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Study on the Optimal Strategy of Missile Interception

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ABSTRACT Missiles have been playing an important role in modern wars. In order to gain advantage in missile contests, the technology of missile interception have been developed for many years. However, constrained by current technology, it is still very difficult to intercept successfully. Thus, it may need to optimize the allocation of intercept missiles to prevent attack missiles from hitting targets. This paper proposes a method to optimize the missile interception strategy (intercept missile allocation) by simulating the games of missile interception, which jointly considering multiple phases, multiple targets, cost and the risk attitude of the defender. This method is developed based on combinatorial theory and cumulative prospect theory. There are two optimization goals considered: the cost and the prospect value. We studied two cases where the detection of attack missiles is perfect or imperfect respectively. Four illustrative examples are presented to illustrate the proposed method and several directions of research in future are pointed.

INDEX TERMS Missile interception, reliability, combinatorial theory, cumulative prospect theory.

I. INTRODUCTION

During cold war, the U.S. and the Soviet Union started an arms race [1]. At the same time, the project of missile defense was largely developed to prevent the attack by intercontinental or long-range missiles. After the cold war, the United States grows to be the only superpower in the world. Some missile-reduction treaties make sweeping cuts, but the arms race is not over by a long shot. Many countries still have enough strategic arms especially missiles of mass destruction in order to maintain deterrence [2]. In this case, to prevent the weapons of mass destruction, U.S. accelerated the research of missile interception technology and the building of guided missile defense system. Thereafter, the missile interception went up on the historical stage of military.

The application of missile interception is complicated and intricate. The Missile Defense (MD) system usually consists of warning system (early-warning satellites, improved warning radars and ground-based radars), ground-based interception (GBI) system (intercept missiles) and battle manage/command control communication and intelligence (BM/C3I) system [3]. Warning system is used to detect missile launches as well as their impact points and launcher points. GBI system and its intercept missiles are the core of MD. For intercept missiles, the Booster rocket sends the interceptor (warhead) near the target. Then the interceptor adjusts its angle and altitude to finally dash and destroy its target. BM/C3I system plays the manager role of MD by connecting the other two systems through computers and communication networks [4]. The intercepting process of the missile defense system is shown as Figure 1.

Figure 1 also clearly illustrates the intercepting process of the missile defense system as well as the function of its subsystems. Warning system is used to detect missile launches as well as their impact points and launcher points. GBI system and its intercept missiles are the core of MD. For intercept missiles, the Booster rocket sends the interceptor (warhead) near the target. Then the interceptor adjusts its angle and altitude to finally dash and destroy its target. BM/C3I system plays the manager role of MD by connecting the other two systems through computers and communication networks [4].

Although the technology of missile interception has been developed for many years, the success rate of intercepting attack missiles is still comparatively low. There are several

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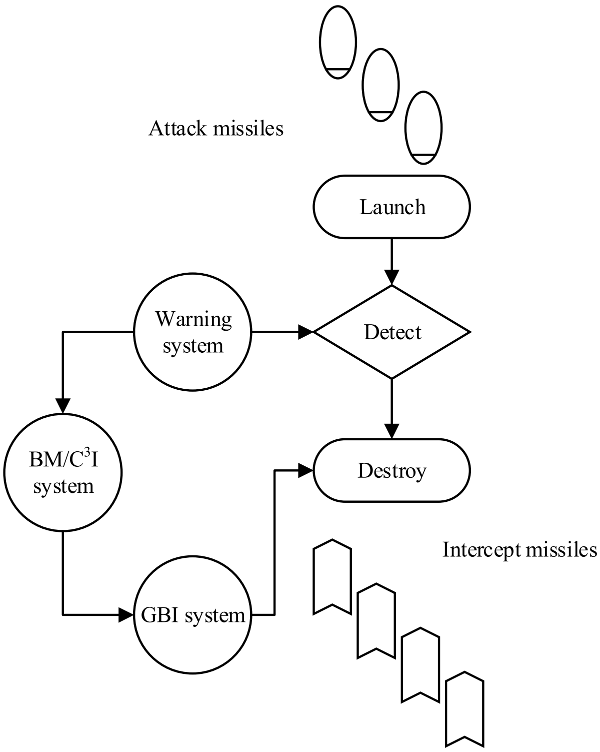

FIGURE 1. The intercept process of the missile defense system.

reasons. First, the missile usually flies extremely fast. Only a few minutes are enough for the missile warhead to hit its target, which leaves the defender no time to respond the attack properly. Second, the cost of missile interception is usually much higher than the cost of missile attack. In this case, it is hard for the defender to undergo an attrition war using intercept missiles so that the amount of intercept missiles prepared may be not enough compared with the amount of attack missiles. Besides, the technology of attack missiles also has been largely developed. For example, it is almost impossible to intercept the multi-warhead missile applied with stealth technology.

Considerable research efforts have been expended in the field of defense strategies [5]-[23], especially the missile intercept [20]–[23]. Shinar *et al.* [20] presented a methodology to assess the probability of successful interception as a function of the parameters of the scenario. Bertsekas *et al.* [21] proposed a solution methodology for a missile defense problem involving the sequential allocation of defensive resources over a series of engagements. Chen *et al.* [22] considered the autopilot for the optimal intercept missile guidance strategies. Garcia *et al.* [23] further extended the study of missile interception to active target defense scenario.

The existing studies have made significant contributions to decision analysis in defense strategy. These studies provided various decision analysis methods for missile interception to support the defender's decision-making. However, there are still three limitations in the field of missile interception strategy: (1) only one single phase for attack missile to be intercepted; (2) only one attack missile is considered to be intercepted; (3) the risk attitudes of defender is ignored. The details of these limitations and the solutions of this study are as follows.

First, few researchers have considered that the interception can be conducted in multiple phases [24]–[30]. Due to technology and specific situation, the interception of a missile may also be made in different phases of the missile projectile. Typically, there are three phases where the interception can be made, being initial interception, midcourse interception, and terminal interception. Second, few researchers have studied missile interception considering multiply targets. In practice, the attacker may launch more than one attack missiles at a time. The two concerns bring an issue about how to allocate the interception resources (intercept missiles) to these attack missiles, which can be addressed by the combinatorial mathematics. In operations research, optimization based on combinatorial mathematics is a topic that consists of finding an optimal object from a finite set of objects [31]. Thus, this study proposed a method to optimize the allocation strategy of interception missiles considering multiple phases and multiple targets by using a combinatorial method.

Third, few researchers have considered the risk attitude of the defender in the missile attack, which means that the defender's behavior is rarely considered. Many psychological studies have proved that there are several psychological characteristics of human behavior under risk and uncertainty, such as reference dependence, loss aversion, and judgmental distortion of likelihood of almost impossible and certain outcomes [32]–[38]. Since decision-making problems in missile interception are usually risky and uncertain, it is necessary to consider the defender's psychological behavior in strategy optimization, which can be incorporated using the cumulative prospect theory (CPT).

Since Tversky and Kahneman [33] proposed prospect theory [32], some behavioral decision-making theories have been developed rapidly. For example, regret theory [39], [40], disappointment theory [40], [41], cumulative prospect theory (CPT) [42], third generation prospect theory [43] and so on. Besides, some decision analysis theories considering multiple factors have been proposed [42]–[44]. Among these theories, CPT has been regarded as the most popular theory [34]–[37]. This is because CPT describes the decision maker's behavioral characteristics well and gives the calculation formulas on values and weights of potential outcomes. Since the formulas have features of clear logic and simple computation process, CPT has been widely used to solve various decision-making problems considering decision maker's behavior. Therefore, this study incorporates CPT into strategy optimization in missile interception to consider the risk attitudes of the defender.

Thus, this study proposes a method to optimize the missile interception strategy (intercept missile allocation) by simulating the games of missile interception, which jointly considering multiple phases, multiple targets and the risk attitude of the defender. This method is developed based on two techniques: (1) combinatorial mathematics which is introduced to consider multiple phases and multiple targets; (2) CPT which is introduced to consider the risk attitude of the defender. In addition, to make the method more general, whether the detection is perfect or not is also discussed. As for the optimization goals, this study considers two aspects: (1) the cost, including expenditure for intercept missiles and the loss due to the failure of interception, which is comparatively objective; (2) the prospect value, calculated according to CPT to measure the outcomes considering the risk attitude of the defender, which is comparatively subjective. The structure of our method is shown in Figure 2.

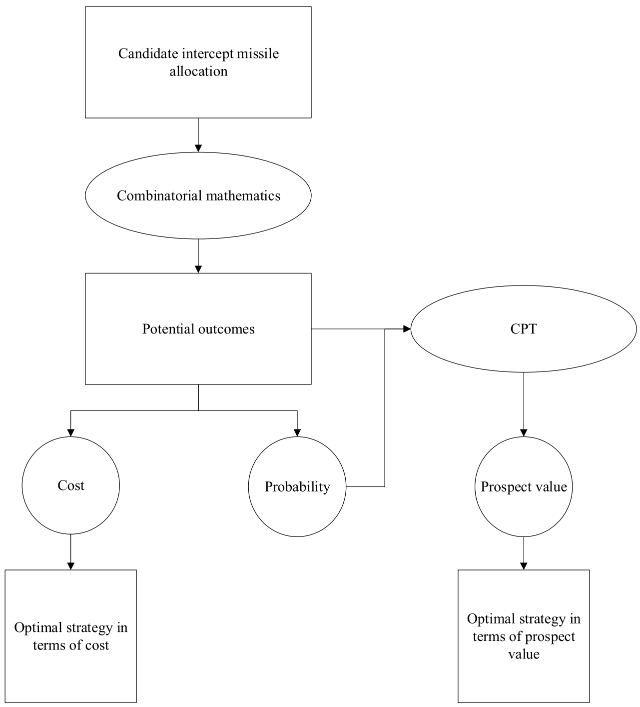

FIGURE 2. The solution procedure of the proposed method.

The remaining of this paper is arranged as follows. Section II describes the procedures of the missile interception and proposes the model to obtain the optimal strategy for the case where the missile interception can only be made in a single phase. Section III further extends the model to the case of multi-phased interception. Section IV introduces CPT into the optimization method. Two numerical examples of single-phased interception are presented in Section V, while Section VI provides two illustrative examples for multi-phased interception. Section VII concludes this study and points out some future works.

II. THE MODEL OF OPTIMAL STRATEGY FOR SINGLE-PHASED INTERCEPTION

We begin with the simple case where the missile interception can only be made in a single phase, such as the midcourse phase. Assume that the defender has N intercept missiles and

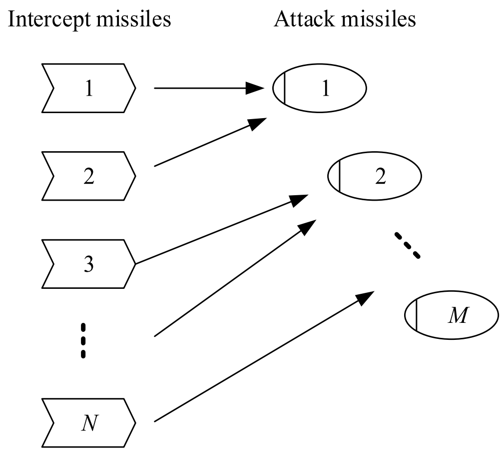

FIGURE 3. Missile interception of Scenario 1.

the attacker has M attack missiles. For each attack missile, the defender can choose to launch zero, one, or multiple intercept missiles to intercept it. In the case of multiple intercept missiles, it is assumed that the success probability of each intercept missile is independent from other intercept missiles. In particular, the probability that each intercept missile can successfully intercept the attack missile i is $P_I(i)$; If the interception is failed, this attack missile will cause a fixed loss $C_L(i)$ to the defender. The cost of each intercept missile is g. Herein, the total cost for the defender includes two parts: the loss due to failed interception and the cost for intercept missiles. The aim of this study is to minimize the defender's total cost by adjusting the allocation of intercept missiles for attack missiles.

A. SCENARIO 1: THE DETECTION OF ATTACK MISSILE IS PERFECT

In this scenario, all attack missiles are detected successfully in time. Assume there are $N(i)$ intercept missiles used to intercept the attack missile i, and $N(1) + \cdots + N(M) \leq N$. Then, the loss due to the intercept failure can be expressed as

$$(1 - P_I(1))^{N(1)} C_L(1) + \cdots + (1 - P_I(M))^{N(M)} C_L(M). \quad (1)$$

In equation (1), $(1 - P_I(\cdot))^{N(\cdot)}$ expresses the probability of a failed interception.

Then, adding up the cost of intercept missiles, we can have the total cost

$$C = \sum_{i=1}^{M} (1 - P_I(i))^{N(i)} C_L(i) + gN(i), \quad \sum_{i=1}^{M} N(i) \leq N, \quad (2)$$

where the first item indicates the loss due to failed interception and the second item indicates the cost of intercept missiles. The defender can obtain the optimal strategy by minimizing the total cost.

This scenario is shown as Figure 3. It can be seen that all attack missiles are detected and known by the defender. The defender can respond to every attack missile.

B. SCENARIO 2: THE DETECTION OF ATTACK MISSILE IS IMPERFECT

In practice, the attack missile may not be detected successfully or timely due to the different conditions. For example, a Chinese-made intercontinental ballistic missile, DF-41, may have the ability to fly with a speed of 30 600 kilometers per hour which is about 25 times as fast as sonic speed. In this case, the defender generally has no time to detect the attack missile and implement the interception. In order to be able to account for such situation, the case where the detection of attack missile is imperfect is also studied in this scenario.

To simplify the model, this study assumes that an attack missile is either not detected until it hits the target or detected at the beginning of its launch. The functions and parameters for this scenario are listed as follows. Let $P_d(i)$ indicate the probability that the attack missile i is detected successfully. Assume there are $N(i)$ intercept missiles used for the attack missile i and $N(1) + \cdots + N(M) \leq N$. Let $D(i)$ indicate whether the attack missile i is detected or not.

Then we can have the joint probability for the different situations of detection as

$$P(D(1), \ldots, D(M)) = \prod_{i=1}^{M} P_d(i)^{D(i)} \prod_{i=1}^{M} \left[(1 - P_d(i))^{(1-D(i))}\right], \tag{3}$$

where $D(i) = 1$ indicates that the detection is successful and $D(i) = 0$ indicates that the detection is failed.

Then we can also obtain the total cost for the allocation of $N(1), \ldots, N(M)$ under given detection outcome $D(1), \ldots, D(M)$ as

$$\begin{aligned} &C(N(1), \ldots, N(M)|D(1), \ldots, D(M)) \\ &= \sum_{i=1}^{M} \left\{C_L(i)(1-P_I(i))^{N(i)}D(i) + C_L(i)(1-D(i)) + gN(i)\right\}, \end{aligned} \tag{4}$$

where the first item indicates the loss due to failed interception, the second item indicates the loss due to failed detection and the third item indicates the cost of intercept missiles. In particular, the first item $(1 - P_I(i))^{N(i)}D(i)$ shows the probability that the attack missile is detected successfully but all its intercept missiles fail to block the attack missile. In contrast, $(1-D(i))$ makes that the second item to be 0 when $D(i) = 1$, and to be positive when $D(i) = 0$, indicating a loss due to the attack missile's hitting target.

The defender will choose $\{N(1), \ldots, N(M)\} = \{N^*(1), \ldots, N^*(M)\}$, which minimizes $C(N(1), \ldots, N(M)|D(1), \ldots, D(M))$. That is, the optimal interception strategy can be expressed as:

$$\begin{aligned} \left\{N^*(1), \ldots, N^*(M)\right\} &= \arg\min_{N(1)+\cdots+N(M)\leq N} \\ &\times C(N(1), \ldots, N(M)|D(1), \ldots, D(M)). \end{aligned} \tag{5}$$

Then, we have the expected total cost as

$$\begin{aligned} C = &\sum_{D(1)=0}^{1} \sum_{D(2)=0}^{1} \cdots \sum_{D(M)=0}^{1} \\ &\times \left[\begin{array}{l} P(D(1), \ldots, D(M)) \\ \cdot C(N^*(1), \ldots, N^*(M)|D(1), \ldots, D(M)) \end{array}\right]. \end{aligned} \tag{6}$$

This scenario is shown as Figure 4. It can be seen that the third attack missile is undetected so that the defender cannot take action to respond to this attack.

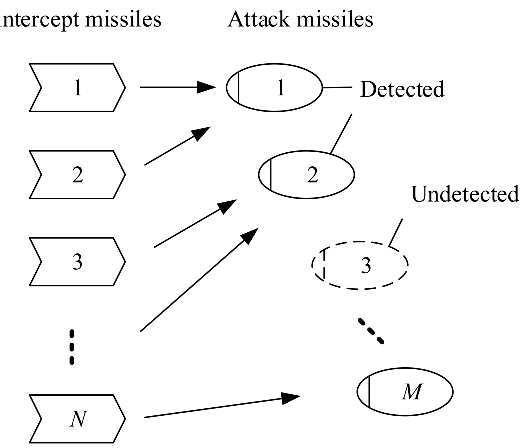

FIGURE 4. **Missile interception of Scenario 2.**

III. THE MODEL OF OPTIMAL STRATEGY FOR MULTI-PHASED INTERCEPTION

From launching to hitting, the flight of missile usually consists of several phases. For instance, there are three phases for the ballistic missile to attack its target: 1) the missile is launched from its launcher and reaches high above the atmosphere of Earth, which is called initial phase; 2) the missile flies to the target area in the outer space, that is, midcourse phase; 3) over the target area the missile returns the atmosphere and hits its target, namely terminal phase. The attacking process of the ballistic missile is shown in Figure 5.

In order to be able to account for such situation, we construct the model for the case where the process of missile attack is multi-phased. Assume that the defender has N intercept missiles and the attacker has M attack missiles in total. The flight of attack missile from launcher to target consists of H phases. For each attack missile, the defender can choose to launch zero, one, or multiple intercept missiles to intercept it during any one or more phases.

In the case of multiple intercept missiles, it is assumed that the success probability of each intercept missile is independent from other intercept missiles. In particular, the probability that each intercept missile can successfully intercept the attack missile i during k-th phase is $P_I(i, k)$.

The fixed loss due to the failed interception of attack missile i is $C_L(i)$. The interception costs at different phases are different. Assume that the cost for each intercept missile used during phase k is $g(k)$. We assume that there are $N(i, k)$

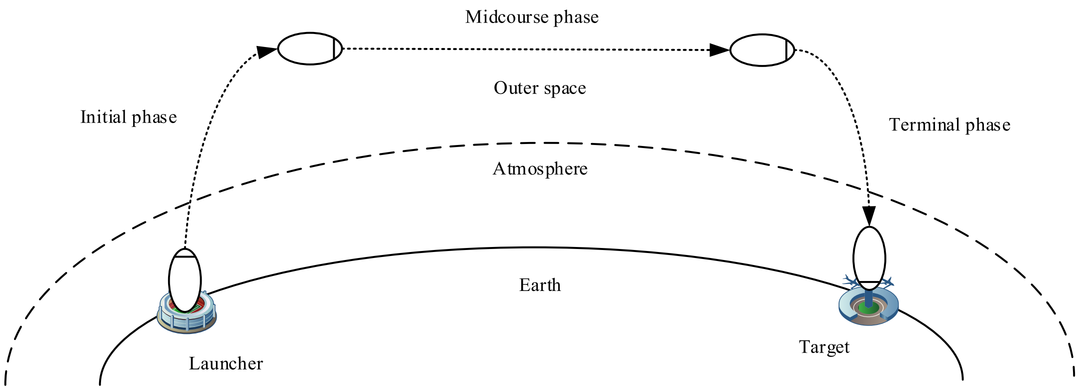

FIGURE 5. Attacking procedure of the ballistic missile.

intercept missiles used to intercept the attack missile i at k-th phase, while $\sum_{i=1}^{M}\left[\sum_{k=1}^{H} N(i,k)\right] \leq N$. Then, let NM indicate the allocation of intercept missiles, that is $NM = \{N(1,1), \ldots, N(1,H), \ldots, N(M,1), \ldots, N(M,H)\}$.

Some will say, if the attack missile is intercepted at the initial phase, there is no need to launch the remaining intercept missiles allocated in advance. However, the time for the missile interception is usually limited, typically up to several minutes. Thus, it is very risky to wait for the result of one intercept missile before launching another one. Therefore, this paper assumes that all the intercept missiles allocated for a single attack missile are launched no matter how early the attack missile is successfully intercepted.

Just as the total cost in single-phased interception, the total cost for the defender in multi-phased interception also includes two parts: the loss due to failed interception and the cost for intercept missiles. Similarly, the aim is to minimize the defender's total cost by adjusting the allocation of intercept missiles for attack missiles.

A. SCENARIO 1: THE DETECTION OF ATTACK MISSILE IS PERFECT

In this scenario, all attack missiles are detected successfully in time. Then, the loss due to the intercept failure can be expressed as

$$\sum_{i=1}^{M}\left\{C_L(i)\prod_{k=1}^{H}\left[(1-P_I(i,k))^{N(i,k)}\right]\right\}. \tag{7}$$

In this equation, $\prod_{k=1}^{H}\left[(1-P_I(i,k))^{N(i,k)}\right]$ expresses the probability that the interception of attack missile i is failed.

Then, adding up the cost for intercept missiles, we can have the total cost

$$C = \sum_{i=1}^{M}\left\{C_L(i)\prod_{k=1}^{H}\left[(1-P_I(i,k))^{N(i,k)}\right] + \sum_{k=1}^{H}[N(i,k)g(k)]\right\}, \quad \sum_{i=1}^{M}\left[\sum_{k=1}^{H} N(i,k)\right] \leq N, \tag{8}$$

where the first item in the brace indicates the total loss due to failed interception and the second item in the brace indicates the cost of intercept missiles. The defender can choose the optimal allocation strategy of intercept missiles NM^* by minimizing the total cost.

B. SCENARIO 2: THE DETECTION OF ATTACK MISSILE IS IMPERFECT

As discussed above, it is hard to detect the attack missile successfully or timely in some conditions. Therefore, we also study the case where the detection of attack missile is imperfect for the multi-phased interception in this scenario. As an initial work on missile interception, here also assumes that an attack missile is either not detected until it hits the target or detected at the beginning of its launch.

The functions and parameters for this scenario are listed as follows. Let $P_d(i)$ indicate the probability that the attack missile i is detected successfully. Let $D(i)$ indicate whether the attack missile i is detected. Then we can have the joint probability for the different situations of detection as

$$P(D(1), \ldots, D(M)) = \prod_{i=1}^{M} P_d(i)^{D(i)} \prod_{i=1}^{M}\left[(1-P_d(i))^{(1-D(i))}\right], \tag{9}$$

where $D(i) = 1$ indicates that the detection is successful and $D(i) = 0$ indicates that the detection is failed.

Then we can obtain the total cost for the allocation of NM under the detecting condition of $D(1), \ldots, D(M)$, which is

$$C(NM|D(1), \ldots, D(M)) = \sum_{i=1}^{M}\left\{D(i)C_L(i)\prod_{k=1}^{H}\left[(1-P_I(i,k))^{N(i,k)}\right] + C_L(i)(1-D(i)) + \sum_{k=1}^{H}[N(i,k)g(k)]\right\}, \tag{10}$$

where the first item in the brace indicates the loss due to failed interception, the second item in the brace indicates the loss due to undetected attack and the third item in the brace indicates the cost of intercept missiles. In detail, when $D(i) = 1$, the first item $(1 - P_I(i))^{N(i)}D(i)$ shows the probability that the attack missile is detected successfully but all its intercept missiles are missed. In contrast, $(1 - D(i))$ makes the second item to be 0 when $D(i) = 1$, and positive when $D(i) = 0$, indicating a loss due to attack missile's hitting target.

The defender will choose

$$\{N(1,1),\ldots,N(1,H),\ldots,N(M,1),\ldots,N(M,H)\} = \left\{N^*(1,1),\ldots,N^*(1,H),\ldots,N^*(M,1),\ldots,N^*(M,H)\right\}, \tag{11}$$

that is, $NM = NM^*$, which minimizes $C(NM|D(1),\ldots, D(M))$. That is, the optimal interception strategy can be expressed as:

$$NM^* = \arg\min_{\sum_{i=1}^{M}\left[\sum_{k=1}^{H} N(i,k)\right]\leq N} C(NM|D(1),\ldots,D(M)). \tag{12}$$

Then, we have the expected total cost as

$$C = \sum_{D(1)=0}^{1}\sum_{D(2)=0}^{1}\cdots\sum_{D(M)=0}^{1}\begin{bmatrix} P(D(1),\ldots,D(M)) \\ \cdot C(NM^*|D(1),\ldots,D(M)) \end{bmatrix}. \tag{13}$$

IV. CUMULATIVE PROSPECT THEORY (CPT)

Cumulative prospect theory (CPT) was first proposed by Tversky and Kahneman [33]. It is a descriptive theory for human decision behavior, which is applied to uncertain as well as to risky prospects with any number of outcomes. CPT is advantageous in analyzing human decision behavior. This theory allows different weighting functions for gains and for losses and invokes two principles, diminishing sensitivity and loss aversion, to explain the characteristic curvature of the value function and the weighting functions. Generally, it can be seen as a combination of the original prospect theory [32] and the rank dependent expected utility model [38].

Thus, in this paper, a decision analysis method based on CPT for solving the risk decision-making problem in missile interception is developed. In this method, defender's behavioral characteristics are considered. Based on CPT, the values of outcomes and weights of probability of outcomes are calculated, respectively. Then, the prospect value of each candidate allocation strategy of intercept missiles is calculated by aggregating the obtained values and weights. Thus, a ranking of all candidate allocation strategies can be determined according to the obtained prospect values. In addition, for the scenarios that the detection is imperfect, an overall prospect value of the interception can be assessed by aggregating the prospect values of all the possible situations through by a total probability formula. The detailed process of the CPT-based method is presented as below.

A. INTERCEPTION DESCRIPTION AND ASSUMPTION

Assume the missile interception result is composed of n potential monetary outcomes $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$. Each outcome is obtained with a corresponding probability, that is, assume x_i is the ith potential outcome then p_i is the probability of potential outcome x_i, $i = 1, 2, \ldots, n$. To assess the value of the missile interception in cognitive psychology, the n outcomes, $x_1, x_2, \ldots, x_n$, are reorganized as a sequence from the largest to the smallest, noted as $x_{(1)} \geq \ldots \geq x_{(t)} \geq 0 \geq x_{(t+1)} \geq \ldots \geq x_n$. In this sequence, $x_{(k)}$ denotes the kth largest one among the n potential outcomes, $k \in \{1, 2, \ldots, n\}$; 0 is the reference point, which denotes the outcome if the defender will not participate in the missile interception or the attacker will not launch attack missiles. According to the sequence of potential outcomes, probabilities $p_1, p_2, \ldots, p_n$ are also reorganized as $p_{(1)}, p_{(2)}, \ldots, p_{(n)}$, where $p(k)$ denotes the probability of potential outcome $x_{(k)}$, $k \in \{1, 2, \ldots, n\}$. To calculate the prospect value, we need to determine the value of potential outcome and the decision weights first.

B. VALUE OF POTENTIAL OUTCOME

The value of potential outcome $x_{(k)}$, $v(x_{(k)})$ can be represented by

$$v(x_{(k)}) = \begin{cases} x_{(k)}^{g}, & x_{(k)} > 0, \\ -\lambda(-x_{(k)})^{l}, & x_{(k)} < 0, \end{cases} \tag{14}$$

where g and l are exponent parameters, and λ is the loss aversion parameter. For $0 < g < 1$, the value function exhibits risk aversion over gains; for $0 < l < 1$, the function exhibits risk seeking over losses. The smaller g is, the greater risk aversion in the gain domain will be. Similarly, the smaller l is, the greater risk seeking in the loss domain will be. It has been widely recognized that loss-aversion factor λ should be greater than 1, which indicates that individuals are more sensitive to losses than gains. Usually, the values of parameters g, l and λ are determined through experiments [33], [35].

C. DECISION WEIGHTS

Assume $\pi_{(k)}^{+}$ is the decision weight for the value of potential gain $x_{(k)}$, $k \in 1, 2, \ldots, t$ and $\pi_{(k)}^{-}$ is the decision weight for the value of potential loss $x_{(k)}$, $k \in 1, 2, \ldots, n$. The decision weights for gains and losses can be expressed as

$$\begin{cases} \pi_{(k)}^{+} = w^{+}\left(\sum_{j=1}^{n} p_{(j)}\right) - w^{+}\left(\sum_{j=k+1}^{n} p_{(j)}\right) \\ \pi_{(k)}^{-} = w^{-}\left(\sum_{j=1}^{k} p_{(j)}\right) - w^{-}\left(\sum_{j=1}^{k-1} p_{(j)}\right) \end{cases} \tag{15}$$

where $w^{+}(\cdot)$ and $w^{-}(\cdot)$ denote the weighting functions for gains and losses, respectively, and they are given by [33], [35]

$$\begin{cases} w^{+}(p) = \dfrac{p^{\chi}}{[p^{\chi} + (1-p)^{\chi}]^{1/\chi}} \\ w^{-}(p) = \dfrac{p^{\delta}}{\left[p^{\delta} + (1-p)^{\delta}\right]^{1/\delta}} \end{cases}, \tag{16}$$

where χ and δ are model parameters. $w^{+}(\cdot)$ and $w^{-}(\cdot)$ are monotonic and exhibit inverse S-shapes for $0.27 < \chi, \delta < 1$. They are adequate for average decision-making behavior (i.e., overweight the outcomes with low probabilities and underweight the outcomes with moderate and high probabilities) [33], [35]. Specially, if $\chi = \delta = 1$, then $\pi^{+}_{(k)} = \pi^{-}_{(k)} = p_{(k)}$, i.e., the decision weights are equal to physical probabilities. The values of parameters χ and δ can also be determined through experiments [33], [35].

D. PROSPECT VALUE

After we have values of outcomes and decision weights, the prospect value of the contest can be calculated as

$$V = \sum_{k=1}^{t} v(x_{(k)})\pi^{+}_{(k)} + \sum_{k=t+1}^{n} v(x_{(k)})\pi^{-}_{(k)}. \tag{17}$$

Thus, the defender chooses the allocation strategy of intercept missiles that maximizes its cumulative prospect value by considering all possible outcomes of the contest.

V. ILLUSTRATIVE EXAMPLES OF THE MODEL FOR SINGLE-PHASED INTERCEPTION

To illustrate the proposed model of single-phased interception, we use two numerical examples to show the calculation process and the application. The first case is for the scenario 1 where the detection is perfect. The second case illustrates the scenario 2 where there is a probability of successfully detecting each attack missile.

For the convenience of comparison, we assume there are two attack missiles (marked as 1 and 2) in all the following different cases. For each attack missile, we have 2 potential outcomes for the defender: the attack missile is intercepted or not. Their potential monetary outcomes are denoted as $x_{1,1}, x_{1,0}, x_{2,1}, x_{2,0}$, where $x_{i,c}$ indicates that the attack missile i is intercepted ($c = 1$) or not ($c = 0$). It is assumed that $x_{1,1} = 300$, $x_{1,0} = -600$, $x_{2,1} = 100$ and $x_{2,0} = -300$ since the failed interception of any one attack missile will cause loss to the defender. As for the values of risk parameters and weighting function parameters in CPT, the settings of a previous literature is adopted, which are $g = 0.85$, $l = 0.85$, $\lambda = 4.1$, $\chi = 0.6$ and $\delta = 0.7$.

A. CASE 1: THE DETECTION OF ATTACK MISSILES IS PERFECT

Assume the number of attack missiles M and the number of intercept missiles N are both 2. The cost of an intercept missile is set as 35. The probability of successful interception and the loss due to failed interception are shown in Table 1. There are totally 6 possible allocations as listed in Table 2.

TABLE 1. The probability of successful interception and the loss due to failed interception for the case of perfect detection and single-phased interception.

Attack missile	$P_I(i)$	$C_L(i)$
1	0.4	190
2	0.6	150

TABLE 2. The possible allocations and outcomes of intercept missiles for the case of perfect detection and single-phased interception, as well as the corresponding cost and prospect value.

Allocations	$N(1)$	$N(2)$	Cost	Prospect value
1	0	2	284	-1053.6
2	0	1	285	-1160.1
3	0	0	340	-1465.2
4	1	1	244	-721.3
5	1	0	299	-1026.3
6	2	0	288.4	-863.4

From Table 1, it is known that the impact of the attack missile 1 is stronger and the corresponding interception is harder compared with the attack missile 2.

Then we can calculate the total costs according to equations (1) and (2), and the prospect value according to equations (14)-(17), for these situations.

When $N(1) = 0$ and $N(2) = 2$,

$$\begin{aligned} C &= \sum_{i=1}^{M}\left[(1-P_I(i))^{N(i)}C_L(i) + gN(i)\right] \\ &= (1-P_I(1))^{N(1)}C_L(1) + (1-P_I(2))^{N(2)}C_L(2) \\ &\quad + gN(1) + gN(2) = (1-0.4)^0 \times 190 \\ &\quad + (1-0.6)^2 \times 150 + 35 \times 0 + 35 \times 2 = 284. \end{aligned}$$

It can be seen that the main cost in this situation comes from the loss caused by the attack missile 1 since all intercept missiles are used to destroy the attack missile 2. The prospect value is -1053.6.

When $N(1) = 0$ and $N(2) = 1$,

$$\begin{aligned} C &= \sum_{i=1}^{M}\left[(1-P_I(i))^{N(i)}C_L(i) + gN(i)\right] \\ &= (1-P_I(1))^{N(1)}C_L(1) + (1-P_I(2))^{N(2)}C_L(2) \\ &\quad + gN(1) + gN(2) = (1-0.4)^0 \times 190 \\ &\quad + (1-0.6)^1 \times 150 + 35 \times 0 + 35 \times 1 = 285. \end{aligned}$$

Compared with the allocation of $N(1) = 0$ and $N(2) = 2$, the attack missile 1 still brings the most loss to the defender, since the cost of one more intercept missile is generally the same as the expected loss deduction. The prospect value is −1160.1.

When $N(1) = 0$ and $N(2) = 0$,

$$C = \sum_{i=1}^{M} \left[(1 - P_I(i))^{N(i)} C_L(i) + gN(i) \right]$$
$$= (1 - P_I(1))^{N(1)} C_L(1) + (1 - P_I(2))^{N(2)} C_L(2) + gN(1) + gN(2) = (1 - 0.4)^0 \times 190 + (1 - 0.6)^0 \times 150 + 35 \times 0 + 35 \times 0 = 340.$$

In this allocation strategy, the defender gives up on intercepting any attack missile so that these attack missiles cause the most loss to the defender. The prospect value is −1465.2.

When $N(1) = 1$ and $N(2) = 1$,

$$C = \sum_{i=1}^{M} \left[(1 - P_I(i))^{N(i)} C_L(i) + gN(i) \right]$$
$$= (1 - P_I(1))^{N(1)} C_L(1) + (1 - P_I(2))^{N(2)} C_L(2) + gN(1) + gN(2) = (1 - 0.4)^1 \times 190 + (1 - 0.6)^1 \times 150 + 35 \times 1 + 35 \times 1 = 244.$$

The defender evenly divides intercept missiles for attack missiles in this situation, which makes both two attack missiles be possible to be destroyed. The prospect value is −721.3.

When $N(1) = 1$ and $N(2) = 0$,

$$C = \sum_{i=1}^{M} \left[(1 - P_I(i))^{N(i)} C_L(i) + gN(i) \right]$$
$$= (1 - P_I(1))^{N(1)} C_L(1) + (1 - P_I(2))^{N(2)} C_L(2) + gN(1) + gN(2) = (1 - 0.4)^1 \times 190 + (1 - 0.6)^0 \times 150 + 35 \times 1 + 35 \times 0 = 299.$$

The defender gives up on intercepting the attack missile 2 so that the main cost comes from the loss due to the second attack. However, because the loss due to the attack missile 1 is higher and its probability of being intercepted is lower compared with the attack missile 2, the total cost for this allocation is still higher than the total cost for the allocation of $N(1) = 0$ and $N(2) = 2$. The prospect value is −1026.3.

When $N(1) = 2$ and $N(2) = 0$,

$$C = \sum_{i=1}^{M} \left[(1 - P_I(i))^{N(i)} C_L(i) + gN(i) \right]$$
$$= (1 - P_I(1))^{N(1)} C_L(1) + (1 - P_I(2))^{N(2)} C_L(2) + gN(1) + gN(2) = (1 - 0.4)^2 \times 190 + (1 - 0.6)^0 \times 150 + 35 \times 2 + 35 \times 0 = 288.4.$$

From this allocation we know that the defender concentrates its resources on the attack missile 1. Accordingly, the loss is mainly from the second attack. However, due to the great impact of the first attack, the corresponding loss also cannot be ignored. The prospect value is −863.4.

It can be seen that the total cost reaches its minimum 244 and the prospect value reaches its maximum −721.3 when $N(1) = 1$ and $N(2) = 1$, that is, deploying one intercept missile for each attack missile.

B. CASE 2: THE DETECTION OF ATTACK MISSILES IS IMPERFECT

Assume the number of attack missiles M and the number of intercept missiles N are both 2. The cost of an intercept missile is set as 35. The probability of successful interception and the loss due to failed interception are shown in Table 3. There are totally 4 possible detection situations, and each situation has several allocation strategies of intercept missiles which are listed in Table 4.

As shown in Table 3, the attack missiles are generally the same as Case 1. In addition, it is more difficult for defender to detect the attack missile 1 than the second one.

Detection Situation 1 ($D(1) = 1$ and $D(2) = 0$): In this situation, the attack missile 1 is detected but the attack missile 2 is not. Herein, the only intercepted attack missile is the attack missile 1 and there are three allocation policies: $[N(1), N(2)] = [0, 0], [1, 0], [2, 0]$.

TABLE 3. The probability of successful detection and interception as well as the loss due to failed interception for the case of perfect detection and single-phased interception.

Attack missile	$P_d(i)$	$P_I(i)$	$C_L(i)$
1	0.5	0.4	190
2	0.7	0.6	150

TABLE 4. The possible scenarios of detection and allocation strategy of attack missiles for the case of perfect detection and single-phased interception.

Scenarios	Detections		Allocations	
	$D(1)$	$D(2)$	$N(1)$	$N(2)$
	1	0		
1			0	0
2			1	0
3			2	0
	0	1		
4			0	0
5			0	1
6			0	2
	1	1		
7			0	2
8			0	1
9			0	0
10			1	1
11			1	0
12			2	0
	0	0		
13			0	0

According to equation (3), the probability of this situation is

$$P(1,0)=\prod_{i=1}^{2}P_d(i)^{D(i)}\prod_{i=1}^{2}\left[(1-P_d(i))^{(1-D(i))}\right]$$
$$=P_d(1)^{D(1)}P_d(2)^{D(2)}(1-P_d(1))^{(1-D(1))}$$
$$\times(1-P_d(2))^{(1-D(2))}=0.5^1\times0.7^0$$
$$\times(1-0.5)^{(1-1)}\times(1-0.7)^{(1-0)}=0.15.$$

Then we can calculate the total cost according to equation (4), and the prospect value according to equations (14)-(17), for this situation.

$$C(0,0|1,0)=\sum_{i=1}^{2}C_L(i)(1-P_I(i))^{N(i)}D(i)$$
$$+\sum_{i=1}^{2}C_L(i)(1-D(i))+\sum_{i=1}^{2}gN(i)$$
$$=C_L(1)(1-P_I(1))^{N(1)}D(1)$$
$$+C_L(2)(1-P_I(2))^{N(2)}D(2)$$
$$+C_L(1)(1-D(1))+C_L(2)(1-D(2))$$
$$+gN(1)+gN(2)=190\times(1-0.4)^0\times1$$
$$+150\times(1-0.6)^0\times0+190\times(1-1)$$
$$+150\times(1-0)+35\times0+35\times0=340.$$

Here $C(0,0|1,0)$ represents the expected cost when the defense strategy is $N(1)=0$, $N(2)=0$ given that the detection result is $D(1)=1$ and $D(2)=0$. The prospect value is $V(0,0|1,0)=-1465.2$. Similarly, we also have $C(1,0|1,0)=299$, $V(1,0|1,0)=-1026.3$ and $C(2,0|1,0)=288.4$, $V(2,0|1,0)=-863.4$.

Finally, according to the result, the defender will choose $N^*(1)=2$ and $N^*(2)=0$, that is, using two intercept missiles to intercept the attack missile 1, in order to minimize the total cost and maximize the prospect value.

Detection Situation 2 ($D(1)=0$ and $D(2)=1$): In this situation, the attack missile 2 is detected but the attack missile 1 is not. Herein, the only intercept target is the attack missile 2 and there are three allocation policies: $[N(1),N(2)]=[0,0],[0,1],[0,2]$.

According to equation (3), the probability of this situation as

$$P(0,1)=\prod_{i=1}^{2}P_d(i)^{D(i)}\prod_{i=1}^{2}\left[(1-P_d(i))^{(1-D(i))}\right]$$
$$=P_d(1)^{D(1)}P_d(2)^{D(2)}(1-P_d(1))^{(1-D(1))}$$
$$\times(1-P_d(2))^{(1-D(2))}=0.5^0\times0.7^1$$
$$\times(1-0.5)^{(1-0)}\times(1-0.7)^{(1-1)}=0.35.$$

Then we can calculate the total cost according to equation (4), and the prospect value according to equations (14)-(17), for this situation.

$$C(0,0|0,1)=\sum_{i=1}^{2}C_L(i)(1-P_I(i))^{N(i)}D(i)$$
$$+\sum_{i=1}^{2}C_L(i)(1-D(i))+\sum_{i=1}^{2}gN(i)$$
$$=C_L(1)(1-P_I(1))^{N(1)}D(1)$$
$$+C_L(2)(1-P_I(2))^{N(2)}D(2)$$
$$+C_L(1)(1-D(1))+C_L(2)(1-D(2))$$
$$+gN(1)+gN(2)=190\times(1-0.4)^0\times0$$
$$+150\times(1-0.6)^0\times1+190\times(1-0)$$
$$+150\times(1-1)+35\times0+35\times0=340.$$

The prospect value is $V(0,0|0,1)=-1465.2$. Similarly, we can also have $C(0,1|0,1)=285$, $V(0,1|0,1)=-1160.1$ and $C(0,2|0,1)=284$, $V(0,2|0,1)=-1053.6$.

Finally, according to the result, the defender will choose $N^*(1)=0$ and $N^*(2)=2$, that is, using two intercept missiles to intercept the attack missile 1, in order to minimize the total cost and maximize the prospect value.

Detection Situation 3 ($D(1)=1$ and $D(2)=1$): In this situation, both the first and attack missile 2s are detected. Herein, there are two attack missiles to be intercepted, which makes this situation to degenerate to case 1. There are six allocation policies: $[N(1),N(2)]=[0,0],[0,1],[0,2],[1,1],[1,0],[2,0]$.

According to equation (3), the probability of this situation is

$$P(1,1)=\prod_{i=1}^{2}P_d(i)^{D(i)}\prod_{i=1}^{2}\left[(1-P_d(i))^{(1-D(i))}\right]$$
$$=P_d(1)^{D(1)}P_d(2)^{D(2)}(1-P_d(1))^{(1-D(1))}$$
$$\times(1-P_d(2))^{(1-D(2))}=0.5^1\times0.7^1$$
$$\times(1-0.5)^{(1-1)}\times(1-0.7)^{(1-1)}=0.35.$$

Then, according to the results of case 1, the defender will choose $N^*(1)=1$ and $N^*(2)=1$, that is, deploying one intercept missile for each attack missile, in order to minimize the total cost and maximize the prospect value.

Detection Situation 4 ($D(1)=0$ and $D(2)=0$): In this situation, both the first and attack missile 2s are undetected. Herein, there is no target attack missile to be intercepted. There is only one allocation strategy: $[N(1),N(2)]=[0,0]$.

According to equation (3), the probability of this situation as

$$P(0,0)=\prod_{i=1}^{2}P_d(i)^{D(i)}\prod_{i=1}^{2}\left[(1-P_d(i))^{(1-D(i))}\right]$$
$$=P_d(1)^{D(1)}P_d(2)^{D(2)}(1-P_d(1))^{(1-D(1))}$$
$$\times(1-P_d(2))^{(1-D(2))}=0.5^0\times0.7^0$$
$$\times(1-0.5)^{(1-0)}\times(1-0.7)^{(1-0)}=0.15.$$

TABLE 5. The cost and prospect value of the possible scenarios.

Scenarios	Cost	CPT
1	340	-1465.2
2	299	-1026.3
3	288.4	-863.4
4	340	-1465.2
5	285	-1160.1
6	284	-1053.6
7	284	-1053.6
8	285	-1160.1
9	340	-1465.2
10	244	-721.3
11	299	-1026.3
12	288.4	-863.4
13	340	-1465.2
Expected	279.06	-970.505

Then we can calculate the total cost according to equation (4), and the prospect value according to equations (14)-(17), for this situation.

$$\begin{aligned} C(0,0|0,0) &= \sum_{i=1}^{2} C_L(i)(1-P_I(i))^{N(i)}D(i) \\ &\quad + \sum_{i=1}^{2} C_L(i)(1-D(i)) + \sum_{i=1}^{2} gN(i) \\ &= C_L(1)(1-P_I(1))^{N(1)}D(1) \\ &\quad + C_L(2)(1-P_I(2))^{N(2)}D(2) \\ &\quad + C_L(1)(1-D(1)) + C_L(2)(1-D(2)) \\ &\quad + gN(1) + gN(2) = 190 \times (1-0.4)^0 \times 0 \\ &\quad + 150 \times (1-0.6)^0 \times 0 + 190 \times (1-0) \\ &\quad + 150 \times (1-0) + 35 \times 0 + 35 \times 0 = 340. \end{aligned}$$

And the prospect value $V(0,0|0,0) = -1465.2$.

The cost and prospect value of the possible scenarios are listed in Table 5.

The Expected Total Cost: Finally, we can obtain the expected total cost, which is

$$\begin{aligned} C &= \sum_{D(1)=0}^{1} \left\{ \sum_{D(2)=0}^{1} \left[\begin{array}{l} P(D(1),D(2)) \\ \cdot C(N^*(1),N^*(2)|D(1),D(2)) \end{array} \right] \right\} \\ &= P(0,0)C(0,0|0,0) + P(0,1) \cdot C(0,2|0,1) \\ &\quad + P(1,0)C(2,0|1,0) + P(1,1)C(1,1|1,1) \\ &= 0.15 \times 340 + 0.35 \times 284 + 0.15 \times 288.4 \\ &\quad + 0.35 \times 244 = 279.06, \end{aligned}$$

and the expected prospect value, which is

$$\begin{aligned} V &= \sum_{D(1)=0}^{1} \left\{ \sum_{D(2)=0}^{1} \left[\begin{array}{l} P(D(1),D(2)) \\ \cdot V(N^*(1),N^*(2)|D(1),D(2)) \end{array} \right] \right\} \\ &= P(0,0)V(0,0|0,0) + P(0,1) \cdot V(0,2|0,1) \\ &\quad + P(1,0)V(2,0|1,0) + P(1,1)V(1,1|1,1) \\ &= 0.15 \times (-1465.2) + 0.35 \times (-1053.6) \\ &\quad + 0.15 \times (-863.4) + 0.35 \times (-721.3) = -970.505. \end{aligned}$$

VI. ILLUSTRATIVE EXAMPLES OF THE MODEL FOR MULTI-PHASED INTERCEPTION

To illustrate the proposed model of multi-phased interception, we use two numerical examples to show the calculation process and the application. The first case is from the scenario 1 where the detection is perfect. The second case illustrates the scenario 2 where there is a probability of successfully detecting each attack missile. We use the practice of ballistic missile to provide examples in this section. As shown in Figure 4, there are 3 phases, and numerical examples are shown as below. To facilitate the comparison, the number of attack missiles, the potential monetary outcomes as well as the values of risk parameters and weighting function parameters in CPT are set as the same as Section IV.

A. CASE 1: THE DETECTION OF ATTACK MISSILES IS PERFECT

Assume the number of attack missiles M is 2 and the number of intercept missiles N is 4. The probability of successful interception at different phases and the loss due to failed interception are shown in Table 6. The costs of an intercept missile used at different phases are shown in Table 7.

TABLE 6. The probability of successful interception and the loss due to failed interception for the case of perfect detection and multi-phased interception.

Attack missile	$P_I(i,1)$	$P_I(i,2)$	$P_I(i,3)$	$C_L(i)$
1	0.4	0.5	0.2	190
2	0.6	0.7	0.3	150

TABLE 7. The costs of an intercept missile used at different phases for the case of perfect detection and multi-phased interception.

$g(1)$	$g(2)$	$g(3)$
35	40	20

From Table 6, it is known that the impact of the attack missile 1 is stronger and the corresponding interception is harder compared with the attack missile 2. In addition, it is the best chance for defender to intercept the attack missile during phase 2, since in this phase the ballistic missile flies steadily compared with other two phases.

As the number of possible intercept allocations is large (146) in this case, we only use one of them to illustrate the calculation process of the proposed model. The allocation strategy *NM* is shown as below,

$$NM = \{1, 1, 0, 1, 1, 0\}.$$

Then we can calculate the total cost according to equation (4), and the prospect value according to equations (14)-(17), for this situation.

$$C = \sum_{i=1}^{2}\left\{C_L(i)\prod_{k=1}^{3}\left[(1-P_I(i,k))^{N(i,k)}\right]+\sum_{k=1}^{3}[N(i,k)g(k)]\right\}$$
$$= \left\{190\prod_{k=1}^{3}\left[(1-P_I(1,k))^{N(1,k)}\right]+\sum_{k=1}^{3}[N(1,k)g(k)]\right\}$$
$$+\left\{150\prod_{k=1}^{3}\left[(1-P_I(i,k))^{N(1,k)}\right]+\sum_{k=1}^{3}[N(i,k)g(k)]\right\}$$
$$= 190\times(1-0.4)\times(1-0.5)+35+40$$
$$+150\times(1-0.6)\times(1-0.7)+35+40=225.$$

The prospect value in this −382.8. After calculating the total costs and the prospect values of all the possible allocations, we find the optimal allocation strategy of the intercept missile. However, the optimal allocation strategies considering total cost or prospect value is different. The optimal allocation strategy that minimizes the total cost is

$$NM_C^* = \{0,2,0,0,1,0\},$$

where its total cost is 212.5.

In contrast, the optimal allocation strategy that maximizes the prospect value is

$$NM_V^* = \{0,3,0,0,1,0\},$$

where its prospect value is −227.9.

B. CASE 2: THE DETECTION OF ATTACK MISSILES IS IMPERFECT

Assume the number of attack missiles M is 2 and the number of intercept missiles N is 4. The probability of successful interception at different phases and the loss due to failed interception are shown in Table 8. The cost of an intercept missile used at different phases are the same as in Case 1 (shown in Table 7).

TABLE 8. The probability of successful interception and the loss due to failed interception for the case of perfect detection and multi-phased interception.

Attack missile	$P_d(i)$	$P_I(i,1)$	$P_I(i,2)$	$P_I(i,3)$	$C_L(i)$
1	0.5	0.4	0.5	0.2	190
2	0.7	0.6	0.7	0.3	150

From Table 7, it is known that the impact of the attack missile 1 is stronger and the corresponding interception is harder compared with the attack missile 2. In addition, it is the best chance for defender to intercept the attack missile during phase 2, since in this phase the ballistic missile flies steadily compared with other two phases.

Detection Situation 1 ($D(1) = 1$ and $D(2) = 0$): In this situation, the attack missile 1 is detected but the attack missile 2 is not. Herein, the only intercept target is the attack missile 1.

According to equation (9), the probability of this situation is

$$P(1,0) = \prod_{i=1}^{2}P_d(i)^{D(i)}\prod_{i=1}^{2}\left[(1-P_d(i))^{(1-D(i))}\right]$$
$$= P_d(1)^{D(1)}P_d(2)^{D(2)}(1-P_d(1))^{(1-D(1))}$$
$$\times(1-P_d(2))^{(1-D(2))} = 0.5^1\times0.7^0$$
$$\times(1-0.5)^{(1-1)}\times(1-0.7)^{(1-0)} = 0.15.$$

There are totally 23 possible allocations of intercept missiles in this situation. Then we can calculate the total cost according to equation (10), and the prospect value according to equations (14)-(17), for these allocations. The results show that when $NM = \{0,2,0,0,0,0\}$, the total cost reaches its minimum which is 277.5 and when $NM = \{0,4,0,0,0,0\}$, the prospect value reaches its maximum which is −570.1. Thus, the defender will choose $NM_C^* = \{0,2,0,0,0,0\}$, that is, using two intercept missiles to intercept the attack missile 1 during phase 2, in order to minimize the total cost, and $NM_V^* = \{0,4,0,0,0,0\}$, that is, using four intercept missiles to intercept the attack missile 1 during phase 2, in order to minimize the prospect value.

Detection Situation 2 ($D(1) = 0$ and $D(2) = 1$): In this situation, the attack missile 2 is detected but the attack missile 1 is not. Herein, the only intercept target is the attack missile 2.

According to equation (9), the probability of this situation is

$$P(0,1) = \prod_{i=1}^{2}P_d(i)^{D(i)}\prod_{i=1}^{2}\left[(1-P_d(i))^{(1-D(i))}\right]$$
$$= P_d(1)^{D(1)}P_d(2)^{D(2)}(1-P_d(1))^{(1-D(1))}$$
$$\times(1-P_d(2))^{(1-D(2))} = 0.5^0\times0.7^1$$
$$\times(1-0.5)^{(1-0)}\times(1-0.7)^{(1-1)} = 0.35.$$

Similar as situation 1, there are also totally 23 possible allocations of intercept missiles in this situation. Then we can calculate the total cost according to equation (10), and the prospect value according to equations (14)-(17), for these allocations. The results show that when $NM = \{0,0,0,0,1,0\}$, the total cost reaches its minimum which is 275 and when $NM = \{0,0,0,0,4,0\}$, the prospect value reaches its maximum which is −918.3. Thus, the defender will choose $NM_C^* = \{0,0,0,0,1,0\}$, that is, using one intercept missiles to intercept the attack missile 2 during phase 2, in order to minimize the total cost, and $NM_V^* = \{0,0,0,0,4,0\}$, that is, using four intercept missiles to intercept the attack missile 2 during phase 2, in order to minimize the prospect value.

Detection Situation 3 ($D(1) = 1$ and $D(2) = 1$): In this situation, both the first and attack missile 2s are detected. Herein, there are two attack missiles to be intercepted, which makes this situation degenerate to case 1. According to equation (9), the probability of this situation is

$$P(1,1) = \prod_{i=1}^{2} P_d(i)^{D(i)} \prod_{i=1}^{2} \left[(1-P_d(i))^{(1-D(i))}\right] = P_d(1)^{D(1)} P_d(2)^{D(2)} (1-P_d(1))^{(1-D(1))} \times (1-P_d(2))^{(1-D(2))} = 0.5^1 \times 0.7^1 \times (1-0.5)^{(1-1)} \times (1-0.7)^{(1-1)} = 0.35.$$

Then, according to the results of case 1, the defender has two optimal allocation strategies. In order to minimize the total cost, the defender will choose $NM_C^* = \{0, 2, 0, 0, 1, 0\}$, that is, deploying two intercept missiles to intercept the attack missile 1 during phase 2, and one intercept missile to intercept the attack missile 2 during phase 2. In this allocation, the total cost is 212.5. In order to maximize the prospect value, the defender will choose $NM_V^* = \{0, 3, 0, 0, 1, 0\}$, that is, using three intercept missiles to intercept the attack missile 1 during phase 2 and one intercept missile to intercept the attack missile 2 during phase 2. In this allocation, the prospect value is −227.9.

Detection Situation 4 $D(1) = 0$ and $D(2) = 0$): In this situation, both the first and attack missile 2s are undetected. Herein, there is no target attack missile to be intercepted. There is only one allocation strategy: $NM = NM^* = \{0, 0, 0, 0, 0, 0\}$.

According to equation (9), the probability of this situation as

$$P(0,0) = \prod_{i=1}^{2} P_d(i)^{D(i)} \prod_{i=1}^{2} \left[(1-P_d(i))^{(1-D(i))}\right] = P_d(1)^{D(1)} P_d(2)^{D(2)} (1-P_d(1))^{(1-D(1))} \times (1-P_d(2))^{(1-D(2))} = 0.5^0 \times 0.7^0 \times (1-0.5)^{(1-0)} \times (1-0.7)^{(1-0)} = 0.15.$$

Then we can calculate the total cost and the prospect value for this allocation, just as the same situation discussed above. The total cost is 340 and the prospect value is −1465.2.

The Expected Total Cost: Finally, we can obtain the expected total cost and the expected prospect value, which are

$$C = \sum_{D(1)=0}^{1} \sum_{D(2)=0}^{1} \cdots \sum_{D(M)=0}^{1} \begin{bmatrix} P(D(1), \ldots, D(M)) \\ \cdot C(NM^* | D(1), \ldots, D(M)) \end{bmatrix}$$
$$= P(0,0) \cdot C(0,0,0;0,0,0|0,0) + P(0,1) \cdot C(0,0,0;0,1,0|0,1) + P(1,0) \cdot C(0,2,0;0,0,0|1,0) + P(1,1) \cdot C(0,2,0;0,1,0|1,1)$$
$$= 0.15 \times 340 + 0.35 \times 275 + 0.15 \times 277.5 + 0.35 \times 212.5 = 263.25,$$

and

$$V = \sum_{D(1)=0}^{1} \sum_{D(2)=0}^{1} \cdots \sum_{D(M)=0}^{1} \begin{bmatrix} P(D(1), \ldots, D(M)) \\ \cdot V(NM^* | D(1), \ldots, D(M)) \end{bmatrix}$$
$$= P(0,0) \cdot V(0,0,0;0,0,0|0,0) + P(0,1) \cdot V(0,0,0;0,4,0|0,1) + P(1,0) \cdot V(0,4,0;0,0,0|1,0) + P(1,1) \cdot V(0,3,0;0,1,0|1,1)$$
$$= 0.15 \times (-1465.2) + 0.35 \times (-918.3) + 0.15 \times (-570.1) + 0.35 \times (-227.9) = -706.465.$$

The cost and prospect value of the possible situations are listed in Table 9.

TABLE 9. The possible combinations of detection and allocation strategy of attack missiles for the case of perfect detection and single-phased interception.

Situations	Cost	CPT
1	277.5	-570.1
2	275	-918.3
3	212.5	-227.9
4	340	-1465.2
Expected	263.25	-706.465

From the results we can see that the optimal allocation strategies respectively under the aims of minimizing the cost and maximizing the prospect value are same in most cases. However, they are still slightly different in some scenarios. The reason for this difference is that the cost is a more objective indicator whereas the prospect value is a more subjective indicator. The reason for this phenomenon is that the optimal strategy to maximize the prospect value considers influence of risk attitudes compared with the optimal strategy to minimize the cost.

VII. CONCLUSION

This study proposed a method to optimize the missile interception strategy (intercept missile allocation) by simulating the games of missile interception, which jointly considering multiple phases, multiple targets, cost and the risk attitude of the defender. Based on combinatorial mathematics and CPT, the proposed method models the process of the missile interception and provides the total cost and prospect value for strategy optimization. Both single-phased and multi-phased interception are considered in this model. Besides, this model includes two scenarios: the detection for attack missiles is perfect or imperfect. In addition, four illustrative examples are provided for the cases where the interception is single-phased or multi-phased and whether the detection is perfect. The results of the examples show that our model can find the optimal strategy of missile intercept by minimizing the expected total cost or the expected prospect value.

This work can be extended in several directions. First, detection probability can be further considered as a random distribution, which reflects the intelligence contest in practice. Second, it is interesting to study the optimal attack strategy given that the defender always responds with the most effective intercept strategy. Third, the optimal deployment of intercept missiles can be studied together with other weapons. For example, it must be interesting to study the deployment of missiles and bombers for the aircraft carrier.

REFERENCES

[1] D. E. Hoffman, *The Dead Hand: The Untold Story of the Cold War Arms Race and Its Dangerous Legacy*. New York, NY, USA: Anchor, 2010.
[2] K. B. Payne, ''Post-cold war deterrence and missile defense,'' *Orbis*, vol. 39, no. 2, pp. 201–233, 1995.
[3] C. L. Glaser and S. Fetter, ''National missile defense and the future of U.S. Nuclear weapons policy,'' *Int. Secur.*, vol. 26, no. 1, pp. 40–92, Jul. 2001.
[4] T. A. Postol, ''The science and technology of the US national Missile defense system,'' in *Proc. APS March Meeting*. American Physical Soc., 2010, Paper X42.014.
[5] W. Bi, C. Chen, and K. Zhang, ''Optimal strategy of attack-defense interaction over load frequency control considering incomplete information,'' *IEEE Access*, vol. 7, pp. 75342–75349, 2019.
[6] R. Peng, G. Levitin, M. Xie, and S. H. Ng, ''Defending simple series and parallel systems with imperfect false targets,'' *Rel. Eng. Syst. Saf.*, vol. 95, no. 6, pp. 679–688, Jun. 2010.
[7] R. Peng, G. Levitin, M. Xie, and S. H. Ng, ''Optimal defence of single object with imperfect false targets,'' *J. Oper. Res. Soc.*, vol. 62, no. 1, pp. 134–141, Jan. 2011.
[8] R. Peng, Q. Q. Zhai, and G. Levitin, ''Defending a single object against an attacker trying to detect a subset of false targets,'' *Rel. Eng. Syst. Saf.*, vol. 149, pp. 137–147, May 2016.
[9] K. Gao, X. Yan, X.-D. Liu, and R. Peng, ''Object defence of a single object with preventive strike of random effect,'' *Rel. Eng. Syst. Saf.*, vol. 186, pp. 209–219, Jun. 2019.
[10] H. Zhang, L. Jiang, S. Huang, J. Wang, and Y. Zhang, ''Attack-defense differential game model for network defense strategy selection,'' *IEEE Access*, vol. 7, pp. 50618–50629, 2019.
[11] X. Chen, X. Liu, L. Zhang, and C. Tang, ''Optimal defense strategy selection for spear-phishing attack based on a multistage signaling game,'' *IEEE Access*, vol. 7, pp. 19907–19921, 2019.
[12] D. Wu, H. Xiao, and R. Peng, ''Object defense with preventive strike and false targets,'' *Rel. Eng. Syst. Saf.*, vol. 169, pp. 76–80, Jan. 2018.
[13] H. Zhang, K. Zheng, X. Wang, S. Luo, and B. Wu, ''Efficient strategy selection for moving target defense under multiple attacks,'' *IEEE Access*, vol. 7, pp. 65982–65995, 2019.
[14] R. Peng, D. Wu, M. Sun, and S. Wu, ''An attack-defense game on interdependent networks,'' *J. Oper. Res. Soc.*, to be published.
[15] D. Wu, X. Yan, R. Peng, and S. Wu, ''Risk-attitude-based defense strategy considering proactive strike, preventive strike and imperfect false targets,'' *Rel. Eng. Syst. Saf.*, vol. 196, Apr. 2020, Art. no. 106778.
[16] C. Lin, H. Xiao, G. Kou, and R. Peng, ''Defending a series system with individual protection, overarching protection, and disinformation,'' *Rel. Eng. Syst. Saf.*, vol. 204, Dec. 2020, Art. no. 107131.
[17] H. Al-Hamadi, I.-R. Chen, D.-C. Wang, and M. Almashan, ''Attack and defense strategies for intrusion detection in autonomous distributed IoT systems,'' *IEEE Access*, vol. 8, pp. 168994–169009, 2020.
[18] H. Xiao, C. Lin, G. Kou, and R. Peng, ''Optimal resource allocation for defending k-out-of-n systems against sequential intentional and unintentional impacts,'' *Rel. Eng. Syst. Saf.*, vol. 201, Sep. 2020, Art. no. 106952.
[19] Q. Zhai, Z.-S. Ye, R. Peng, and W. Wang, ''Defense and attack of performance-sharing common bus systems,'' *Eur. J. Oper. Res.*, vol. 256, no. 3, pp. 962–975, Feb. 2017.
[20] J. Shinar, Y. Lipman, and M. Zarkh, ''Mixed strategies in missile versus missile interception scenarios,'' in *Proc. Amer. Control Conf. (ACC)*, 1995, pp. 4116–4120.
[21] D. P. Bertsekas, M. L. Homer, D. A. Logan, S. D. Patek, and N. R. Sandell, ''Missile defense and interceptor allocation by neuro-dynamic programming,'' *IEEE Trans. Syst., Man, Cybern. A, Syst. Humans*, vol. 30, no. 1, pp. 42–51, Jan. 2000.
[22] R. H. Chen, J. L. Speyer, and D. Lianos, ''Optimal intercept missile guidance strategies with autopilot lag,'' *J. Guid., Control, Dyn.*, vol. 33, no. 4, pp. 1264–1272, Jul. 2010.
[23] E. Garcia, D. W. Casbeer, and M. Pachter, ''Active target defense using first order missile models,'' *Automatica*, vol. 78, pp. 139–143, Apr. 2017.
[24] H. Wang, W. Wang, and R. Peng, ''A two-phase inspection model for a single component system with three-stage degradation,'' *Rel. Eng. Syst. Saf.*, vol. 158, pp. 31–40, Feb. 2017.
[25] C. Wang, L. Xing, R. Peng, and Z. Pan, ''Competing failure analysis in phased-mission systems with multiple functional dependence groups,'' *Rel. Eng. Syst. Saf.*, vol. 164, pp. 24–33, Aug. 2017.
[26] R. Peng, Q. Zhai, L. Xing, and J. Yang, ''Reliability of demand-based phased-mission systems subject to fault level coverage,'' *Rel. Eng. Syst. Saf.*, vol. 121, pp. 18–25, Jan. 2014.
[27] G. Wang, R. Peng, and L. Xing, ''Reliability evaluation of unrepairable K-out-of-N: G systems with phased-mission requirements based on record values,'' *Rel. Eng. Syst. Saf.*, vol. 178, pp. 191–197, Oct. 2018.
[28] S. Zhang, S. Sun, S. Si, and P. Wang, ''A decision diagram based reliability evaluation method for multiple phased-mission systems,'' *Eksploatacja i Niezawodnosc-Maintenance Rel.*, vol. 19, no. 3, pp. 485–492, Jun. 2017.
[29] T. Yu, W. Cui, B. Song, and S. Wang, ''Reliability growth estimaiton for unmanned aerial vechicle during flight-testing phases,'' *Eksploatacja i Niezawodnosc-Maintenance Rel.*, vol. 2, pp. 43–47, Jan. 2010.
[30] X. Wu and X. Wu, ''Mission reliability modeling and evaluation of multi-mission phased mission system based on an extended object-oriented Petri net,'' *Eksploatacja i Niezawodnosc-Maintenance Rel.*, vol. 19, no. 2, pp. 244–253, Mar. 2017.
[31] E. Nikolova, ''Approximation algorithms for reliable stochastic combinatorial optimization,'' in *Proc. Int. Conf. Int. Conf. Approximation*, 2010, pp. 338–351.
[32] D. Kahneman and A. Tversky, ''Prospect theory: An analysis of decision under risk,'' *Econ., J. Econ. Soc.*, vol. 47, no. 2, pp. 263–291, 1979.
[33] A. Tversky and D. Kahneman, ''Advances in prospect theory: Cumulative representation of uncertainty,'' *J. Risk Uncertainty*, vol. 5, no. 4, pp. 297–323, Oct. 1992.
[34] U. Schmidt, C. Starmer, and R. Sugden, ''Third-generation prospect theory,'' *J. Risk Uncertainty*, vol. 36, no. 3, pp. 203–223, Jun. 2008.
[35] M. Abdellaoui, ''Parameter-free elicitation of utilities and probability weighting functions,'' *Manage. Sci.*, vol. 46, no. 11, pp. 1497–1512, 2000.
[36] U. Schmidt and H. Zank, ''Risk aversion in cumulative prospect theory,'' *Manage. Sci.*, vol. 54, no. 1, pp. 208–216, Jan. 2008.
[37] H. Bleichrodt, U. Schmidt, and H. Zank, ''Additive utility in prospect theory,'' *Manage. Sci.*, vol. 55, no. 5, pp. 863–873, May 2009.
[38] J. Quiggin, ''Comparative statics for rank-dependent expected utility theory,'' *J. Risk Uncertainty*, vol. 4, no. 4, pp. 339–350, Dec. 1991.
[39] D. E. Bell, ''Regret in decision making under uncertainty,'' *Oper. Res.*, vol. 30, no. 5, pp. 961–981, Oct. 1982.
[40] G. Loomes and R. Sugden, ''Regret theory: An alternative theory of rational choice under uncertainty,'' *Econ. J.*, vol. 92, no. 368, pp. 805–824, Dec. 1982.
[41] D. E. Bell, ''Disappointment in decision making under uncertainty,'' *Oper. Res.*, vol. 33, no. 1, pp. 1–27, 1985.
[42] G. Loomes and R. Sugden, ''Disappointment and dynamic consistency in choice under uncertainty,'' *Rev. Econ. Stud.*, vol. 53, no. 2, pp. 271–282, Apr. 1986.

[43] T. L. Saaty, *Theory and Applications of the Analytic Network Process: Decision Making With Benefits, Opportunities, Costs, and Risks*. Pittsburgh, PA, USA: RWS, 2005.
[44] A. P. Tchangani, "Considering bipolarity of attributes with regards to objectives in decisions evaluation," *Inzinerine Ekonomika-Eng. Econ.*, vol. 21, no. 5, pp. 475–484, 2010.

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