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Input-to-State Stability and Stabilization of Sampled-Data Systems Under Aperiodic Sampling and Random Sampling

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ABSTRACT In this article, input-to-state stability (ISS) and stabilization are examined for sampled-data systems under deterministic aperiodic sampling and random sampling, respectively. Using the direct design method, the sampled-data systems are transformed into switched systems with switched time-varying delays. First, the ISS definition and criterion appropriate for these systems are provided. Based on this, the ISS criterion on sampled-data systems under deterministic aperiodic sampling is given. Second, after the stochastic ISS (SISS) definition and criterion are provided for switched nonlinear systems with randomly switching delays, the SISS criterion for the sampled-data systems under random sampling is provided. All of the ISS and SISS definitions are given in the form of $\mathcal{KL}$ function that is quite elegant and easy to work with. Then, sufficient conditions for input-to-state stabilization are obtained for sampled-data linear systems under deterministic aperiodic sampling and random sampling, respectively, via the Lyapunov-Krasovskii method. Finally, based on the criteria, a piecewise controller is designed by the matrix inequality approach for a sampled-data linear time invariant system, and simulation results are provided to illustrate our design method. The main conclusion of this article is that sampling intervals will affect the controller design of the systems, and the ISS properties are maintained using a piecewise controller.

INDEX TERMS Sampled-data system, input-to-state stability, aperiodic sampling, random sampling.

I. INTRODUCTION

Since the performance of actual control system is affected by unmodeled dynamics, parameter perturbation, exogenous disturbance, measurement error and other uncertainties, research on the robustness of control system has been playing an important role in the development of control theory and technology. For nonlinear control system robustness analysis, a new method from the perspective of input-to-state stability (ISS), input-to-output stability (IOS) and integral input-to-state stability (iISS) has been developed and a series of basic theoretical results focusing on ISS-, IOS-lyapunov functions have been obtained by many researchers such as Angeli, Liberzon, Lin, Praly, and Sontag ([2], [26]–[28], [34]–[40]). The ISS concept has also been applied in many fields such as robotic systems [1], signal processing [3], tracking control [4], and swarm formation [32]. Additionally, the ISS theory has been applied to the study of continuous-time systems [35], [36], discrete-time systems [20], deterministic systems, stochastic systems [43], [49], switched systems [29], [44], [49], and impulsive systems [15]. For sampled-data systems, the ISS and IOS have also been studied in [8], [18] and [30], respectively.

Starting in late 1950s, computer control system have appeared and developed rapidly and can be viewed as a kind of hybrid system. In control engineering, such a hybrid system is traditionally called the sampled-data system. For the sampled-data system, the control action can only be updated in the sampling instants. In [23] and [47], three common methods of sampled-data control system research have been summarized by Lamnabhi-Lagarrigue *et al.*: (i) digital redesign: a continuous time controller is designed and then

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implemented in discrete time; (ii) discrete design: a discrete time controller is designed by using the discrete time model of the controlled object; and (iii) direct design: the controller is designed by using the precise sampled-data model of the controlled object to satisfy the system's stability and required performance indexes. Unlike the other two methods, although the direct design method requires the most advanced design techniques and is relatively difficult to apply, it does not involve an approximate process and can better maintain the stability and performance indexes. In recent years, for the sampled-data control of switched system or time-delay systems, Li *et al.* have reported many results (See [24], [25] and references therein). For the ISS study of the sampled-data systems, only in [30], Nesic and Laila designed the input-to-state stabilizing controller using the discrete design method. Even in [42], although the input-to-state stability analysis was performed by direct method, the controller design was also based on the discrete design method. In addition, it must be pointed out that all of the above studies on ISS focused on the sampled-data control systems with a constant sampling period.

However, due to sampling jitter, data loss or fluctuation when the control algorithm and real-time scheduling protocol interact with each other in a network control system, studies of the modeling of aperiodic sampled-data system began to appear and research on these systems has becoming increasingly popular. While the traditional sampling control theory has been well-established, it cannot be used to deal with the aperiodic sampled-data control system with time-varying sampling interval. As pointed out in [50], for example, the control of an aperiodic sampled-data system cannot simply use the upper or lower bounds of the sampling period to achieve robust control. Using several examples, Zhang and Yu [48] and Gu *et al.* [14] respectively pointed out that the evolution of sampling interval length will affect the stability of the system and that the sampling interval length can be used as a control parameter to stabilize the system. For the stability and stabilization of sampled-data control system with aperiodic sampling, Tang *et al.* [41] studied the random stabilization using the input-delay method but only obtained a common feedback controller; Hu *et al.* [17] studied the robust sampled-data control problem for Markovian jump linear systems, and found that the feedback gains of the controllers depend on the Markovian jump parameters but not on the different sampling periods; Yang *et al.* [46] designed multiple stabilization controllers that depend on the different sampling periods for networked control systems, but the delta operator approach was used that is a discrete design method. Many recent developments in the studies of the stability of systems with aperiodic sampling are reviewed in [16]. For the ISS and stabilization of aperiodic sampled-data control systems, further study is necessary to accurately represent the influence of the length of the time-varying sampling interval.

In this article, the ISS and stabilization of sampled-data systems under deterministic aperiodic sampling and random sampling will be considered, respectively. First, the definition of ISS and the criterion of switched nonlinear systems with switched time-varying delays will be given. Based on these definitions, the ISS criterion for the sampled-data systems under deterministic aperiodic sampling will be provided. Second, after providing the SISS definition and criterion for switched nonlinear systems with randomly switching time-delays, the SISS criterion for sampled-data systems under random sampling will be provided. All of the ISS and SISS definitions will be given in the form of the $\mathcal{KL}$ function that is quite elegant and easier to work with ([49]). Then, sufficient conditions for input-to-state stabilization will be obtained for sampled-data linear systems under deterministic aperiodic sampling and random sampling via the Lyapunov-Krasovskii method and the matrix inequality approach. Finally, a piecewise controller for input-to-state stabilization will be designed due to the different sampling periods.

The remainder of this article is organized as follows: In Section II, after the aperiodic sampled-data systems are transformed into switched nonlinear systems with switched time-varying delays, the corresponding ISS definition and criterion will be provided. Based on these definitions, the ISS criterion on sampled-data systems under deterministic aperiodic sampling will be provided. In Section III, after the SISS definition and criterion are provided for switched nonlinear systems with randomly switching time-delays, the SISS criterion for the sampled-data systems under random sampling will be provided. Section IV presents sufficient conditions for input-to-state stabilization of the sampled-data linear systems under deterministic aperiodic sampling and random sampling via the Lyapunov-Krasovskii method and the matrix inequality approach. In Section V, for a sampled-data linear system, a piecewise controller is designed due to the different sampling periods using the matrix inequality approach, and simulation results are provided to illustrate our design method. Section V includes some concluding remarks.

Note: Throughout the paper, $\mathbb{R}_+$ and $\mathbb{N}_0$ denote the set of all nonnegative real numbers and natural numbers including zero, respectively; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the Euclidean norm $|x| = (\sum_{i=1}^{n} x_i^2)^{1/2}$; $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P > 0 (< 0)$, for $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive (negative) definite. The superscript T stands for matrix transposition. In symmetric block matrices, we use $*$ as an ellipsis for terms that are induced by symmetry. $\mathcal{C}([-r, 0]; \mathbb{R}^n)$ denotes the set of the continuous functions that are defined on $[-r, 0]$ and take values in $\mathbb{R}^n$. By $L_2([-r, 0]; \mathbb{R}^n)$ denotes the space of square integrable functions $\phi : [-r, 0] \to \mathbb{R}^n$. A function u is said to be essentially bounded if $ess \sup_{t\geq t_0} |u(t)| < \infty$, and the essential supremum norm is denoted by the symbol $\|\cdot\|_\infty$. $E(x)$ denotes the expectation of the stochastic variable x. Finally, we denote the composition of two functions $\varphi : A \to B$ and $\psi : B \to C$ by $\psi \circ \varphi : A \to C$.

II. SAMPLED-DATA SYSTEMS WITH APERIODIC SAMPLING

A. DEFORMATION OF SAMPLED-DATA SYSTEMS WITH APERIODIC SAMPLING

Consider the nonlinear system

$$\dot{x} = f(t, x(t), u(t), w(t)), \tag{1}$$

where $f(\cdot)$ is a nonlinear function, $x(t) \in \mathbb{R}^n$ is the system state, $w(t) \in \mathbb{R}^p$ is an exogenous signal, and $u(t) = u_c(x)$ is ISS controller which has been designed. For state-feedback sampled-data stabilization, only discrete measurement of $x(t)$ can be used for the control, that is we only have the measurements $x(t_k)$ at the sampling instants t_k with $0 = t_0 < t_1 < \cdots < t_k < \cdots < \lim_{k\to\infty} t_k = \infty$.

$$u(t) = u(x(t_k)), \quad \forall t \in [t_k, t_{k+1}), \tag{2}$$

$$\dot{x} = f(t, x(t), u(x(t_k)), w(t)), \quad \forall t \in [t_k, t_{k+1}). \tag{3}$$

For $t \in [t_k, t_{k+1})$, define input delay $\eta(t) = t - t_k$, and the sampling interval length $\eta_k = t_{k+1} - t_k$. It is observed that $\dot{\eta}(t) = 1$. It is assumed that $\eta_k \in \{\bar{\eta}_1, \bar{\eta}_2, \cdots, \bar{\eta}_N\}$. Just as pointed out in [50], the control of the aperiodic sampled-data system will be affected by the different sampling periods. Therefore, different controllers or controller gains are more appropriate corresponding to the different sampling periods. In this case, the closed-loop system of (3) can be reformulated as the following switched system with time-varying input delays

$$\dot{x} = f(t, x(t), u_{\sigma(t)}(x(t - \eta_{\sigma(t)}(t))), w(t)), \quad \forall t \in [t_0, \infty), \tag{4}$$

where the switching law $\sigma(t) = i \in \mathcal{N} = \{1, 2, \cdots, N\}$, $u_i(\cdot)$ and $\eta_i(t) = t - t_{i,k} = t - t_k$ respectively denote the different controllers and the input delay $\eta(t)$ with the corresponding sampling interval length $\eta_k = t_{i,k+1} - t_k = t_{k+1} - t_k = \bar{\eta}_i$. Then, the input delay $\eta(t)$ is described as a piecewise function. For every sampling interval, the closed-loop system is related to an input-delay submodel

$$\dot{x} = f(t, x(t), u_i(x(t - \eta_i(t))), w(t)), \quad \forall t \in [t_k, t_{k+1}), \tag{5}$$

where $k = 1, 2, \cdots$, $\eta_k = \bar{\eta}_i$, and the whole closed-loop system is equivalent to a switched system with infinite submodels. When the submodel (5) is uniformly stable, the stability of the sampled-data system can be determined by the following switched nonlinear system with a finite mode set $\mathcal{N}$:

$$\dot{x} = f(t, x(t), u_i(x(t - \eta_i(t))), w(t)), \forall t \in [t_k, t_{k+1}), \eta_k = \bar{\eta}_i. \tag{6}$$

Since the sampling period takes only finitely many values, Zeno behavior does not occur for system (6).

B. ISS OF NONLINEAR SYSTEMS WITH TIME-VARYING DELAY

To study the ISS of aperiodic sampled-data control systems, we need the support of some basic theories of ISS. Here, we consider the following time-varying delay system

$$\dot{x} = f(t, x(t), x(t - \tau(t)), w(t)), \tag{7}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $w(t) \in \mathbb{R}^p$ is an exogenous signal, $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a continuously differentiable function, $f(t, 0, 0, 0) = 0$ for any $t \in \mathbb{R}$, and $\tau(t)$ is a piecewise-continuous time delay that satisfies $0 \le \tau(t) \le r$. Given a measurable locally essentially bounded input w, system (7) with initial condition $x_{t_0} = \phi \in \mathcal{C}([-r, 0]; \mathbb{R}^n)$ has a unique solution ([22], chapter 3, Section 2.4).

The space of functions $\phi : [-r, 0] \to \mathbb{R}^n$ that are absolutely continuous on $[-r, 0)$, have a finite $\lim_{\theta\to 0^-} \phi(\theta)$ and have square integrable first-order derivatives is denoted by W with the norm

$$\|\phi\|_W = \left(\phi^T(0)\phi(0) + \int_{-r}^{0} \dot{\phi}^T(\tau)\dot{\phi}(\tau)d\tau\right)^{1/2}.$$

We also denote $x_t(\theta) = x(t + \theta)$ $(\theta \in [-r, 0])$.

For convenience, we shall introduce the following definitions and a lemma:

Definition 1 [21]: A function $\varphi(v)$ is said to belong to the class $\mathcal{K}$ if $\varphi \in \mathcal{C}(\mathbb{R}_+)$, $\varphi(0) = 0$ and $\varphi(v)$ is strictly increasing in v. $\mathcal{K}_\infty$ is the subset of $\mathcal{K}$ functions that are unbounded. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$, if $\beta(\cdot, t)$ is of class $\mathcal{K}$ in the first argument for each fixed $t \ge 0$ and $\beta(s, t)$ decreases to 0 as $t \to +\infty$ for each fixed $s \ge 0$.

Definition 2 [12]: System (7) is said to be uniformly globally input-to-state stable (ISS) if there exist a $\mathcal{KL}$ function β and a $\mathcal{K}$ function γ such that for any initial time t_0 and any initial state $x_{t_0} = \phi \in \mathcal{C}([-r, 0]; \mathbb{R}^n)$ and any measurable, essentially bounded input w, the solution $x(t, t_0, \phi)$ exists for all $t \ge t_0$ and furthermore it satisfies

$$|x(t, t_0, \phi)| \le \beta(\|\phi\|_W, t - t_0) + \gamma(\|w\|_\infty).$$

Given a continuous functional $V : \mathbb{R} \times W \times L_2([-r, 0], \mathbb{R}^n) \to \mathbb{R}^+$, define (see e.g., [22])

$$\dot{V}(t, \phi, \dot{\phi}) = \limsup_{h\to 0^+} \frac{1}{h}[V(t + h, x_{t+h}(t, \phi), \dot{x}_{t+h}(t, \phi)) - V(t, \phi, \dot{\phi})],$$

where $x_t(t_0, \phi)$, for $t \ge t_0$, is the solution of system (7) with the initial condition $x_{t_0} = \phi \in W$.

Lemma 1: ([12]) Let there exists a locally Lipschitz with respect to the second and the third argument functional $V : \mathbb{R} \times W \times L_2([-r, 0], \mathbb{R}^n) \to \mathbb{R}^+$ such that the function $v(t) = V(t, x_t, \dot{x}_t)$ is absolutely continuous for essentially bounded measurable w. If there also exist functions

$\underline{\alpha}, \overline{\alpha}$ of class $\mathcal{K}_\infty$, and functions α, χ of class $\mathcal{K}$ such that

$$\begin{aligned}&\text{(i) } \underline{\alpha}(|x_t(0)|) \le V(t, x_t, \dot{x}_t) \le \overline{\alpha}(\|x_t\|_W),\\&\text{(ii) } \dot{V}(t, x_t, \dot{x}_t) \le -\alpha(\|x_t\|_W)\\&\quad \text{for } V(t, x_t, \dot{x}_t) \ge \chi(\|w\|_\infty),\end{aligned}$$

then system (7) is a uniformly globally ISS with $\gamma = \underline{\alpha}^{-1} \circ \chi$.

C. ISS OF SAMPLED-DATA SYSTEMS UNDER DETERMINISTIC APERIODIC SAMPLING

The Lemma 1 is an ISS criterion for a nonlinear time-delay system (7). However, considering the switching in the system (6), Lemma 1 is not sufficient for the study of ISS of nonlinear sampled-data systems with aperiodic sampling intervals. For convenience, we introduce the following general nonlinear systems with time varying time-delays and switching parameters as follows

$$\dot{x} = f_{\sigma(t)}(t, x(t), x(t - \eta_{\sigma(t)}(t)), w(t)), \tag{8}$$

where $\sigma(t)$ is the switching signals with the values in $\mathcal{N} = \{1, 2, \cdots, N\}$. The switching between subsystems is synchronous with the switching between the time delays. This is a more general form of closed-loop system (4). The maximum of the time delays induced by sampling, $\max_{i\in\mathcal{N}}\{\eta_i(t)\}$, is also denoted as r similar to the time delay in Subsection II-B. Using the multiple Lyapunov functions method, the following ISS criterion can be obtained for the further studies.

Theorem 1: Let there exist a set of locally Lipschitz with respect to the second and the third arguments functionals $V_i : \mathbb{R} \times W \times L_2([-r, 0], \mathbb{R}^n) \to \mathbb{R}^+$ $(i \in \mathcal{N})$ such that the functions $v_i(t) = V_i(t, x_t, \dot{x}_t)$ are absolutely continuous for essentially bounded measurable w. If additionally there exist functions $\underline{\alpha}_i, \overline{\alpha}_i$ of class $\mathcal{K}_\infty$, and functions α_i, χ_i of class $\mathcal{K}$ such that

$$\begin{aligned}&\text{(i) } \underline{\alpha}_i(|x_t(0)|) \le V_i(t, x_t, \dot{x}_t) \le \overline{\alpha}_i(\|x_t\|_W),\\&\text{(ii) } \dot{V}_i(t, x_t, \dot{x}_t) \le -\alpha_i(V_i(t, x_t, \dot{x}_t))\\&\quad \text{for } V_i(t, x_t, \dot{x}_t) \ge \chi_i(\|w\|_\infty),\\&\text{(iii) at each switching instant } t_l, (l = 0, 1, 2 \cdots),\\&\max\{v_{i_{l-1}}(t_l), \overline{\alpha} \circ \chi(\|w\|_\infty)\} \ge v_{i_l}(t_l),\\&\text{where } i_l \in \mathcal{N} \text{ means that the } i_l\text{-th subsytem be}\\&\text{activiated in } [t_l, t_{l+1}),\end{aligned}$$

then system (8) is uniformly globally ISS with $\gamma = \underline{\alpha}^{-1} \circ \chi$, where $\underline{\alpha} = \inf_{i\in\mathcal{N}} \underline{\alpha}_i$, $\overline{\alpha} = \sup_{i\in\mathcal{N}} \overline{\alpha}_i$, $\chi = \sup_{i\in\mathcal{N}} \chi_i$.

Proof: Denote $\Lambda_\chi = \{x_t \in W : \|x_t\|_W < \chi_{\sigma(t)}(\|w\|_\infty)\}$, where the space W is defined before Definition 1. Let τ be the first time when the solution $x_t = x_t(t_0, \phi)$ enters Λ_χ. For all $t \in [t_0, \tau]$,

$$\dot{v}_{\sigma(t)}(t) \le -\alpha_{\sigma(t)}(v_{\sigma(t)}(t)).$$

Now let us construct a $\mathcal{KL}$ function to be a bound of $v_{\sigma(t)}$. Let us suppose the above inequality holds for $t \in [t_0, \infty)$. At any time interval $[t_l, t_{l+1})(l \in \mathbb{N}_0)$, the i_l-th subsystem is supposed to be activated, and the comparison principle in [21] can be used. Then, there exists a class of $\mathcal{KL}$ functions β_i $(i \in \mathcal{N})$ such that

$$\begin{aligned}&v_{i_0}(t) \le \beta_{i_0}(v_{i_0}(t_0), t - t_0), \quad t \in [t_0, t_1);\\&v_{i_1}(t) \le \beta_{i_1}(v_{i_1}(t_1), t - t_1), \quad t \in [t_1, t_2);\\&\vdots\\&v_{i_l}(t) \le \beta_{i_l}(v_{i_l}(t_l), t - t_l), \quad t \in [t_l, t_{l+1});\\&\vdots\end{aligned}$$

By condition (iii), the above inequalities and the properties of $\mathcal{KL}$ function ([21]),

$$\tilde{\beta}(v_0, t - t_0) \triangleq \begin{cases}\beta(v_0, t - t_0), & t \in [t_0, t_1);\\ \beta(\beta(v_0, t_1 - t_0), t - t_1), & t \in [t_1, t_2);\\ \vdots & \\ \underbrace{\beta(\cdots\beta}_{l+1}(v_0, t_1 - t_0), t - t_1), & t \in [t_l, t_{l+1});\\ \vdots & \end{cases} \tag{9}$$

is well defined such that

$$v_{\sigma(t)}(t) \le \tilde{\beta}(v_0, t - t_0), \ \forall t \in [t_0, \infty),$$

where $\beta = \max\{\beta_i, i \in \mathcal{N}\}$, $v_0 = v_{i_0}(x_0, t_0)$. Due to the β satisfying $\beta(\Delta, 0) = \Delta$ and $\beta \in \mathcal{KL}$, $\tilde{\beta}$ is continuous and $\tilde{\beta} \in \mathcal{KL}$. Now, let us constrain the above inequality on $[t_0, \tau]$. Therefore,

$$v_{\sigma(t)}(t) \le \tilde{\beta}(v_0, t - t_0), \quad \forall t \in [t_0, \tau].$$

Define $\beta(r, s) = \underline{\alpha}^{-1} \circ \tilde{\beta}(\overline{\alpha}(r), s)$, by condition (ii),

$$|x(t, t_0, \phi)| \le \beta(\|\phi\|_W, t - t_0), \quad \forall t \in [t_0, \tau], \tag{10}$$

where $\beta(r, s) \in \mathcal{KL}$ can be known from Lemma 3.2 in [21].

On the other hand, when $t \in (\tau, \infty)$, since $\dot{v}_{\sigma(t)}(t)$ is negative for x_t outside the set Λ_χ, we have $v_{\sigma(t)}(t) < \chi_{\sigma(t)}(\|w\|_\infty)$, and

$$|x(t, t_0, \phi)| < \underline{\alpha}^{-1} \circ \chi_{\sigma(t)}(\|w\|_\infty). \tag{11}$$

Combining (10) and (11), we have, for any $t \in [t_0, \infty)$,

$$|x(t, t_0, \phi)| \le \beta(\|\phi\|_W, t - t_0) + \gamma(\|w\|_\infty). \tag{12}$$

Therefore, system (8) is ISS with $\gamma = \underline{\alpha}^{-1} \circ \chi$. ■

Remark 1: Unlike Lemma 1 in [12], Theorem 1 can be applied to the judgment of the ISS property of a time-varying delay systems with switching parameters. It can be applied to the ISS analysis of our sampled-data control systems with switching controllers or controller gains.

III. SAMPLED-DATA SYSTEMS UNDER RANDOM SAMPLING

For system (1), the sampling instants t_k with $0 = t_0 < t_1 < \cdots < t_k < \cdots < \lim_{k\to\infty} t_k = \infty$ are random. To stress the presence of the time delay due to sampling, we rewrite (4) as

$$\dot{x}(t) = f(t, x(t), u_{\sigma(t)}(x(t - \eta_{\sigma(t)}(t))), w(t)), \quad (13)$$

where $\eta_{\sigma(t)}(t) = t - t_k$, for $t \in [t_k, t_{k+1})$, and the sampling interval length $\eta_k = t_{k+1} - t_k$. Due to the randomness of t_k, the input delay $\eta_{\sigma(t)}(t)$ and the sampling intervals are random. Only the assumption that $\eta_k \in \{\bar{\eta}_1, \bar{\eta}_2, \cdots, \bar{\eta}_N\}$ holds and is deterministic.

A. ISS OF NONLINEAR SYSTEMS WITH RANDOM DELAY

As a basis of ISS of sampled-data systems under random sampling, the ISS of nonlinear systems with random delay will be considered in this subsection. For convenience, some definitions and preliminary results will be given first.

Consider the following nonlinear system with random delay

$$\dot{x} = f(t, x(t), x(t - \tau(t)), w(t)), \quad (14)$$

where $x(t) \in \mathbb{R}^n$ is the system state, $w(t) \in \mathbb{R}^p$ is an exogenous signal, $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a continuously differentiable function, $f(t, 0, 0, 0) = 0$ for any $t \in \mathbb{R}$, and $\tau(t)$ is random time delay that satisfies $0 \le \tau(t) \le r$. Given a locally essentially bounded measurable input w, system (14) with initial condition $x_{t_0} = \phi \in \mathcal{C}([-r, 0]; \mathbb{R}^n)$ has a unique solution ([22]). From Definition 4 in [19], $\{(t, x_t, \tau(t)) : t \ge t_0\}$ is a right-continuous strong Markov process. The weak infinitesimal generator of this process can be defined using the following average right-derivative:

$$\mathcal{L}V(t, x_t, \tau(t)) = \lim_{h\to 0^+} \frac{1}{h} E_{tx\tau}\{V(t+h, x_{t+h}, \tau(t+h)) - V(t, x_t, \tau(t))\}$$

for all continuous functionals V mapping, where the subscript denotes conditional expectation conditioned on the "present" state, $(t, x_t, \tau(t))$, and for which the limit function is itself right-continuous in the mean with respect to $E_{tx\tau}$.

Remark 2: Generally, when considering a stochastic differential equation, $\mathcal{W}_t$ is used to denote the filtration generated by Brownian motion processes or "white noise" processes. However, here, the random delay itself is a process that is measurable with respect to some filtration denoted by $\mathcal{T}_t$. $\mathcal{T}_t$ may have a different form from $\mathcal{W}_t$. In this article, the expectation (condition expectation) and infinitesimal generator is taken based on the filtration $\mathcal{T}_t$.

Moreover, we indicate with the symbol $\|\cdot\|_{M_2}$ the well-known norm (see [5], [9], [13] and [33]) induced by the inner product in the Hilbert space $M_2 = \mathbb{R}^n \times L_2([-r, 0]; \mathbb{R}^n)$, given by, for $Y = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$, with $y_0 \in \mathbb{R}^n$, $y_1 \in L_2([-r, 0]; \mathbb{R}^n)$,

$$\|Y\|_{M_2} = \left(y_0^T y_0 + \int_{-r}^0 y_1^T(\tau) y_1(\tau) d\tau\right)^{1/2}.$$

Definition 3: System (14) is said to be

(i) input-to-state stable in mean (ISSiM), if there exist functions $\beta \in \mathcal{KL}$ and $\alpha, \gamma \in \mathcal{K}_\infty$ such that

$$\alpha(E|x(t)|) \le \beta(E\|\phi\|_{M_2}, t - t_0) + \gamma(\|w\|_\infty);$$

(ii) stochastic input-to-state stable (SISS), if for any given $\varepsilon > 0$, there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that

$$P\{|x(t)| \le \beta(E\|\phi\|_{M_2}, t - t_0) + \gamma(\|w\|_\infty)\} \ge 1 - \varepsilon. \quad (15)$$

For system (14) with $w(t) \equiv 0$, the SISS means globally asymptotically stability in probability (GASiP). Further, if inequality (15) holds independently of t_0, SISS means uniformly GASiP. The following lemma is a criterion for GASiP of system (14) with $w(t) \equiv 0$.

Lemma 2: Let there exist a locally Lipschitz with respect to the second arguments functional $V : \mathbb{R} \times \mathcal{C}([-r, 0], \mathbb{R}^n) \times \mathbb{R} \to \mathbb{R}^+$ such that the function $v(t) = V(t, x_t, \tau(t))$ is absolutely continuous for system (14) with $w(t) \equiv 0$. If there also exist functions $\underline{\alpha}(\cdot), \overline{\alpha}(\cdot)$ of class $\mathcal{K}_\infty$, and function α of class $\mathcal{K}$ such that $\alpha \circ \overline{\alpha}^{-1}$ is convex and for all $x \in \mathbb{R}^n$ and $t \ge t_0$

$$\underline{\alpha}(|x_t(0)|) \le V(t, x_t, \tau(t)) \le \overline{\alpha}(\|x_t\|_{M_2}), \quad (16)$$

$$\mathcal{L}V(t, x_t, \tau(t)) \le -\alpha(\|x_t\|_{M_2}), \quad (17)$$

then system (14) with $w(t) \equiv 0$ is uniformly GASiP.

Proof: Define the time $\tau_l = \inf\{t \ge t_0 : \|x_t\| \ge l\}$. It is a Markov time of the $tx\tau$-process. Thus, the minimum of t and τ_r, denoted $t_l = \min\{t, \tau_l\}$ is also a Markov time for this process.

Employing Dynkin's Formula ([31]), we obtain

$$EV(t_l, x_{t_l}, \tau(t_l)) = EV_0 + E\left[\int_{t_0}^{t_l} \mathcal{L}V(s, x_s, \tau(s)) ds\right].$$

Upon letting $l \to \infty$, $t \to \infty$, using Fatou's lemma ([31]) on the left and monotone convergence on the right, and by Fubini's theorem ([31]), we obtain

$$EV(t, x_t, \tau(t)) = EV_0 + E\left[\int_{t_0}^{t} \mathcal{L}V(s, x_s, \tau(s)) ds\right] = EV_0 + \int_{t_0}^{t} E\left[\mathcal{L}V(s, x_s, \tau(s))\right] ds.$$

Therefore, $EV(t, x_t, \tau(t))$ is (locally) absolutely continuous, for any $t \ge t_0$. It is easy to see that $EV(t, x_t, \tau(t)) \ge 0$. Moreover, by (16) and (17), we have

$$\mathcal{L}V(t, x_t, \tau(t)) \le -\alpha \circ \overline{\alpha}^{-1}(V(t, x_t, \tau(t))).$$

Then, similar to the proof of Theorem 1 in [49], for any $\varepsilon \in (0, 1)$, there exists a $\mathcal{KL}$ function β such that

$$P\{|x(t)| < \beta(E\|\phi\|_{M_2}, t - t_0)\} \ge 1 - \varepsilon, \ \forall t \in [t_0, \infty).$$

Therefore, system (14) with $w(t) \equiv 0$ is uniformly GASiP. ■

The key to Lemma 2 is the fact that the $tx\tau$-process is a right-continuous strong Markov process, enabling the use of Dynkin's formula. Based on Lemma 2, we can further give the criterion on SISS of system (14) as the following lemma.

Lemma 3: Let there exist a locally Lipschitz with respect to the second arguments functional $V : \mathbb{R} \times \mathcal{C}([-r, 0], \mathbb{R}^n) \times \mathbb{R} \rightarrow \mathbb{R}^+$ such that the function $v(t) = V(t, x_t, \tau(t))$ is absolutely continuous for essentially bounded measurable w. If additionally there exist functions $\underline{\alpha}(\cdot), \overline{\alpha}(\cdot)$ of class $\mathcal{K}_\infty$, and functions α, χ of class $\mathcal{K}$ such that $\alpha \circ \overline{\alpha}^{-1}$ is convex and for all $x \in \mathbb{R}^n$ and $t \geq t_0$

$$\underline{\alpha}(|x_t(0)|) \leq V(t, x_t, \tau(t)) \leq \overline{\alpha}(\|x_t\|_{M_2}),$$
$$\|x_t\|_{M_2} \geq \chi(\|w\|_\infty) \Rightarrow \mathcal{L}V(t, x_t, \tau(t)) \leq -\alpha(\|x_t\|_{M_2}),$$

then system (14) is uniformly globally SISS with $\gamma = \underline{\alpha}^{-1} \circ \chi$.

Proof: Based on Lemma 2, from Theorem 1 in [49] and Lemma 1 in [12], this proof is readily available. It is omitted here. ■

B. ISS OF SAMPLED-DATA SYSTEMS UNDER RANDOM SAMPLING

Based on the results of Subsection III-A, we can give the SISS criterion on the closed-loop system (13).

Theorem 2: Let there exist a family of locally Lipschitz with respect to the second argument functionals $V_i : \mathbb{R} \times \mathcal{C}([-r, 0], \mathbb{R}^n) \times \mathbb{R} \rightarrow \mathbb{R}^+ (i \in \mathcal{N})$ such that the function $v_i(t) = V_i(t, x_t, \eta_i(t))$ is absolutely continuous for measurable essentially bounded w. If additionally there exist functions $\underline{\alpha}_i, \overline{\alpha}_i$ of class $\mathcal{K}_\infty$, and functions α_i, χ_i of class $\mathcal{K}$ such that $\alpha_i \circ \overline{\alpha}_i^{-1}$ is convex and for all $x \in \mathbb{R}^n$ and $t \geq t_0$

(i) $\underline{\alpha}_i(|x_t(0)|) \leq V_i(t, x_t, \eta_i(t)) \leq \overline{\alpha}_i(\|x_t\|_{M_2})$,

(ii) $\mathcal{L}V_i(t, x_t, \eta_i(t)) \leq -\alpha_i(V_i(t, x_t, \eta_i(t)))$ for $V_i(t, x_t, \eta_i(t)) \geq \chi_i(\|w\|_\infty)$,

(iii) *at each switching instant* $t_l (l = 0, 1, 2 \cdots)$,
$\max\{E[V_{i_{l-1}}(t_l)], \overline{\alpha} \circ \chi(\|w\|_\infty)\} \geq E[V_{i_l}(t_l)]$,
where $i_l \in \mathcal{N}$ means that the i_l-th subsytem be activiated in $[t_l, t_{l+1})$,

then system (13) is uniformly globally SISS with $\gamma = \underline{\alpha}^{-1} \circ \chi$, where $\underline{\alpha} = \inf_{i \in \mathcal{N}} \underline{\alpha}_i$, $\overline{\alpha} = \sup_{i \in \mathcal{N}} \overline{\alpha}_i$ $\chi = \sup_{i \in \mathcal{N}} \chi_i$.

Proof: From Theorem 4 in [49], Lemma 3 and the proof of Theorem 1, this proof is readily available. It is omitted here. ■

Remark 3: Theorem 2 is a generalization of Lemma 1 in [12]. It can be applied to the systems with random delay and switching parameters, based on which the ISS of sampled-data systems with piecewise controller and random sampling can be studied.

Remark 4: In section III, the norm $\|\cdot\|_{M_2}$ for the functional x_t is used, whereas in section II, the norm $\|\cdot\|_W$ is used. The use of the two types of norm arises from the demand on the functional x_t as can be observed from their definitions. This point can be clearly found in [12] and [33]. Because different Lyapunov functionals are chosen, leading to the different demands on the functional x_t, so that two different norms are used in [12] and [33], respectively. In this section, if the functional x_t is absolutely continuous on $[-r, 0)$, have a finite $\lim_{\theta \to 0^-} x_t(\theta)$ and have square integrable first-order derivatives, the definition and results of this section also hold with the norm $\|\cdot\|_{M_2}$ be replaced by $\|\cdot\|_W$. In the next section, due to the existence of the requirement of $\dot{x}_t$ in the Lyapuonov-Krasovskii functionals, norm $\|\cdot\|_W$ will be used.

IV. ISS STABILIZATION OF SAMPLED-DATA LINEAR SYSTEMS UNDER APERIODIC SAMPLING

In the above sections, for the sampled-data systems under deterministic aperiodic sampling and random sampling, the ISS problems are considered. Some criteria on ISS and SISS are provided. In this section, based on the matrix inequalities approach ([12]), we will consider the examples of the ISS stabilization problem for sampled-data linear systems under deterministic aperiodic sampling and random sampling, respectively. To adapt to the characteristics of aperiodic sampling and piecewise controller, a class of new Lyapunov-Krasovskii functionals will be given.

A. CASE OF DETERMINISTIC APERIODIC SAMPLING

Consider a class of sampled-data linear systems with control $u(t) \in \mathbb{R}^m$ and disturbance $w(t) \in \mathbb{R}^p$ given by

$$\dot{x}(t) = Ax(t) + B_2u(t) + B_1w(t), \tag{18}$$

where $x(t) \in \mathbb{R}^n$, A, B_2, and B_1 are matrices with appropriate dimensions. Given a set of sampling state feedback control

$$u(t) = K_ix(t_k), \quad t \in [t_k, t_{k+1}), \tag{19}$$

or in other way,

$$u(t) = K_ix(t - \eta_i(t)), \quad t \in [t_k, t_{k+1}), \tag{20}$$

where $\eta_i(t) = t - t_{i,k} = t - t_k$ denotes the input delay $\eta_{\sigma(t)}(t)$ with the corresponding sampling interval length $\eta_k = t_{i,k+1} - t_k = t_{k+1} - t_k = \overline{\eta}_i \ (i \in \mathcal{N})$.

We assume that K_i satisfy $|K_i|^2 \leq \overline{K}_i$, with some constants $\overline{K}_i > 0 \ (i \in \mathcal{N})$. We apply the relation

$$K_ix(t - \eta_i(t)) = K_ix(t) - \int_{t-\eta_i(t)}^{t} K_i\dot{x}(s)ds$$

and represent the closed-loop system (18) and (20) in the form

$$\dot{x}(t) = Ax(t) + B_2K_ix(t) - B_2 \int_{t-\eta_i(t)}^{t} K_i\dot{x}(s)ds + B_1w(t). \tag{21}$$

We note that (21) is equivalent to (18) and (20).

Consider the following Lyapunov-Krasovskii functional

$$V_i(t) = x^T(t)Px(t) + (t_{i,k+1} - t) \int_{t-\eta_i(t)}^{t} \dot{x}^T(s)K_i^TR_iK_i\dot{x}(s)ds, \tag{22}$$

where $t \in [t_k, t_{k+1})$, P and R_i are symmetric and positive definite matrices. Since $t - \eta_i(t) = t_{i,k}$ and $t_{i,k+1} - t_{i,k+1}^- = 0$, where $t_{i,k+1}^-$ is the left limit of t to $t_{i,k+1}$, it is observed that

$V_i(t_{i,k}) = x^T(t_{i,k})Px(t_{i,k})$, $V_i(t^-_{i,k+1}) = x^T(t^-_{i,k+1})Px(t^-_{i,k+1})$ and $V_i(t_{i,k}) = V_j(t^-_{i,k})$. So, condition (iii) of Theorem 1 holds.

$$(t_{i,k+1}-t)\int_{t-\eta_i(t)}^{t} \dot{x}^T(s)K_i^T R_i K_i \dot{x}(s)ds \leq \bar{\eta}_i\lambda_{\max}(R_i)\overline{K}_i\int_{t-\eta_i(t)}^{t}|\dot{x}(s)|^2 ds,$$

where $\lambda_{\max}(R_i)$ denote the largest eigenvalue of the symmetric matrix R_i.

Since $\lambda_{\min}(P)|x(t)|^2 \leq x^TPx \leq \lambda_{\max}(P)|x(t)|^2$, we conclude that assumption (i) of Theorem 1 is satisfied with $\underline{\alpha}(|x|) = \lambda_{\min}(P)|x|^2$ and with $\overline{\alpha}(s) = c_i s^2$, where $\lambda_{\min}(P)$ denote the smallest eigenvalue of the symmetric matrix P, $c_i = \max\{\lambda_{\max}(P), \bar{\eta}_i\lambda_{\max}(R_i)\overline{K}_i\}$.

To satisfy the assumption (ii) of Theorem 1, we will find conditions such that for some constant $\alpha_i > 0$,

$$\dot{V}_i(t) + \alpha_i V_i(t) < 0 \text{ for } V_i(t) \geq |w(t)|^2.$$

For this to be true, by applying the $\mathcal{S}$ procedure ([6], [45]), we only need to find some positive constants λ_i such that the following inequality holds.

$$U_i = \dot{V}_i(t) + \alpha_i V_i(t) + \lambda_i(V_i(t) - |w(t)|^2) < 0. \tag{23}$$

Hence differentiating $V_i(t)$ along the closed-loop system (21), we obtain

$$\begin{aligned}U_i \leq\ & 2x^T(t)P\Big(Ax(t) + B_2K_ix(t) + B_1\, w(t) \\ & -B_2\int_{t-\eta_i(t)}^{t} K_i\dot{x}(s)ds\Big) + (\alpha_i+\lambda_i)x^T(t)Px(t) \\ & +(\alpha_i+\lambda_i)(t_{i,k+1}-t)\int_{t-\eta_i(t)}^{t}\dot{x}^T(s)K_i^TR_iK_i\dot{x}(s)ds \\ & -\lambda_i|w(t)|^2 - \int_{t-\eta_i(t)}^{t}\dot{x}^T(s)K_i^TR_iK_i\dot{x}(s)ds \\ & +(t_{i,k+1}-t)\dot{x}^T(t)K_i^TR_iK_i\dot{x}(t).\end{aligned}$$

Then, applying Jensen's inequality ([14]),

$$\int_{t-\eta_i(t)}^{t}\dot{x}^T(s)K_i^TR_iK_i\dot{x}(s)ds \geq \bar{\eta}_i\zeta_{2i}^TR_i\zeta_{2i},$$

where $\zeta_{2i} = \frac{1}{\bar{\eta}_i}\int_{t-\eta_i(t)}^{t}K_i\dot{x}(s)ds$, we find that

$$U_i \leq \zeta^T(t)\Psi_i\zeta(t) + \bar{\eta}_i\dot{x}^T(t)K_i^TR_iK_i\dot{x}(t), \tag{24}$$

where

$$\zeta^T(t) = \left[x^T(t),\ \tfrac{1}{\bar{\eta}_i}\int_{t-\eta_i(t)}^{t}[K_i\dot{x}(s)]^Tds,\ w^T(t)\right],$$

$$\Psi_i = \begin{bmatrix}\Psi_{11i} & -\bar{\eta}_iPB_2 & PB_1\\ * & -\bar{\eta}_iR_i(1-(\alpha_i+\lambda_i)\bar{\eta}_i) & 0\\ * & * & -\lambda_iI_p\end{bmatrix},$$

$$\Psi_{11i} = PA + A^TP + PB_2K_i + K_i^TB_2^TP + (\alpha_i+\lambda_i)P.$$

Setting the right-hand side of (24) for $\dot{x}(t)$ into (23) and applying Schur complements formula, we finally verify that $U_i < 0$ for $x(t) \neq 0$ if the following nonlinear matrix inequality (NLMI) holds:

$$\begin{bmatrix}M_{11i} & -\bar{\eta}_iPB_2 & PB_1 & M_{14i}\\ * & M_{22i} & 0 & M_{24i}\\ * & * & -\lambda_iI_p & M_{34i}\\ * & * & * & -\bar{\eta}_iR_i\end{bmatrix} < 0, \tag{25}$$

where

$$\begin{aligned}M_{11i} &= P(A+B_2K_i) + (A+B_2K_i)^TP + (\alpha_i+\lambda_i)P,\\ M_{14i} &= \bar{\eta}_i(A+B_2K_i)^TK_i^TR_i,\\ M_{22i} &= -\bar{\eta}_iR_i(1-(\alpha_i+\lambda_i)\bar{\eta}_i),\\ M_{24i} &= -\bar{\eta}_i^2B_2^TK_i^TR_i,\\ M_{34i} &= \bar{\eta}_iB_1^TK_i^TR_i.\end{aligned}$$

Then, along the trajectory of the submodel (21), it follows that for $V_i(t) \geq |w(t)|^2$, $V_i(t) \leq e^{-\alpha_i(t-t_{i,k})}V_i(t_k),\ t \in [t_k, t_{k+1})$.

Inequality (25) may be considered as a NLMI, where α_i and λ_i are tuning parameters. From the above procedure, conditions of Theorem 1 hold. We then have the following lemma.

Lemma 4: Given the controller gain K_i, if there exist matrices $P = P^T > 0$, $R_i = R_i^T > 0$ and constants $\lambda_i, \alpha_i > 0$ that solve (25), the i-th submodel in system (21) is ISS with $\gamma(s) = \frac{s^2}{\lambda_{\min}(P)}$.

Based on Lemma 4, we give the stability criterion of sampled-data system (18)-(20) in the following.

Theorem 3: For system (21), given the controller gains K_i ($i \in \mathcal{N}$), if there exist matrices $P = P^T > 0$, $R_i = R_i^T > 0$ and constants $\lambda_i, \alpha_i > 0$ that solve (25), sampled-data system (18)-(20) is ISS with $\gamma(s) = \frac{s^2}{\lambda_{\min}(P)}$.

Proof: For any $t \in [t_0, \infty)$, there exists $k \in \mathbb{N}_0$, such that $t \in [t_k, t_{k+1})$. With $V_i(t)$ in (22), it follows that $V_{\sigma(t_k)}(t_k) = V_{\sigma(t_{k-1})}(t_k^-)$ for any k, where $\sigma(t_k)$ and $\sigma(t_{k-1}) \in \mathcal{N}$. Then, from Lemma 4, it follows that for $V_{\sigma(t)}(t) \geq |w(t)|^2$,

$$\begin{aligned}&V_{\sigma(t_k)}(t)\\ &\quad\leq e^{-\alpha_{\sigma(t_k)}(t-t_k)}V_{\sigma(t_k)}(t_k) = e^{-\alpha_{\sigma(t_k)}(t-t_k)}V_{\sigma(t_{k-1})}(t_k^-)\\ &\quad\leq e^{-\alpha_{\sigma(t_k)}(t-t_k)} \times e^{-\alpha_{\sigma(t_{k-1})}(t_k^- - t_{k-1})}V_{\sigma(t_{k-1})}(t_{k-1})\\ &\quad\ \ \vdots\\ &\quad\leq e^{-\alpha_{\min}(t-t_0)}V_{\sigma(t_0)}(t_0),\end{aligned}$$

where $\alpha_{\min} = \min\limits_{i\in\mathcal{N}}\{\alpha_i\}$. So, from $\underline{\alpha}(|x|) = |x|^2\lambda_{\min}(P)$, it is obtained that for any $t \in [t_0, +\infty)$,

$$\begin{aligned}|x(t)|^2 &\leq \frac{1}{\lambda_{\min}(P)}e^{-\alpha_{\min}(t-t_0)}V_{\sigma(t_0)}(t_0) + \frac{\|w\|_\infty^2}{\lambda_{\min}(P)}\\ &\leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}e^{-\alpha_{\min}(t-t_0)}|x(t_0)|^2 + \frac{\|w\|_\infty^2}{\lambda_{\min}(P)}.\end{aligned}$$

Therefore, the closed-loop system (18)-(20) is ISS with $\gamma(s) = \frac{s^2}{\lambda_{\min}(P)}$. ■

Furthermore, using the descriptor approach ([11]), a new sufficient condition for ISS of the i-th submodel in system

(21) may be derived that in some cases leads to less restrictiveness. We add the right-hand side of $0 = 2[x^T(t)P_2^T + \dot{x}^T(t)P_3^T]\delta_i$ to U_i given by (23) where

$$\delta_i = -\dot{x}(t) + Ax(t) + B_2K_ix(t) - B_2\int_{t-\eta_i(t)}^{t} K_i\dot{x}(s)ds + B_1\, w(t).$$

We can obtain that

$$\begin{aligned} U_i &\leq 2x^T(t)P\dot{x}(t) + 2[x^T(t)P_2^T + \dot{x}^T(t)P_3^T]\Big[-\dot{x}(t) \\ &\quad + Ax(t) + B_2K_ix(t) + B_1\, w(t) - B_2\int_{t-\eta_i(t)}^{t} K_i\dot{x}(s)ds\Big] \\ &\quad + (\alpha_i+\lambda_i)x^T(t)Px(t) - \lambda_i|w(t)|^2 \\ &\quad + (\alpha_i+\lambda_i)(t_{i,k+1}-t)\int_{t-\eta_i(t)}^{t} \dot{x}^T(s)K_i^TR_iK_i\dot{x}(s)ds \\ &\quad - \int_{t-\eta_i(t)}^{t} \dot{x}^T(s)K_i^TR_iK_i\dot{x}(s)ds \\ &\quad + (t_{i,k+1}-t)\dot{x}^T(t)K_i^TR_iK_i\dot{x}(t). \end{aligned}$$

Then, applying Jensen's inequality, we arrive to (24), where

$$\zeta^T(t) = \Big[x^T(t),\ \dot{x}^T(t),\ \tfrac{1}{\bar{\eta}_i}\textstyle\int_{t-\eta_i(t)}^{t}[K_i\dot{x}(s)]^T ds,\ w^T(t)\Big],$$

$$\Psi_i = \begin{bmatrix} \Psi_{11i} & \Psi_{12i} & -\bar{\eta}_iP_2^TB_2 & P_2^TB_1 \\ * & -P_3^T - P_3 & -\bar{\eta}_iP_3^TB_2 & P_3^TB_1 \\ * & * & -\bar{\eta}_iR_i(1-(\alpha_i+\lambda_i)\bar{\eta}_i) & 0 \\ * & * & * & -\lambda_iI_p \end{bmatrix},$$

and where

$$\Psi_{11i} = P_2^T(A+B_2K_i) + (A^T + K_i^TB_2^T)P_2 + (\alpha_i+\lambda_i)P,$$
$$\Psi_{12i} = P - P_2^T + (A^T + K_i^TB_2^T)P_3.$$

From Schur complements formula, it follows that $U_i < 0$ for $x(t) \neq 0$ if the following NLMI holds:

$$\begin{bmatrix} \Psi_{11i} & \Psi_{12i} & -\bar{\eta}_iP_2^TB_2 & P_2^TB_1 & 0 \\ * & -P_3^T - P_3 & -\bar{\eta}_iP_3^TB_2 & P_3^TB_1 & \bar{\eta}_iK_i^TR_i \\ * & * & -\bar{\eta}_iR_i(1-(\alpha_i+\lambda_i)\bar{\eta}_i) & 0 & 0 \\ * & * & * & -\lambda_iI_p & 0 \\ * & * & * & * & -\bar{\eta}_iR_i \end{bmatrix} < 0. \quad (26)$$

Furthermore, choosing $P_3 = \epsilon P_2$, where ϵ is a tuning scalar parameter (which may be restrictive) and defining

$$Q_2 = P_2^{-1},\quad \bar{P} = Q_2^TPQ_2,\ \bar{R}_i = R_i^{-1},\ Y_i = K_iQ_2.$$

Multiplying (26) by $diag\{P_2^{-1}, P_2^{-1}, R_i^{-1}, I_p, R_i^{-1}\}$ and its transpose, from the right and the left, respectively, we obtain:

Theorem 4: Consider system (18) with sampling state feedback controller (19). If for some tuning scalar parameter ϵ there exist $n\times n$ matrix $0 < \bar{P}$, Q_2, $m\times m$ matrices $\bar{R}_i$, $m\times n$ matrices Y_i and constants $\lambda_i, \alpha_i > 0$ such that for any $i \in \mathcal{N}$, the following NLMIs are satisfied

$$\begin{bmatrix} \Gamma_{11i} & \Gamma_{12i} & -\bar{\eta}_iB_2\bar{R}_i & B_1 & 0 \\ * & -\epsilon Q_2^T - \epsilon Q_2 & -\epsilon\bar{\eta}_iB_2\bar{R}_i & \epsilon B_1 & \bar{\eta}_iY_i^T \\ * & * & -\bar{\eta}_i\bar{R}_i(1-(\alpha_i+\lambda_i)\bar{\eta}_i) & 0 & 0 \\ * & * & * & -\lambda_iI_p & 0 \\ * & * & * & * & -\bar{\eta}_i\bar{R}_i \end{bmatrix} < 0, \quad (27)$$

where

$$\Gamma_{11i} = (AQ_2 + B_2Y_i) + (Q_2^TA^T + Y_i^TB_2^T) + (\alpha_i+\lambda_i)\bar{P},$$
$$\Gamma_{12i} = \bar{P} - Q_2 + \epsilon(Q_2^TA^T + Y_i^TB_2^T),$$

the sampled-data system (18)-(19) is ISS with $\gamma(s) = \frac{s^2}{\lambda_{\min}(P)}$ and $K_i = Y_iQ_2^{-1}$.

Remark 5: Based on the ISS criterion (Theorem 1), Theorem 4 provides a sampling controller design method for a sampled-data system (18)-(19) under deterministic aperiodic sampling. The designed sampling controller is not a common controller for all of the sampling intervals. For different sampling intervals, different controller gains are designed that fully reflect the influence of the time delays introduced by the sampling on the system.

B. CASE OF RANDOM SAMPLING

For a sampled-data linear system (18) with sampling state feedback control (19) or (20), t_k $(k = 1, 2, \cdots)$ are assumed to be random sampling instants, and correspondingly, $\eta_i(t) = t - t_{i,k} = t - t_k$ denotes the random time delay $\eta_{\sigma(t)}(t)$ with the corresponding sampling interval length $\eta_k = t_{i,k+1} - t_k = t_{k+1} - t_k = \bar{\eta}_i$ $(i \in \mathcal{N})$.

Similar to the procedure of Subsection IV-A, choosing the Lyapunov-Krasovskii functional (22), we obtain that along the trajectory of the submodel (21), if (25) or (27) holds, it follows that for $V_i(t) \geq |w(t)|^2$,

$$V_i(t) \leq e^{-\alpha_i(t-t_{i,k})}V_i(t_k),\ a.s.\quad t\in[t_k, t_{k+1}).$$

For sampled-data system (18)-(19) under random sampling, iterating the last inequality from $k = 0$ to $k = N_\sigma(t)$ for an arbitrary time $t > t_0$, when $V_{\sigma(t)}(t) \geq |w(t)|^2$, we arrive at,

$$V_{\sigma(t)}(t) \leq e^{-\alpha_{\min}(t-t_0)}V_{\sigma(t_0)}(t_0),\ a.s.\quad t\in[t_0, \infty),$$

where $\alpha_{\min} = \min_{i\in\mathcal{N}}\{\alpha_i\}$, $N_\sigma(t)$ denotes the number of samplings on the interval $[t_0, t)$. Since the initial condition is deterministic, taking expectations on both sides of the previous inequality, we obtain $E[V_{\sigma(t)}(t)] \leq e^{-\alpha_{\min}(t-t_0)}V_{\sigma(t_0)}(t_0)$, $t \in [t_0, \infty)$. From $\underline{\alpha}(|x|) = |x|^2\lambda_{\min}(P)$, it is obtained that for any $t\in[t_0, +\infty)$,

$$\begin{aligned} &E[|x(t)|^2] \\ &\leq \frac{1}{\lambda_{\min}(P)}e^{-\alpha_{\min}(t-t_0)}V_{\sigma(t_0)}(t_0) + \frac{\|w\|_\infty^2}{\lambda_{\min}(P)} \\ &\leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}e^{-\alpha_{\min}(t-t_0)}|x(t_0)|^2 + \frac{\|w\|_\infty^2}{\lambda_{\min}(P)}. \end{aligned}$$

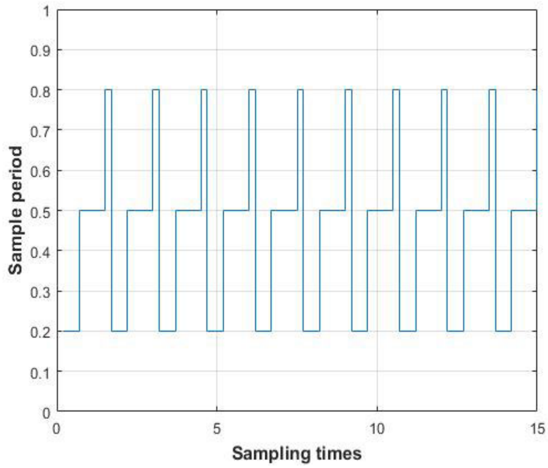

FIGURE 1. Sampling periods of deterministic aperiodic sampling.

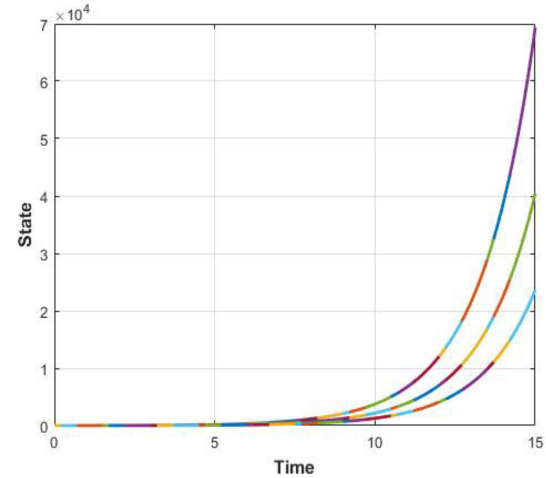

FIGURE 2. State response of system (18) without control input under deterministic aperiodic sampling.

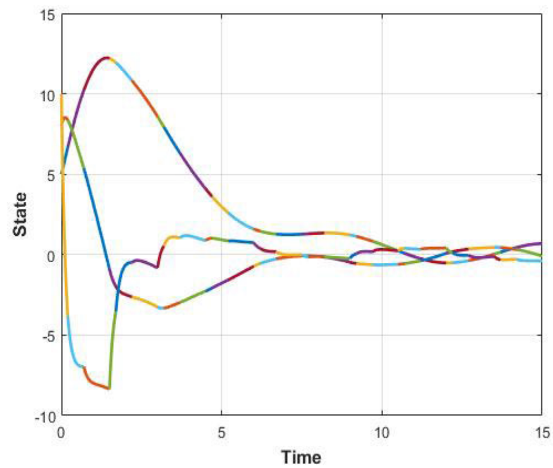

FIGURE 3. State response of system (18)-(19) under deterministic aperiodic sampling.

Therefore, the closed-loop system (18)-(19) is ISS in mean square with $\gamma(s) = \frac{s^2}{\lambda_{\min}(P)}$. From the proof of Theorem 2 in [49], the closed-loop system (18)-(19) is also SISS. Therefore, we reach the following conclusion.

Theorem 5: If the conditions of Theorem 4 hold, the sampled-data system (18)-(19) under random sampling is SISS.

Remark 6: In [7], the stability analysis and stabilization of randomly switched systems were studied. For the randomly switched systems, discrete switches are triggered by a stochastic process. When the multi-Lyapunov functions were chosen for stability analysis, each Lyapunov function has a proportional relationship with the scaling factor $\mu > 1$. Due to the existence of the scaling factor, the stochastic process that triggered the discrete switches will affect the stability of the randomly switched systems. Here, since the selected multi-Lyapunov functions are equal at the sampling instants, the scaling factor μ can be considered to be 1. Therefore, for any random sampling, our Theorem 5 holds.

V. SIMULATION EXAMPLE

Example 1: Consider the sampled-data linear system (18)-(19) with $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -7 \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and K_i to be determined. The sampling periods $\bar{\eta}_i$ take values in the set $\{0.2, 0.5, 0.8\}$ and are given at each time step k just as in Fig. 1. Without control input, the system (18) will not be stable. It is observed from Fig. 2 that the state of the system with initial state $x_0 = [5, 8, 10]^T$ is divergent.

Using the Matlab LMI Control Toolbox, we solve NLMIs (27) in Theorem 4 with tuning of the scalar parameter $\epsilon = 1$, all the convergence indexes α_i and parameters λ_i set to 0.1; then, for every sampling period, we obtain the feedback gain as follows

$$K_i = \begin{cases} K_1 = \begin{pmatrix} -3.7164 & -7.5588 & 0.6983 \end{pmatrix}, & \bar{\eta}_1 = 0.2; \\ K_2 = \begin{pmatrix} -3.5941 & -7.5230 & -0.0852 \end{pmatrix}, & \bar{\eta}_2 = 0.5; \\ K_3 = \begin{pmatrix} -3.1137 & -6.6080 & -0.4003 \end{pmatrix}, & \bar{\eta}_3 = 0.8, \end{cases}$$

and the closed-loop system (18)-(19) is ISS. It is observed from Fig. 3 that when the sampling periods of 0.2, 0.5 and 0.8 are selected periodically as shown in Fig. 1, the system state with initial $x_0 = [5, 8, 10]^T$ will be ultimately bounded. From Theorem 5, we also know that the system (18) under random sampling is SISS. For random sampling period signal shown as in Fig. 4, where the sampling periods of 0.2, 0.5 and 0.8 are chosen with a uniform distribution, the system state with the initial state $x_0 = [5, 8, 10]^T$ will be ultimately bounded as shown in Fig. 5.

Remark 7: According to the theory of Lyapunov stability, the Lyapunov function can be regarded as "energy". Here, our Lyapunov functional selects a common P matrix and a different R_i matrix. The term that contain R_i in the Lyapunov functional can be viewed as the energy of the control input that is necessary to stabilize the system. In Example 1, the system is required to have the same relative stability parameters. This is due to the different sampling interval

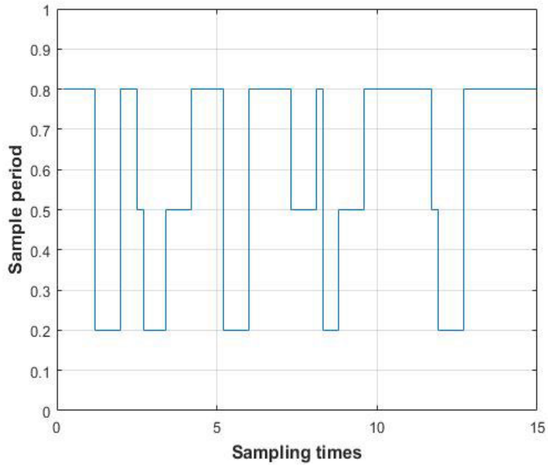

FIGURE 4. Sampling periods of random aperiodic sampling.

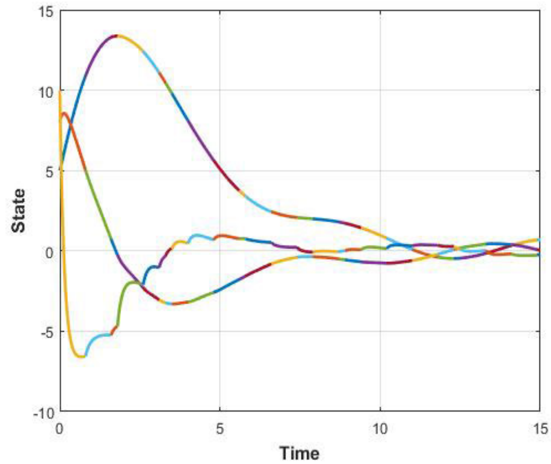

FIGURE 5. State response of system (18)-(19) under random sampling.

lengths that require the design of different feedback controller gains. As pointed out in [50] using an example, the control of an aperiodic sampled-data system cannot simply use the upper or lower bounds of the sampling period to control it robustly. Even if the common controller exists, our Example 1 shows that we do not need to consume the control input energy that is suitable for all sampling intervals to stabilize the aperiodic sampling system. Greater energy savings may be obtained when different sampling intervals correspond to different feedback controller gains. Since only stability and stabilization are considered in this article, in the future, we can further consider how much energy can be saved by using our piecewise controller design method compared with the common-controller design method from the perspective of optimal control.

VI. CONCLUSION

In this article, ISS and input-to-state stabilization of sampled-data systems under deterministic aperiodic sampling and random sampling have been studied, respectively. Using the direct design method for sampled-data systems, the systems are transformed into switched nonlinear systems with switched time-varying delays and randomly switching delays, respectively. For these delay systems, the ISS, SISS definitions and corresponding criteria have been provided. Based on these definitions and criteria, the ISS and SISS properties of the sampled-data systems under deterministic aperiodic sampling and random sampling are studied, respectively. As was pointed out, sampling periods will affect the control of the sampled-data systems. It is found that by using different controller gains for different sampling intervals in the systems, the ISS and SISS properties can also be guaranteed.

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