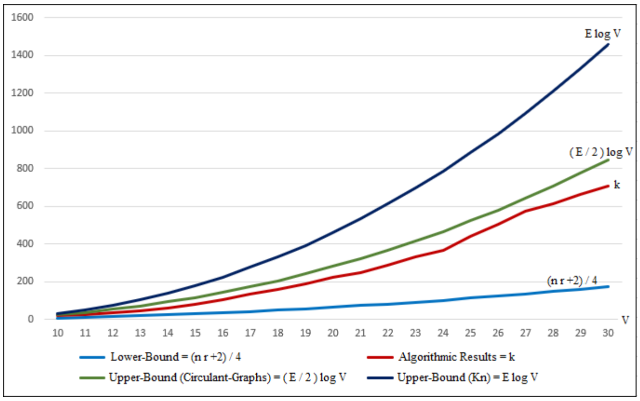


FIGURE 3. Comparison of $es(C_n(a_2, a_3, \ldots, a_m))$ with its asymptotic bounds.

It can be observed clearly that by deleting one generator sequentially, the graph is decomposed from 9-regular to 3-regular and the value of the largest label is also reduced from $k = 47$ to $k = 14$.

Examples of circulant graphs given in Figure 2, in Table 2 and the chart given in Figure 3 for higher order circulant graphs, prove the claim of this study. It can be observed as mathematical inequality that

$$\max\left\{\left\lceil\frac{nr+2}{4}\right\rceil, r\right\} \leq \frac{nr}{4}\log_2(n) \leq \frac{nr}{2}\log_2(n),$$

that completes the proof.

IV. CONCLUDING REMARKS

Using the values of k for circulant graphs calculated with the help of algorithm, and suggested upper bound that is definitely smaller than upper bound of complete graph because decomposed circulant graphs are subgraphs of K_n. Line chart given in Figure 3 shows the comparison between the curves of a lower bound for any graph, algorithmic results of circulant graphs as k, suggested upper- bound of $es(C_n(a_2, a_3, \ldots, a_m)) \leq |E|/2\log_2(|V|) \leq es(K_n) = |E|\log_2|V|$.

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