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Model Order Reduction Based on Approximate Cross Gramian and Laguerre Series for Linear Input-Output Systems

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ABSTRACT In this paper, we first focus on the topic of calculation of cross gramians for linear square systems, which are constructed approximately from the Laguerre series expansion coefficient vectors that are obtained by solving a recurrence formula instead of solving Sylvester equations directly. Then based on such approximate cross gramian, the reduced-order models (ROMs) are produced by truncating the states that are associated with the smaller approximate Hankel singular values (HSVs). In addition, combining with the idea of dominant subspace projection method, we modify our proposed algorithm to obtain a ROM that preserves the stability. What's more, our algorithms are extended to non-square case successfully. The main properties of ROMs are discussed as well. Finally, some numerical simulations are provided to illustrate the effectiveness of our proposed algorithms in the views of accuracy and computational cost.

INDEX TERMS Model order reduction, Laguerre functions, cross Gramian, balanced truncation, stability.

I. INTRODUCTION

Model order reduction (MOR) was first developed in the area of system and control theory, which studies properties of dynamical systems in application for reducing their complexity while preserving their input-output behaviors as well as essential properties like stability and passivity. Nowadays, such technique is widely applied in simulation and design of very large-scale integrated (VLSI) electrical circuits, weather prediction, air quality simulation, system analysis, virtual synchronous machine (VSM) and many other engineering fields [1]–[5].

In system and control theory, balanced truncation (BT) is a robust MOR tool [6] and is now commonly used in many dynamical systems [7]–[11]. It produces a stable-preserving reduced-order model (ROM) by truncating the states that are associated with smaller Hankel singular values (HSVs), and provides a global error bound. However, such method needs to solve two Lyapunov equations that have the computational cost of $\mathcal{O}(n^3)$, where n is the dimension of the system. So exact BT method is expensive to implement for dealing with large-scale systems. As a result, BT method based on approximate controllability and observability gramians is an alternative choice which aims to obtain an approximate balanced system in a numerically efficient way, such as low rank square root method (LRSRM), dominant subspaces projection model reduction (DSPMR), BT-based MOR via low rank decomposition of controllability and observability gramians using Legendre polynomials [12]–[14].

The definition of cross gramian for single-input-single-output (SISO) linear time-invariant (LTI) systems first appeared in [15] which is the solution to a Sylvester equation. Then such definition was extended to multi-input-multi-output (MIMO) symmetric systems successfully in [16]. Due to the fact that the cross gramian contains the controllability and observability information at the same time, a varieties of classical BT-based methods were proposed. However, a major constraint of the cross gramian is that it is

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computed strictly for square systems and exhibits the core property [16] only for symmetric systems. So it is originally concerned with MOR for the symmetric case. In order to extend its applications in MOR, a series of constructive results on improving such technique have been achieved. At first, such approach was extended the use of the cross gramian from symmetric systems to some special general symmetric systems such as orthogonally symmetric system [17], whose cross gramian satisfies the core property. Then some strategies on symmetrization were taken into account. In [18], relies on a symmetrizer matrix and uses an embedding technique, a symmetric system was produced. In [19], it proposed the definition of cross gramian for non-square systems, which converts the original system to the so-called SISO "average" system. While based on multiple decompositions, a non-square system is converted into a coupled system with a series of SISO subsystems [20]. Based on these techniques, the MOR based on cross gramian has been extend to non-square case successfully. Besides, some algorithms have been proposed to improve the efficiency of calculating the low-rank decomposition of cross gramians, such as the Laguerre polynomials [21], the matrix sign functions [22] and the hierarchical approximate proper orthogonal decomposition (HAPOD) [23]. Based on these low-rank factors of cross gramian, several BT-related MOR methods were proposed. In order to extend the usage of cross gramians, in [24], the empirical cross gramians for nonlinear systems and parameterized systems were introduced based on the trajectories of the underlying system with perturbations in the input as well as initial state and the Galerkin projection was generated by performing the singular value decomposition (SVD) on it. Moreover, in recent years, some new MOR methods based on cross gramian were proposed, such as H_2 optimal technique in combination with the Stiefel manifold [25], the DSPMR method that needs a single HAPOD instead of multiple decomposition [26].

Motivated by improving the efficiency of BT methods, our paper proposes a series of MOR algorithms based on approximate cross gramians via Laguerre functions, which have been successfully applied to MOR for linear and nonlinear systems [27]–[29]. Our approach aims to calculate the low-rank decomposition of cross gramians, whose factors are constructed directly from the expansion coefficients of impulse responses in the space spanned by Laguerre functions by solving a recurrence formula. Combined ideas from LRSRM, an associated ROM is produced by truncating the states that are corresponding to the smaller approximate HSVs. Compared with BT methods, our proposed algorithm just needs to solve sparse linear equations instead of solving Sylvester equations and only one SVD technique for a low-dimensional matrix is implemented, which makes it more flexible and computationally efficient. Moreover, in combination with the dominant subspace projection method, our algorithm is improved to alleviate the shortcoming, which may unexpectedly lead to an unstable ROM even if the original one is stable. In addition, referred to the "average" system, our proposed algorithms are extended to non-square case for dealing with MOR.

This paper is organized as follows. In Section II, the cross gramians of square LTI systems and Laguerre functions are introduced briefly. In Section III, the calculation of cross gramians whose low-rank decomposition factors are constructed via Laguerre series as well as the associated algorithms based on LRSRM is proposed for square and non-square LTI systems respectively. Then a modification based on DSPMR of the proposed algorithms is also introduced. The main properties of the ROMs, such as the relation to BT-based methods and stability preservation, are discussed as well in this section. In Section IV, some numerical simulations are provided to indicate the efficiency of our proposed methods. In Section V, some conclusions on our related work are given.

Throughout our paper, these following notations are used. I denotes an identity matrix that has proper dimension. For a matrix $A \in \mathbb{R}^{n\times n}$, $A > 0$ means A is a positive definite matrix, while $A < 0$ denotes a negative definite matrix. $\lambda_i(A)$ is the ith eigenvalue of A.

II. PRELIMINARIES

A. CROSS GRAMIANS FOR LTI SYSTEMS

Consider a stable LTI input-output system as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{q\times n}$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^q$ is the output and $u(t) \in \mathbb{R}^p$ is the input. In general, such system can be simply indicated by $\{A, B, C\}$ as well. Without loss of generality, we assume that $x(0) = 0$. The transfer function $H(s)$ of system (1) is $H(s) = C(sI - A)^{-1}B$. It is noted that system (1) is square if $p = q$ and is symmetric if $H(s)$ is symmetric.

The cross gramian of square system (1) is defined as

$$W_X = \int_0^{+\infty} e^{At} BC e^{At} \mathrm{d}t,$$

which is the solution to the following Sylvester equation

$$AW_X + W_X A + BC = 0. \tag{2}$$

What's more, the relevant controllability gramian W_C and observability gramian W_O are given by

$$W_C = \int_0^{+\infty} e^{At} BB^{\mathrm{T}} e^{A^{\mathrm{T}}t} \mathrm{d}t$$

and

$$W_O = \int_0^{+\infty} e^{A^{\mathrm{T}}t} C^{\mathrm{T}} C e^{At} \mathrm{d}t.$$

The HSVs σ_i of system (1) are given by

$$\sigma_i = \sqrt{\lambda_i(W_C W_O)}, \quad i = 1, 2, \cdots, n.$$

The core property of the cross gramian [16] in MOR is

$$W_X^2 = W_C W_O. \tag{3}$$

Note that the above equation (3) holds only for symmetric LTI systems. As a result, the HSVs of symmetric LTI system $\{A, B, C\}$ can be analogously given by the eigenvalues of its cross gramian W_X:

$$\sigma_i = |\lambda_i(W_X)|, \quad i = 1, 2, \cdots, n.$$

It should be pointed out that the cross gramians of non-symmetric, square MIMO LTI systems can be obtained by solving the Sylvester equation (2) as well, however without any theoretical background as for the symmetric case, so there is no guarantee for obtaining the ROM with appropriate quality [30].

In addition, from the definition of cross gramian, W_X can be interpreted as cross covariance matrix of the system's impulse response and adjoint system's impulse response. As originally in [6], these impulse responses are trajectories,

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B\delta(t) \Rightarrow x(t) = e^{At}B,\\ \dot{z}(t) &= A^{\mathrm{T}}z(t) + C^{\mathrm{T}}\delta(t) \Rightarrow z(t) = e^{A^{\mathrm{T}}t}C^{\mathrm{T}}.\end{aligned}$$

As a results, from the definition of W_C, W_O and W_X, it has

$$\begin{aligned}W_C &= \int_0^{+\infty} x(t)x^{\mathrm{T}}(t)\mathrm{d}t,\\ W_O &= \int_0^{+\infty} z(t)z^{\mathrm{T}}(t)\mathrm{d}t,\\ W_X &= \int_0^{+\infty} x(t)z^{\mathrm{T}}(t)\mathrm{d}t.\end{aligned} \tag{4}$$

B. LAGUERRE FUNCTIONS

The Laguerre polynomials are defined as follows:

$$l_i(t) = \frac{e^t}{i!}\frac{\mathrm{d}^i(e^{-t}t^i)}{\mathrm{d}t^i}, \quad i = 0, 1, 2, \cdots.$$

Then Laguerre functions are given by Laguerre polynomials as follows:

$$\phi_i^{\alpha}(t) = \sqrt{2\alpha}e^{-\alpha t}l_i(2\alpha t), \quad i = 0, 1, 2, \cdots,$$

where α is a positive real constant [27]. The Laplace transform of $\phi_i^{\alpha}(t)$ is

$$\Phi_i^{\alpha}(s) = \mathcal{L}(\phi_i^{\alpha}(t)) = \frac{\sqrt{2\alpha}}{s+\alpha}\left(\frac{s-\alpha}{s+\alpha}\right)^i, \quad i = 0, 1, 2, \cdots,$$

where $\mathcal{L}$ is the Laplace transform. According to [27], the sequence of $\{\phi_i^{\alpha}\}$ forms a uniformly bounded orthogonal basis for the Hilbert space $L_2(\mathbb{R}^+)$; thus the impulse response function $h(t)$ of system $\{A, B, C\}$ admits the Fourier-Laguerre expansion as:

$$h(t) = \sum_{i=0}^{+\infty} F_i\phi_i^{\alpha}(t). \tag{5}$$

Applying Laplace transform on both sides of (5), it holds

$$H(s) = \sum_{i=0}^{+\infty} F_i\Phi_i^{\alpha}(s).$$

According to [27], $H(s)$ can be optimally approximated in the H_2 norm sense by the truncated Fourier-Laguerre expansion

$$H(s) \approx \sum_{i=0}^{m-1} F_i\Phi_i^{\alpha}(s).$$

III. MOR BASED ON CROSS GRAMIAN VIA LAGUERRE SERIES

In this section, a series of MOR algorithms based on cross gramian and Laguerre series for LTI systems, as well as the main properties of ROMs, like the stability, will be fully discussed.

A. LOW-RANK APPROXIMATION TO CROSS GRAMIANS FOR SQUARE LTI SYSTEMS

Consider an SISO LTI system as follows:

$$\begin{cases}\dot{x}(t) = Ax(t) + bu(t),\\ y(t) = cx(t),\end{cases} \tag{6}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^{1\times n}$. According to (4) with impulse response of system (6) and its adjoint system, as well as from Parseval's theorem, the cross gramian in the frequency domain is given by

$$\begin{aligned}W_X &= \int_0^{+\infty} x(t)z^{\mathrm{T}}(t)\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\mathrm{i}\omega)Z^{\mathrm{H}}(\mathrm{i}\omega)\mathrm{d}\omega\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} (\mathrm{i}\omega I - A)^{-1}bc(-\mathrm{i}\omega I - A)^{-1}\mathrm{d}\omega.\end{aligned} \tag{7}$$

For $X(\mathrm{i}\omega)$, expand it in the form of Laguerre series as follows:

$$X(\mathrm{i}\omega) = (\mathrm{i}\omega I - A)^{-1}b = \sum_{i=0}^{+\infty} f_i\Phi_i^{\alpha}(\mathrm{i}\omega),$$

where $f_i \in \mathbb{R}^n$ is the Fourier coefficient vectors. Let $u = \frac{\mathrm{i}\omega-\alpha}{\mathrm{i}\omega+\alpha}$, then $X(\mathrm{i}\omega)$ can be represented as

$$\begin{aligned}X(\mathrm{i}\omega) &= (1-u)[\alpha(1+u)I - (1-u)A]^{-1}b\\ &= (1-u)[u(A+\alpha I) - (A-\alpha I)]^{-1}b\\ &= \frac{1}{\sqrt{2\alpha}}(1-u)\sum_{i=0}^{+\infty} f_iu^i.\end{aligned}$$

Namely, it holds

$$\begin{aligned}\sqrt{2\alpha}(uE_u - A_u)^{-1}b &= -\sqrt{2\alpha}(I - uA_u^{-1}E_u)^{-1}A_u^{-1}b\\ &= -\sqrt{2\alpha}\sum_{i=0}^{+\infty}(A_u^{-1}E_u)^iA_u^{-1}bu^i\\ &= \sum_{i=0}^{+\infty} f_iu^i,\end{aligned}$$

where $A_u = A - \alpha I$ and $E_u = A + \alpha I$. Hence, it leads to

$$f_i = -\sqrt{2\alpha}(A_u^{-1}E_u)^iA_u^{-1}b, \quad i = 0, 1, 2, \cdots.$$

Let $X(\mathrm{i}\omega) \approx \sum_{i=0}^{m-1} f_i\Phi_i^{\alpha}(\mathrm{i}\omega)$ and $\tilde{b} = -\sqrt{2\alpha}b$, where m is the desired approximation terms of the Laguerre series and

is about $10 \sim 30$ for an acceptable accuracy in general, then f_i can be calculated by a recurrence formula as follows:

$$A_u f_0 = \tilde{b},\ A_u f_i = E_u f_{i-1}, \quad i = 1, 2, \cdots, m-1,$$

which are represented as the following linear equations

$$\mathcal{P}\mathcal{F} = \mathcal{B}, \tag{8}$$

where

$$\mathcal{P} = \begin{bmatrix} A_u & 0 & 0 & \cdots & 0 & 0 \\ E_u & -A_u & 0 & \cdots & 0 & 0 \\ 0 & E_u & -A_u & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & -A_u & 0 \\ 0 & 0 & 0 & \cdots & E_u & -A_u \end{bmatrix},$$

$$\mathcal{F} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{m-2} \\ f_{m-1} \end{bmatrix},\ \mathcal{B} = \begin{bmatrix} \tilde{b} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

For the state $z(t)$ with impulse response of the adjoint system of system (6), we expand its transform $Z(\mathrm{i}\omega)$ as Laguerre series:

$$Z(\mathrm{i}\omega) = (\mathrm{i}\omega I - A^{\mathrm{T}})c^{\mathrm{T}} \approx \sum_{i=0}^{m-1} g_i \Phi_i^{\alpha}(\mathrm{i}\omega),$$

where $g_i \in \mathbb{R}^n$ is the Fourier coefficient vector. Analogously to (8), g_i can be given by

$$\mathcal{Q}\mathcal{G} = \mathcal{C}, \tag{9}$$

where

$$\mathcal{Q} = \begin{bmatrix} A_u^{\mathrm{T}} & 0 & 0 & \cdots & 0 & 0 \\ E_u^{\mathrm{T}} & -A_u^{\mathrm{T}} & 0 & \cdots & 0 & 0 \\ 0 & E_u^{\mathrm{T}} & -A_u^{\mathrm{T}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & -A_u^{\mathrm{T}} & 0 \\ 0 & 0 & 0 & \cdots & E_u^{\mathrm{T}} & -A_u^{\mathrm{T}} \end{bmatrix},$$

$$\mathcal{G} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_{m-2} \\ g_{m-1} \end{bmatrix},\ \mathcal{C} = \begin{bmatrix} \tilde{c}^{\mathrm{T}} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

and $\tilde{c}^{\mathrm{T}} = -\sqrt{2\alpha}c^{\mathrm{T}}$.

Finally, substituting the approximations of $X(\mathrm{i}\omega)$ and $Z(\mathrm{i}\omega)$ into (7), it has

$$\begin{aligned} W_X &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\mathrm{i}\omega)Z^{\mathrm{H}}(\mathrm{i}\omega)\mathrm{d}\omega \\ &\approx \frac{1}{2\pi}\int_{-\infty}^{+\infty} [\sum_{i=0}^{m-1} f_i \Phi_i^{\alpha}(\mathrm{i}\omega)][\sum_{i=0}^{m-1} g_i \Phi_i^{\alpha}(\mathrm{i}\omega)]^{\mathrm{H}}\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} [\sum_{i=0}^{m-1} f_i \Phi_i^{\alpha}(\mathrm{i}\omega)][\sum_{i=0}^{m-1} g_i^{\mathrm{T}} \overline{\Phi_i^{\alpha}(\mathrm{i}\omega)}]\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} [\sum_{i,j=0}^{m-1} f_i g_j^{\mathrm{T}} \Phi_i^{\alpha}(\mathrm{i}\omega)\overline{\Phi_j^{\alpha}(\mathrm{i}\omega)}]\mathrm{d}\omega \\ &= \sum_{i=0}^{m-1} f_i g_i^{\mathrm{T}}. \end{aligned}$$

Let

$$F = \begin{bmatrix} f_0\ f_1\ \cdots\ f_{m-1} \end{bmatrix} \in \mathbb{R}^{n\times m}$$

and

$$G = \begin{bmatrix} g_0\ g_1\ \cdots\ g_{m-1} \end{bmatrix} \in \mathbb{R}^{n\times m},$$

then the cross gramian W_X of SISO LTI system (6) is approximately given by

$$W_X \approx FG^{\mathrm{T}}, \tag{10}$$

which can be regard as the low-rank decomposition form of the W_X.

On the other hand, a square MIMO LTI system is taken into account as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{11}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{p\times n}$. Then the structures of B and C can be decomposed as

$$B = \begin{bmatrix} b_1 & b_2 & \cdots & b_p \end{bmatrix}$$

and

$$C^{\mathrm{T}} = \begin{bmatrix} c_1^{\mathrm{T}} & c_2^{\mathrm{T}} & \cdots & c_p^{\mathrm{T}} \end{bmatrix},$$

where b_i is the ith column of B and c_i is the ith row of C. From the definition of cross gramian W_X, it leads to

$$\begin{aligned} W_X &= \int_0^{+\infty} e^{At}BCe^{At}\mathrm{d}t = \int_0^{+\infty} e^{At}(\sum_{i=1}^{p} b_i c_i)e^{At}\mathrm{d}t \\ &= \sum_{i=1}^{p}\int_0^{+\infty} e^{At}b_i c_i e^{At}\mathrm{d}t. \end{aligned}$$

Let W_X^i is the cross gramian of the ith subsystem $\{A, b_i, c_i\}$, then it holds

$$W_X = \sum_{i=1}^{p} W_X^i.$$

While similar to Section II.A, W_X^i can be interpreted as cross covariance matrix of the ith subsystem's impulse response and its adjoint system's impulse response, which are trajectories,

$$\begin{aligned} \dot{x}_i(t) &= Ax(t) + b_i\delta(t) \Rightarrow x_i(t) = e^{At}b_i, \\ \dot{z}_i(t) &= A^{\mathrm{T}}z(t) + c_i^{\mathrm{T}}\delta(t) \Rightarrow z_i(t) = e^{A^{\mathrm{T}}t}c_i^{\mathrm{T}}. \end{aligned}$$

As a result, we obtain

$$W_X = \sum_{i=1}^{p} W_X^i = \sum_{i=1}^{p} \int_0^{+\infty} x_i(t) z_i^{\mathrm{T}}(t) \mathrm{d}t$$
$$= \frac{1}{2\pi} \sum_{i=1}^{p} \int_{-\infty}^{+\infty} X_i(\mathrm{i}\omega) Z_i^{\mathrm{H}}(\mathrm{i}\omega) \mathrm{d}\omega.$$

Analogously to (10), W_X^i can be given by $W_X^i \approx F_i G_i^{\mathrm{T}}$, where $F_i = \left[f_{i,0}\ f_{i,1}\ \cdots\ f_{i,m-1} \right]$ and $G_i = \left[g_{i,0}\ g_{i,1}\ \cdots\ g_{i,m-1} \right]$ that are calculated from (8) and (9) respectively for the subsystem $\{A, b_i, c_i\}$. If we choose $F = \left[F_1\ F_2\ \cdots\ F_p \right]$ and $G = \left[G_1\ G_2\ \cdots\ G_p \right]$, then we obtain

$$W_X = \sum_{i=1}^{p} W_X^i \approx \sum_{i=1}^{p} F_i G_i^{\mathrm{T}} = F G^{\mathrm{T}}. \tag{12}$$

Obviously, the cross gramian of square MIMO case can reduce to the classic cross gramian in case of an SISO system.

B. BASIC ALGORITHMS FOR SQUARE LTI SYSTEMS

For the square LTI system $\{A, B, C\}$, after obtaining its low-rank decomposition of cross gramian W_X from (10) or (12), an SVD technique is applied to $G^{\mathrm{T}}F$:

$$G^{\mathrm{T}}F = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^{\mathrm{T}} \\ V_2^{\mathrm{T}} \end{bmatrix},$$

where $\Sigma_1 = \mathrm{diag}\{\tilde{\sigma}_1, \tilde{\sigma}_2, \cdots, \tilde{\sigma}_r\}$ and $\Sigma_2 = \mathrm{diag}\{\tilde{\sigma}_{r+1}, \tilde{\sigma}_{r+2}, \cdots, \tilde{\sigma}_{r_N}\}$ with $r_N = rank(G^{\mathrm{T}}F)$. The projections S_L and S_R can be constructed as follows:

$$S_L = G U_1 \Sigma_1^{-\frac{1}{2}},\ S_R = F V_1 \Sigma_1^{-\frac{1}{2}}.$$

Obviously, it has $S_L^{\mathrm{T}} S_R = I$. As a result, the resulted ROM of system $\{A, B, C\}$ is given by

$$\begin{cases} \dot{x}_r(t) = A_r x_r(t) + B_r u(t), \\ y_r(t) = C_r x_r(t), \end{cases} \tag{13}$$

where $A_r = S_L^{\mathrm{T}} A S_R \in \mathbb{R}^{r\times r}$, $B_r = S_L^{\mathrm{T}} B \in \mathbb{R}^{r\times p}$, $C_r = C S_R \in \mathbb{R}^{p\times r}$. Since we truncate the smaller singular values of $G^{\mathrm{T}}F$, then referring to the square-root method, for a given approximate error tolerance "tol", the order r of the ROM (13) is adaptively chosen by the following approximate error indicator

$$\delta = 2 \sum_{k=r+1}^{r_N} \tilde{\sigma}_k < tol,$$

where $\{\tilde{\sigma}_1, \tilde{\sigma}_2, \cdots, \tilde{\sigma}_{r_N}\}$ is in decreasing order. The above procedure of obtaining ROM (13) is summarized as Algorithm 1.

Remark 1: In Algorithm 1, only one SVD technique is applied to an $m \times m$ dimensional matrix with $m \ll n$ and the computational cost is $\mathcal{O}(nm^2)$ except Step (1), so the total complexity is related to the numerical cost of solving linear equations (8) and (9), which strongly depends on the structure of A. Generally speaking, for a sparse matrix,

Algorithm 1 Cross Gramian Based on LRSRM for Square Systems

Input: A, B, C, tol, α, m;

1) Compute low-rank factors of F and G from (8) and (9) respectively;
2) Compute the SVD of $G^{\mathrm{T}}F$: $G^{\mathrm{T}}F = U\Sigma V^{\mathrm{T}}$, $U_r = U(:, 1:r)$, $\Sigma_r = \Sigma(1:r, 1:r)$, $V_r = V(:, 1:r)$, where r satisfies $\delta = 2 \sum_{k=r+1}^{r_N} \tilde{\sigma}_k < tol$;
3) Compute $S_L = G U_r \Sigma_r^{-\frac{1}{2}}$, $S_R = F V_r \Sigma_r^{-\frac{1}{2}}$;
4) Compute the ROM: $A_r = S_L^{\mathrm{T}} A S_R$, $B_r = S_L^{\mathrm{T}} B$, $C_r = C S_R$.

Output: A_r, B_r, C_r, r.

the computational cost is $\mathcal{O}(c^2 mn)$, where c is the average number of nonzero elements per row/column of such matrix [31]. On the other hand, for relatively large models, some fast iterative methods like generalized minimal residual (GMRES) [32] and biconjugate gradients stabilized (BI-CGSTAB) method [33], are alternative ways to obtain the solutions to (8) and (9).

Although Algorithm 1 is similar to exact BT-based methods, however, a main drawback is that it is a Petrov-Galerkin projection ($S_L \neq S_R$), which may lead to numerical errors and instabilities. In order to alleviate such shortcoming, we combine the idea from DSPMR method to modify our algorithm.

In DSPMR method [12], the order r of the ROM may be still larger than a desired number even though r is much smaller than n since $r = rank([F\ G])$. In order to reach a desired order r that is arbitrary small ($r \ll n$), we need a modification on DSPMR method. At first, we compute the SVDs of

$$\frac{1}{\|F\|_F} F = U_F \Sigma_F V_F^{\mathrm{T}}$$

and

$$\frac{1}{\|G\|_F} G = U_G \Sigma_G V_G^{\mathrm{T}},$$

where $\|\cdot\|_F$ denotes the Frobenius norm. The scalar factors $\frac{1}{\|F\|_F}$ and $\frac{1}{\|G\|_F}$ are the so-called weighting factors which aim to equilibrate the influence of controllability and observability that may be skewed, i.e., due to different scaling of B and C. Then we choose the first k columns of U_F and U_G to construct a matrix and apply the 'economy size' SVD to it

$$\left[U_F(:, 1:k)\ U_G(:, 1:k) \right] = U \Sigma V^{\mathrm{T}}.$$

Finally, we choose U as the projection, then the corresponding ROM is given by

$$\begin{cases} \dot{x}_r(t) = A_r x_r(t) + B_r u(t), \\ y_r(t) = C_r x_r(t), \end{cases} \tag{14}$$

where $A_r = U^{\mathrm{T}} A U$, $B_r = U^{\mathrm{T}} B$ and $C_r = CU$. In general, k is often chosen as $k \leqslant \min\{rank(F), rank(G)\}$. The above procedure of MOR is summarized as Algorithm 2.

Algorithm 2 Cross Gramian Based on DSPMR for Square Systems

Input: A, B, C, k, α, m;

1) Compute low-rank factors of F and G from (8) and (9) respectively;
2) Compute the SVDs:
$$\frac{1}{\|F\|_F}F = U_F\Sigma_F V_F^{\mathrm{T}},\ \frac{1}{\|G\|_F}G = U_G\Sigma_G V_G^{\mathrm{T}};$$
3) Compute the 'economy size' SVD:
$$\left[\, U_F(:, 1:k)\ U_G(:, 1:k)\,\right] = U\Sigma V^{\mathrm{T}};$$
4) Compute the ROM: $A_r = U^{\mathrm{T}}AU$, $B_r = U^{\mathrm{T}}B$, $C_r = CU$, $r = rank(U)$.

Output: A_r, B_r, C_r, r.

Remark 2: In the above algorithms, the choice of the parameter α is important since it directly affects the accuracy of the approximation of the ROM. Such topic for linear systems has been discussed in [27], [34], where more details about how to choose the parameter α can be found. For example, a good choice of α is $4\mathrm{B} \leqslant \alpha \leqslant \pi^2\mathrm{B}$, where B is the bandwidth measured in Hz of the original system (6) or (11). What's more, compared with the moment matching approach, since α is a real number, the matrices in the reduction remain real during projection, which makes it suitable for circuit synthesis.

C. MAIN PROPERTIES

One hand, from (4), according to Parseval's theorem, the controllability gramian W_C and observability gramian W_O of the square system $\{A, B, C\}$ satisfy

$$\begin{aligned} W_C &= \int_0^{+\infty} x(t)x^{\mathrm{T}}(t)\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\mathrm{i}\omega)X(\mathrm{i}\omega)^{\mathrm{H}}\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} (\mathrm{i}\omega I - A)^{-1}BB^{\mathrm{T}}(-\mathrm{i}\omega I - A^{\mathrm{T}})^{-1}\mathrm{d}\omega \end{aligned}$$

and

$$\begin{aligned} W_O &= \int_0^{+\infty} z(t)z^{\mathrm{T}}(t)\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Z(\mathrm{i}\omega)Z(\mathrm{i}\omega)^{\mathrm{H}}\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} (\mathrm{i}\omega I - A^{\mathrm{T}})^{-1}C^{\mathrm{T}}C(-\mathrm{i}\omega I - A)^{-1}\mathrm{d}\omega. \end{aligned}$$

According to the structures of B and C, we have

$$W_C = \sum_{i=1}^{p} W_C^i,\ W_O = \sum_{i=1}^{p} W_O^i,$$

where W_C^i and W_O^i are controllability and observability gramians of SISO subsystem $\{A, b_i, c_i\}$ respectively.

As a result, analogously to (12), it holds

$$W_C = \sum_{i=1}^{p} W_C^i \approx \sum_{i=1}^{p} F_iF_i^{\mathrm{T}} = FF^{\mathrm{T}} \tag{15}$$

and

$$W_O = \sum_{i=1}^{p} W_O^i \approx \sum_{i=1}^{p} G_iG_i^{\mathrm{T}} = GG^{\mathrm{T}}, \tag{16}$$

where F_i and G_i are calculated from (8) and (9) for the subsystem $\{A, b_i, c_i\}$, while $F = \left[\, F_1\ F_2\ \cdots\ F_p\,\right]$ and $G = \left[\, G_1\ G_2\ \cdots\ G_p\,\right]$. As a result, the approximation (15) and (16) can be regarded as the low-rank decomposition of W_C and W_O either for square case or SISO case. Then comparing with the LRSRM algorithm in [12], we have the following property.

Property 1: The ROM (13) obtained by Algorithm 1 is the same as the ROM obtained by LRSRM [12] that is based on approximate controllability and observability gramians (15) and (16).

Remark 3: The equivalence of the ROMs obtained by BT based on exact gramians only holds for symmetric systems, however, fortunately, we extend such equivalence relation based on approximate gramians to the square case successfully under some certain conditions.

On the other hand, from what was mentioned in [35], if $A + A^{\mathrm{T}} < 0$ and $E > 0$, the LTI system

$$E\dot{x} = Ax(t) + Bu(t),\ y(t) = Cx(t),$$

is stable. As a result, for the stability preservation, we have the following property.

Property 2: For the square LTI system (11), if $A + A^{\mathrm{T}} < 0$, the ROM $\{A_r, B_r, C_r\}$ obtained by Algortihm 2 is stable.

It should be pointed out that a stable LTI system may not satisfy $A + A^{\mathrm{T}} < 0$. As a result, two strategies are alternative to retain the stability. One hand, according to [34], if A is continuous-stable, by solving the following Lyapunov equation

$$A^{\mathrm{T}}Q + QA + I = 0$$

and applying the Cholesky decomposition to Q: $Q = LL^{\mathrm{T}}$, an equivalent system $\{TAT^{-1}, TB, CT^{-1}\}$ of $\{A, B, C\}$ is produced with $T = L^{\mathrm{T}}$, which makes $\hat{A} + \hat{A}^{\mathrm{T}} < 0$ with $\hat{A} = TAT^{-1}$. Using Algorithm 2 to system $\{TAT^{-1}, TB, CT^{-1}\}$, it leads to a stable ROM.

On the other hand, according to [36], for a stable matrix A, if it exists a positive definite matrix J such that

$$JA + A^{\mathrm{T}}J < 0,$$

and $W^{\mathrm{T}} = (U^{\mathrm{T}}JU)^{-1}U^{\mathrm{T}}J$, where U is calculated from Algorithm 2, then the ROM $\{W^{\mathrm{T}}AU, W^{\mathrm{T}}B, CU\}$ is stable as well.

D. NON-SQUARE LTI SYSTEMS

Finally, we consider the non-square LTI system as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{17}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{q\times n}$, $p \neq q$. Obviously, the cross gramian of such system can not be

calculated from Sylvester equation (2) since the dimensions of B and C are incompatible. So at first, we partition B and C as

$$B = \begin{bmatrix} b_1 & b_2 & \cdots & b_p \end{bmatrix},\ b_i \in \mathbb{R}^{n\times 1},$$

and

$$C^{\mathrm{T}} = \begin{bmatrix} c_1^{\mathrm{T}} & c_2^{\mathrm{T}} & \cdots & c_q^{\mathrm{T}} \end{bmatrix},\ c_j \in \mathbb{R}^{1\times n}.$$

For the SISO subsystem $\{A, b_i, c_j\}$, the related gramians are given by

$$W_C^i = \int_0^{+\infty} e^{At} b_i b_i^{\mathrm{T}} e^{A^{\mathrm{T}}t}\mathrm{d}t,$$
$$W_O^j = \int_0^{+\infty} e^{A^{\mathrm{T}}t} c_j^{\mathrm{T}} c_j e^{At}\mathrm{d}t,$$
$$W_X^{ij} = \int_0^{+\infty} e^{At} b_i c_j e^{At}\mathrm{d}t.$$

It is clear that $W_C = \sum_{i=1}^{p} W_C^i$ and $W_O = \sum_{j=1}^{q} W_O^j$. As a result, from the core property (3), it produces

$$W_C W_O = (\sum_{i=1}^{p} W_C^i)(\sum_{j=1}^{q} W_O^j) = \sum_{i=1}^{p}\sum_{j=1}^{q}(W_X^{ij})^2.$$

For square systems, its cross gramian W_X satisfies

$$W_X = \sum_{i=1}^{p} W_X^{ii}. \tag{18}$$

According to [19], the cross gramian of non-square system (17) is defined as the sum of cross gramians of the pq SISO subsystems, i.e.,

$$W_X = \sum_{i=1}^{p}\sum_{j=1}^{q} W_X^{ij}.$$

Obviously, such cross gramian does not retain the property (3) and it should be emphasized that the cross gramian of non-square case does not reduce to the classic cross gramian in case of a square MIMO system (compare (18)).

What's more, we can obtain

$$\begin{aligned} W_X &= \sum_{i=1}^{p}\sum_{j=1}^{q} W_X^{ij} = \sum_{i=1}^{p}\sum_{j=1}^{q}\int_0^{+\infty} e^{At} b_i c_j e^{At}\mathrm{d}t \\ &= \int_0^{+\infty} e^{At}(\sum_{i=1}^{p}\sum_{j=1}^{q} b_i c_j) e^{At}\mathrm{d}t \\ &= \int_0^{+\infty} e^{At}(\sum_{i=1}^{p} b_i)(\sum_{j=1}^{q} c_j) e^{At}\mathrm{d}t. \end{aligned}$$

It is seen that the cross grmaian W_X of system (17) equals to the cross gramian of the SISO system $\{A, \sum_{i=1}^{p} b_i, \sum_{j=1}^{q} c_j\}$. As a result, analogously to Algorithm 1, we have the following algorithm for non-square case. (see Algorithm 3)

Algorithm 3 Cross Gramian Based on LRSRM for Non-Square Systems

Input: A, B, C, tol, α, m;

1) Compute low-rank factors of F and G of the SISO system $\{A, \sum_{i=1}^{p} B(:,i), \sum_{j=1}^{q} C(j,:)\}$ from (8) and (9);
2) Compute the SVD of $G^{\mathrm{T}}F$: $G^{\mathrm{T}}F = U\Sigma V^{\mathrm{T}}$, $U_r = U(:, 1:r)$, $\Sigma_r = \Sigma(1:r, 1:r)$, $V_r = V(:, 1:r)$, where r satisfies $\delta = 2\sum_{k=r+1}^{r_N}\tilde{\sigma}_k < tol$;
3) Compute $S_L = GU_r\Sigma_r^{-\frac{1}{2}}$, $S_R = FV_r\Sigma_r^{-\frac{1}{2}}$;
4) Compute the ROM: $A_r = S_L^{\mathrm{T}}AS_R$, $B_r = S_L^{\mathrm{T}}B$, $C_r = CS_R$.

Output: A_r, B_r, C_r, r.

Algorithm 4 Cross Gramian Based on DSPMR for Non-Square Systems

Input: A, B, C, k, α, m;

1) Compute low-rank factors of F and G of the SISO system $\{A, \sum_{i=1}^{p} B(:,i), \sum_{j=1}^{q} C(j,:)\}$ from (8) and (9);
2) Compute the SVDs:
$$\frac{1}{\|F\|_F}F = U_F\Sigma_F V_F^{\mathrm{T}},\ \frac{1}{\|G\|_F}G = U_G\Sigma_G V_G^{\mathrm{T}};$$
3) Compute the 'economy size' SVD:
$$\begin{bmatrix} U_F(:,1:k) & U_G(:,1:k) \end{bmatrix} = U\Sigma V^{\mathrm{T}};$$
4) Compute the ROM: $A_r = U^{\mathrm{T}}AU$, $B_r = U^{\mathrm{T}}B$, $C_r = CU$, $r = rank(U)$.

Output: A_r, B_r, C_r, r.

Meanwhile, analogously to Algorithm 2, we also have Algorithm 4 based on DSPMR for non-square systems.

Since the non-square system (17) and its associated "average" system have the same matrix A, which is related to the stability of the system, thus the results about stability preservation in Section III.C can be extended to non-square case based on Algorithm 4 successfully.

Remark 4: Comparing with the methods for non-square systems in [18] and [20], Algorithm 3 and Algorithm 4 may be more efficient, especially when there are more input and output nodes.

IV. NUMERICAL SIMULATIONS

In this section, three numerical examples are provided to demonstrate the effectiveness of our proposed algorithms in different cases. We make a comparison with the classical BT method [6] and cross-gramian-based dominant subspaces (CG-DS) method [26]. All examples are performed in MATLAB(R2016b) and we use MATLAB's ode15s to solve differential equations in this paper.

TABLE 1. Computational cost and stability of ROMs in Example 1.

Systems	Time (second)	A_r	$A_r + A_r^{\mathrm{T}}$
S-1	0.5444	stable	not negative definite
S-2	0.5198	stable	negative definite
S-3	0.7252	stable	negative definite
S-4	1.1323	stable	negative definite

Example 1: The first example is the (full order model) FOM benchmark that comes from [12]. This is an SISO LTI system (with $E = I$) of the structure:

$$\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t),$$

with order $n = 1006$ and the component matrices are given by the following forms:

$$A_1 = \begin{bmatrix} -1 & 100 \\ -100 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} -1 & 200 \\ -200 & -1 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -1 & 400 \\ -400 & -1 \end{bmatrix},$$
$$A_4 = \begin{bmatrix} -1 & & & \\ & -2 & & \\ & & \ddots & \\ & & & -1000 \end{bmatrix},$$
$$A = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & A_3 & \\ & & & A_4 \end{bmatrix},\ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 10 \\ \vdots \\ 10 \end{bmatrix} \in \mathbb{R}^6,\ B_2 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^{1000},\ C = B^{\mathrm{T}}.$$

Numerical results show that A is stable and $A + A^{\mathrm{T}} < 0$.

We calculate four ROMs: S-1is obtained by Algorithm 1, S-2 is obtained by Algorithm 2, S-3 is obtained by the CG-DS method and S-4 is obtained by the classical BT method. The input is $u(t) = e^{-t}$. Here the parameters are $\alpha = 100$, $tol = 10^{-6}$ and $m = 20$.

Fig. 1 shows the transient responses of four ROMs as well as relative errors, while Fig. 2 shows the comparison about HSVs of different ROMs with the ones of original system. Table 1 shows the computational cost of calculating different ROMs and the stability.

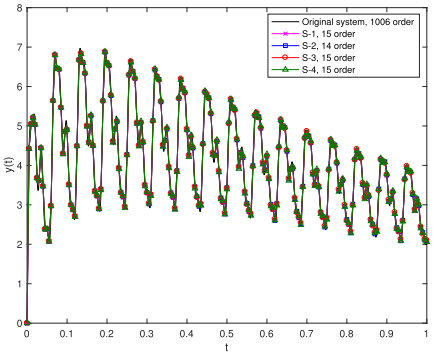
From Fig. 1, Fig. 2 and Table 1, it is seen that: (i) all of the four ROMs obtained by these methods have a good performance in approximating the output behavior of original system; (ii) our proposed algorithms provide a competitive approximation with BT and are slightly better than CG-DS; (iii) the corresponding computational time consumed by our proposed algorithms are a little less than the other methods; (iv) these ROMs well approximately match the first several HSVs of the original system.

Example 2: The second example is a model of compact disc (CD) player [37]. The model describes the dynamics between the lens actuator and the radial arm position of a portable CD player. This square system is described by the following model that has 120 states with two inputs and two outputs

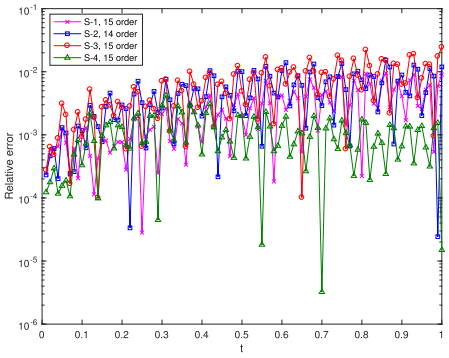
$$\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t).$$

Numerical results show that A is stable and $A + A^{\mathrm{T}} < 0$.

We calculate four ROMs: S-1 is obtained by Algorithm 1, S-2 is obtained by Algorithm 2, S-3 is obtained by the CG-DS method and S-4 is obtained by the classical BT method. The input is $u(t) = \begin{bmatrix} e^{-t} & 0 \end{bmatrix}^{\mathrm{T}}$. Here we choose $\alpha = 50$, $tol = 10^{-6}$ and $m = 16$. Numerical results show that the first component $y_1(t)$ of $y(t)$ is well approximated when the order



(a) Transient responses of $y(t)$.



(b) Relative errors.

FIGURE 1. Results of responses in Example 1.

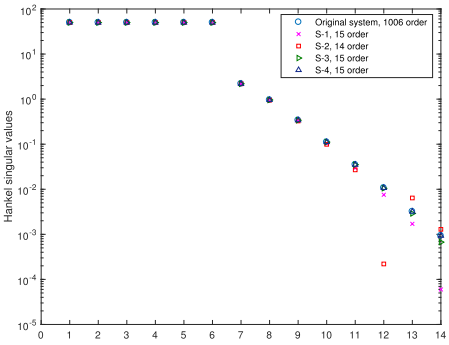
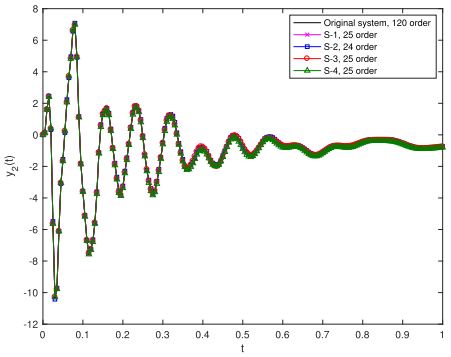
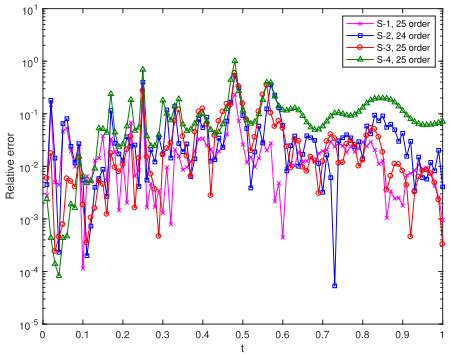


FIGURE 2. Results of HSVs in Example 1.



(a) Transient responses of $y_2(t)$.



(b) Relative errors.

FIGURE 3. Results of responses in Example 2.

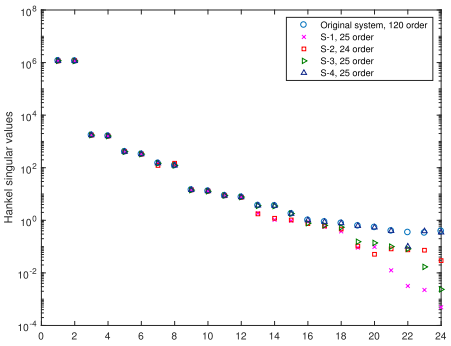


FIGURE 4. Results of HSVs in Example 2.

decreases to 10. Thus we only compare the second component $y_2(t)$ of different ROMs.

In Fig. 3 the transient responses of four ROMs are compared with the original system as well as relative errors, and in Fig. 4 the HSVs of different ROMs are compared with original system. In Table 2, the computational cost of calculating different ROMs as well as the stability is compared.

According to the results in Fig. 3, Fig. 4 and Table 2, it is seen that: (i) all the ROMs can approximate the transient response of original system very well; (ii) our proposed algorithms have a superior performance to BT, especially Algorithm 1, and are competitive with CG-DS; (iii) our proposed

TABLE 2. Computational cost and stability of ROMs in Example 2.

Systems	Time (second)	A_r	$A_r + A_r^{\mathrm{T}}$
S-1	0.2087	stable	not negative definite
S-2	0.1863	stable	negative definite
S-3	0.2139	stable	negative definite
S-4	0.3073	stable	negative definite

TABLE 3. Computational cost and stability of ROMs in Example 3.

Systems	Time (second)	A_r	$A_r + A_r^{\mathrm{T}}$
S-1	3.2786	stable	not negative definite
S-2	3.2860	stable	negative definite
S-3	2.4836	stable	negative definite
S-4	4.6817	stable	negative definite

algorithms need a little less computational cost of calculating ROMs than those of CG-DS and BT; (iv) these ROMs can also closely match the first several HSVs of the original system.

Example 3: At last, we consider a 2D model for a tunable optical filter, which is a single-input-five-output system with the order $n = 1668$ [38]. The non-square system realization is described as

$$E\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t),$$

where E is diagonal with $\det(E) \neq 0$. Then the state equation can be converted to

$$\dot{x}(t) = \hat{A}x(t) + \hat{B}u(t),$$

where $\hat{A} = E^{-1}A$ and $\hat{B} = E^{-1}B$. Numerical result shows that such system is stable but $\hat{A}+\hat{A}^{\mathrm{T}}$ is not a negative definite matrix. Solving the Lyapunov equation as follows:

$$A^{\mathrm{T}}Q + QA + I = 0$$

and applying the Cholesky decomposition to Q : $Q = LL^{\mathrm{T}}$. Let $T = L^{\mathrm{T}}$, then an equivalent system is produced

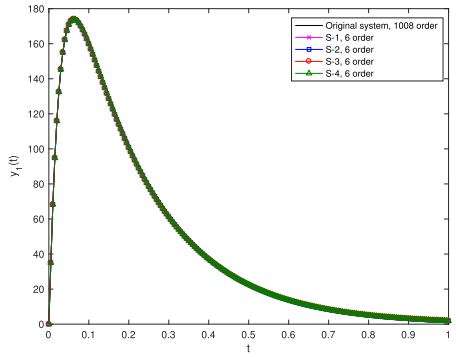
$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t),\ y(t) = \tilde{C}\tilde{x}(t),$$

where $\tilde{A} = T\hat{A}T^{-1}$, $\tilde{B} = T\hat{B}$ and $\tilde{C} = CT^{-1}$. In this case, we can see that $\tilde{A} + \tilde{A}^{\mathrm{T}} < 0$.

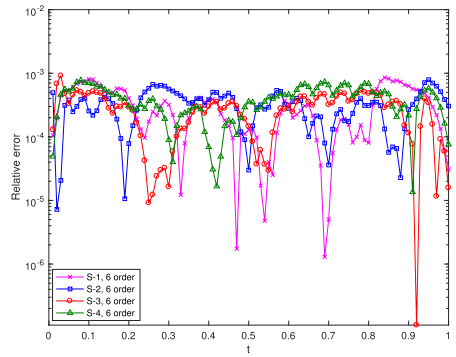
We calculate four ROMs: S-1 is obtained by Algorithm 3, S-2 is obtained by Algorithm 4, S-3 is obtained by the CG-DS method and S-4 is obtained by the classical BT method. The input is $u(t) = e^{-5t}$. The parameters are $\alpha = 1045$, $tol = 10^{-6}$ and $m = 10$. Numerical experiment result shows that all the outputs are almost the same, so only the first component $y_1(t)$ of different systems are compared.

The transient responses of ROMs obtained by four methods are compared with the original system in Fig. 5, as well as the corresponding relative errors. In Fig. 6, the HSVs of four ROMs are compared with the ones of original system. The computational cost of calculating four ROMs and the corresponding stability are shown in Table 3.

From the results in Fig. 5, Fig. 6 and Table 3, it is seen that: (i) all the ROMs have better approximations to the transient response of original system; (ii) compared with CG-DS and BT, our proposed algorithms for non-square case provide a slightly better performance; (iii) Although CG-DS needs the least time to calculate the ROM, our algorithms are more



(a) Transient responses of $y_1(t)$.



(b) Relative errors.

FIGURE 5. **Results of responses in Example 3.**

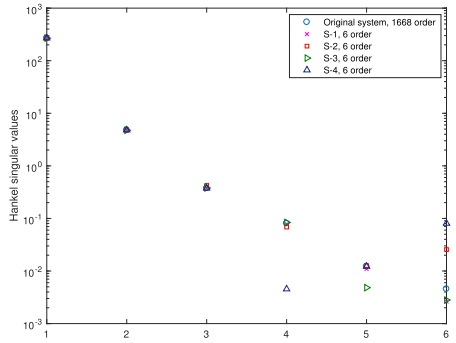


FIGURE 6. **Results of HSVs in Example 3.**

efficient than BT; (iv) the ROMs obtained by Algorithm 3 and Algorithm 4 can more or less match the first several HSVs of the original system, which implies that the definition of cross gramian for non-square systems in [19] is reasonable.

In addition, it should be pointed out that even though the ROM S-1 is stable in these three examples, the stability preservation is not guaranteed in general either for Algorithm 1 or Algorithm 3.

V. CONCLUSION

In this paper, we first discuss the calculation of approximate cross gramian via Laguerre series. Then a series of MOR algorithms based on the low-rank decomposition of approximate cross gramians for LTI systems are proposed. The cross gramian is approximately calculated from the expansion coefficients of Laguerre functions that are obtained by a recurrence formula instead of solving the Sylvester equation directly, which makes it more flexible and computationally efficient. What's more, we modify our algorithm to alleviate the shortcoming, which may lead to an unstable ROM for the original stable systems. Meanwhile, the ROM retains the stability under some certain conditions and is equivalent to the one obtained by LRSRM based on approximate controllability and observability gramians in square case. In addition, our algorithms are extended to non-square LTI systems successfully, which also retain the stability. Finally, the results of numerical simulations illustrate the effectiveness of our proposed methods. In order to extend the applications of our algorithms, the topic on calculating cross gramians for nonlinear systems will be taken into account in our future work.

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