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Research on the Update Method of Attitude Quaternion for Strapdown Inertial Navigation

DEBIAO ZHANG [1], JIE LI [1,2], YANG LIU [1], XIAOKAI WEI [1], AND KAIQIANG FENG [1]

[1]Key Laboratory of Instrumentation Science & Dynamic Measurement, North University of China, Taiyuan 030051, China
[2]National Key Laboratory for Electronic Measurement Technology, North University of China, Taiyuan 030051, China

Corresponding author: Jie Li (lijie@nuc.edu.cn)

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ABSTRACT In the strapdown inertial navigation system, the accuracy of the solving attitude and position is closely related to the quaternion update algorithm. For this reason, what method is used to update the quaternion has become an important project. At present, the primary method is to update the quaternion by solving the equivalent rotation vector. A new quaternion update method is proposed in this paper. By using the third-order and fifth-order Taylor series to approximate the solution of the quaternion differential equation, two attitude quaternion update equations are derived. For single sample and two samples two cases, the derived formula is completely consistent with the common formula derived based on the equivalent rotation vector method. For the three samples in the coning motion environment, the fifth-order formula is derived in this paper, the error analysis method is applied to assess the accuracy of the formula, the two coefficients of the formula correction is optimized, and the drift rate error and drift rate are calculated, which are compared with the drift rate errors of the optimized three-sample equivalent rotation vector algorithm. The expressions for calculating the drift rate errors of the two are the same, and the calculation results are the same. In order to simplify the calculation, the established fifth-order formula is simplified to fourth-order one in this paper, but the drift rate error and drift rate remain unchanged. Finally, simulations and experiments demonstrate the correctness of the attitude quaternion update method proposed in this paper. The attitude quaternion update method proposed in this paper has certain theoretical research and application value for the attitude and position calculation of strapdown inertial navigation application, as well as the further studies of integrated navigation algorithm, and enriches the theoretical system of inertial navigation.

INDEX TERMS Attitude quaternion, Taylor series, algorithm, drift error.

I. INTRODUCTION

The navigation algorithm of strapdown inertial navigation system is composed of attitude algorithm, speed algorithm and position algorithm, among of which, attitude algorithm is always the most important research content. This is because the attitude algorithm not only directly determines the accuracy of the navigation attitude angle, but also has a vital influence on the output accuracy of speed and position. The commonly used attitude algorithms include Euler method, directional cosine method, quaternion algorithm, etc.

The Euler angle algorithm directly calculates the course angle, pitch and roll angle by solving the Euler angle differential equation, which has the simple and clear relationship and the intuitive concept, easy to understand. The orthogonalization is not required during the calculation process, but it contains trigonometric operation, which brings certain difficulties to real-time calculation. When the pitch angle is close to 90°, the equation degenerates, which is equivalent to the locking of the inertial platform in the platform inertial navigation. Hence, this method is only suitable for the situation where the horizontal attitude changes little, not for the attitude determination of the full attitude carrier [1].

The directional cosine method solves the differential equation of attitude matrix, avoiding the degeneration of the equation in the Euler angle algorithm, and can work in full attitude. However, the differential equation of attitude matrix is essentially the linear differential equation set containing 9 unknowns. Compared with the quaternion method, it has the

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large calculation amount and difficult real-time calculation, so it is not practical.

The quaternion method is only required to solve the linear differential equation set with four unknowns, with smaller calculation amount than the directional cosine method, and the simple algorithm, easy to operate. It is a practical engineering method. However, the quaternion method is actually the single-sample algorithm in the rotation vector method, with the insufficient degree of compensation for non-commutativity errors caused by limited rotation, so it is only suitable for the attitude calculation of low dynamic carrier. For highly dynamic carrier, the algorithm drift in attitude calculation will be serious. In order to solve these problems, scholars introduced the concept of rotation vector and conducted a lot of studies on attitude algorithm based on rotation vector [2], [3].

Chelnokov consider equations and algorithms describing the operation of strapdown inertial navigation systems (SINS) intended for determining the inertial attitude parameters (the Rodrigues-Hamilton (Euler) parameters) and the apparent velocity of a moving object. The construction of these equations and algorithms is based on the Kotelnikov-Study transference principle, Hamiltonian quaternions and Clifford biquaternions, and differential equations in four-dimensional (quaternion and biquaternion) orthogonal operators [4]. For the navigation algorithm of the strapdown inertial navigation system, by comparing to the equations of the dual quaternion and quaternion, the superiority of the attitude algorithm based on dual quaternion over the ones based on rotation vector in accuracy is analyzed in the case of the rotation of navigation frame [5]. In Ghanbarpourasl's study, due to the observability problem, a new robust multiplicative quaternion Kalman filter is designed for the alignment of a stationary platform [6]. The traditional velocity integration algorithms in a strapdown inertial navigation system are to approximate the rotational vector which related to the transformation matrix and then integrate the transformed specific force vector over the velocity update interval. In order to eliminate approximate integration error existed in the traditional algorithms, a new velocity integration algorithm with a time-varying slew rate vector is developed in Yueyang's paper [7]. Quaternion equations are proposed for the ideal operation of spatial inertial navigation systems with an azimuthally stabilized platform and a gyrostabilized platform that keeps its orientation invariant in inertial space, quaternion equations for the ideal operation of strapdown inertial navigation systems in the regular four-dimensional Kustaanheimo-Stiefel variables with consideration of zonal, tesseral, and sectorial harmonics of the Earth's gravitational field [8].

Bortz at first proposed the equation for solving the rotation vector in 1971 [9]–[11], and on this basis, in 1983 Miller proposed a three-sample cone optimization algorithm with optimal performance under coning motion [12], and then foreign scholars have gradually proposed other improved cone algorithms to further enhance the accuracy of the algorithm, such as: enhanced three-sample cone algorithm [13], four-sample cone algorithm [14], N-sample cone algorithm general expression [15], the cone algorithm to determine the optimal coefficient according to the frequency response characteristics of the gyroscope [16], Mao presents a framework for a strapdown Inertial Navigation System (INS) algorithm design by using Lie group and Lie algebra [17] etc. The rotation vector method can adopt multi-sample algorithm to achieve effective compensation for non-commutativity errors. The algorithm relationship is simple and easy to operate. With the optimization of coefficients, the algorithm drift is minimized in the same sample algorithm. Therefore, it is especially suitable for attitude update of carrier with frequent angle maneuvering and severe angular vibration.

Originally demonstrated by Euler and later derived by Rodriguez, every two finite rotations in space can be represented by a single equivalent rotation. Zarrouk presents a new geometric method to derive both the coordinates of the vector direction and the angle of the equivalent rotation of a rigid body that undergoes two successive finite rotations [18].

Both the quaternion method and the rotation vector method realize the attitude update by calculating the attitude quaternion. However, the quaternion method directly solves the attitude quaternion differential equation, but the rotation vector method solves the attitude quaternion by solving the rotation vector. Hence, the algorithm concepts of the two are not the same. The accuracy of the solving attitude and position is closely related to the quaternion update algorithm. Therefore, how to update the quaternion has become an important issue.

Among these algorithms, the quaternion algorithm is most preferred due to the fewer calculation parameters and no singularity. The key to the issue is to implement the attitude quaternion update algorithm to overcome the non-commutativity error caused by finite rotation. The equivalent rotation vector algorithm adopts the second term of Equation (1) (the third term is also used in some expressions) [19] to calculate a correction to deal with the non-commutativity error, and it has achieved excellent results.

$$\dot{\boldsymbol{\Phi}} = \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\Phi} \times \boldsymbol{\omega} + A\boldsymbol{\Phi} \times (\boldsymbol{\Phi} \times \boldsymbol{\omega}) \tag{1}$$

wherein, $\boldsymbol{\Phi}$ is the equivalent rotation vector corresponding to the angular position change of the carrier coordinate system from t_k to t_{k+1}, and $\boldsymbol{\omega}$ is the angular velocity of the carrier in this period of time.

This paper attempts to use Taylor series to approximate the solution of the quaternion differential equation to derive the attitude quaternion update expression to deal with this non-commutativity error. For the two cases of the current collected gyro output $\Delta\boldsymbol{\theta}(T)$ plus the gyro output sample $\Delta\boldsymbol{\theta}(T-h)$ in previous period and the two samples ($\Delta\boldsymbol{\theta}_{n1}$, $\Delta\boldsymbol{\theta}_{n2}$), the third-order Taylor series expansion is adopted in this paper to derive a quaternion expression for attitude change. For the three samples under the cone motion environment, the fifth-order Taylor series expansion is adopted in this paper to derive a quaternion expression for attitude change. The error analysis method is used to evaluate the accuracy of the expression, the two coefficients of correction value of the expression are optimized, and the drift rate error and drift rate are calculated. In order to simplify the calculation, the established fifth-order expression is simplified to fourth-order one in this paper, but the drift rate error and drift rate remain unchanged. The attitude quaternion update method proposed in this paper has certain theoretical research and application value for the attitude and position calculation of strapdown inertial navigation application, as well as the further studies of integrated navigation algorithm, and enriches the theoretical system of inertial navigation.

II. THE THIRD-ORDER TAYLOR SERIES EXPRESSION OF ATTITUDE CHANGE QUATERNION

The attitude quaternion is updated from $\boldsymbol{Q}(t_k)$ to $\boldsymbol{Q}(t_{k+1})$, which can be shown in Equation (2).

$$\boldsymbol{Q}(t_{k+1}) = \boldsymbol{Q}(t_k) \otimes \boldsymbol{q}(h) \tag{2}$$

wherein,

$$\boldsymbol{q}(h) = \cos\frac{\Phi}{2} + \frac{\boldsymbol{\Phi}}{\Phi}\sin\frac{\Phi}{2} \tag{3}$$

wherein, $\boldsymbol{Q}(t_k)$ and $\boldsymbol{Q}(t_{k+1})$ is the attitude quaternion at t_k and t_{k+1}, respectively; $h = t_{k+1} - t_k$, $\boldsymbol{q}(h)$ is the attitude change quaternion during $[t_k, t_{k+1}]$, $\Phi = |\boldsymbol{\Phi}|$.

The current mainstream method is to update the quaternion by solving the equivalent rotation vector. The new quaternion update method proposed in this article approximates the solution of the quaternion differential equation with the third-order and fifth-order Taylor series to derive the attitude quaternion update formula.

This attitude change quaternion $\boldsymbol{q}(h)$ meets the differential equation, shown in Equation (4).

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{q}(t) \otimes \frac{\boldsymbol{\omega}_{nb}^{b}}{2} \tag{4}$$

wherein, $\boldsymbol{\omega}_{nb}^{b}$ is the carrier angular velocity, which represents the projection of the carrier coordinate system's rotational angular velocity relative to the navigation coordinate system in the carrier coordinate system. The simplified zero scalar quaternion $\boldsymbol{\omega} = \begin{bmatrix} 0 & \omega_x & \omega_y & \omega_z \end{bmatrix}^T$ is used to replace $\boldsymbol{\omega}_{nb}^{b}$, and change the above expression to uppercase.

$$\dot{\boldsymbol{Q}} = \boldsymbol{Q} \otimes \frac{\boldsymbol{\omega}}{2} \tag{5}$$

At the interval $[T, T+h]$, the third-order Taylor series expansion that approximates the solution of Equation (5) is shown in Equation (6).

$$\boldsymbol{Q}(T+h) = \boldsymbol{Q}(T) + h\dot{\boldsymbol{Q}}(T) + \frac{h^2}{2!}\ddot{\boldsymbol{Q}}(T) + \frac{h^3}{3!}\dddot{\boldsymbol{Q}}(T) \tag{6}$$

Solving Equation (5) respectively for the second, third, fourth, and fifth-order derivatives are

$$\begin{aligned}\ddot{\boldsymbol{Q}} &= \frac{1}{2}(\boldsymbol{Q} \otimes \boldsymbol{\omega})^{(1)} = \frac{1}{2}[\dot{\boldsymbol{Q}} \otimes \boldsymbol{\omega} + \boldsymbol{Q} \otimes \dot{\boldsymbol{\omega}}] \\ &= \boldsymbol{Q} \otimes (\frac{\boldsymbol{\omega} \otimes \boldsymbol{\omega}}{4} + \frac{\dot{\boldsymbol{\omega}}}{2})\end{aligned} \tag{7}$$

$$\dddot{\boldsymbol{Q}} = \frac{1}{2}(\boldsymbol{Q} \otimes \boldsymbol{\omega})^{(2)} = \begin{bmatrix} \frac{1}{2}[\ddot{\boldsymbol{Q}} \otimes \boldsymbol{\omega} + 2\dot{\boldsymbol{Q}} \otimes \dot{\boldsymbol{\omega}} + \boldsymbol{Q} \otimes \ddot{\boldsymbol{\omega}}] \\ = \boldsymbol{Q} \otimes \left((\frac{\boldsymbol{\omega} \otimes \boldsymbol{\omega}}{4} + \frac{\dot{\boldsymbol{\omega}}}{2}) \otimes \frac{\boldsymbol{\omega}}{2} + \frac{1}{2}(\boldsymbol{\omega} \otimes \dot{\boldsymbol{\omega}} + \ddot{\boldsymbol{\omega}}) \right) \end{bmatrix} \tag{8}$$

For higher-order derivative, the Leibniz solution expression can be applied to obtain

$$\begin{aligned}(uv)^{(n)} = [&C_n^0 u^{(n)} v + C_n^1 u^{(n-1)} \dot{v} + \ldots + C_n^{n-1} \dot{u} v^{(n-1)} \\ &+ C_n^n u v^{(n)}]\end{aligned} \tag{9}$$

$$\begin{aligned}\boldsymbol{Q}^{(4)} &= \frac{1}{2}(\boldsymbol{Q} \otimes \boldsymbol{\omega})^{(3)} \\ &= \frac{1}{2}[C_3^0 \dddot{\boldsymbol{Q}} \otimes \boldsymbol{\omega} + C_3^1 \ddot{\boldsymbol{Q}} \otimes \dot{\boldsymbol{\omega}} + C_3^2 \dot{\boldsymbol{Q}} \otimes \ddot{\boldsymbol{\omega}} + C_3^3 \boldsymbol{Q} \otimes \dddot{\boldsymbol{\omega}}] \\ &= \boldsymbol{Q} \otimes \begin{bmatrix} \frac{\boldsymbol{\omega} \otimes \boldsymbol{\omega} \otimes \boldsymbol{\omega} \otimes \boldsymbol{\omega}}{16} + \frac{\dot{\boldsymbol{\omega}} \otimes \boldsymbol{\omega} \otimes \boldsymbol{\omega}}{8} \\ + \frac{\boldsymbol{\omega} \otimes \dot{\boldsymbol{\omega}} \otimes \boldsymbol{\omega}}{4} + \frac{\ddot{\boldsymbol{\omega}} \otimes \boldsymbol{\omega}}{4} + \frac{3\boldsymbol{\omega} \otimes \boldsymbol{\omega} \otimes \dot{\boldsymbol{\omega}}}{8} \\ + \frac{3\dot{\boldsymbol{\omega}} \otimes \dot{\boldsymbol{\omega}}}{4} + \frac{3\boldsymbol{\omega} \otimes \ddot{\boldsymbol{\omega}}}{4} + \frac{\dddot{\boldsymbol{\omega}}}{2} \end{bmatrix}\end{aligned} \tag{10}$$

$$\begin{aligned}\boldsymbol{Q}^{(5)} &= \frac{1}{2}(\boldsymbol{Q} \otimes \boldsymbol{\omega})^{(4)} \\ &= \boldsymbol{Q} \otimes \begin{bmatrix} \frac{\boldsymbol{\omega} \otimes \boldsymbol{\omega} \otimes \boldsymbol{\omega} \otimes \boldsymbol{\omega} \otimes \boldsymbol{\omega}}{32} + \frac{\dot{\boldsymbol{\omega}} \otimes \boldsymbol{\omega} \otimes \boldsymbol{\omega} \otimes \boldsymbol{\omega}}{16} \\ + \frac{\boldsymbol{\omega} \otimes \dot{\boldsymbol{\omega}} \otimes \boldsymbol{\omega} \otimes \boldsymbol{\omega}}{8} + \frac{\ddot{\boldsymbol{\omega}} \otimes \boldsymbol{\omega} \otimes \boldsymbol{\omega}}{8} \\ + \frac{\boldsymbol{\omega} \otimes \boldsymbol{\omega} \otimes \boldsymbol{\omega} \otimes \dot{\boldsymbol{\omega}}}{4} + \frac{\dot{\boldsymbol{\omega}} \otimes \boldsymbol{\omega} \otimes \dot{\boldsymbol{\omega}}}{2} \\ + \boldsymbol{\omega} \otimes \dot{\boldsymbol{\omega}} \otimes \dot{\boldsymbol{\omega}} + \ddot{\boldsymbol{\omega}} \otimes \dot{\boldsymbol{\omega}} + \frac{3\boldsymbol{\omega} \otimes \boldsymbol{\omega} \otimes \ddot{\boldsymbol{\omega}}}{4} \\ + \frac{3\boldsymbol{\omega} \otimes \ddot{\boldsymbol{\omega}}}{2} + \frac{\boldsymbol{\omega} \otimes \dddot{\boldsymbol{\omega}}}{2} + \frac{\dddot{\boldsymbol{\omega}}}{2} \end{bmatrix}\end{aligned} \tag{11}$$

Simplify the above two equations according to the quaternion operation rules to obtain

$$\boldsymbol{Q}^{(4)} \approx \boldsymbol{Q} \otimes \left[\frac{(\Delta\boldsymbol{\theta}_0)^4}{16}\boldsymbol{1} + \frac{\bar{\boldsymbol{\omega}} \times \ddot{\bar{\boldsymbol{\omega}}}}{2}\right] \tag{12}$$

$$\boldsymbol{Q}^{(5)} \approx \boldsymbol{Q} \otimes \left[\frac{(\Delta\theta_0)^4}{32}[\Delta\boldsymbol{\theta}] + \frac{\dot{\bar{\boldsymbol{\omega}}} \times \ddot{\bar{\boldsymbol{\omega}}}}{2}\right] \tag{13}$$

Substitute Equation (5), (7) and (8) into Equation (6) and sort out to obtain

$$\begin{aligned}&\boldsymbol{Q}(T+h) \\ &= \boldsymbol{Q}(T) \otimes \begin{bmatrix} 1 + \frac{1}{2}\left((\boldsymbol{\omega}h + \frac{1}{2}\dot{\boldsymbol{\omega}}h^2 + \frac{1}{6}\ddot{\boldsymbol{\omega}}h^3) \right) \\ + \frac{1}{8}\boldsymbol{\omega} \otimes \boldsymbol{\omega}h^2 + \frac{1}{48}\boldsymbol{\omega} \otimes \boldsymbol{\omega} \otimes \boldsymbol{\omega}h^3 \\ + \frac{1}{24}\dot{\boldsymbol{\omega}} \otimes \boldsymbol{\omega}h^3 + \frac{1}{12}\boldsymbol{\omega} \otimes \dot{\boldsymbol{\omega}}h^3 \end{bmatrix}\end{aligned} \tag{14}$$

According to the quaternion algorithm, calculate each item in the equation separately

$$\frac{1}{2}\left(\boldsymbol{\omega}h+\frac{1}{2}\dot{\boldsymbol{\omega}}h^2+\frac{1}{6}\ddot{\boldsymbol{\omega}}h^3\right) = \frac{1}{2}\int_0^h\left(\boldsymbol{\omega}+\dot{\boldsymbol{\omega}}\tau+\frac{1}{2}\ddot{\boldsymbol{\omega}}\tau^2\right)d\tau = \frac{1}{2}[\Delta\boldsymbol{\theta}] \tag{15}$$

$$\boldsymbol{\omega}\otimes\boldsymbol{\omega}h^2 = \begin{bmatrix}(0+\omega_x i+\omega_y j+\omega_z k)\otimes(0+\omega_x i+\omega_y j+\omega_z k)h^2 \\ = -(\omega_x^2+\omega_y^2+\omega_z^2)h^2 = -\left|\boldsymbol{\omega}h\right|^2 = -\Delta\boldsymbol{\theta}_0^2\boldsymbol{I}\end{bmatrix} \tag{16}$$

$$\dot{\boldsymbol{\omega}}\otimes\boldsymbol{\omega}h^3 = \begin{bmatrix}(0+\dot{\omega}_x i+\dot{\omega}_y j+\dot{\omega}_z k)\otimes(0+\omega i_x+\omega_y j+\omega_z k)h^3 \\ = -(\dot{\omega}_x\omega_x+\dot{\omega}_y\omega_y+\dot{\omega}_z\omega_z)h^3+\dot{\bar{\boldsymbol{\omega}}}\times\bar{\boldsymbol{\omega}}h^3\end{bmatrix} \tag{17}$$

Equation (18) and (19) can be obtained in same way as

$$\boldsymbol{\omega}\otimes\boldsymbol{\omega}\otimes\boldsymbol{\omega}h^3 = -(\Delta\boldsymbol{\theta}_0)^2[\Delta\boldsymbol{\theta}] \tag{18}$$

$$\boldsymbol{\omega}\otimes\dot{\boldsymbol{\omega}}h^3 = -(\omega_x\dot{\omega}_x+\omega_y\dot{\omega}_y+\omega_z\dot{\omega}_z)h^3+\bar{\boldsymbol{\omega}}\times\dot{\bar{\boldsymbol{\omega}}}h^3 \tag{19}$$

Given $\dot{\bar{\boldsymbol{\omega}}}\times\bar{\boldsymbol{\omega}} = -\bar{\boldsymbol{\omega}}\times\dot{\bar{\boldsymbol{\omega}}}$, perform the transformation as

$$\frac{1}{24}\dot{\boldsymbol{\omega}}\otimes\boldsymbol{\omega}h^3+\frac{1}{12}\boldsymbol{\omega}\otimes\dot{\boldsymbol{\omega}}h^3 = -\frac{1}{24}(\omega_x\dot{\omega}_x+\omega_y\dot{\omega}_y+\omega_z\dot{\omega}_z)h^3+\frac{1}{24}\bar{\boldsymbol{\omega}}\times\dot{\bar{\boldsymbol{\omega}}}h^3 \tag{20}$$

In actual engineering applications, h takes small value, h^3 can be considered as a high-level small amount of h^2, which can be ignored, so there is Equation (21) as

$$\left[\frac{1}{8}(\omega_x^2+\omega_y^2+\omega_z^2)h^2-\frac{1}{24}(\omega_x\dot{\omega}_x+\omega_y\dot{\omega}_y+\omega_z\dot{\omega}_z)h^3\right] \approx \frac{1}{8}(\Delta\boldsymbol{\theta}_0)^2\boldsymbol{I} \tag{21}$$

Substitute Equation (15) to (21) into (14), and perform approximation to obtain the quaternion attitude update expression as

$$\boldsymbol{Q}(T+h) = \left[(1-\frac{(\Delta\boldsymbol{\theta}_0)^2}{8})\boldsymbol{I}+(\frac{1}{2}-\frac{(\Delta\boldsymbol{\theta}_0)^2}{48})[\Delta\boldsymbol{\theta}]+\frac{1}{24}(\bar{\boldsymbol{\omega}}\times\dot{\bar{\boldsymbol{\omega}}})h^3\right]\times\boldsymbol{Q}(T) \tag{22}$$

Change Equation (22) to lowercase as

$$\boldsymbol{q}(T+h) = \left[(1-\frac{(\Delta\boldsymbol{\theta}_0)^2}{8})\boldsymbol{I}+(\frac{1}{2}-\frac{(\Delta\boldsymbol{\theta}_0)^2}{48})[\Delta\boldsymbol{\theta}]+\frac{1}{24}(\bar{\boldsymbol{\omega}}\times\dot{\bar{\boldsymbol{\omega}}})h^3\right]\times\boldsymbol{q}(T) \tag{23}$$

Or

$$\boldsymbol{q}(n+1) = \left[(1-\frac{(\Delta\boldsymbol{\theta}_0)^2}{8})1+(\frac{1}{2}-\frac{(\Delta\boldsymbol{\theta}_0)^2}{48})[\Delta\boldsymbol{\theta}]+\frac{1}{24}(\bar{\boldsymbol{\omega}}\times\dot{\bar{\boldsymbol{\omega}}})h^3\right]\times\boldsymbol{q}(n) \tag{24}$$

wherein, $[\Delta\boldsymbol{\theta}] = \begin{bmatrix}0 & -\Delta_x\theta & -\Delta\theta_y & -\Delta\theta_z \\ \Delta\theta_x & 0 & \Delta\theta_z & -\Delta\theta_y \\ \Delta\theta_y & -\Delta\theta_z & 0 & \Delta\theta_x \\ \Delta\theta_z & \Delta\theta_y & -\Delta\theta_x & 0\end{bmatrix}$, $\Delta\theta_x$, $\Delta\theta_y$, $\Delta\theta_z$ is the angular increment of gyro at x, y and z axis, respectively. $\frac{1}{24}(\bar{\boldsymbol{\omega}}\times\dot{\bar{\boldsymbol{\omega}}})h^3$ is one motion correction.

III. COMPARISON WITH THE COMMONLY USED ATTITUDE CHANGE QUATERNION EXPRESSION

A. COMPARISON OF SINGLE-SAMPLE EXPRESSION

In the current period, only one set of three gyro outputs of the system is sampled once. In order to enhance the calculation accuracy, the gyro output collected in the previous period is also used to form a one-plus-one condition where the gyro angular velocity output is considered constant $\bar{\boldsymbol{\omega}} = a$, then $\bar{\boldsymbol{\omega}}\times\dot{\bar{\omega}}h^3 = [\Delta\boldsymbol{\theta}(T-h)\times\Delta\boldsymbol{\theta}(T)]$. Then Equation (23) becomes

$$\boldsymbol{q}(T+h) = \left[(1-\frac{(\Delta\boldsymbol{\theta}_0)^2}{8})\boldsymbol{I}+(\frac{1}{2}-\frac{(\Delta\boldsymbol{\theta}_0)^2}{48})[\Delta\boldsymbol{\theta}] + \frac{1}{24}[\Delta\boldsymbol{\theta}(T-h)\times\Delta\boldsymbol{\theta}(T)]\right]\boldsymbol{q}(T) \tag{25}$$

Or

$$\boldsymbol{q}(n+1) = \left[(1-\frac{(\Delta\boldsymbol{\theta}_0)^2}{8})\boldsymbol{I}+(\frac{1}{2}-\frac{(\Delta\boldsymbol{\theta}_0)^2}{48})[\Delta\boldsymbol{\theta}] + \frac{1}{24}[\Delta\boldsymbol{\theta}(n-1)\times\Delta\boldsymbol{\theta}(n)]\right]\boldsymbol{q}(n) \tag{26}$$

In this case, Equation (27) is provided in Ref[1]

$$\boldsymbol{q}(t) = \boldsymbol{q}(t-T_{AUS})\cdot\left\{1+\frac{1}{2}\Delta\underline{\boldsymbol{\phi}_b}(t)-\frac{1}{8}\Delta\boldsymbol{\phi}^2-\frac{1}{48}\Delta\boldsymbol{\phi}^2\Delta\underline{\boldsymbol{\phi}_b}(t) + \frac{1}{24}\Delta\underline{\boldsymbol{\phi}_b}(t-T_{AUS})\times\Delta\underline{\boldsymbol{\phi}_b}(t)\right\} \tag{27}$$

Equation (28) is provided in Ref[1]

$$\boldsymbol{q}(n+1) = \left\{(1-\frac{(\Delta\boldsymbol{\theta}_0)^2}{8})\boldsymbol{I}+(\frac{1}{2}-\frac{(\Delta\boldsymbol{\theta}_0)^2}{48})[\Delta\boldsymbol{\theta}] + \frac{1}{24}[\Delta\boldsymbol{\theta}(n-1)\times\Delta\boldsymbol{\theta}(n)]\right\}\boldsymbol{q}(n) \tag{28}$$

It can be seen that there is no difference in the above three equations except the slightly different text symbols and writing methods used. The correctness of the quaternion update method proposed in this paper under the single-sample case is proved.

B. COMPARISON OF TWO-SAMPLE EXPRESSION

The two samples is to collect the gyro output twice at an equal interval $h/2$ in an update period h. In this case, the gyro angular velocity output is considered as the linear function of time: $\boldsymbol{\omega} = \boldsymbol{a}+2\boldsymbol{b}\tau$. The angular increment of gyro output and relevant expression is shown as follows:

$$\begin{cases} \Delta\boldsymbol{\theta}_n = \Delta\boldsymbol{\theta}_{n1} + \Delta\boldsymbol{\theta}_{n2} \\ \Delta\boldsymbol{\theta}_{n1} = \int_0^{\frac{h}{2}} (\boldsymbol{a} + 2\boldsymbol{b}\tau)d\tau = \frac{1}{2}\boldsymbol{a}h + \frac{1}{4}\boldsymbol{b}h^2 \\ \Delta\boldsymbol{\theta}_{n2} = \int_{\frac{h}{2}}^{h} (\boldsymbol{a} + 2\boldsymbol{b}\tau)d\tau = \frac{1}{2}\boldsymbol{a}h + \frac{3}{4}\boldsymbol{b}h^2 \end{cases} \tag{29}$$

Equation (30) can be obtained from Equation (29).

$$\begin{cases} \boldsymbol{b}\mathrm{h}^2 = 2(\Delta\boldsymbol{\theta}_{n2} - \Delta\boldsymbol{\theta}_{n1}) \\ \boldsymbol{a}h = 3\Delta\boldsymbol{\theta}_{n1} - \Delta\boldsymbol{\theta}_{n2} \end{cases} \tag{30}$$

Considering

$$\begin{aligned} \bar{\boldsymbol{\omega}} \times \dot{\boldsymbol{\omega}}h^3 &= [(\boldsymbol{a} + 2\boldsymbol{b}\tau) \times 2\boldsymbol{b}] = 2(\boldsymbol{a} \times \boldsymbol{b}) \\ &= (6\Delta\boldsymbol{\theta}_{n1} - 2\Delta\boldsymbol{\theta}_{n2}) \times (2\Delta\boldsymbol{\theta}_{n2} - 2\Delta\boldsymbol{\theta}_{n1})h^3 \\ &= 8(\Delta\boldsymbol{\theta}_{n1} \times \Delta\boldsymbol{\theta}_{n2}) \end{aligned} \tag{31}$$

Substitute the result to Equation (24) to obtain

$$\begin{aligned} &\boldsymbol{q}(n+1) \\ &= \left[(1 - \frac{(\Delta\boldsymbol{\theta}_0)^2}{8})\boldsymbol{I} + (\frac{1}{2} - \frac{(\Delta\boldsymbol{\theta}_0)^2}{48})[\Delta\boldsymbol{\theta}] \right. \\ &\quad \left. + \frac{1}{3}(\Delta\boldsymbol{\theta}_{n1} \times \Delta\boldsymbol{\theta}_{n2})\right]\boldsymbol{q}(n) \end{aligned} \tag{32}$$

The expression provided in Ref[1] is as follows

$$\begin{aligned} &\boldsymbol{q}(n+1) \\ &= \left[(1 - \frac{(\Delta\boldsymbol{\theta}_0)^2}{8})\boldsymbol{I} + (\frac{1}{2} - \frac{(\Delta\boldsymbol{\theta}_0)^2}{48})[\Delta\boldsymbol{\theta}] \right. \\ &\quad \left. + \frac{1}{3}(\Delta\boldsymbol{\theta}_{n1} \times \Delta\boldsymbol{\theta}_{n2})\right]\boldsymbol{q}(n) \end{aligned} \tag{33}$$

Equation (32) are identical to (33), so the correctness of the quaternion update method proposed in this paper under the two samples is proved.

IV. THREE-SAMPLE ATTITUDE CHANGE QUATERNION EXPRESSION

A. THE FIFTH-ORDER TAYLOR SERIES APPROXIMATION OF THE SOLUTIONS OF QUATERNION DIFFERENTIAL EQUATION

At the interval $[T, T+h]$, the third-order Taylor series expansion that approximates the solution of Equation (5) is shown in Equation (34).

$$\begin{aligned} \boldsymbol{Q}(T+h) = \boldsymbol{Q}(T) + h\dot{\boldsymbol{Q}}(T) + \frac{h^2}{2!}\ddot{\boldsymbol{Q}}(T) + \frac{h^3}{3!}\dddot{\boldsymbol{Q}}(T) \\ + \frac{h^4}{4!}\boldsymbol{Q}^{(4)}(T) + \frac{h^5}{5!}\boldsymbol{Q}^{(5)}(T) \end{aligned} \tag{34}$$

Substitute Equation (5), (7)-(13) to Equation (34), sort out and simplify to obtain

$$\begin{aligned} &\boldsymbol{Q}(T+h) \\ &= \begin{bmatrix} (1 - \frac{(\Delta\boldsymbol{\theta}_0)^2}{8} + \frac{(\Delta\boldsymbol{\theta}_0)^4}{384})1 + (\frac{1}{2} - \frac{(\Delta\boldsymbol{\theta}_0)^2}{48} \\ + \frac{(\Delta\boldsymbol{\theta}_0)^4}{3840})[\Delta\boldsymbol{\theta}] + \frac{1}{24}(\bar{\boldsymbol{\omega}} \times \dot{\boldsymbol{\omega}})h^3 + \frac{1}{48}(\bar{\boldsymbol{\omega}} \times \ddot{\boldsymbol{\omega}})h^4 \\ + \frac{1}{240}(\dot{\boldsymbol{\omega}} \times \ddot{\boldsymbol{\omega}})h^5 \end{bmatrix} \\ &\quad \times \boldsymbol{Q}(T) \end{aligned} \tag{35}$$

wherein, $\frac{1}{24}(\bar{\boldsymbol{\omega}} \times \dot{\boldsymbol{\omega}})h^3 + \frac{1}{48}(\bar{\boldsymbol{\omega}} \times \ddot{\boldsymbol{\omega}})h^4 + \frac{1}{240}(\dot{\boldsymbol{\omega}} \times \ddot{\boldsymbol{\omega}})h^5$ is called the quaternion correction of the three-sample attitude changes of the carrier cone motion.

B. CALCULATION FOR QUATERNION CORRECTION OF ATTITUDE CHANGE

At the interval $[t, t+h]$, the gyro angular velocity output can be fitted with a parabola:

$$\boldsymbol{\omega}_{nb}^b(t_n + \tau) = \boldsymbol{a} + 2\boldsymbol{b}\tau + 3\boldsymbol{c}\tau^2, \quad 0 \le \tau \le h \tag{36}$$

The angular increment of gyro output is

$$\Delta\boldsymbol{\theta}(\tau) = \int_0^{\tau} \boldsymbol{\omega}(t_n + \tau)d\tau \tag{37}$$

The so-called three samples refers to the three angular increments of gyro output at the equal interval, that is

$$\begin{cases} \Delta\boldsymbol{\theta}_1 = \int_0^{\frac{h}{3}} (\boldsymbol{a}+2\boldsymbol{b}\tau+3\boldsymbol{c}\tau^2)d\tau = \frac{1}{3}\boldsymbol{a}h + \frac{1}{9}\boldsymbol{b}h^2 + \frac{1}{27}\boldsymbol{c}h^3 \\ \Delta\boldsymbol{\theta}_2 = \int_{\frac{h}{3}}^{\frac{2h}{3}} (\boldsymbol{a}+2\boldsymbol{b}\tau+3\boldsymbol{c}\tau^2)d\tau = \frac{1}{3}\boldsymbol{a}h + \frac{3}{9}\boldsymbol{b}h^2 + \frac{7}{27}\boldsymbol{c}h^3 \\ \Delta\boldsymbol{\theta}_3 = \int_{\frac{2h}{3}}^{h} (\boldsymbol{a}+2\boldsymbol{b}\tau+3\boldsymbol{c}\tau^2)d\tau = \frac{1}{3}\boldsymbol{a}h + \frac{5}{9}\boldsymbol{b}h^2 + \frac{19}{27}\boldsymbol{c}h^3 \end{cases} \tag{38}$$

Solving the simultaneous equations (38) to obtain

$$\begin{cases} \boldsymbol{a}h = \frac{11}{2}\Delta\boldsymbol{\theta}_1 - \frac{7}{2}\Delta\boldsymbol{\theta}_2 + \Delta\boldsymbol{\theta}_3 \\ \boldsymbol{b}h = -9\Delta\boldsymbol{\theta}_1 + \frac{27}{2}\Delta\boldsymbol{\theta}_2 - \frac{9}{2}\Delta\boldsymbol{\theta}_3 \\ \boldsymbol{c}h = \frac{9}{2}\Delta\boldsymbol{\theta}_1 - 9\Delta\boldsymbol{\theta}_2 + \frac{9}{2}\Delta\boldsymbol{\theta}_3 \end{cases} \tag{39}$$

Based on Equation (36), we can get

$$\begin{cases} \frac{1}{24}(\bar{\boldsymbol{\omega}} \times \dot{\boldsymbol{\omega}})h^3 = \frac{1}{24}(\boldsymbol{a}h) \times (2\boldsymbol{b}h^2) = \frac{1}{12}(\boldsymbol{a} \times \boldsymbol{b})h^3 \\ \frac{1}{48}(\bar{\boldsymbol{\omega}} \times \ddot{\boldsymbol{\omega}})h^4 = \frac{1}{48}(\boldsymbol{a}h) \times (6\boldsymbol{c}h^3) = \frac{1}{8}(\boldsymbol{a} \times \boldsymbol{c})h^4 \\ \frac{1}{240}(\dot{\boldsymbol{\omega}} \times \ddot{\boldsymbol{\omega}})h^5 = \frac{1}{240}(2\boldsymbol{b}h^2) \times (6\boldsymbol{c}h^3) = \frac{1}{20}(\boldsymbol{b} \times \boldsymbol{c})h^5 \end{cases} \tag{40}$$

Substitute a, b and c obtained from Equation (39) to (40) to obtain the correction as

$$\begin{aligned} &\frac{1}{24}(\bar{\boldsymbol{\omega}} \times \dot{\boldsymbol{\omega}})h^3 + \frac{1}{48}(\bar{\boldsymbol{\omega}} \times \ddot{\boldsymbol{\omega}})h^4 + \frac{1}{240}(\dot{\boldsymbol{\omega}} \times \ddot{\boldsymbol{\omega}})h^5 \\ &= \frac{1}{12}(\boldsymbol{a} \times \boldsymbol{b}) + \frac{1}{8}(\boldsymbol{a} \times \boldsymbol{c}) + \frac{1}{20}(\boldsymbol{b} \times \boldsymbol{c}) \end{aligned}$$

$$= \frac{57}{160}(\Delta\boldsymbol{\theta}_1\times\Delta\boldsymbol{\theta}_2)+\frac{33}{160}(\Delta\boldsymbol{\theta}_1\times\Delta\boldsymbol{\theta}_3)+\frac{57}{160}(\Delta\boldsymbol{\theta}_2\times\Delta\boldsymbol{\theta}_3)$$
$$= \frac{33}{160}(\Delta\boldsymbol{\theta}_1\times\Delta\boldsymbol{\theta}_3)+\frac{57}{160}[\Delta\boldsymbol{\theta}_2\times(\Delta\boldsymbol{\theta}_3-\Delta\boldsymbol{\theta}_1)] \quad (41)$$

Substitute Equation (41) to (35) to derive the fifth-order Taylor series expansion of the quaternion for attitude update as

$$\boldsymbol{Q}(T+h) = \begin{bmatrix} (1-\frac{(\Delta\boldsymbol{\theta}_0)^2}{8}+\frac{(\Delta\boldsymbol{\theta}_0)^4}{384})1+(\frac{1}{2}-\frac{(\Delta\boldsymbol{\theta}_0)^2}{48} \\ +\frac{(\Delta\boldsymbol{\theta}_0)^4}{3840})[\Delta\boldsymbol{\theta}]+\frac{33}{160}(\Delta\boldsymbol{\theta}_1\times\Delta\boldsymbol{\theta}_3)+\frac{57}{160} \\ \times[\Delta\boldsymbol{\theta}_2\times(\Delta\boldsymbol{\theta}_3-\Delta\boldsymbol{\theta}_1)] \end{bmatrix} \times\boldsymbol{Q}(T) \quad (42)$$

Compare Equation (42) and (A-6) to obtain the quaternion expression for the three-sample attitude change as

$$\boldsymbol{q}(h) = \begin{bmatrix} (1-\frac{(\Delta\boldsymbol{\theta}_0)^2}{8}+\frac{(\Delta\boldsymbol{\theta}_0)^4}{384})1+(\frac{1}{2}-\frac{(\Delta\boldsymbol{\theta}_0)^2}{48}+\frac{(\Delta\boldsymbol{\theta}_0)^4}{3840}) \\ \times[\Delta\boldsymbol{\theta}]+\frac{33}{160}(\Delta\boldsymbol{\theta}_1\times\Delta\boldsymbol{\theta}_3)+\frac{57}{160} \\ \times[\Delta\boldsymbol{\theta}_2\times(\Delta\boldsymbol{\theta}_3-\Delta\boldsymbol{\theta}_1)] \end{bmatrix} \quad (43)$$

wherein, $\frac{33}{160}(\Delta\boldsymbol{\theta}_1\times\Delta\boldsymbol{\theta}_3)+\frac{57}{160}[\Delta\boldsymbol{\theta}_2\times(\Delta\boldsymbol{\theta}_3-\Delta\boldsymbol{\theta}_1)]$ is correction. The coefficients $k_1 = 33/160$ and $k_2 = 55/160$ are to be analyzed and optimized.

Equation (43) is rewritten into matrix form as

$$\boldsymbol{q}(h) = \begin{bmatrix} (1-\frac{(\Delta\boldsymbol{\theta}_0)^2}{8}+\frac{(\Delta\boldsymbol{\theta}_0)^4}{384}) \\ (\frac{1}{2}-\frac{(\Delta\boldsymbol{\theta}_0)^2}{48}+\frac{(\Delta\boldsymbol{\theta}_0)^4}{3840})\Delta\theta_x+k_1\Delta\theta'_x+k_2\Delta\theta''_x \\ (\frac{1}{2}-\frac{(\Delta\boldsymbol{\theta}_0)^2}{48}+\frac{(\Delta\boldsymbol{\theta}_0)^4}{3840})\Delta\theta_y+k_1\Delta\theta'_y+k_2\Delta\theta''_y \\ (\frac{1}{2}-\frac{(\Delta\boldsymbol{\theta}_0)^2}{48}+\frac{(\Delta\boldsymbol{\theta}_0)^4}{3840})\Delta\theta_z+k_1\Delta\theta'_z+k_2\Delta\theta''_z \end{bmatrix} \quad (44)$$

wherein, $k_1\Delta\theta'+k_2\Delta\theta''$ is correction item. According to Equation (A-11) and (A-12), Equation (45) and (46) can be obtained.

$$k_1\Delta\boldsymbol{\theta}' = k_1\begin{bmatrix}\Delta\theta'_x\\ \Delta\theta'_y\\ \Delta\theta'_z\end{bmatrix} = k_1\begin{bmatrix} F^2\sin(\frac{2\omega h}{3}) \\ -2FE\sin(\frac{\omega h}{3})\sin\omega(t+\frac{h}{2}) \\ 2FE\sin(\frac{\omega h}{3})\cos\omega(t+\frac{h}{2}) \end{bmatrix} \quad (45)$$

$$k_2\Delta\boldsymbol{\theta}'' = k_2\begin{bmatrix}\Delta\theta'_x\\ \Delta\theta'_y\\ \Delta\theta'_z\end{bmatrix} = k_2\begin{bmatrix} 2F^2\sin(\frac{\omega h}{3}) \\ -2FE\sin(\frac{\omega h}{3})\sin\omega(t+\frac{h}{2}) \\ 2FE\sin(\frac{\omega h}{3})\cos\omega(t+\frac{h}{2}) \end{bmatrix} \quad (46)$$

C. QUATERNION ERROR ANALYSIS OF ATTITUDE CHANGE AND OPTIMIZATION OF CORRECTION

Assume that $\boldsymbol{q}(h)$ is the true attitude change quaternion, $\hat{\boldsymbol{q}}(h)$ is the attitude change quaternion estimated by Equation (44), and $\tilde{\boldsymbol{q}}(h)$ is the attitude change error quaternion. $\tilde{\boldsymbol{q}}(h) = \boldsymbol{q}(h)\otimes\hat{\boldsymbol{q}}^{-1}(h)$, then

$$\tilde{\boldsymbol{q}}(h) = \boldsymbol{q}(h)\otimes\hat{\boldsymbol{q}}^{*}(h) = M[\boldsymbol{q}(h)]\hat{\boldsymbol{q}}^{*}(h) \quad (47)$$

That is

$$\begin{bmatrix}\tilde{q}_0\\ \tilde{q}_1\\ \tilde{q}_2\\ \tilde{q}_3\end{bmatrix} = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \times \begin{bmatrix} (1-\frac{(\Delta\boldsymbol{\theta}_0)^2}{8}+\frac{(\Delta\boldsymbol{\theta}_0)^4}{384}) \\ -[(\frac{1}{2}-\frac{(\Delta\boldsymbol{\theta}_0)^2}{48}+\frac{(\Delta\boldsymbol{\theta}_0)^4}{3840})\Delta\theta_x+k_1\Delta\theta'_x+k_2\Delta\theta''_x] \\ -[(\frac{1}{2}-\frac{(\Delta\boldsymbol{\theta}_0)^2}{48}+\frac{(\Delta\boldsymbol{\theta}_0)^4}{3840})\Delta\theta_y+k_1\Delta\theta'_y+k_2\Delta\theta''_y] \\ -[(\frac{1}{2}-\frac{(\Delta\boldsymbol{\theta}_0)^2}{48}+\frac{(\Delta\boldsymbol{\theta}_0)^4}{3840})\Delta\theta_y+k_1\Delta\theta'_y+k_2\Delta\theta''_y] \end{bmatrix} \quad (48)$$

wherein, $\boldsymbol{q}(h)$ is the quaternion of the real attitude change represented by Equation (A-6). If the estimated attitude change quaternion $\hat{\boldsymbol{q}}(h)$ does not contain error, the product of the above Equation should be 1. If $\hat{\boldsymbol{q}}(h)$ contains errors, it will cause an error in the updated attitude quaternion $\boldsymbol{Q}(T+h)$. The important factor affecting the error is the low-order item in $\hat{\boldsymbol{q}}(h)$, Thus, only the low-order term need to be considered, and the above formula is simplified into the following formula

$$\begin{bmatrix}\tilde{q}_0\\ \tilde{q}_1\\ \tilde{q}_2\\ \tilde{q}_3\end{bmatrix} = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & q_2 & q_1 & q_0 \end{bmatrix} \times \begin{bmatrix} 1 \\ -[\frac{1}{2}\Delta\theta_x+\Delta\theta'_x+\Delta\theta''_x] \\ -[\frac{1}{2}\Delta\theta_y+\Delta\theta'_y+\Delta\theta''_y] \\ -[\frac{1}{2}\Delta\theta_y+\Delta\theta'_y+\Delta\theta''_y] \end{bmatrix} \quad (49)$$

Observing Equation (A-6), it can be seen that q_2, q_3 in $\boldsymbol{q}(h)$ are periodic variable whose frequency is equal to the frequency of cone motion. Observing Equation (A-9) and Equation (46) and (49), it can be seen that $\Delta\theta_y$, $\Delta\theta_z$, $\Delta\theta'_y$, $\Delta\theta'_z$ and $\Delta\theta''_y$, $\Delta\theta''_z$ are also the periodic variable whose frequency is equal to the frequency of cone motion. Correspondingly, $\tilde{q}_2, \tilde{q}_3$ in $\tilde{\boldsymbol{q}}(h)$ are also periodic variables. For Equation(49), only $\boldsymbol{q}(h)$ can cause $\boldsymbol{Q}(T+h)$ to produce drift rate error, that is the DC component with aperiodic term in Equation (A-6).

$$\begin{cases} q_0 = 1 - 2\sin^2(\frac{a}{2})\sin^2(\frac{\omega h}{2}) \\ q_1 = -\sin^2(\frac{a}{2})\sin(\omega h) \end{cases} \tag{50}$$

Reflected in the quaternion $\tilde{\boldsymbol{q}}(h)$ of the attitude change error, which is

$$\begin{cases} \tilde{q}_0 = q_0 + q_1[\frac{1}{2}\Delta\theta_x + \Delta'\theta_x + \Delta\theta''_x] \\ \tilde{q}_1 = q_1 - q_0[\frac{1}{2}\Delta\theta_x + \Delta'\theta_x + \Delta\theta''_x] \end{cases} \tag{51}$$

Because a is a small amount, $2\sin^2(\frac{a}{2})\sin^2(\frac{\omega h}{2}) \ll 1$, then $q_0 = 1$. Therefore, the main component of the error quaternion can be derived as

$$\tilde{q}_1 = q_1 - [\frac{1}{2}\Delta\theta_x + \Delta'\theta_x + \Delta\theta''_x] \tag{52}$$

The quaternion error of attitude change is written in standard form as

$$\tilde{q}_1(h) = \frac{\Delta\theta_{\varepsilon x}}{\Delta\theta_\varepsilon}\sin(\frac{\Delta\theta_\varepsilon}{2}) \tag{53}$$

wherein, $\Delta\theta_{\varepsilon x}$ is the drift rate error on the x axis; $\Delta\theta_\varepsilon$ is the modulus of the drift error. According to Equation (53), the drift rate error is concentrated on x axis. $\Delta\theta_{\varepsilon x}/\Delta\theta_\varepsilon \approx 1$ The error $\Delta\theta_\varepsilon$ is small amount, so

$$\tilde{q}_1(h) = \frac{\Delta\theta_{\varepsilon x}}{\Delta\theta_\varepsilon}\sin(\frac{\Delta\theta_\varepsilon}{2}) = \sin(\frac{\Delta\theta_\varepsilon}{2}) = \frac{\Delta\theta_\varepsilon}{2} \tag{54}$$

Then

$$\Delta\theta_\varepsilon = 2\tilde{q}_1 \tag{55}$$

Substitute Equation (52) into (55) to obtain

$$\Delta\theta_\varepsilon = 2q_1 - [\Delta\theta_x + 2\Delta\theta'_x + 2\Delta\theta''_x] \tag{56}$$

Substitute the corresponding $\Delta\theta_x$, $\Delta\theta'_x$, $\Delta\theta''_x$ parameters in Equation (A-9), (45), (46) and (50) into Equation (56) to obtain the drift rate error expression as

$$\Delta\theta_\varepsilon = \left| \begin{array}{l} -2\sin^2(\frac{a}{2})\sin(\omega h) + 2\omega h\sin^2(\frac{a}{2}) \\ -2F^2k_1\sin(\frac{2\omega h}{3}) - 4F^2k_2\sin(\frac{\omega h}{3}) \end{array} \right| \tag{57}$$

wherein, $F = 2\sin(a)\sin(\frac{\omega h}{6})$. The above expression can be sorted as

$$\begin{aligned} \Delta\theta_e &= \left| \begin{array}{l} -2\sin^2(\frac{a}{2})\sin(\omega h) + 2\omega h\sin^2(\frac{a}{2}) - \\ 2F^2k_1\sin(\frac{2\omega h}{3}) - 4F^2k_2\sin(\frac{\omega h}{3}) \end{array} \right| \\ &= \left| \begin{array}{l} -\frac{1}{2}a^2\sin(\omega h) + \frac{1}{2}a^2\omega h - \\ 4k_1a^2\sin(\frac{2\omega h}{3})\sin^2(\frac{\omega h}{6}) - 8k_2a^2\sin(\frac{\omega h}{3}) \\ \times\sin^2(\frac{\omega h}{6}) \end{array} \right| \\ &= a^2\left| \begin{array}{l} -\frac{1}{2}\sin(\omega h) + \frac{1}{2}\omega h - 4k_1\sin(\frac{2\omega h}{3}) \\ \times(1-\cos(\frac{\omega h}{3}) - 8k_2\sin(\frac{\omega h}{3})(1-\cos(\frac{\omega h}{3}) \end{array} \right| \\ &= a^2\left| \begin{array}{l} -\frac{1}{2}(\omega h) + \frac{1}{2}\omega h - 4k_1[(\frac{2\omega h}{3}) - \frac{1}{2}(\omega h) - \\ \frac{1}{2}(\frac{\omega h}{3})] - 8k_2[(\frac{\omega h}{3}) - (\frac{\omega h}{3})] \end{array} \right| \end{aligned} \tag{58}$$

Expand the sine function in the above expression in increasing order to ωh series, and solve the coefficients of each power ωh

$$\begin{aligned} &(\omega h)^1 : 0 \\ &(\omega h)^3 : (-\frac{1}{3!})\left\{-\frac{1}{2} + \frac{8}{9}(k_1+k_2)\right\} \\ &(\omega h)^5 : (-\frac{1}{5!})\left\{-\frac{1}{2} + \frac{40}{27}k_1 + \frac{40}{81}k_2)\right\} \\ &(\omega h)^7 : (\frac{1}{7!})\left(-\frac{1}{2} + \frac{7728}{54\times 81}k_1 + \frac{1008}{54\times 81}k_2\right) \end{aligned}$$

Because $(\omega h) \ll 1$, in order to make $\Delta\theta_e$ as small as possible, the low power coefficient of ωh is required to be zero. For this reason, the coefficient of $(\omega h)^3$ and $(\omega h)^5$ is selected to be zero, to obtain

$$\begin{cases} \frac{8}{9}(k_1+k_2) = \frac{1}{2} \\ \frac{40}{27}k_1 + \frac{40}{81}k_2 = \frac{1}{2} \end{cases} \tag{59}$$

Solve the simultaneous expressions to get the optimization coefficient of the correction:

$$\begin{cases} k_1 = \frac{9}{40} \\ k_2 = \frac{27}{80} \end{cases} \tag{60}$$

Substitute k_1, k_2 into Equation (43) to obtain the final expression of attitude change quaternion as

$$\boldsymbol{q}(h) = \left[\begin{array}{l} (1 - \frac{(\Delta\boldsymbol{\theta}_0)^2}{8} + \frac{(\Delta\boldsymbol{\theta}_0)^4}{384})\boldsymbol{I} \\ +(\frac{1}{2} - \frac{(\Delta\boldsymbol{\theta}_0)^2}{48} + \frac{(\Delta\boldsymbol{\theta}_0)^4}{3840})[\Delta\boldsymbol{\theta}] \\ +\frac{9}{40}(\Delta\boldsymbol{\theta}_1 \times \Delta\boldsymbol{\theta}_3) \\ +\frac{27}{80}[\Delta\boldsymbol{\theta}_2 \times (\Delta\boldsymbol{\theta}_3 - \Delta\boldsymbol{\theta}_1)] \end{array} \right] \tag{61}$$

And the residual error of the quaternion expression for attitude change as

$$\Delta\theta_e = a^2\left|(\omega h)^7(-\frac{1}{7!})\left(-\frac{1}{2} + \frac{1}{54\times 81}(7728\times\frac{18}{80} + 1008\times\frac{27}{80}\right)\right|$$

$$= a^2 \left|(\omega h)^7 \frac{1}{5040}(-\frac{1}{2} + \frac{1}{54 \times 81}(7728 \times \frac{18}{80} + 1008 \times \frac{27}{80})\right|$$

$$= \frac{a^2(\omega h)^7}{204120} \tag{62}$$

The drift rate of the formula in the h interval is

$$\Delta\dot{\theta}_\varepsilon = \frac{a\omega^7 h^6}{204120} \approx a\omega^7 h^6 \times 10^{-5} \tag{63}$$

D. THE IMPROVED FOURTH-ORDER APPROXIMATE CALCULATION

The quaternion expression for attitude change is derived in Ref [12] as

$$\begin{aligned} \boldsymbol{q}(h) &= \boldsymbol{C}, \quad \phi \boldsymbol{S} \\ \boldsymbol{C} &= \cos(1/2\boldsymbol{\phi}_0), \quad \boldsymbol{S} = (1/\boldsymbol{\phi}_0)\sin(1/2\boldsymbol{\phi}_0), \\ \boldsymbol{\phi}_0 &= (\boldsymbol{\phi}.\boldsymbol{\phi})^{1/2} \\ \boldsymbol{\phi} &= \boldsymbol{\theta} + \frac{9}{20}\boldsymbol{\theta}_1 \times \boldsymbol{\theta}_3 + \frac{27}{40}\boldsymbol{\theta}_2 \times (\boldsymbol{\theta}_3 - \boldsymbol{\theta}_1) \end{aligned} \tag{64}$$

The fourth-order approximation is performed in Ref [12] to the above expression to obtain

$$\boldsymbol{C} = 1 - \frac{\boldsymbol{\phi}_0^2}{8} + \frac{\boldsymbol{\phi}_0^4}{480}, \quad \boldsymbol{S} = \frac{1}{2} - \frac{\boldsymbol{\phi}_0^2}{48} \tag{65}$$

In Ref [12], if $\boldsymbol{C} = 1 - \frac{\boldsymbol{\phi}_0^2}{2^2 2!} + k\frac{\boldsymbol{\phi}_0^4}{2^4 4!}$, $\boldsymbol{S} = \frac{1}{2} - \frac{\boldsymbol{\phi}_0^2}{48}$, it can be found that if $k = 0.8$, the coefficient of $\boldsymbol{\phi}_0^4$ can be "0," the calculation proves that $384/480 = 0.8 = k$.

Based on the above information, "0" in Ref[12] means that $k\frac{1}{2^4 4!} = 0.8 \times \frac{1}{384} = 2.0833 \times 10^{-3}$. Compared to the improvement of Ref[12], in Equation (61) derived, the coefficient $1/3840 = 2.6 \times 10^{-4}$ can be regarded as "0." So Equation (61) can be approximated as

$$\boldsymbol{q}(h) = \begin{bmatrix} (1 - \frac{(\Delta\boldsymbol{\theta}_0)^2}{8} + \frac{(\Delta\boldsymbol{\theta}_0)^4}{480})\boldsymbol{I} + (\frac{1}{2} - \frac{(\Delta\boldsymbol{\theta}_0)^2}{48})[\Delta\boldsymbol{\theta}] \\ + \frac{9}{40}(\Delta\boldsymbol{\theta}_1 \times \Delta\boldsymbol{\theta}_3) + \frac{27}{80}[\Delta\boldsymbol{\theta}_2 \times (\Delta\boldsymbol{\theta}_3 - \Delta\boldsymbol{\theta}_1)] \end{bmatrix} \tag{66}$$

V. SIMULATION AND EXPERIMENTAL VERIFICATION

In order to prove the correctness of the attitude quaternion update method proposed in this paper, according to the derivation process of this paper, the attitude quaternion update equation derived from the method of approximating the solution of the quaternion differential equation with the third and fifth order Taylor series is used in this paper. The simulation experiment and ground sports car experiment were carried out respectively. The attitude update algorithm proposed in this paper is compared with the fourth-order Runge-Kutta method to solve quaternion differential equations and the fourth-order Runge-Kutta method to solve Bortz equation. It verifies the superiority of the proposed attitude quaternion update method. Has a smaller error.

A. SIMULATION VERIFICATION

The single-sample, two-sample, and three-sample algorithms of attitude update assume that the angular velocity of the carrier between two adjacent sampling points is a constant, a straight line, and a parabola. The angular velocity of the carrier is not really as assumed, and the coefficients determined in the formulas cannot ensure that the algorithm drift is minimized. For strapdown inertial navigation attitude update, cone motion is the worst working environment condition, which will induce serious drift of the mathematical platform. Therefore, when the algorithm is optimized, the cone movement is often used as the environmental condition. This means that if the algorithm drift can be ensured to be the smallest under the environmental conditions of the cone movement, then the algorithm drift can be ensured to be the smallest under other environmental conditions. Therefore, the cone motion environment is selected to simulate and verify the method proposed in this paper.

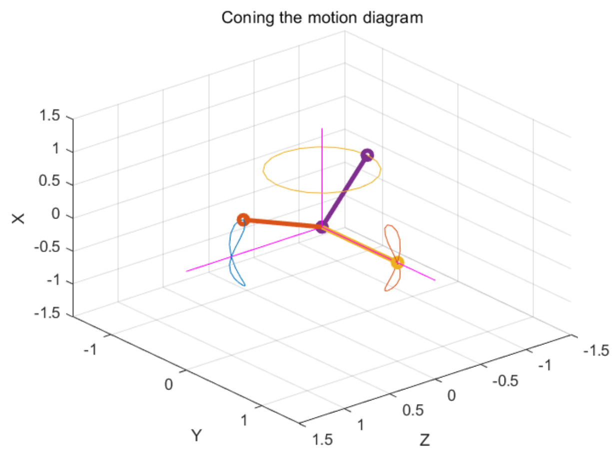

FIGURE 1. Coning the motion diagram.

In order to demonstrate the effectiveness of the method proposed in this paper, simulations are carried out in a cone motion environment. The schematic diagram of cone movement is shown in Fig.1. The simulation scene is set to cone movement around the X axis, the half-apex angle is 1.5708°, the coning frequency is 2 Hz, and the sampling interval is 0.01s.

For the three sub-samples in the cone motion environment, this paper derives the fifth-order formula, applies the error analysis method to evaluate the accuracy of the formula, optimizes the two coefficients of the formula correction, and calculates the drift rate error and drift rate. In order to simplify the calculation, this article simplifies the established fifth-order formula to fourth-order, but the drift rate error and drift rate remain unchanged. Fig.2 shows the three-axis attitude angle changes and coning noncommutativity drift under the condition of cone motion. Fig.3 is the comparison of the three axial attitude misalignment angles obtained after optimization of the two coefficients of the correction amount of the formula derived in this paper and the three axial attitude

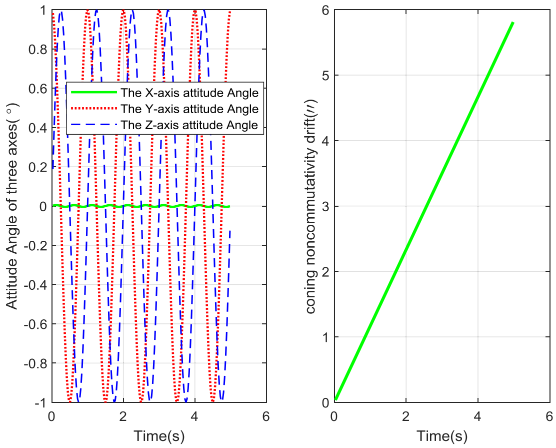

FIGURE 2. Triaxial attitude Angle and coning noncommutativity drift under cone motion condition.

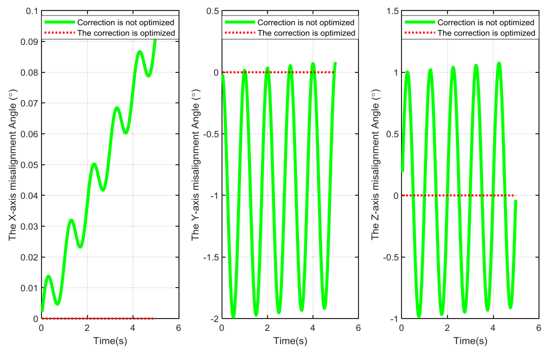

FIGURE 3. Modification optimization and non - optimization comparison.

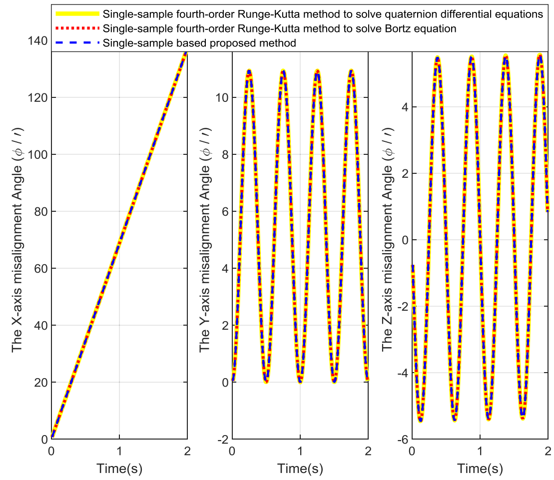

FIGURE 4. Comparison of three attitude updating algorithms of single sample.

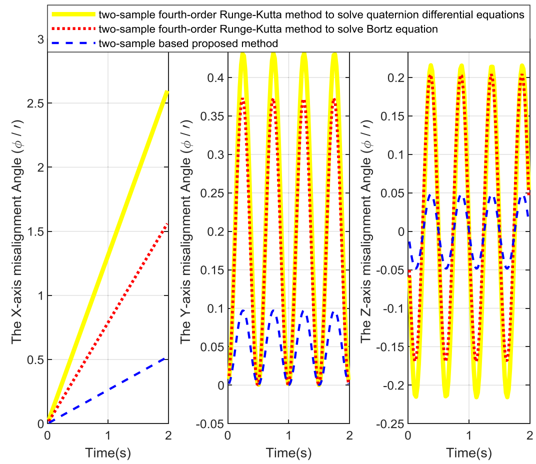

FIGURE 5. Comparison of three attitude updating algorithms of two sample.

misalignment angles obtained without optimization. It can be seen from Fig.3 that when the two coefficients of the correction amount are not optimized, the pitch angle error, yaw angle error, and roll angle error appear as divergent oscillation and unilateral offset over time. These two phenomena respectively represent the oscillation error and drift error of the attitude error, and the oscillation error frequency is the same as the cone movement frequency. After the two coefficients of the correction amount are optimized, the misalignment angles of the three axes are greatly reduced, which also shows the effectiveness of the optimization method proposed in this paper.

In order to illustrate the superiority of the algorithm proposed in this paper, the algorithm proposed in this paper was compared with the fourth-order Runge-Kutta method to solve quaternion differential equations and the fourth-order Runge-Kutta method to solve Bortz equation. Fig.4 shows the comparison of the misalignment angles in the three axes of the three attitude update algorithms in the case of a single sample. It can be seen from Fig.4 that the three axis misalignments calculated by the three attitude update algorithms in the case of a single sample. The misalignments angles are basically the same, that is, the results of the three algorithms are basically the same. Fig.5 shows the comparison of the three-axis misalignment angles of the three attitude update algorithms in the case of the tow sample. It can be seen from Fig.5 that in the case of the two sample, the proposed attitude calculation algorithm compares The fourth-order Runge-Kutta method to solve quaternion differential equations and fourth-order Runge-Kutta method to solve Bortz equation have a small misalignment angle, which shows that the attitude update algorithm proposed in this paper has more accurate results., Has a more superior solution performance.

For the three sub-samples in the cone motion environment, this paper derives the fifth-order formula, applies the error analysis method to assess the accuracy of the formula, optimizes the two coefficients of the formula correction, and calculates the drift rate error and drift rate, in order to simplify the calculation, This article simplifies the established fifth-order formula to fourth-order, but the drift rate error and drift rate remain unchanged. The proposed algorithm is simulated and verified under cone motion conditions. Fig.6 is a comparison of the three-axis misalignment angles of the three

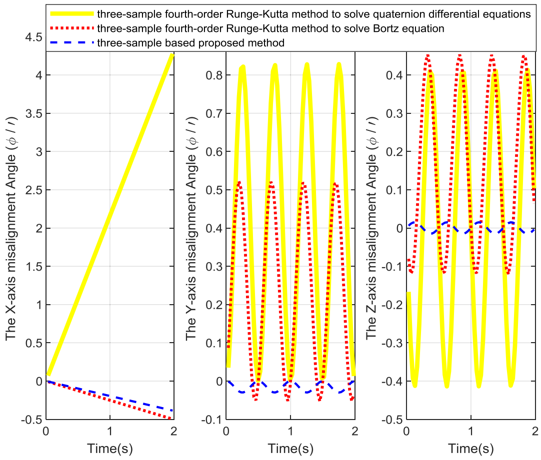

FIGURE 6. Comparison of three attitude updating algorithms of three sample.

attitude update algorithms in the three-sample case. It can be seen from Fig.6 that in the three-sample case, the proposed attitude update algorithm on the three axes Compared with the fourth-order Runge-Kutta method to solve quaternion differential equations and the fourth-order Runge-Kutta method to solve Bortz equation, the misalignment angle is smaller, which shows that the attitude update algorithm proposed in this paper is still under cone motion conditions. It has accurate solution results and superior performance.

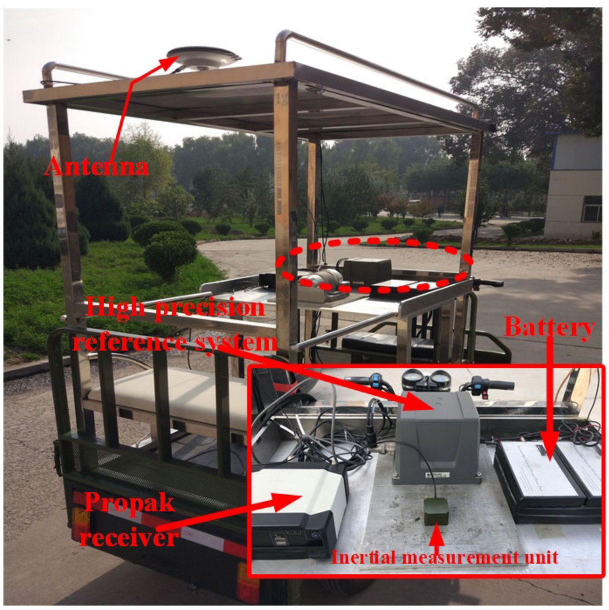

FIGURE 7. The car-mounted experimental platform used in this experiment.

B. EXPERIMENTAL VERIFICATION

In order to illustrate the practicability of the method proposed in this paper, the algorithm proposed in this paper is verified through ground sports car experiments. Fig.7 shows the vehicle-mounted experimental platform used in the experiment, which mainly includes a self-developed miniature inertial measurement unit containing a three-axis gyroscope and a three-axis accelerometer and a high-precision reference integrated navigation system. Among them, the sensor configuration description in the self-developed miniature inertial measurement unit is shown in Table 1. The high-precision reference integrated navigation system is composed of LCI-1 strategic-level IMU (inertial measurement unit), Propak satellite receiver and two GNSS antennas, and is used to provide high-precision attitude, speed and position reference for the miniature inertial measurement unit. The attitude, speed and position accuracy provided by the high-precision reference integrated navigation system are 0.01°, 0.05m/s and 0.1m respectively. The total test time is 50s, and the sports car test trajectory is shown in Fig.8.

TABLE 1. Sensor parameter configuration.

Parameter	Parameter Value
INS out frequency	$200Hz$
Gyro range	$\pm 300^{\circ}/s$
Gyro bias	$12^{\circ}/h$
Accelerometer range	$\pm 10g$
Accelerometer bias	$5mg$

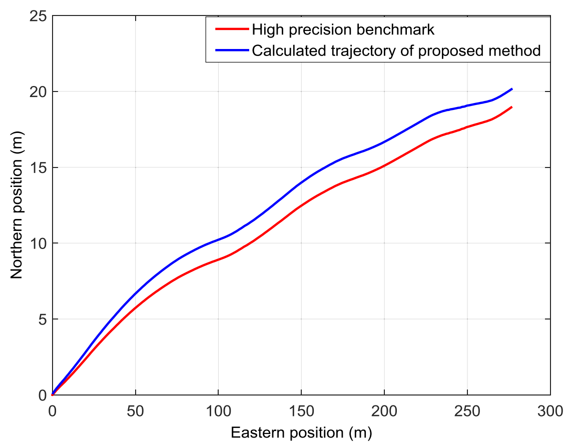

FIGURE 8. The test trajectory in the car-mounted experiment.

After the sports car experiment, the experimental data of the inertial measurement unit and the high-precision reference integrated navigation system can be obtained. Use the attitude update method proposed in this article to solve the experimental data, and the results of the fourth-order Runge-Kutta method to solve quaternion differential equations and fourth-order Runge-Kutta method to solve Bortz equation comparing. The solution results of the three attitude update methods are compared with the experimental results of the high-precision reference integrated navigation system, and the pitch angle error, roll angle error and yaw angle error

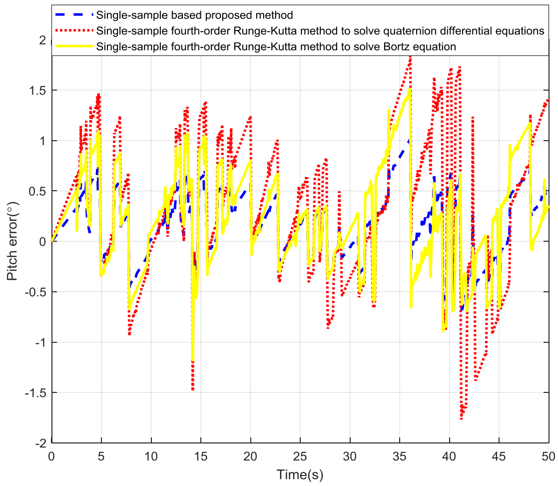

FIGURE 9. Comparison of pitch angle error.

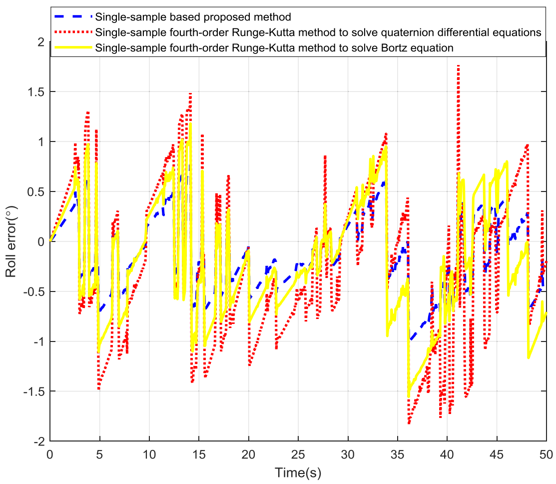

FIGURE 10. Comparison of roll angle error.

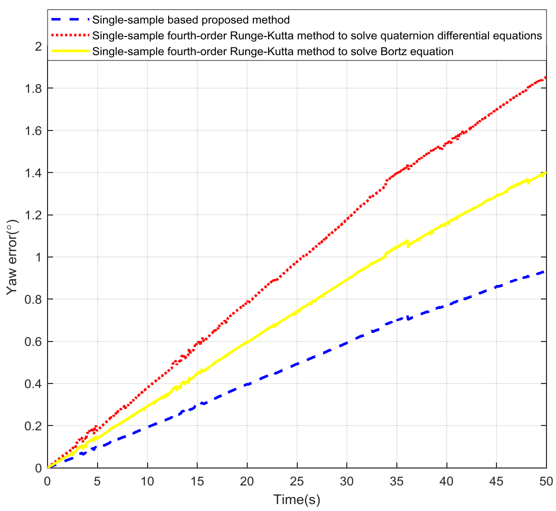

FIGURE 11. Comparison of yaw angle error.

of the three methods are obtained. Fig.9, Fig.10, Fig.11 are the pitch angle error, roll angle error, and yaw angle error of the three attitude update methods.

It can be seen from the figure that the fourth-order Runge-Kutta method to solve quaternion differential equations has the largest attitude angle error and the lowest attitude solution accuracy. The attitude angle error of the fourth-order Runge-Kutta method to solve Bortz equation is smaller than that of the fourth-order Runge-Kutta method to solve quaternion differential equations. The accuracy of the attitude calculation is relatively good, but the accuracy improvement is limited. The attitude update method proposed in this paper has the smallest attitude angle errors in the three axes. Therefore, the experimental results can show that the attitude quaternion update method proposed in this paper has higher attitude resolution accuracy, and also proves the effectiveness of the attitude update method proposed in this paper.

VI. CONCLUSION

The new quaternion update method proposed in this paper has conducted innovations in theory, and enriched the knowledge system of inertial navigation theory. This paper mainly focuses on the theoretical derivation, compares the results of the deduction and existing calculation methods, and verified its correctness. The method proposed and the rotation vector method both realize the attitude update by calculating the attitude quaternion. By using the third-order and fifth-order Taylor series to approximate the solution of the quaternion differential equation. The rotation vector method solves the equivalent rotation vector to solve attitude quaternion. The calculation of two methods are different. For the two cases of the current period output sample of the gyro, and the previous period output sample and the two samples, the derived formula is completely consistent with the common formula derived based on the equivalent rotation vector method. For the three samples in the coning motion environment, the fifth-order formula is derived in this paper, the error analysis method is applied to assess the accuracy of the formula, the two coefficients of the formula correction is optimized, and the drift rate error and drift rate are calculated, which are compared with the drift rate errors of the optimized three-sample equivalent rotation vector algorithm. The expressions for calculating the drift rate errors of the two are the same, and the calculation results are the same. In order to simplify the calculation, the established fifth-order formula is simplified to fourth-order one in this paper, but the drift rate error and drift rate remain unchanged. Finally, through simulation and ground sports car experiments, the three-dimensional attitude, pitch angle, roll angle and yaw angle errors are calculated in the case of single, two and three samples. And compared with two attitude update methods of fourth-order Runge-Kutta method to solve quaternion differential equations and fourth-order Runge-Kutta method to solve Bortz equation. The results show that the attitude quaternion update method proposed in this paper has higher posture resolution accuracy.

APPENDIX

Assuming that the carrier coordinate system *o-xyz* and the reference coordinate system *o-XYZ* coincide at t, the reference coordinate system rotates the angle a of the ray 0L relative to the carrier coordinate system, which can be described by the following rotation quaternion

$$\boldsymbol{Q}(t)=\cos\frac{a}{2}+\boldsymbol{u}_L^R\sin\frac{a}{2} \tag{A-1}$$

wherein, u_L^R is the unit vector along the direction of the ray, and its expression is

$$\boldsymbol{u}_L^R=\begin{bmatrix}0\\ \cos(\omega t)\\ \sin(\omega t)\end{bmatrix} \tag{A-2}$$

Then

$$\boldsymbol{Q}(t)=\begin{bmatrix}\cos(\frac{a}{2})\\ 0\\ \sin(\frac{a}{2})\cos(\omega t)\\ \sin(\frac{a}{2})\sin(\omega t)\end{bmatrix} \tag{A-3}$$

Assuming that $q(h)$ is the quaternion of attitude change that connects the quaternion $\boldsymbol{Q}(t)$ at t and the quaternion $\boldsymbol{Q}(t+h)$ at $t+h$, then

$$\boldsymbol{Q}(t+h)=\boldsymbol{Q}(t)\otimes\boldsymbol{q}(h) \tag{A-4}$$

Or

$$\boldsymbol{q}(h)=\boldsymbol{Q}^{-1}(h)\otimes\boldsymbol{Q}(t+h)=\boldsymbol{M}[\boldsymbol{Q}^*(t)]\boldsymbol{Q}(t+h) \tag{A-5}$$

Expand this expression to get the quaternion of the attitude change during the update period as

$$\boldsymbol{q}(h)=\begin{bmatrix}1-2\left(\sin^2(\frac{a}{2})\sin^2(\frac{\omega h}{2})\right)\\ -\sin^2(\frac{a}{2})\sin(\omega h)\\ -\sin(a)\sin(\frac{\omega h}{2})\sin\omega(t+\frac{h}{2})\\ \sin(a)\sin(\frac{\omega h}{2})\cos\omega(t+\frac{h}{2})\end{bmatrix} \tag{A-6}$$

According to the quaternion differential equation at t

$$\dot{\boldsymbol{Q}}(t)=\frac{1}{2}\boldsymbol{Q}(t)\otimes[\boldsymbol{\omega}(t)]^b \tag{A-7}$$

To obtain the angular velocity of the moving body as

$$[\boldsymbol{\omega}(t)]^b=2\boldsymbol{Q}^{-1}(t)\otimes\dot{\boldsymbol{Q}}(t)=\begin{bmatrix}-2\omega\sin^2(\frac{a}{2})\\ -\omega\sin(a)\sin(\omega t)\\ \omega\sin(a)\cos(\omega t)\end{bmatrix} \tag{A-8}$$

the angular increment of the three gyros output in the interval of time h

$$\begin{aligned}\Delta\boldsymbol{\theta}&=\int_t^{t+h}\boldsymbol{\omega}^b(\tau)d\tau=\begin{bmatrix}\Delta\theta_x\\ \Delta\theta_y\\ \Delta\theta_z\end{bmatrix}\\ &=\begin{bmatrix}-2\omega h\sin^2(\frac{a}{2})\\ -2\sin(a)\sin(\frac{\omega h}{2})\sin\omega(t+\frac{h}{2})\\ 2\sin(a)\sin(\frac{\omega h}{2})\cos\omega(t+\frac{\omega h}{2})\end{bmatrix}\end{aligned} \tag{A-9}$$

and the angular increments of the three gyros output in trisection $h/3$ time interval:

$$\begin{aligned}\Delta\boldsymbol{\theta}_i&=\int_{t+\frac{i-1}{3}h}^{t+\frac{i}{3}h}\boldsymbol{\omega}^b(\tau)d\tau=\begin{bmatrix}\Delta\theta_{ix}\\ \Delta\theta_{iy}\\ \Delta\theta_{iz}\end{bmatrix}\\ &=\begin{bmatrix}-\frac{2}{3}\omega h\sin^2(\frac{a}{2})\\ -2\sin(a)\sin(\frac{\omega h}{6})\sin\omega(t+\frac{2i-1}{6}h)\\ 2\sin(a)\sin(\frac{\omega h}{6})\cos\omega(t+\frac{2i-1}{6}h)\end{bmatrix}\\ &=\begin{bmatrix}-E\\ -F\sin\omega(t+\frac{2i-1}{6}h)\\ F\cos\omega(t+\frac{2i-1}{6}h)\end{bmatrix},\quad i=1,2,3\end{aligned} \tag{A-10}$$

wherein, $E=\frac{2}{3}\omega h\sin^2(\frac{a}{2})$, $F=2\sin(a)\sin(\frac{\omega h}{6})$ then

$$\begin{aligned}&\Delta\boldsymbol{\theta}_1\otimes\Delta\boldsymbol{\theta}_3\\ &=\begin{bmatrix}0&-\Delta\theta_{1z}&\Delta\theta_{1y}\\ \Delta\theta_{1z}&0&-\Delta\theta_{1x}\\ -\Delta\theta_{1y}&\Delta\theta_{1x}&0\end{bmatrix}\begin{bmatrix}\Delta\theta_{3x}\\ \Delta\theta_{3y}\\ \Delta\theta_{3z}\end{bmatrix}\\ &=\begin{bmatrix}-\Delta\theta_{1z}\Delta\theta_{3y}+\Delta\theta_{1y}\Delta\theta_{3z}\\ \Delta\theta_{1z}\Delta\theta_{3x}-\Delta\theta_{1x}\Delta\theta_{3z}\\ -\Delta\theta_{1y}\Delta\theta_{3x}+\Delta\theta_{1x}\Delta\theta_{3y}\end{bmatrix}\\ &=\begin{bmatrix}F^2\cos\omega(t+\frac{h}{6})\sin\omega(t+\frac{5h}{6})\\ -F^2\sin\omega(t+\frac{h}{6})\cos\omega(t+\frac{5h}{6})\\ -FE\cos\omega(t+\frac{h}{6})+EF\cos\omega(t+\frac{5h}{6})\\ -FE\sin\omega(t+\frac{h}{6})+EF\sin\omega(t+\frac{5h}{6})\end{bmatrix}\\ &=\begin{bmatrix}F^2[\frac{1}{2}\sin\omega(2t+h)-\frac{1}{2}\sin(\frac{-2\omega h}{3})\\ -\frac{1}{2}\sin\omega(2t+h)-\frac{1}{2}\sin(\frac{-2\omega h}{3})]\\ FE[-2\sin(\frac{\omega h}{3})\sin\omega(t+\frac{h}{2})]\\ FE[2\sin(\frac{\omega h}{3})\cos\omega(t+\frac{h}{2})]\end{bmatrix}\\ &=\begin{bmatrix}F^2\sin(\frac{2\omega h}{3})\\ -2FE\sin(\frac{\omega h}{3})\sin\omega(t+\frac{h}{2})\\ 2FE\sin(\frac{\omega h}{3})\cos\omega(t+\frac{h}{2})\end{bmatrix}\end{aligned} \tag{A-11}$$

Write

$$\begin{bmatrix}\Delta\theta'_x\\ \Delta\theta'_y\\ \Delta\theta'_z\end{bmatrix}=\begin{bmatrix}F^2\sin(\frac{2\omega h}{3})\\ -2FE\sin(\frac{\omega h}{3})\sin\omega(t+\frac{h}{2})\\ 2FE\sin(\frac{\omega h}{3})\cos\omega(t+\frac{h}{2})\end{bmatrix} \tag{A-12}$$

In a similar way

$$\begin{bmatrix} \Delta\theta_x'' \\ \Delta\theta_y'' \\ \Delta\theta_z'' \end{bmatrix} = \begin{bmatrix} 2F^2\sin(\frac{\omega h}{3}) \\ -2FE\sin(\frac{\omega h}{3})\sin\omega(t+\frac{h}{2}) \\ 2FE\sin(\frac{\omega h}{3})\cos\omega(t+\frac{h}{2}) \end{bmatrix} \quad \text{(A-13)}$$

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DEBIAO ZHANG was born in Jiaxiang, Shandong, China, in June 1989. He is currently pursuing the Ph.D. degree in instrument science and technology with the School of Instrument and Electronics, North University of China. He currently studies circuits and systems, signal acquisition, and dynamic testing techniques in complex and harsh environments.

JIE LI was born in Shanxi, China, in 1976. He received the Ph.D. degree in navigation, guidance, and control from the Department of Automatic Control, Beijing Institute of Technology, Beijing, China, in 2005. From April 2006 to June 2009, he was a Postdoctoral Research Associate with the North University of China, Taiyuan, China, where he works on navigation, guidance, and control. Since 2007, he has been with the School of Instrument and Electronics, North University of China, first as an Associate Professor, and then as a Professor, since 2013. His current research interests include strapdown inertial navigation, integrated navigation, and intelligent information processing.

YANG LIU was born in Siping, Jilin, China, in December 1995. He is currently pursuing the master's degree in instrument science and technology with the School of Instrument and Electronics, North University of China. He currently studies high-speed data transmission and storage and integrated navigation.

XIAOKAI WEI was born in Nei Monggol, China, in 1992. He received the B.E. degree in weapon system and launch engineering from the College of Mechatronics Engineering, North University of China, Taiyuan, China, in 2015, where he is currently pursuing the Ph.D. degree in instrument science and technology. His current research interests include inertial navigation, integrated navigation, and adaptive control.

KAIQIANG FENG was born in Shanxi, China, in 1989. He received the B.E. degree from the School of Instrument and Electronics, North University of China, Taiyuan, China, in 2013, where he is currently pursuing the Ph.D. degree in navigation, guided, and control. His current research interests include inertial navigation, inertial-based integrated navigation systems, and state estimation theory.