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# Two-Dimensional Separable Gridless Direction-of-Arrival Estimation Based on Finite Rate of Innovation

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**ABSTRACT** In order to solve the problem that the gridless DOA estimation algorithms based on generalized finite rate of innovation (FRI) signal reconstruction model are not suitable for two-dimensional DOA estimation using planar array, a separable gridless DOA estimation algorithm exploiting bi-orthogonal sparse linear array (BSLA) structure is proposed in this article, which is called 2D-SGFRI. The 2D-SGFRI algorithm firstly recovers the covariance data of the virtual array formed by BSLA through the matrix completion method, so as to obtain the complete covariance data vectors about two independent parameters respectively. Next, since the covariance data vector satisfies the constraints of annihilation filter equations, the generalized FRI signal reconstruction model can be utilized to retrieve DOA from the covariance data vector. Compared with the existing DOA estimation algorithms based on generalized FRI signal reconstruction model, the 2D-SGFRI algorithm can be can be effectively applied to two-dimensional DOA estimation, and can obtain stable estimation results. At the same time, due to the reduction of the dimension of positive semidefinite matrix, the 2D-SGFRI algorithm can significantly reduce the computational complexity compared with the two-dimensional DOA estimation algorithms based on atomic norm minimization (ANM). A series of simulation experiments are shown to verify the effectiveness and superiority of 2D-SGFRI algorithm.

**INDEX TERMS** Two-dimensional DOA estimation, bi-orthogonal sparse linear array (BSLA), finite rate of innovation (FRI), matrix completion, annihilation filter.

## I. INTRODUCTION

As the key and difficult issue in DOA estimation, two-dimensional DOA estimation has been widely concerned and studied [1], [2]. The subspace-based one-dimensional DOA estimation algorithms can be effectively extended to two-dimensional cases, such as 2D-MUSIC [3], 2D-ESPRIT [4] and matrix enhancement matrix pencil (MEMP) [5]. With the development of compressed sensing (CS) and sparse signal reconstruction theory [6]–[8], a series of DOA estimation algorithms based on CS have emerged in recent years [9], [10]. Compared with the subspace-based DOA estimation algorithms, CS-based DOA estimation algorithms have better estimation performance under some demanding conditions, such as less snapshots, low signal to noise ratio (SNR) and coherent signals.

However, the deviation between the DOAs of real incident sources and the pre-set spatial angle discrete grid will substantially reduce the performance of CS-based DOA estimation algorithms, which is called grid mismatch problem [11]. In addition, in the two-dimensional DOA estimation problem, the dimensions of the over complete dictionary formed by the spatial angle discrete grid and correlation between adjacent atoms will increase immensely, which lead to the serious degradation of the estimation performance of CS-based DOA estimation algorithms.

In order to overcome the grid mismatch problem, the gridless DOA estimation algorithms have been proposed by introducing the concept of atomic norm of infinite dimensional atomic set in continuous domain [12]–[15]. These algorithms

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transform the DOA estimation problem into a convex optimization problem by minimizing the atomic norm of the observation vector, which is called atomic norm minimization (ANM) problem. Due to the properties of a series of semidefinite matrices, the ANM problem has been proved to be equivalent to a semidefinite programming (SDP) process, which can be implemented effectively through many state-of-the-art convex optimization solvers. Furthermore, the ANM-based gridless DOA estimation algorithm can be well extended to the case of two-dimensional because it does not need the discretization process of two-dimensional angle space. The Vandermonde decomposition theory of multi-dimensional Toeplitz matrices [16] provides a basis for the application of ANM in two-dimensional and even multi-dimensional estimation problems, which is first realized in [17]. A decoupled ANM model is established to reduce the dimension of positive semidefinite matrix, and successfully applied in the field of spectrum estimation, which can be regarded as the DOA estimation problem in the case of single snapshot [18]. In [19], the decoupled ANM model is extended to the case of multi snapshots, and combined with coprime planar array (CPPA), which can effectively increase the degree of freedom (DOF) and improve the estimation accuracy as a result of expanding the array aperture.

The performance of ANM-based gridless DOA estimation algorithm completely depends on SDP and Vandermonde decomposition of toeplitz matrix. However, in two-dimensional DOA estimation problems, especially when CPPA is used, the dimensions of above two processes will be very large, which will lead to a rapid increase in the complexity of the algorithm. Alternatively, the gridless DOA estimation algorithm based on finite rate of innovation (FRI) does not need to deal with SDP process [20]–[22]. The FRI-based gridless DOA estimation algorithm uses Bessel function expansion formula to project a plane array of arbitrary geometry onto a virtual continuous uniform linear array, so as to obtain the signal model satisfying the annihilation equations [23]. The FRIDA-V algorithm proposed in [24] directly processes the multi snapshots data received by the planar array, and no longer depends on the covariance data after vectorization, making the algorithm suitable for coherent signals. In [25], the generalized FRI signal reconstruction model is extended to two-dimensional and even higher dimensional cases, but it is not specifically applied to the problem of two-dimensional DOA estimation.

Generally speaking, for the generalized FRI signal reconstruction model, the most urgent problem is that there is no algorithm that can be effectively applied in the field of two-dimensional DOA estimation using planar antenna array. Therefore, a FRI-based two-dimensional separable gridless DOA estimation algorithm using bi-orthogonal sparse linear array (BSLA) is proposed in this article, which is called 2D-SGFRI. Firstly, the 2D-SGFRI algorithm recovers the complete covariance data vectors of the received signal of two continuous virtual uniform linear arrays in two different directions using the matrix completion theory, which are both satisfied the constraints of annihilation equations about two independent parameters respectively. Then, a simplified generalized FRI signal reconstruction model can be established based on the covariance data vector mentioned above. Finally, the final DOA estimation results are obtained by the pairing strategy in [22].

Compared with other two-dimensional gridless DOA estimation algorithms, 2D-SGFRI algorithm has the following two advantages:

(a) Extension in application scenarios: Due to the introduction of BSLA, 2D-SGFRI algorithm transforms a two-dimensional DOA estimation problem into two independent one-dimensional DOA estimation problems, which effectively solves the problem that the gridless DOA estimation algorithm based on generalized FRI signal reconstruction model cannot be applied in the field of two-dimensional DOA estimation using planar antenna array;

(b) Reduction in algorithm complexity: Compared with the two-dimensional gridless DOA estimation algorithm based on ANM, the dimension of SDP problem in 2D-SGFRI is lower, which saves the computational cost. In addition, the 2D-SGFRI algorithm replaces Vandermonder decomposition of two-dimensional Toeplitz matrix with two one-dimensional FRI signal reconstruction processes, which can also reduce the complexity.

The rest of this article is organized as follows. In Section 2, the principle and process of 2D-SGFRI algorithm are described in detail, including two-dimensional signal model of DOA estimation based on BSLA, covariance data recovery based on matrix completion theory and simplified separate two-dimensional generalized FRI signal reconstruction model. Some simulation results and analysis are presented in Section 3 before the paper is concluded in Section 4.

The notations used in this article are introduced as follows. The lower (upper) case bold font represents the vector (matrix). The superscripts $(\bullet)^{\mathrm{T}}$ and $(\bullet)^{\mathrm{H}}$ denote transpose and conjugate transpose of $\bullet$. $\mathbb{Z}_+$, $\mathbb{R}_+$ and $\mathbb{C}$ is the set of non-negative integer, non-negative real numbers and complex numbers, respectively. $\mathbb{E}\{\bullet\}$ is the expectation of $\bullet$. $\langle\bullet\rangle_{\mathrm{i}}$ is the i-th element in a vector. $\langle\bullet\rangle_{\mathrm{i,j}}$ represents the element at i-th row and j-th column of a matrix. $|\bullet|$ denotes the cardinality of a set. $\|\bullet\|_2$ is the $\ell_2$ norm of the vector $\bullet$. $\|\bullet\|_{\mathrm{F}}$ and $\|\bullet\|_*$ stand for the matrix norm and kernel norm of a matrix. The symbol toep$(\bullet)$ is used to represent a matrix that satisfies both Hermitian and Toeplitz structures with the vector $\bullet$ as the first row. The expression $\mathbf{A} \succeq 0$ means that matrix $\mathbf{A}$ is an SDP matrix.

## II. 2D-SGFRI ALGORITHM

In this section, we first introduce the two-dimensional signal model of DOA estimation based on BSLA, and then the matrix completion theory is exploited to recover the complete covariance data of received signal satisfying the annihilation filtering equation. Finally, the DOA information can be revived from the complete covariance data of received signal

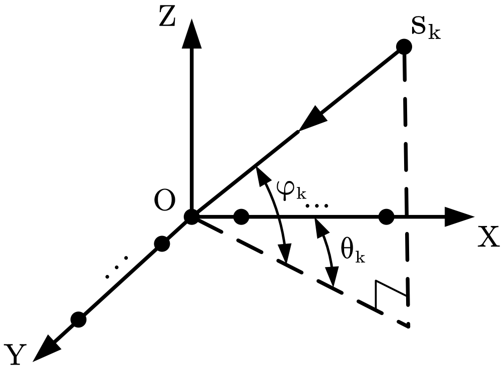


**FIGURE 1.** The spatial angles of the *k*-th incident source.

in terms of the simplified generalized FRI signal reconstruction model.

### A. TWO-DIMENSIONAL SIGNAL MODEL OF DOA ESTIMATION BASED ON BSLA

The BSLA exploited in this article is a special planar array geometry, which consists of two orthogonal sparse linear arrays. Before introducing the signal model, the spatial angles of the incident sources are defined, as shown in Figure 1. $\theta_k \in [0°, 360°]$ and $\varphi_k \in [0°, 90°]$ represent the azimuth and elevation angle of the k-th incident source, respectively.

Consider a BSLA composed of two orthogonal sparse linear arrays. One of them contains M elements regarded as X-axis, and the other contains N elements, which is regarded as Y-axis. The position sets of the antennas on two axes are recorded as $\mathbb{D}_x = \{dx_1, \ldots, dx_M\} \lambda/2$ and $\mathbb{D}_y = \{dy_1, \ldots, dy_N\} \lambda/2$, respectively. $\lambda$ is the wavelength of the incident sources, $dx_i, dy_j \in \mathbb{Z}_+$, $1 \leqslant i \leqslant M$ and $1 \leqslant j \leqslant N$. For the convenience of later description, these two sets are considered to be arranged in ascending.

Assume that there are K far-field, narrow-band and uncorrelated signals in the space impinging on the BSLA from directions $\theta_k$ and $\varphi_k$ with $k = 1, \ldots, K$.

The single snapshot model of the BSLA received signal in two axes $\mathbf{x}(t) \in \mathbb{C}^M$ and $\mathbf{y}(t) \in \mathbb{C}^N$ can be expressed as

$$\mathbf{x}(t) = \sum_{k=1}^{K} \mathbf{a}_x(\theta_k, \varphi_k) s_k(t) + \mathbf{n}_x(t),$$
$$\mathbf{y}(t) = \sum_{k=1}^{K} \mathbf{a}_y(\theta_k, \varphi_k) s_k(t) + \mathbf{n}_y(t), \quad (1)$$

respectively. $s_k(t) \in \mathbb{C}$ represents the complex amplitude of the k-th signal at t-th snapshot. $\mathbf{n}_x(t) \in \mathbb{C}^M$ and $\mathbf{n}_y(t) \in \mathbb{C}^M$ are zero mean additive Gaussian white noise with the same variance $\sigma^2$, $\sigma \in \mathbb{R}_+$. The steering vectors of the k-th signal along with the X-axis $\mathbf{a}_x(\alpha_k) \in \mathbb{C}^M$ and Y-axis $\mathbf{a}_y(\beta_k) \in \mathbb{C}^N$ are

$$\mathbf{a}_x(\alpha_k) = \left[e^{-j\pi\alpha_k dx_1}, \ldots, e^{-j\pi\alpha_k dx_M}\right]^T,$$
$$\mathbf{a}_y(\beta_k) = \left[e^{-j\pi\beta_k dy_1}, \ldots, e^{-j\pi\beta_k dy_N}\right]^T, \quad (2)$$

where $\alpha_k = \cos(\theta_k)\cos(\varphi_k)$ and $\beta_k = \sin(\theta_k)\cos(\varphi_k)$.

The covariance matrices of the received signal of two orthogonal sparse linear arrays $\mathbf{R}_x \in \mathbb{C}^{M\times M}$ and $\mathbf{R}_y \in \mathbb{C}^{N\times N}$ are:

$$\mathbf{R}_x = \mathbb{E}\left\{\mathbf{x}(t)\mathbf{x}^H(t)\right\} = \sum_{k=1}^{K} p_k \mathbf{a}_x(\alpha_k)\mathbf{a}_x^H(\alpha_k) + \sigma^2\mathbf{I},$$
$$\mathbf{R}_y = \mathbb{E}\left\{\mathbf{y}(t)\mathbf{y}^H(t)\right\} = \sum_{k=1}^{K} p_k \mathbf{a}_y(\beta_k)\mathbf{a}_y^H(\beta_k) + \sigma^2\mathbf{I}, \quad (3)$$

where $p_k \in \mathbb{R}_+$ represents the power of the k-th signal.

In fact, due to the limited number of snapshots, the exact covariance matrices of the BSLA received signal as shown in formula (3) cannot be obtained. It is generally approximated as follows

$$\widehat{\mathbf{R}_x} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}(t)\mathbf{x}^H(t),$$
$$\widehat{\mathbf{R}_y} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{y}(t)\mathbf{y}^H(t), \quad (4)$$

where $T \in \mathbb{Z}_+$ is the total number of snapshots.

Through the above analysis, it is not difficult to see that the two-dimensional DOA estimation model can be transformed into two independent one-dimensional DOA estimation models by using the special geometric structure of BSLA, which provides support for the promotion of one-dimensional DOA estimation algorithm in two-dimensional estimation problems.

### B. COVARIANCE DATA RECOVERY BASED ON MATRIX COMPLETION THEORY

For the sparse linear array, there are missing parts in the covariance data, which leads to the covariance matrix does not satisfy the Toeplitz structure. As a common method to recover complete covariance data, matrix completion theory is generally used to fill the discontinuous part of virtual array formed by sparse linear array represented by coprime array. Before the completion of the matrix, we need to process the data in the covariance matrix. Obviously, there are a large number of redundancy data in $\mathbf{R}_x$ and $\mathbf{R}_y$, which is not helpful to improve the accuracy of subsequent DOA estimation, but will increase the burden of operation. Therefore, the covariance data need to be removed redundant and rearranged in a certain order. It is worth mentioning that the next analysis takes $\mathbf{R}_x$ as an example, which is completely consistent with the processing of the data in $\mathbf{R}_y$.

First of all, the specific form of data in $\mathbf{R}_x$ is

$$\langle\mathbf{R}_x\rangle_{i,j} = \sum_{k=1}^{K} p_k e^{-j\pi(dx_i - dx_j)\alpha_k} + \sigma^2\delta_{ij}, \quad (5)$$

where $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ in other cases. Based on the above analysis, it can be seen that the values of different elements in $\mathbf{R}_x$ only depend on the spacing between two antennas $dx_i - dx_j$, that is, the array element position of the virtual array. Therefore, the sets of array element positions along with X-axis in the continuous virtual array with the largest aperture that can be formed by BSLA is $\widetilde{\mathbb{D}}_x = \{-dx_M, \ldots, dx_M\}$. It is worth noting that the array element position here omits the half wavelength of the signal $\lambda/2$, that is to say, the location of the m-th virtual element in X-axis should be $(-dx_M + m - 1)\lambda/2$. At the same time, a series of covariance data collections composed by the data in $\mathbf{R}_x$ are established as follows

$$\mathbb{S}_\nu = \left\{ \langle \widehat{\mathbf{R}}_x \rangle_{i,j} \left| dx_i - dx_j = \nu, 1 \leqslant i, j \leqslant M \right. \right\}, \nu \in \widetilde{\mathbb{D}}_x. \quad (6)$$

Next, according to the set $\mathbb{S}_\nu$ defined in formula (6), the covariance matrix of the received signal along with the X-axis direction after removing redundancy, zero padding and ascending arrangement $\widetilde{\mathbf{R}}_x \in \mathbb{C}^{(dx_M+1)\times(dx_M+1)}$ can be expressed as

$$\langle \widetilde{\mathbf{R}}_x \rangle_{i,j} = \begin{cases} 0, & \text{if } \left|\mathbb{S}_{i-j}\right| = 0, \\ \frac{1}{\left|\mathbb{S}_{i-j}\right|} \sum \mathbb{S}_{i-j}, & \text{otherwise}, \end{cases} \quad (7)$$

where $1 \leqslant i, j \leqslant dx_M + 1$ and $\sum \mathbb{S}_{i-j}$ is the sum of all elements in $\mathbb{S}_{i-j}$.

Finally, since the covariance matrix of the received signal of ideal virtual uniform linear array satisfies the semi-definite positive and Toeplitz structure, the complete covariance matrix can be recovered by exploiting the following matrix completion theory based on matrix kernel norm minimization

$$\begin{aligned} &\arg\min_{\widehat{\mathbf{r}}_x} \frac{1}{2} \left\| \widetilde{\mathbf{R}}_x - \mathrm{toep}(\widehat{\mathbf{r}}_x) \circ \mathbf{G}_x \right\|_F^2 + \tau \left\| \mathrm{toep}(\widehat{\mathbf{r}}_x) \right\|_*, \\ &\text{s.t.,} \ \ \mathrm{toep}(\widehat{\mathbf{r}}_x) \succeq 0, \end{aligned} \quad (8)$$

where $\tau \in \mathbb{R}_+$ is the regularization parameter and $\mathbf{G}_x \in \mathbb{Z}^{(dx_M+1)\times(dx_M+1)}$ is a binary selection matrix whose positions of zero elements are the same as those of zero elements in $\widetilde{\mathbf{R}}_x$.

The SDP problem described in formula (8) can be effectively solved by CVX toolbox, and the optimal result $\mathrm{toep}(\widehat{\mathbf{r}}_x)$ is the complete covariance matrix corresponding to the virtual uniform linear array. Similarly, the missing data in the covariance matrix of the received signal of the sparse linear array located in Y-axis direction can also be recovered by a consistent method as follow

$$\begin{aligned} &\arg\min_{\widehat{\mathbf{r}}_y} \frac{1}{2} \left\| \widetilde{\mathbf{R}}_y - \mathrm{toep}(\widehat{\mathbf{r}}_y) \circ \mathbf{G}_y \right\|_F^2 + \tau \left\| \mathrm{toep}(\widehat{\mathbf{r}}_y) \right\|_*, \\ &\text{s.t.,} \ \ \mathrm{toep}(\widehat{\mathbf{r}}_y) \succeq 0. \end{aligned} \quad (9)$$

### C. SIMPLIFIED SEPARATER TWO-DIMENSIONAL GENERALIZED FRI SIGNAL RECONSTRUCTION MODEL

From the previous analysis, it is not difficult to see that the covariance matrix after the completion $\mathrm{toep}(\widehat{\mathbf{r}}_x)$ only depends on the value of its first row $\widehat{\mathbf{r}}_x$. In the case of ignoring noise, the ideal mathematical model $\mathbf{r}_x \in \mathbb{C}^{(dx_M+1)}$ can be expressed as

$$\mathbf{r}_x = \left[ \sum_{k=1}^{K} p_k, \sum_{k=1}^{K} p_k e^{j\pi\alpha_k}, \ldots, \sum_{k=1}^{K} p_k e^{j\pi\alpha_k dx_M} \right]^T. \quad (10)$$

The element position parameters are usually prior, so no matter how large the scale of $\mathbf{r}_x$ is, its value is only related to $p_k$ and $\alpha_k$, that is, 2K parameters. Such signals are called FRI signals. As the key to the reconstruction processing of FRI signal, an annihilation filter (AF) polynomial is established as follows

$$C(z) = \prod_{k=1}^{K} \left(1 - g_k^{-1} z\right) = \sum_{q=1}^{K+1} c_q z^{q-1}, \quad (11)$$

where $g_k = e^{j\pi\alpha_k}$ with $k = 1, .., K$ and $c_q \in \mathbb{C}$ is the coefficient of the polynomial $C(z)$ with $q = 1, \ldots, K+1$. It is worth noting that $c_1 = 1$.

Therefore, $g_k$ is the zero point of polynomial $C(z)$, that is, $C(g_k) = 0$. In this way, we can transform the estimation of parameter $\alpha_k$ to the estimation of polynomial coefficient $c_q$ through AF equation as follow

$$\begin{aligned} \sum_{q=1}^{K+1} c_q \langle \mathbf{r}_x \rangle_{q+i} &= \sum_{q=1}^{K+1} c_q \sum_{k=1}^{K} p_k e^{j\pi\alpha_k(q+i-1)} \\ &= \sum_{k=1}^{K} p_k g_k^i \sum_{q=1}^{K+1} c_q g_k^{q-1} = \sum_{k=1}^{K} p_k g_k^i C(g_k) = 0, \end{aligned} \quad (12)$$

where $i = 0, 1, \ldots, dx_M - K$.

The equation shown in formula (12) is written in the form of matrix

$$\mathbf{T}(\mathbf{r}_x)\mathbf{c} = \mathbf{R}(\mathbf{c})\mathbf{r}_x = \mathbf{0}, \quad (13)$$

where $\mathbf{c} = [c_1, \ldots, c_{K+1}]^T$ and the specific form of matrix $\mathbf{T}(\mathbf{r}_x) \in \mathbb{C}^{(dx_M-K+1)\times(K+1)}$ and $\mathbf{R}(\mathbf{c}) \in \mathbb{C}^{(dx_M-K+1)\times(dx_M+1)}$ is

$$\mathbf{T}(\mathbf{r}_x) = \begin{bmatrix} \langle \mathbf{r}_x \rangle_1 & \langle \mathbf{r}_x \rangle_2 & \cdots & \langle \mathbf{r}_x \rangle_{K+1} \\ \langle \mathbf{r}_x \rangle_2 & \langle \mathbf{r}_x \rangle_3 & \cdots & \langle \mathbf{r}_x \rangle_{K+2} \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{r}_x \rangle_{dx_M-K+1} & \langle \mathbf{r}_x \rangle_{dx_M-K+2} & \cdots & \langle \mathbf{r}_x \rangle_{dx_M+1} \end{bmatrix},$$

$$\mathbf{R}(\mathbf{c}) = \begin{bmatrix} c_1 & \cdots & c_{K+1} & 0 & 0 & \cdots & 0 \\ 0 & c_1 & \cdots & c_{K+1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & c_1 & \cdots & c_{K+1} & 0 \\ 0 & \cdots & 0 & 0 & c_1 & \cdots & c_{K+1} \end{bmatrix}. \quad (14)$$

After the AF equation is obtained, the reconstruction processing of FRI signal can be described as the following optimization process

$$\begin{aligned} &\arg\min_{\mathbf{r}_x, \mathbf{c}} \left\| \widehat{\mathbf{r}}_x - \mathbf{r}_x \right\|_2^2, \\ &\text{s.t.,} \ \begin{cases} \mathbf{R}(\mathbf{c})\mathbf{r}_x = 0 \\ \mathbf{c}_1 = 1. \end{cases} \end{aligned} \quad (15)$$

Although the problem in formula (15) is not a convex optimization process, it can be accurately solved by the double iteration strategy given in reference [22], but the two linear equations in the iterative process are simplified to the following form

$$\begin{bmatrix} \mathbf{0} & \mathbf{T}^{H}(\widehat{\mathbf{r}_{x}}) & \mathbf{e} \\ \mathbf{T}(\widehat{\mathbf{r}_{x}}) & -\mathbf{R}(\mathbf{c}_{n-1})\mathbf{R}^{H}(\mathbf{c}_{n-1}) & \mathbf{0} \\ \mathbf{e}^{T} & \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \eta \\ \mu \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ 1 \end{bmatrix},$$
$$\begin{bmatrix} \mathbf{I} & \mathbf{R}^{H}(\mathbf{c}) \\ \mathbf{R}(\mathbf{c}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{r}_{x} \\ \lambda \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{r}_{x}} \\ \mathbf{0} \end{bmatrix}, \quad (16)$$

where $\mathbf{e} = [1, 0, \ldots, 0]^{T} \in \mathbb{Z}_{+}^{K+1}$. The detailed derivation of formula (16) is given in the appendix of this article.

It can be seen from formula (16) that compared with the existing generalized FRI signal reconstruction model, the computational burden of solving the bi-variate optimization problem in the proposed algorithm becomes lighter. This is due to the introduction of the matrix completion process, which makes the received signal of the antenna array in each dimension become uniform. This is one of the main contributions of this article.

A polynomial is constructed by taking the optimal solution $\mathbf{c}$ of formula (16) as the coefficient, then the K zeros of the polynomial exactly correspond to $e^{\jmath\pi\alpha_k}$. In this way, we can get the estimation of the parameter $\alpha_k$. Similarly, the estimation result of $\beta_k$ is obtained by the following optimization problem

$$\arg\min_{\mathbf{r}_y,\mathbf{h}} \left\| \widehat{\mathbf{r}_y} - \mathbf{r}_y \right\|_2^2,$$
$$\text{s.t.,} \begin{cases} \mathbf{R}(\mathbf{h})\,\mathbf{r}_y = 0 \\ \mathbf{h}_1 = 1. \end{cases} \quad (17)$$

After obtaining the estimation results of $\alpha_k$ and $\beta_k$ parameters, the two groups of parameters are paired according to the pairing strategy in reference [22], and then the final DOA parameters $\theta_k$ and $\varphi_k$ are estimated according to the following angle translation formulas

$$\theta_k = \begin{cases} \arctan\left(\frac{\beta_k}{\alpha_k}\right), \text{if} \alpha_k > 0 \text{and} \beta_k > 0, \\ \arctan\left(\frac{\beta_k}{\alpha_k}\right) + \pi, \text{if} \alpha_k < 0, \\ \arctan\left(\frac{\beta_k}{\alpha_k}\right) + 2\pi, \text{if} \alpha_k > 0 \text{and} \beta_k < 0, \end{cases}$$
$$\varphi_k = \arccos\left(\sqrt{\alpha_k^2 + \beta_k^2}\right). \quad (18)$$

In Table 1, the detailed process of 2D-SGFRI algorithm is given, and the double iterative reconstruction algorithm of FRI signal is shown in Table 2.

## III. SIMULATIONS AND RESULT ANALYSIS

In this section, several simulation experiments are shown to verify the effectiveness and superiority of 2D-SGFRI algorithm. An eight elements coprime array is adopted in both X-axis and Y-axis directions with the set of antenna positions

$$\mathbb{D}_x = \mathbb{D}_y = \{0, 4, 5, 8, 10, 12, 15, 16\}\,\lambda/2, \quad (19)$$

which imply that $M = N = 8$ and $dx_M = dy_N = 16$.

**TABLE 1.** The process of 2D-SGFRI algorithm.

| | |
|---|---|
| **Input** | the position sets of antenna array $\mathbb{D}_X$ and $\mathbb{D}_y$, the received signals matrix $\mathbf{X}$ and $\mathbf{Y}$ |
| **Initialize** | the regularization parameter $\tau$; |
| **Step 1** | calculate the covariance matrix $\widehat{\mathbf{R}_X}$ and $\widehat{\mathbf{R}_Y}$ by Formula (4); |
| **Step 2** | the elements in set $\mathbb{S}_{\nu_X}$ and $\mathbb{S}_{\nu_Y}$ can be determined according to Formula (6); |
| **Step 3** | calculate $\widetilde{\mathbf{R}_X}$ and $\widetilde{\mathbf{R}_Y}$ according to Formula (7); |
| **Step 4** | the SDP problems shown in Formula (8) and Formula (9) can be solved respectively by using CVX toolbox to obtaine $\widehat{\mathbf{r}_X}$ and $\widehat{\mathbf{r}_Y}$; |
| **Step 5** | according to the double iterative reconstruction algorithm of FRI signal, the bi-variate optimization problems shown in Formula (15) and Formula (17) can be solved respectively to obtain $\alpha_k$ and $\beta_k$; |
| **Step 6** | the final DOA estimation results $\theta_k$ and $\varphi_k$ can be obtained according to the angle transformation method as shown in formula (18) and the pairing strategy proposed in reference [22]. |

**TABLE 2.** The process of double iterative reconstruction algorithm of FRI signal.

| | |
|---|---|
| **Input** | the observation vector $\widehat{\mathbf{r}}$ |
| **Initialize** | the maximum number of initializations $\mathrm{Max_{ini}}$, the maximum number of internal iterations $\mathrm{Max_{ite}}$; |
| | for loop $\leftarrow$ 1 to $\mathrm{Max_{ini}}$ : |
| **Step 1** | calculate the matrix $\mathbf{T}(\widehat{\mathbf{r}})$ by Formula (14); |
| **Step 2** | random initialization $\mathbf{c}_0$; |
| | for n $\leftarrow$ 1 to $\mathrm{Max_{ite}}$ : |
| **Step 3** | establish the matrix $\mathbf{R}(\mathbf{c}_{n-1})$ according to Formula (14); |
| **Step 4** | solve the first system of linear equations shown in formula (16) to update $\mathbf{c}_n$; |
| **Step 5** | establish the matrix $\mathbf{R}(\mathbf{c}_n)$ and solve the second system of linear equations to update $\mathbf{r}$; |
| | end |
| | end |
| **Step 6** | construct the polynomial $\mathbf{C}(z)$ by taking the iterative result $\mathbf{c}_n$ as the coefficient; |
| **Step 7** | calculate the zeros of polynomial $\mathbf{C}(z)$ denoted as $z_k$ and the result of parameter estimation is $\alpha_k = \mathrm{Im}(\ln(z_k))/\pi$. |

### A. EFFECTIVENESS

In this experiment, we use the BSLA with 16 elements described above to distinguish four incident sources in two-dimensional angle space. The azimuth angle and elevation angle are selected randomly in $\theta_k \in [0^\circ, 360^\circ]$ and $\varphi_k \in [0^\circ, 90^\circ]$ respectively, and the minimum angle interval is $10^\circ$. In 2D-SGFRI, regularization parameter $\tau = 0.5$, the maximum number of initializations and internal iterations are both 20. When SNR is 20dB and the number of snapshots is 100, the simulation result of 2D-SGFRI algorithm to distinguish four independent signals is shown in Fig. 2. It can be seen from the simulation results that 2D-SGFRI can successfully distinguish four incident sources.

### B. ESTIMATION ACCURACY

In this simulation, the 2D-SGFRI algorithm is compared with three other two-dimensional DOA estimation algorithms,

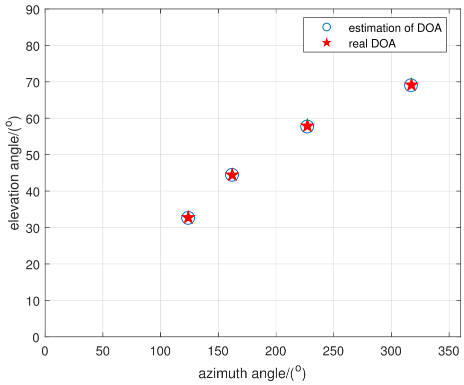


**FIGURE 2.** Estimation results of 2D-SGFRI algorithm.

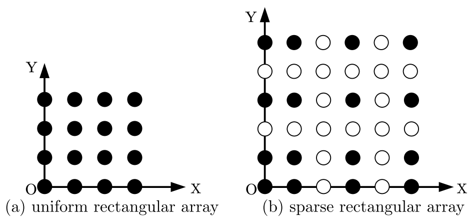


**FIGURE 3.** Antenna array structure.

including 2D-MUSIC algorithm for planar array with arbitrary geometry [3], decoupled atomic norm minimization algorithm for uniform rectangular array (2D-DANM) [18] and 2D Toeplitz matrix complete algorithm for sparse planar array (2D-TMC) [26]. Among them, 2D-MUSIC is suitable for the bi-orthogonal sparse linear array used in this article, but 2D-DANM algorithm is only suitable for uniform rectangular array and 2D-TMC algorithm is suitable for uniform rectangular array and sparse rectangular array. To ensure that the number of elements is the same, the 2D-DANM algorithm uses a uniform rectangular array of $4 \times 4$, that is, the total number of elements is 16. Similarly, the 2D-TMC algorithm utilizes a sparse rectangular array of $4 \times 4$. The specific structures of the above arrays are shown in Figure 3. In Figure 3, the solid circles represent the actual antenna positions, and the hollow circles represent that there is no antenna at this position. In the direction of X-axis and Y-axis, the distances between adjacent circles are all half of the incident source wavelength.

In addition, in this article, the root mean square error (RMSE) for azimuth angle is defined as

$$\mathrm{RMSE} = \sqrt{\frac{1}{\mathrm{KL}} \sum_{\mathrm{l}=1}^{\mathrm{L}} \sum_{\mathrm{k}=1}^{\mathrm{K}} \left(\hat{\theta}_{\mathrm{k},\mathrm{l}} - \theta_{\mathrm{k}}\right)^2}, \tag{20}$$

where $\mathrm{L} = 500$ is the number of Monte-Carlo experiments. The definition of RMSE for elevation angle is similar to formula (20).

The above four algorithms are used to distinguish four independent incident sources in space, and the selection of incident angles is the same as the previous experiment.

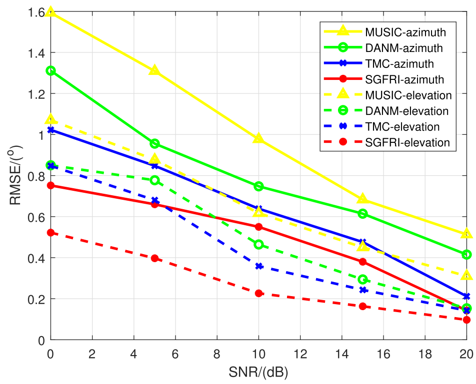


**FIGURE 4.** RMSE vs. SNR.

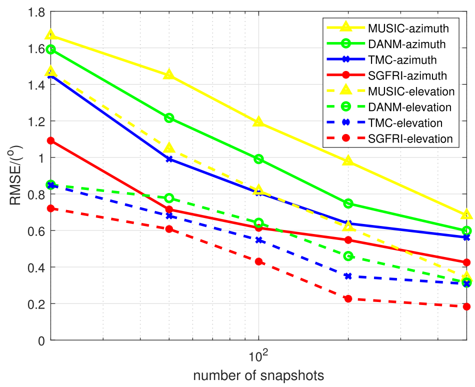


**FIGURE 5.** RMSE vs. the number of snapshots.

When the number of snapshots is fixed at 200, the RMSE statistical results of the above four 2D-DOA estimation algorithms under different SNR are shown in Fig. 4. Similarly, when the SNR is fixed at 10dB, the RMSE statistical results under different numbers of snapshots are shown in Fig. 5. The simulation results show that the proposed 2D-SGFRI algorithm has the best estimation performance under the same conditions.

It is worth mentioning that the traditional DOA estimation algorithm based on the generalized FRI signal reconstruction model [22]–[25] cannot be directly applied to the plane array DOA estimation problem. Therefore, in the simulation experiment of this article, there is no comparison with this kind of algorithm.

### *C. RESOLUTION*

In this simulation, we consider two closely spaced targets to examine the angular resolution of the above four 2D-DOA estimation algorithms. The azimuth angles are assumed as $\theta - \Delta\theta$ and $\theta + \Delta\theta$. The generation method of elevation angles is the same as that of azimuth angle. Set SNR and the number of snapshots as 10 dB and 100. When the deviations between the estimated angles and the real angles are all within 2 degrees, it is considered that this experiment is successful. The angular resolution probability is depicted in Fig. 6, where 500 independent trials are repeated. It can be seen that the

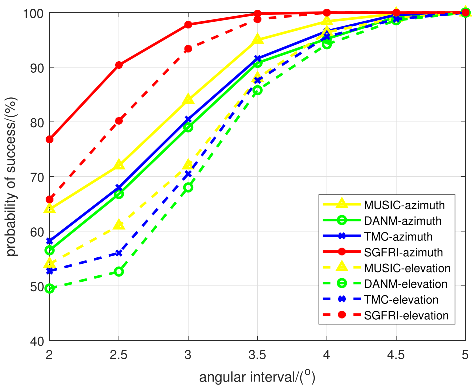


**FIGURE 6.** Angular resolution.

2D-SGFRI algorithm has the best resolution, and has nearly 100% estimated success probability when the angle interval is greater than 3.5°.

It can be seen from the simulation results that although 2D-MUSIC algorithm and 2D-SGFRI algorithm use the same array geometry, there is a certain gap in both estimation accuracy and angle resolution. There are many reasons for this situation. Firstly, the 2D-MUSIC algorithm requires that the received signal of the antenna array meet the two-dimensional positive semi-definite Toeplitz structure. However, because the BSLA exploited in this article does not have this structure, and lacks the process of matrix completion or other means, so even under better experimental conditions, the performance of 2D-MUSIC algorithm still has a certain gap with 2D-SGFRI algorithm. Secondly, compared with 2D-SGFRI algorithm, 2D-MUSIC algorithm is more sensitive to SNR and snapshot number, so the estimation performance gap between them will be larger under poor experimental conditions.

### *D. ALGORITHM COMPLEXITY*

At last, in order to more clearly show the differences with other methods in the computational complexity, specific analysis is given in this section.

In 2D-MUSIC algorithm [3], the parts that play a leading role in computing complexity include the calculation of covariance matrix, its eigenvalue decomposition, and the process of peak searching. Their corresponding computational complexity are as follows: $O(M^2L)$, $O(M^3)$, and $O(M^2N_xN_y)$, where $N_x$ and $N_y$ represent the number of grids in the azimuth and elevation directions respectively.

The computational complexity of 2D-DANM algorithm and 2D-TMC algorithm have been given respectively in references [18] and [26]. Similarly, the complexity of the proposed 2D-SGFRI algorithm is not difficult to give based on the analysis of SDP process complexity in the above two references. To sum up, the computational complexity of the above four algorithms are given in Table 3. In Table 3, $M_x$ and $M_y$ represent the maximum number of elements in the x-axis and Y-axis directions, respectively. $\varepsilon$ is the desired recovery precision.

**TABLE 3.** The computational complexity of the above four algorithms.

| Algorithm | Complexity |
|---|---|
| 2D-MUSIC | $O(M^2L + M^3 + M^2N_xN_y)$ |
| 2D-DANM | $O((M_x + M_y)^{3.5}log(1/\varepsilon)$ |
| 2D-TMC | $O((2M_x + 2M_y)^{3.5}log(1/\varepsilon)$ |
| 2D-SGFRI | $O((M_x^{3.5} + M_y^{3.5})log(1/\varepsilon))$ |

## IV. CONCLUSION

In this article, a two-dimensional gridless DOA estimation algorithm named 2D-SGFRI is proposed by exploiting bi-orthogonal sparse linear array. The 2D-SGFRI algorithm transforms a two-dimensional DOA estimation problem into two one-dimensional DOA problems in independent directions through the distinctive geometry structure of BSLA, thus avoiding the spectrum peak search in two-dimensional angle space and Vandermonder decomposition processing of two-dimensional Toeplitz matrix. In each one-dimensional DOA estimation problem, the 2D-SGFRI algorithm first uses the matrix completion theory to recover the complete covariance data, which brings the advantage of improving the estimation accuracy. Then, through the simplified generalized FRI signal reconstruction model, the estimation results of incident source direction parameters can be obtained from the complete covariance data. Finally, through the processing of pairing and angle conversion, the DOA information of the sources can be estimated. The simulation results also prove the effectiveness and superiority of 2D-SGFRI algorithm.

## APPENDIX
## THE SOLUTION PROCESS OF OPTIMIZATION PROBLEM

For a fixed parameter $\mathbf{c}$, the optimization problem as shown in formula (15) will be simplified to a quadratic minimization problem only with respect to the parameter $\mathbf{r}_x$. The associated Lagrangian is

$$\mathrm{L}(\mathbf{r}_x, \lambda) = \frac{1}{2}\|\widehat{\mathbf{r}_x} - \mathbf{r}_x\|_2^2 + \lambda^H\mathbf{R}(\mathbf{c})\mathbf{r}_x, \tag{21}$$

where $\lambda \in \mathbb{C}^{dx_M - K+1}$ is the Lagrangian multiplier.

According to the fact that the derivative of Langrangian function to variable $\mathbf{r}_x$ and $\lambda$ is zero, we can have

$$\begin{cases} \dfrac{\partial \mathrm{L}}{\partial \mathbf{r}_x} = \mathbf{0}, \\ \dfrac{\partial \mathrm{L}}{\partial \lambda} = \mathbf{0}. \end{cases} \Leftrightarrow \begin{cases} \mathbf{r}_x - \widehat{\mathbf{r}_x} + \mathbf{R}^H(\mathbf{c})\lambda = \mathbf{0}, \\ \mathbf{R}(\mathbf{c})\mathbf{r}_x = \mathbf{0}. \end{cases} \tag{22}$$

It can be obtained from the above formula

$$\mathbf{R}(\mathbf{c})\mathbf{R}^H(\mathbf{c})\lambda = \mathbf{R}(\mathbf{c})\widehat{\mathbf{r}_x}. \tag{23}$$

According to the specific form shown in formula (14), it is not difficult to see that $\mathbf{R}(\mathbf{c})$ matrix is a row full rank matrix, so there are

$$\lambda = \left[\mathbf{R}(\mathbf{c})\mathbf{R}^H(\mathbf{c})\right]^{-1}\mathbf{R}(\mathbf{c})\widehat{\mathbf{r}_x}. \tag{24}$$

By substituting formula (24) into the first equation in formula (22), we can obtain

$$\mathbf{r}_{\mathrm{x}} = \widehat{\mathbf{r}_{\mathrm{x}}} - \mathbf{R}^{\mathrm{H}}(\mathbf{c})\left[\mathbf{R}(\mathbf{c})\mathbf{R}^{\mathrm{H}}(\mathbf{c})\right]^{-1}\mathbf{R}(\mathbf{c})\widehat{\mathbf{r}_{\mathrm{x}}}. \quad (25)$$

When the parameter $\mathbf{c}$ is determined, the above formula can be used to solve the parameter $\mathbf{r}_{\mathrm{x}}$. This result naturally satisfies the constraint of AF equation. In addition, in order to avoid the complex and unstable operation process such as matrix inversion, formula (25) can be rewritten into the form of linear equations, that is, the second linear equations in formula (16).

Then, the above results are brought back to the objective function of the optimization problem

$$\begin{aligned}\|\widehat{\mathbf{r}_{\mathrm{x}}} - \mathbf{r}_{\mathrm{x}}\|_2^2 &= \left\|\mathbf{R}^{\mathrm{H}}(\mathbf{c})\left[\mathbf{R}(\mathbf{c})\mathbf{R}^{\mathrm{H}}(\mathbf{c})\right]^{-1}\mathbf{R}(\mathbf{c})\widehat{\mathbf{r}_{\mathrm{x}}}\right\|_2^2 \\ &= \widehat{\mathbf{r}_{\mathrm{x}}}^{\mathrm{T}}\mathbf{R}^{\mathrm{H}}(\mathbf{c})\left\{\left[\mathbf{R}(\mathbf{c})\mathbf{R}^{\mathrm{H}}(\mathbf{c})\right]^{-1}\right\}^{\mathrm{T}}\mathbf{R}(\mathbf{c})\mathbf{R}^{\mathrm{H}}(\mathbf{c}) \\ &\quad \left[\mathbf{R}(\mathbf{c})\mathbf{R}^{\mathrm{H}}(\mathbf{c})\right]^{-1}\mathbf{R}(\mathbf{c})\widehat{\mathbf{r}_{\mathrm{x}}} \\ &= \widehat{\mathbf{r}_{\mathrm{x}}}^{\mathrm{H}}\mathbf{R}^{\mathrm{H}}(\mathbf{c})\left[\mathbf{R}(\mathbf{c})\mathbf{R}^{\mathrm{H}}(\mathbf{c})\right]^{-1}\mathbf{R}(\mathbf{c})\widehat{\mathbf{r}_{\mathrm{x}}} \\ &= \mathbf{c}^{\mathrm{H}}\mathbf{T}^{\mathrm{H}}(\widehat{\mathbf{r}_{\mathrm{x}}})\left[\mathbf{R}(\mathbf{c})\mathbf{R}^{\mathrm{H}}(\mathbf{c})\right]^{-1}\mathbf{T}(\widehat{\mathbf{r}_{\mathrm{x}}})\mathbf{c}. \quad (26)\end{aligned}$$

Since the process of minimizing the above objective function of parameter $\mathbf{c}$ is very complicated, an iterative optimization strategy is adopted to determine the optimal solution of parameter $\mathbf{c}$. Specifically, it is to solve the following quadratic minimization problem

$$\begin{aligned}&\arg\min_{\mathbf{c}} \mathbf{c}^{\mathrm{H}}\mathbf{T}^{\mathrm{H}}(\widehat{\mathbf{r}_{\mathrm{x}}})\left[\mathbf{R}(\mathbf{c}_{\mathrm{n}-1})\mathbf{R}^{\mathrm{H}}(\mathbf{c}_{\mathrm{n}-1})\right]^{-1}\mathbf{T}(\widehat{\mathbf{r}_{\mathrm{x}}})\mathbf{c} \\ &\text{s.t.,}\ \mathbf{c}_1 = 1. \quad (27)\end{aligned}$$

The Lagrangian function corresponding to the above quadratic minimization problem is

$$\begin{aligned}\mathrm{L}(\mathbf{c},\mu) = \frac{1}{2}\mathbf{c}^{\mathrm{H}}\mathbf{T}^{\mathrm{H}}(\widehat{\mathbf{r}_{\mathrm{x}}})\left[\mathbf{R}(\mathbf{c}_{\mathrm{n}-1})\mathbf{R}^{\mathrm{H}}(\mathbf{c}_{\mathrm{n}-1})\right]^{-1}\mathbf{T}(\widehat{\mathbf{r}_{\mathrm{x}}})\mathbf{c} \\ +\mu(\mathbf{c}_1 - 1), \quad (28)\end{aligned}$$

where $\mu \in \mathbb{R}$ is Lagrangian multiplier.

Similarly, according to the first derivative of the Lagrangian function corresponding to the optimal solution of the quadratic optimization problem is zero, we can obtain

$$\begin{cases}\mathbf{T}^{\mathrm{H}}(\widehat{\mathbf{r}_{\mathrm{x}}})\left[\mathbf{R}(\mathbf{c}_{\mathrm{n}-1})\mathbf{R}^{\mathrm{H}}(\mathbf{c}_{\mathrm{n}-1})\right]^{-1}\mathbf{T}(\widehat{\mathbf{r}_{\mathrm{x}}})\mathbf{c} + \mu\mathbf{e} = \mathbf{0}, \\ \mathbf{c}_1 = 1,\end{cases} \quad (29)$$

where $\mathbf{e} \in \mathbb{R}^{\mathrm{K}+1}$ is an unit column vector where the first element is 1 and the remaining elements are 0.

The above results can be rewritten into

$$\begin{cases}\mathbf{T}^{\mathrm{H}}(\widehat{\mathbf{r}_{\mathrm{x}}})\eta + \mu\mathbf{e} = \mathbf{0}, \\ \mathbf{T}(\widehat{\mathbf{r}_{\mathrm{x}}})\mathbf{c} - \mathbf{R}(\mathbf{c}_{\mathrm{n}-1})\mathbf{R}^{\mathrm{H}}(\mathbf{c}_{\mathrm{n}-1})\eta = \mathbf{0}, \\ \mathbf{c}_1 = 1,\end{cases} \quad (30)$$

where $\eta = \left[\mathbf{R}(\mathbf{c}_{\mathrm{n}-1})\mathbf{R}^{\mathrm{H}}(\mathbf{c}_{\mathrm{n}-1})\right]^{-1}\mathbf{T}(\widehat{\mathbf{r}_{\mathrm{x}}})\mathbf{c}$ is an auxiliary variable. Similarly, formula (30) can also be written in matrix form, that is, the first linear system of equations in formula (16).

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