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Opinion Dynamics With Competitive Relationship and Switching Topologies

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ABSTRACT In this paper, the discrete-time opinion dynamics model with competitive relationship and switching topologies is investigated. Different from the usual DeGroot model, competition between individuals and the switching topologies are considered in a social network. Furthermore, the structurally balanced and unbalanced network topologies are investigated simultaneously. It is shown that if all network topologies are structurally balanced and include a spanning tree, then there may appear polarization, neutralization or fluctuation. And we also find conditions under which the group is split into two clusters at most, specifically, opinions will reach polarization (bipartite consensus) or all individuals will remain neutrality (consensus). If there exist the structurally unbalanced network topologies in social networks, we obtain the condition in which all individuals will finally hold the neutral attitude on the discussed topic. Two examples are provided to illustrate the effectiveness of the obtained results.

INDEX TERMS Opinion dynamics, consensus, polarization, DeGroot model, signed graph.

I. INTRODUCTION

Recently, opinion dynamics, which has been studying the exchange and discussion of opinions between individuals which may occur in mediums and situations as varied as company boardrooms, elementary school classrooms and online social media, has drawn considerable attention from control theory, sociology, physics, economics, biology and so on [1]–[6]. The so-called opinion dynamics focuses on the basic problem of social networks: how individuals (agents) are influenced by the presence of others in a social group. Exploring the evolutionary laws of group opinions and behaviors in social networks is an important issue in the process of human self-understanding, and can reveal the basic laws of human society and animal population development in order to promote the development of society in a more harmonious direction [7]–[11].

For the sake of investigating the evolution of opinions and individual interpersonal influence, many models have been proposed in the past few decades [12]–[19]. The early representative model was proposed by DeGroot and later called the DeGroot model [12]. DeGroot model is the most basic and classic static-neighborhood model where neighbors are defined according to an unchanging directed graph. In this case, individuals update their opinions by averaging their own and other's displayed opinions according to the weighted adjacency matrix. If the network topology is strongly connected and aperiodic, DeGroot model always reaches an opinion consensus for the arbitrary initial opinions [12], i.e., all agents make an agreement about a topic. Subsequently, the consensus conditions of DeGroot model were examined intensively [20]–[23]. Authors of the literature [20] pointed out that the condition of aperiodic and connectivity can be reduced to include a spanning tree if each node has self-loop. Furthermore, this result was generalized to the case of switching topologies based on results on convergence of infinite products of stochastic matrices [20]. For changing bidirectional and cut-balanced network topologies, necessary and sufficient conditions for consensus boiled down to repeated joint connectivity of the graphs [21], [22].

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For general directed network topologies with self-loop, the sufficient conditions for consensus were closely related to the uniform quasi-strong connectivity of the graphs [23].

In the past few decades, various deformations of the DeGroot model have also been extensively studied [9], [14]–[16], [24], [25]. Combining the updated rules of the DeGroot model with the idea of "homophily" [26] which means that agents readily adopt opinions of like-minded individuals but accept the more deviant opinions with discretion, bounded confidence (BC) models [26], [27] were proposed to investigate the opinion evolution. In BC models, agents do not completely consider opinions outside their confidence intervals. In other words, in BC models, People can influence each other when their opinions are close enough to each other, i.e., the distance of the opinion states $|x_i - x_j|$ is less than a certain confidence threshold. Compared with the DegGoot model, to a certain extent, BC models are more able to capture the characteristic of opinion evolution and can model many phenomena, for example, animal flocking [28]. Actually many opinion phenomena found in social networks such as consensus, disagreement and polarization and so on were presented in BC models [16]. In [16], [29], [30], homogeneous BC models were investigated where all agents were assumed to have the same confidence threshold. In fact, in real social networks, the different agents often have the different confidence threshold [31]. heterogeneous BC model was proposed to model this phenomenon where dynamic behavior is more complicated [31], [32]. In BC models, the network topologies are dependent on the system states, which makes it more difficult to perform a strict mathematical analysis on it. So far, their rigorous mathematical analysis is still an open problem. For instance, it is very difficult to predict the structure of opinion clusters for a given initial condition [16], [29].

The DeGroot model has many extensions in different views of dynamics and psychology. The Friedkin-Johnsen (FJ) model, as a typical example, is induced by the stubborn agents [33], [34]. The so-called stubborn agents refer to the agents with prejudices that have the willing to maintain their initial opinions. Some relevant works [35]–[37] have shown that the prejudice in social networks is often hidden but can continuously influence agent behavior. Different from the DeGroot model [12], in which each actor is completely open to interpersonal influence, the stubborn agents in the FJ model always consider their initial opinions for every iteration of opinions. Therefore, it is difficult to reach opinion consensus for F-J models. In fact, agents often form multiple clusters in F-J models [33], [34], [36], [38]. But if peer pressure between individuals was considered in FJ model, the modified FJ model can obtain consensus if peer pressure is increasing and unbounded [14].

It should be pointed out that the possible competitive or confrontational relationship between individuals were not consider in most of the references mentioned above. However, antagonism, competition, indifference or distrust between individuals are ubiquitous in the real world [17], [39]–[42]. These phenomena are usually modeled by repulsive couplings or negative ties [41] among the agents, i.e., the signed graph, where the positive edges represent friendly and cooperative interactions and negative edges correspond to antagonistic counterpart. Recently, opinion dynamics with antagonism or competition has attracted significant attention [14], [17], [41]–[47]. As shown in [44], the evolution of opinions on signed graphs may be more complicated due to the antagonistic interaction. In fact, under different opinion protocols, opinion dynamics on signed graphs may result in clusters, polarity, consensus, neutrality or fluctuation [44]. In [7], the continuous-time DeGroot model on signed graphs was studied and a necessary and sufficient condition of polarization was obtained based on structurally balanced graphs. It has been shown that the bipartite consensus of opinions is usually closely related to the structurally balanced graphs [7]. In [47], the bipartite consensus problem for high-order opinion dynamics systems was investigated. In [17], [40], opinion dynamics with switching topologies and confrontational relationship was researched. It should be pointed out that it was implicitly assumed in most of the literature mentioned above that the network topologies are structurally balanced. In [48], [49], the structurally unbalanced graphs were considered. However, in [48], [49], authors only focused on the signs of opinions and ignored their size, i.e., the sign-consensus. In the structurally unbalanced social networks, the relationship between individuals is more complicated, which makes it more difficult to analyze the model mathematically. Therefore, when the structurally balanced and unbalanced networks exist at the same time, how opinions will evolve is still an open topic. In this article, we will make efforts to it.

On the other hand, in the literature on the signed networks mentioned above, it is always assumed that the diagonal entries of the adjacency matrix are non-negative. However, in this paper, the diagonal entries of the adjacency matrix are allowed to take negative values. If the diagonal entry $a_{ii} < 0$, we call the i-th agent as the non-confidence agent which lacks confidence in the signs of their own opinions. Some relevant works [50]–[52] showed that the non-confidence agents do not tend to freely expresses their opinions, but often exist in social networks. The non-confidence agent finally tends to hold a neutral attitude [51]–[53], which is consistent with the conclusions we will obtain.

The main contributions of this paper are as follow. Firstly, a novel opinion dynamics model, where "competition", "switching" and "confidence" are considered simultaneously, is proposed. Secondly, when all network topologies are structurally balanced, a few conditions are obtained to guarantee that opinions converge to two clusters at most; When the structurally balanced and unbalanced network topologies coexist, by using the spectrum analysis and the matrix theory, a consensus condition is gotten. This implies that the networks may still achieve consensus in the presence of competition. At last, we find that if there exist the non-confidence agents in a social network all agents will have a neutral attitude towards events under certain conditions.

The rest of the paper is organized as follows: basic definitions and properties of graphs and models are recalled in Section II; The dynamic behavior in our models are discussed in III; In Section IV, we give two examples to illustrate the effectiveness of the obtained results. Finally, in Section V, we give our conclusion.

II. PRELIMINARIES

In this section, model and mathematical preliminaries are provided to derive the main results of this paper.

A. NOTATIONS

Throughout this paper, $\mathbb{R}^{m\times n}$ and $\mathbb{R}^n$ denote, respectively, the $m \times n$ real matrix and the n-dimensional real space. Suppose $A \in \mathbb{R}^{m\times n}$ or $A \in \mathbb{R}^n$, $A \geq 0$ means all elements of A are not less than 0, and A^T denotes its transpose. Let $A = [a_{ij}]$ and $B = [a_{ij}]$ be two real $n \times n$ matrices, $A \geq B$ means $a_{ij} \geq b_{ij}$ for all $1 \leq i, j \leq n$. If B is an arbitrary complex matrix, then $|B|$ denotes the nonnegative matrix with entries $|b_{ij}|$. $\rho(A)$ denotes the spectral radius of A. $\mathbf{1}_n$ denotes the n-dimensional vector $[1, 1, \cdots, 1]^T$. We use $\mathrm{diag}[m_1, m_2, \cdots, m_n]$ to denote the diagonal matrix whose diagonal entries are $m_1, m_2, \cdots, m_n$. The notation $[-1, 1]^n$ denotes the set $\{x|x_i \in [-1, 1], x \in \mathbb{R}^n\}$.

B. GRAPH THEORY

Consider a set of n nodes (agents or individuals) denoted by $V = \{1, 2, \cdots, n\}$ and the subset $E \subset V \times V$, $G = (V, E)$ is called a digraph with the set of nodes (or vertices) V and the set of edges E. A path from a vertex i to another vertex j is a sequence of distinct vertices starting with i and ending with j, in which each vertex is adjacent to its next vertex. We say that the digraph $G = (V, E)$ contains a spanning tree if there is a vertex i such that there exists a path from i to every other vertex in $G = (V, E)$ where node i is called root node. Furthermore, if each node is a root node, then a digraph $G = (V, E)$ is said to be strongly connected. The neighbor set of the vertex i is defined by $\mathbb{N}_i = \{j \in V|(j, i) \in E\}$. We say that $G = (V, E)$ is an undirected graph if $\forall j \in \mathbb{N}_i$ having $i \in \mathbb{N}_j$. For an undirected graph, the strong connectivity means connectivity. Suppose a matrix $A \in \mathbb{R}^{n\times n}$ satisfies: $a_{ij} \neq 0 \iff (j, i) \in E$, then matrix A is called the weighted adjacency matrix of the graph $G(A)$. If the adjacency matrix A is assumed to take both positive and negative values, then it is called the signed adjacency matrix and its associated graph is called the signed graph $G(A)$. As pointed out in [1], [12], when individuals make a decision, almost all individuals will always consider the opinions of themselves and their neighbors comprehensively. So, in this paper, we assume $a_{ii} \neq 0$ for all $i \in V$, i.e., each vertex has a self-loop. The state of agent $i \in V$ at k is a continuous value $x_i(k) \in [-1, 1]$ that represents opinion or position on a topic. The sign of x_i determines whether the individual is in favor or against about the topic, and the size of x_i determines the degree of opposition or support. If $x_i = 0$, We say that agent i is currently neutral on the topic. Furthermore, if $a_{ii} > 0$, agent i is called the confidence agent which has a positive attitude towards his current opinion, i.e., he has confidence about the signs of his current opinions and desires to maintain the signs of his opinions; If $a_{ii} < 0$, agent i is called the non-confidence agent which is skeptical of his current opinion, i.e., he lacks confidence on the signs of the current opinions and hopes to change the signs of his opinions. Such as individuals are suspicious by nature, finally they often remain neutral opinions on the topic. Competition, antagonism or distrust between agents are modeled by negative edges among the agents. The positive edges among the agents mean that the cooperative and friendly relationship between agents. So in this paper, we assume that $G(A)$ is the signed graph.

C. MODEL DESCRIPTION

Inspired by the literature [7], [12], in this paper, the following opinion dynamics model with n individuals is examined:

$$x_i(k+1) = \sum_{j=1}^{n} a_{ij}(\sigma(k))x_j(k), \quad i = 1, 2, \ldots, n, \qquad (1)$$

where $\sigma(k) : \mathbb{N} \to \mathfrak{M} = \{1, 2, \ldots, m\}$ is the switching signal. If $a_{ij}(\sigma(k)) \neq 0$, it means that at k instant the agent i can receive information from the agent j. In other words, the agent j is a neighbor of the agent i. When making a decision the agent i will consider opinions of the agent j, i.e., the agent i will be affected by the agent j. $|a_{ij}|$ indicates the degree of impact and $\sum_{j=1}^{n} |a_{ij}(\sigma(k))| = 1$ is assumed for all $i = 1, 2, \ldots, n$ [17]. Similar to [7], [14], [17], $a_{ij} > 0$ represents the cooperative relationship and $a_{ij} < 0$ corresponds to confrontation and competition.

Let $A(\sigma(k)) = [a_{ij}(\sigma(k))]$, $x(k) = [x_1(k), x_2(k), \ldots.., x_n(k)]^T$, then the model (1) can be rewritten by the following compact form:

$$x(k+1) = A(\sigma(k))x(k), \qquad (2)$$

where $|A(\sigma(k))|$ is the row random matrix.

Remark 1: The usual DeGroot model only considers the cooperative relationship between individuals, and ignores the possible competitive relationship between individuals. In our model, cooperation and competition are considered at the same time. On the other hand, for the usual DeGroot model, the network topology is fixed, and our model considers the impact of switching topology on the evolution of opinions. Therefore, to a certain extent, our model is more general than the usual DeGroot model.

Remark 2: Different from the most works about opinion dynamics on the signed graph [14], [17], [39]–[43], [47], in our model the signed graph is allowed to be structurally unbalanced. Furthermore, as far as the authors know, in the investigation about the signed graph, it was implied that the diagonal entries of the adjacency matrix are nonnegative. Here the negative diagonal entries are allowed which correspond to the non-confidence agents. This means that the methods of the above articles cannot be directly applied to our model.

D. MATHEMATICAL PRELIMINARIES

The following definitions and lemmas are needed for the derivation of our main results in this paper.

Definition 1 [7], [17]: If $\lim_{k\to\infty} |x_i(k)| = \alpha > 0$ for the arbitrary initial opinions and all $i \in V$, and there exist i and j satisfying that $\lim_{k\to\infty} x_i(k) = -\lim_{k\to\infty} x_j(k)$, then we say that the system (1) can achieve bipartite consensus or polarization. In particular, if $\alpha = 0$, we claim that the system (1) reaches consensus. At this time, all agents remain neutralization.

Definition 2 [7]: Suppose $A \in \mathbb{R}^{n\times n}$. If there exist two sets $V_1 \bigcap V_2 = \emptyset$, where $\emptyset$ denotes the empty set and $V_1 \bigcup V_2 = V$ such that $a_{ij} \geq 0$ for $\forall i, j \in V_l (l \in \{1, 2\})$ and $a_{ij} \leq 0$ for $\forall i \in V_p, j \in V_q, p \neq q, (p, q \in \{1, 2\})$, we claim that the signed graph $G(A)$ (or the matrix A) is structurally balanced. In other words, the signed graph $G(A)$ is structurally balanced if and only if there exists a diagonal matrix Γ =diag$[\tau_1, \tau_2, \ldots, \tau_n]$ such that $\Gamma A \Gamma \geq 0$ where $\tau_i \in \{1, -1\}$.

Lemma 1 [20]: Assume that $A(p) \geq 0$ is a row random matric and its diagonal elements are greater than zero for all $p \in \mathfrak{M}$. If $G(A(p))$ contains a spanning tree for all $p \in \mathfrak{M}$, then the system (1) can reach consensus for the arbitrary switching signal.

Lemma 2 [54]: Let $A \geq 0$ be an irreducible matrix, and let B be a complex matrix with $|B| \leq A$. If β is any eigenvalue of B, then $|\beta| \leq \rho(A)$. Furthermore, equality is valid,i.e, $\beta = \rho(A)e^{i\phi}$ for some real ϕ, if and only if $|B| = A$, and where B has the form

$$B = e^{i\phi} DAD^{-1} \tag{3}$$

and where D is a diagonal matrix whose diagonal entries have modulus unity.

Lemma 3: Let $A \in \mathbb{R}^{n\times n}$ be an irreducible matrix, and there exists at leat a diagonal entry $a_{ii} > 0$, then $\rho(A) = \rho(|A|)$ if and only if $G(A)$ is structurally balanced.

Proof: Obviously, $|A| \geq 0$ is an irreducible matrix. According to Lemma 2, $\rho(A) = \rho(|A|)$ if and only if there exist a real number ϕ and a diagonal matrix D whose diagonal entries have modulus unity such that $|A| = e^{i\phi} DAD^{-1}$. Let's assume D =diag$[e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}]$, then D^{-1} =diag$[e^{-i\theta_1}, e^{-i\theta_2}, \ldots, e^{-i\theta_n}]$. According to $|A| = e^{i\phi} DAD^{-1}$, one can obtain $|a_{pq}| = e^{i\phi} e^{i(\theta_p - \theta_q)} a_{pq}$ for all $p, q = 1, 2, \ldots, n$. In particular, $|a_{pp}| = e^{i\phi} a_{pp}$. Because there exist diagonal entries more than 0, it is obvious that $e^{i\phi} = 1$. At this time, $|a_{pq}| = e^{i(\theta_p - \theta_q)} a_{pq}$. According to the connectivity $G(A)$, there exists $q_1 \neq 1$ satisfying $a_{1q_1} \neq 0$. So $|a_{1q_1}| = e^{i(\theta_1 - \theta_{q_1})} a_{1q_1}$. It is obtained that $e^{i(\theta_1 - \theta_{q_1})} \in \{1, -1\}$. Therefore, $e^{i\theta_{q_1}} = \alpha_{q_1} e^{i\theta_1}$ where $\alpha_{q_1} \in \{1, -1\}$. For all $a_{1q_j} \neq 0$, $e^{i\theta_{q_j}} = \alpha_{q_j} e^{i\theta_1}$. For $a_{q_j q_l} \neq 0$, $e^{i\theta_{q_l}} = \alpha_{q_l} e^{i\theta_1}$ with $\alpha_{q_l} \in \{1, -1\}$. According to the connectivity $G(A)$, this process can go on continuously, i.e., q_j can take all points in the set $\{2, 3, .., n\}$. So the matrix D has the form $D = e^{i\theta_1}$diag$[\alpha_1, \alpha_2, \ldots, \alpha_n]$ with $\alpha_j \in \{1, -1\}$. Noticing $D^{-1} = e^{-i\theta_1}$diag$[\alpha_1, \alpha_2, \ldots, \alpha_n]$, so

$$\begin{aligned} |A| &= e^{i\phi} DAD^{-1} \\ &= \text{diag}[\alpha_1, \alpha_2, \ldots, \alpha_n] A \text{diag}[\alpha_1, \alpha_2, \ldots, \alpha_n] \geq 0. \end{aligned} \tag{4}$$

According to Definition 2, this implies that $G(A)$ is structurally balanced. The proof of Lemma 3 has been completed.

Remark 3: In Lemma 3, It is important that matrix A has diagonal elements greater than 0, otherwise, the conclusion of Lemma 3 will not necessarily hold. For example, for $A > 0$, $-A$ is an irreducible matrix. It is obvious that $\rho(-A) = \rho(|-A|)$. But the graph $G(-A)$ is not structurally balanced. Lemma 3 plays a major role in obtaining the main results of this paper.

According to Lemma 3, if $|A|$ is the row random matrix, we have the following corollary.

Corollary 1: Assume that $A \in \mathbb{R}^{n\times n}$ is an irreducible matrix with $a_{ii} > 0$ for some i, $1 \leq i \leq n$, and $|A|$ is the row random matrix. If $G(A)$ is structurally unbalanced, then $\rho(A) < 1$.

III. MAIN RESULTS

In this section, the convergence of model (1) is investigated. Firstly, we examine the opinion evolution of model (1) when $\sigma(k_1) = \sigma(k_2)$ for all $k_1, k_2 \in \mathbb{N}$, i.e., the network topology is fixed. In other words, we firstly consider the following model:

$$x(k+1) = Ax(k). \tag{5}$$

Then we will consider the impact of switching on the evolution of individual opinions. For the system (5), we have the following results.

Theorem 1: For the system (5), it is assumed that $G(A)$ is structurally balanced. If $G(A)$ has a spanning tree with $a_{ii} > 0$ for all $i \in \{1, 2, \ldots, n\}$, then the system will achieve bipartite consensus or consensus; Assume that $G(A)$ is structurally unbalanced. If $G(A)$ is connected and there exists a diagonal entry $a_{ii} > 0$, then $\lim_{k\to\infty} x_i(k) = 0$ for all $i \in \{1, 2, \ldots, n\}$, i.e., the system (5) reaches consensus.

Remark 4: For the usual DeGroot model in which there exists only the cooperative relationship between individuals, if $G(A)$ has a spanning tree with the positive self-loop, then all individuals form a cluster, i.e., consensus. Theorem 1 shows that if cooperation and competition between individuals exist at the same time, then the result is relatively more complicated. In fact, if $G(A)$ is structurally balanced, this means that the group can be divided into two parts. The individuals of each part are cooperative relationship, and the individuals between the different parts are competitive relationship. Therefore, agreement between individuals in each part can be reached. It is difficult for individuals between the different parts to reach agreement due to competition unless that all individuals maintain a neutral attitude. When $G(A)$ is structurally unbalanced, the relationship between individuals is more complicated. For example, the following situation may arise: individual i believes that he is a cooperative relationship with individual j, while individual j believes that he is a competitive relationship with individual i. In this case, it seems that the agreement can be accepted by the group that only if all individuals are neutral. This is consistent with the results of Theorem 1.

Proof: I) the first case: the structurally balanced graph.

When $G(A)$ is structurally balanced, according to Definition 2, there exists a diagonal matrix Γ such that $\Gamma A \Gamma \geq 0$. Let $y(k) = \Gamma x(k)$, one can easily obtain that

$$\begin{aligned} y(k+1) &= \Gamma x(k+1) \\ &= \Gamma A x(k) \\ &= \Gamma A \Gamma y(k). \end{aligned} \tag{6}$$

It is obvious that $G(\Gamma A \Gamma)$ has a spanning tree and its all diagonal elements are positive when $G(A)$ contains a spanning tree with $a_{ii} > 0$ for all $i \in \{1, 2, \ldots, n\}$. According to Lemma 1, for the arbitrary initial value, $\lim_{k\to\infty} y(k) = \mathbf{I}_n \alpha$ for $\alpha \in \mathbb{R}$. So $\lim_{k\to\infty} x(k) = \Gamma \mathbf{I}_n \alpha$. If $\alpha = 0$, the system (5) reaches consensus. The final opinions of all individuals remain neutrality. If $\alpha \neq 0$, the system (5) achieves bipartite consensus.

II) the second case: the structurally unbalanced graph.

Assume that $G(A)$ is connected and structurally unbalanced. If A has the positive diagonal entries, noticing that $|A|$ is the row random matrix, according to Corollary 1, $\rho(A) < 1$. Therefore $\lim_{k\to\infty} A^k = \mathbf{0}$ where $\mathbf{0}$ represents a matrix whose all elements are zero. It implies that $\lim_{k\to\infty} x_i(k) = 0$ for the arbitrary initial value, i.e., the system (5) achieves consensus. ◇

Corollary 2: For the system (5), suppose $G(A)$ is a connect graph. If there exist the confidence agents and the non-confidence agents, then the final opinions of all individuals converge to 0.

Proof: Since the confidence agents and the non-confidence agents simultaneously exist, $G(A)$ is structurally unbalanced. According to Theorem 1, the conclusion is clearly established.

Remark 5: Some relevant works show that individuals, who lack of the confidence, tend to maintain a neutral attitude towards topics [51], [52]. Corollary 2 shows that if the network is connected, the non-confident individuals can influence the confident agents, thus affect the group's decision to a certain extent.

Now, let's consider the impact of switching on the evolution of opinions. For this purpose, the matrices are firstly classified. If there exists a matrix Γ such that $\Gamma A_1 \Gamma \geq 0$ and $\Gamma A_2 \Gamma \geq 0$, we claim that A_1 and A_2 are the same structure. We also say that the networks $G(A_1)$ and $G(A_2)$ have the same topology structure. We use $V_\Gamma = \{A | \Gamma A \Gamma \geq 0\}$ to represent a set of matrices like this. Obviously, if $A \in V_\Gamma$, then $G(A)$ is structurally balanced. Furthermore, if $A(p) \in V_\Gamma$ for all $p \in \mathfrak{M}$, then, when the network topologies are switched, the cooperation or competition between individuals will not change.

Theorem 2: Assume $G(A(p))$ contains a spanning tree with the positive diagonal entries for all $p \in \mathfrak{M}$. If $A(p) \in V_\Gamma$ for all $p \in \mathfrak{M}$, i.e., all A_p have the same structure, then for the arbitrary switching signal, the system (1) will achieve bipartite consensus or consensus, i.e., the group forms at most two clusters.

Proof: Since all $A(p) \in V_\Gamma$, similar to the proof of Theorem 1, we firstly let $y(k) = \Gamma x(k)$, then we have

$$\begin{aligned} y(k+1) &= \Gamma x(k+1) \\ &= \Gamma A(\sigma(k)) x(k) \\ &= \Gamma A(\sigma(k)) \Gamma y(k). \end{aligned} \tag{7}$$

Obviously, $G(\Gamma A(\sigma(k))\Gamma)$ has a spanning tree and its all diagonal elements are positive. According to Lemma 1 and the proof of Theorem 1, it is obtained that the system (1) will achieve bipartite consensus or consensus.

Remark 6: In Theorem 2, $A(p)$ is assumed to belong to V_Γ for all $p \in \mathfrak{M}$. In fact, a few $A(p) \notin V_\Gamma$ are allowed. Under this case, Theorem 2 still holds as long as there exists a positive integer k_s such that $A(\sigma(k)) \in V_\Gamma$ for all $k \geq k_s$.

In Theorem 2, every $G(A(p))$ is assumed to contain a spanning tree. In fact, this condition can be weaker. So we have the following results.

Theorem 3: Assume that matrix $A(p)$ has positive diagonal entries and $A(p) \in V_\Gamma$ for all $p \in \mathfrak{M}$. If there exists an infinite sequence of uniformly bounded, nonoverlapping time intervals $[k_j, k_j + q_j), j = 1.2\ldots.$ starting $k_1 = 0$ such that each interval $[k_j + q_j, k_{j+1})$ is uniformly bounded and the union of a group of directed graphs $\{G(A(\sigma(k_j + q_j))), G(A(\sigma(k_j + q_j + 1))), \ldots, G(A(\sigma(k_{j+1} - 1)))\}$ in each interval $[k_j + q_j, k_{j+1})$ contains a spanning tree where the union of graphs is a directed graph with edge set given by the union of the edge sets of $G(A(\sigma(k_j + q_j))), G(A(\sigma(k_j + q_j+1))), \ldots, G(A(\sigma(k_{j+1}-1)))$, then the system (1) achieves bipartite consensus or consensus.

In Theorem 3, all $G(A(p)) \in V_\Gamma$ is requested. According to definition of the structurally balanced graph and the proof of Theorem 3.10 in [20], Theorem 3 can be proofed easily. Here the specific proof is omitted.

Remark 7: Noticing that the condition of all $A(p) \in V_\Gamma$ implies that the relationship between individuals has not changed with the change of topology. It means that the sign of $a_{ij}(\sigma(k))$ does not change over time. Otherwise, the conclusion of the Theorem 2 will not necessarily hold. We will illustrate this point through numerical simulation later.

In Theorem 2 and Theorem 3, we assume that all graphs $G(A(p))$ are structurally balanced. Next we will discuss how opinion will evolve under the condition of the existence of structurally unbalanced graphs. We have the following results when the structurally unbalanced graphs are considered.

Theorem 4: Denote the matrix set $\Omega = \{A(p) | \exists a_{ii}(p) > 0$ and $G(A(p))$ is connected and structurally unbalanced, $p \in \mathfrak{M}\}$. Assume $A^T(p) = A(p)$ for all $p \in \mathfrak{M}$ and Ω is not an empty set. If there exist the time series $0 \leq k_1 < k_2 <, \ldots, < k_j <, \ldots$ such that $G(A(\sigma(k_j))) \in \Omega$ and $\lim_{j\to\infty} k_j = \infty$, then $\lim_{k\to\infty} x_i(k) = 0$ for all $i \in V$. In particular, if $A(p) = A^T(p) \in \Omega$ for all $p \in \mathfrak{M}$, then $\lim_{k\to\infty} x_i(k) = 0$ for the arbitrary switching signal.

Proof: In the proof of this theorem, we use the Euclidean norm of the vector $x \in \mathbb{R}^n$, i.e.,

$$\|x\| = (x^T x)^{\frac{1}{2}} \tag{8}$$

For $A \in \mathbb{R}^{n\times n}$, we define the matrix norm

$$\|A\| = \sup_{\|x\|=1} \|Ax\| \tag{9}$$

For (10) and (9), we have $\|Ax\| \leq \|A\|\|x\|$. Furthermore, if $A^T = A$, then $\|A\| = \rho(A)$ [54]. Denote $\varrho = \max\{\rho(A)|A \in \Omega\}$. According to lemma 3, $\varrho < 1$. For any positive integer $k \geq k_1$, there exist integer k_j and k_{j+1} satisfying $k_j \leq k < k_{j+1}$. Noticing $\rho(A(p)) \leq 1$ for all $p \in \mathfrak{M}$, when $k \geq k_1$ then we have that

$$\begin{aligned}
\|x(k)\| &= \|A(\sigma(k-1))x(k-1)\| \\
&\leq \|A(\sigma(k-1))\|\|x(k-1)\| \\
&\leq \|A(\sigma(k-1))\|\|A(\sigma(k-2))\|\|x(k-2)\| \\
&\quad . \\
&\quad . \\
&\quad . \\
&\leq \|A(\sigma(k-1))\| \dots \|A(\sigma(0))\|\|x(0)\| \\
&\leq \rho(A(\sigma(k-1))) \dots \rho(A(\sigma(0)))\|x(0)\| \\
&\leq \varrho^j \|x(0)\|
\end{aligned} \tag{10}$$

Noticing $\varrho < 1$ and $j \to \infty$ when $k \to \infty$, we have $\lim_{k\to\infty} \|x(k)\| = 0$. If $A(p) = A^T(p) \in \Omega$ for all $p \in \mathfrak{M}$, for the the arbitrary switching signal, there exist integer k_j and k_{j+1} satisfying $k_j \leq k < k_{j+1}$. So the conclusion of the theorem is still valid under this case. The proof of the theorem has been completed. ◇

Remark 8: In Theorem 4, both structurally balanced and unbalanced network topologies are allowed to coexist, and we do not make any requirements for the connectivity of structurally balanced network topologies. In this case, as long as the structurally unbalanced network topologies satisfying the conditions can appear frequently, all agents will eventually remain neutrality to topics. It means that the final opinions of the group are determined to a certain extent by the structurally unbalanced network topologies.

Remark 9: In the Theorem 4, if we discard the condition $A(p) = A^T(p)$ and keep other conditions unchanged, then the results of theorem may not necessarily true even if all network topologies are connected. We will illustrate this point through the numerical simulation later.

Remark 10: In this article, the impact of switching topology on the evolution of opinions is discussed, but Theorem 4 is not limited to the switching topology. From the proof of Theorem 4, it can be seen that if $\mathfrak{M}$ is an infinite set, the conclusion of Theorem 4 is still valid as long as Ω is a finite set. This means that Theorem 4 also can be applied to dynamic changing topologies. In [20], Degroot model with dynamic changing topologies was investigated. It is worth pointing out that in [20], only the cooperative relationship between individuals was considered and the possible competition between individuals was ignored. In Theorem 4, we assume that cooperation and competition exist at the same time. On the other hand, in [17], [55], DeGroot model with dynamic changing topologies and competition was examined where the structurally balanced and unbalanced network topologies were considered. It should be pointed out the diagonal entries of the adjacency matrix are not allowed to take negative values in [17], [55]. However, there is no such restriction in our Theorem 4. Therefore, this paper promotes the results of literature [17], [20], [55] to some extent.

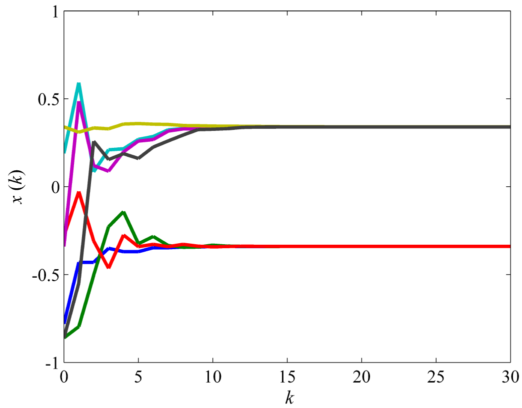

FIGURE 1. The system (1) reaches bipartite consensus in Example 1.

IV. NUMERICAL EXAMPLE

In this section, we will give two examples to observe the state evolution of systems in order to verify our obtained results.

Example 1: Consider a network with 7 individuals. The network topologies are shown as follow:

$$A(1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.8 & 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0.7 & 0 & 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.9 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & -0.8 & 0 & 0.2 & 0 & 0 \\ -0.2 & 0 & 0 & 0 & 0 & 0.8 & 0 \\ 0 & -0.6 & 0 & 0 & 0 & 0 & 0.4 \end{bmatrix},$$

$$A(2) = \begin{bmatrix} 0.3 & 0 & 0.7 & 0 & 0 & 0 & 0 \\ 0.8 & 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0.4 & 0 & -0.5 & 0 & 0 \\ 0 & -0.6 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & -0.6 & 0 & 0.2 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0 & 0.4 \end{bmatrix},$$

$$A(3) = \begin{bmatrix} 0.5 & 0 & 0.4 & 0 & -0.1 & 0 & 0 \\ 0.1 & 0.2 & 0 & 0 & -0.7 & 0 & 0 \\ 0 & 0.8 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & -0.3 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0.1 & 0 & 0 \\ 0 & 0 & -0.2 & 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0.4 \end{bmatrix}.$$

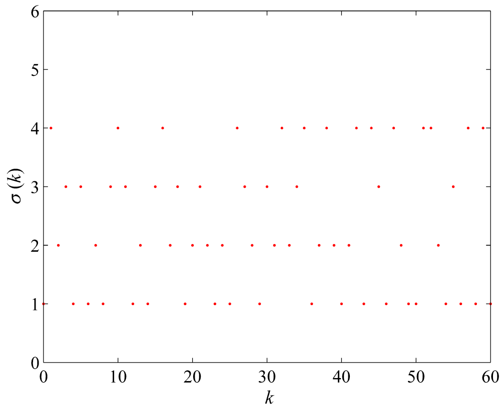

FIGURE 2. The switching signal I in Example 1.

It is obvious that matrices $A(1)$, $A(2)$ and $A(3)$ have a spanning tree and are structurally balanced. Let Γ $=\mathrm{diag}[1, 1, 1, -1, -1, -1, -1]$, then $\Gamma A(p)\Gamma \geq 0$ for $p = 1, 2, 3$. So $A(1)$, $A(2)$ and $A(3)$ have the same structure. According to Theorem 2, for the arbitrary switching signal, the group will finally form two cluster at most. By FIGURE 1, one can find that all individuals are split into two clusters. In the following, we focus on how opinions will evolve if there exist the matrices which are not the same structure with $A(1)$. For this purpose, two structurally balanced matrices are added as follows:

$$A(4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.8 & 0.2 & 0 & 0 & 0 & 0 & 0 \\ -0.8 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ -0.6 & 0 & 0 & 0.4 & 0 & 0 & 0 \\ 0.7 & 0 & 0 & 0 & 0.3 & 0 & 0 \\ 0.3 & 0 & 0 & 0 & 0 & 0.7 & 0 \\ -0.2 & 0 & 0 & 0 & 0 & 0 & 0.8 \end{bmatrix},$$

$$A(5) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1 & 0.9 & 0 & 0 & 0 & 0 & 0 \\ 0.7 & 0 & 0.3 & 0 & 0 & 0 & 0 \\ -0.6 & 0 & 0 & 0.4 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ -0.7 & 0 & 0 & 0 & 0 & 0.3 & 0 \\ -0.9 & 0 & 0 & 0 & 0 & 0 & 0.1 \end{bmatrix}.$$

It can be easily verified that $\Gamma A(4)\Gamma \geq 0$ does not hold. The matrices $A(4)$ and $A(1)$ are not the same structure. Obviously, $A(5)$ and $A(1)$ are the same structure. For the random switching signal shown in FIGURE 2, the evolution of opinions is shown in FIGURE 3. According to FIGURE 3, the system achieves consensus. Furthermore, for the switching signal shown in FIGURE 4, the evolution of opinions is illustrated in FIGURE 5. By FIGURE 5, the system does not converge for the switching signal II. So if $A(p) \in V_\Gamma$ for all $p \in \mathfrak{M}$ can not be satisfied, the conclusion of the theorem 2 will not necessarily hold. In summary, by Example 1, if all network topologies are structurally balanced, opinions may present polarization, neutralization or fluctuation due to competitive relationship and switching topologies.

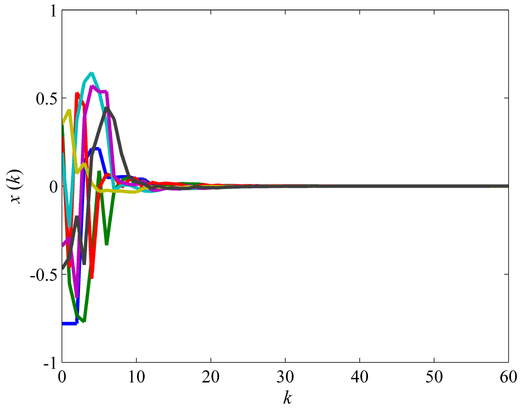

FIGURE 3. The system (1) reaches consensus for the switching signal I in Example 1.

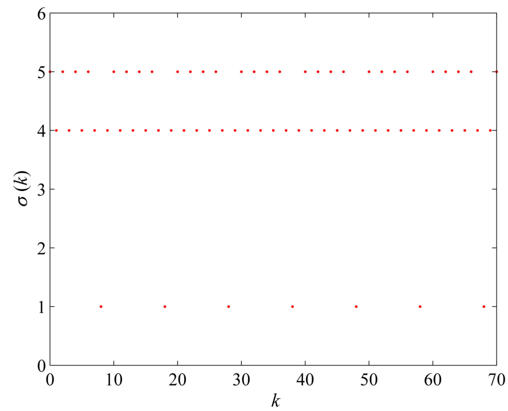

FIGURE 4. The switching signal II in Example 1.

Example 2: In this example, we assume that the structurally balanced and unbalanced network topologies exist simultaneously. We still consider a network with 7 individuals. The network topologies are shown as follow:

$$A(1) = \begin{bmatrix} 0.1 & 0.2 & -0.7 & 0 & 0 & 0 & 0 \\ 0.2 & 0.7 & -0.1 & 0 & 0 & 0 & 0 \\ -0.7 & -0.1 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0.3 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.1 & 0.5 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0.6 \end{bmatrix},$$

$$A(2) = \begin{bmatrix} -0.5 & -0.2 & 0 & -0.3 & 0 & 0 & 0 \\ -0.2 & -0.4 & -0.3 & 0 & 0.1 & 0 & 0 \\ 0 & -0.3 & -0.6 & -0.1 & 0 & 0 & 0 \\ -0.3 & 0 & -0.1 & -0.6 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0.8 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.1 & 0.5 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0.6 \end{bmatrix},$$

$$A(3) = \begin{bmatrix} 0.3 & 0.3 & 0.4 & 0 & 0 & 0 & 0 \\ 0.3 & -0.5 & -0.2 & 0 & 0 & 0 & 0 \\ 0.4 & -0.2 & -0.2 & 0.1 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & -0.2 & 0.7 & 0 & 0 \\ 0 & 0 & 0.1 & 0.7 & 0.1 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.1 & 0.5 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & -0.6 \end{bmatrix},$$

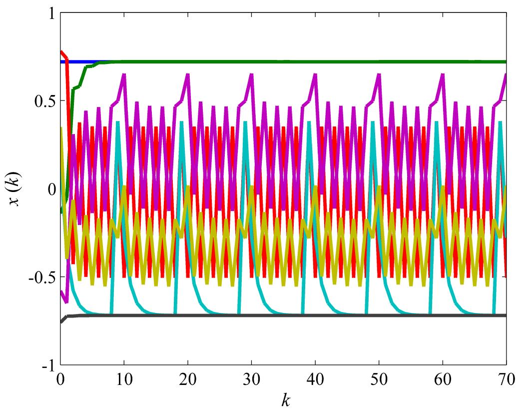

FIGURE 5. **The system (1) does not converge for the switching signal II in Example 1.**

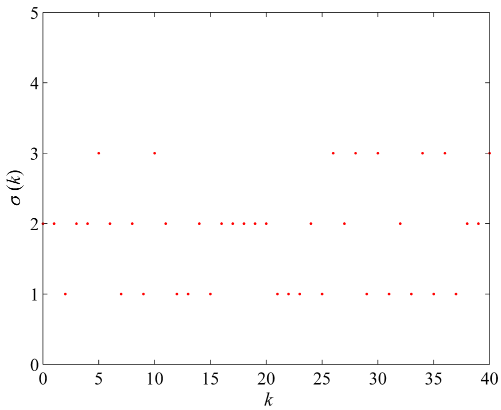

FIGURE 6. **The switching signal I in Example 2.**

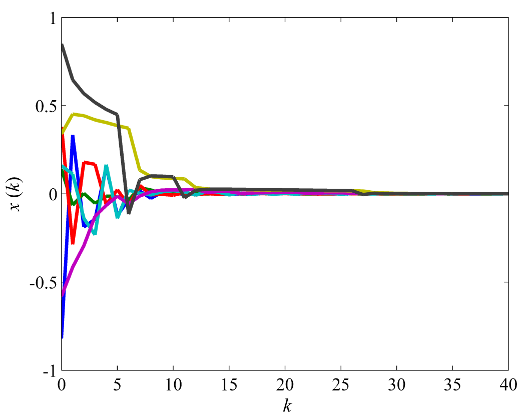

FIGURE 7. **The system (1) reaches consensus for the switching signal I in Example 2.**

It is obvious that $A^T(p) = A(p)$ for $p = 1, 2, 3$. $A(1)$ is structurally balanced. $A(2)$ and $A(3)$ are connected and structurally unbalanced. One can easily find that $A(2), A(3) \in \Omega$. It should be pointed out that $\rho(A(1)) = 1$. According to Theorem 4, if the network topologies $A(2)$ and $A(3)$ appear frequently, then opinions of all individuals will converge to 0. We adopt the switching signal illustrated in FIGURE 6. The evolution of opinion is shown in FIGURE 7. By FIGURE 7, all opinions converge to 0. Note that if the condition of symmetry cannot be satisfied, the conclusion of the theorem may not true. To illustrate this point, we use the following matrices $A(4)$ and $A(5)$:

$$A(4) = \begin{bmatrix} 0.2 & 0.2 & 0.1 & 0.2 & 0.3 & 0 & 0 \\ 0 & -0.7 & -0.2 & -0.1 & 0 & 0 & 0 \\ 0.2 & 0.3 & 0.2 & 0 & 0.2 & 0 & 0.1 \\ 0.6 & 0 & 0 & 0.1 & 0.3 & 0 & 0 \\ 0 & 0.1 & 0.2 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & -0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.8 & 0.2 \end{bmatrix},$$

$$A(5) = \begin{bmatrix} 0.2 & -0.5 & 0 & 0 & 0.2 & 0 & 0.1 \\ 0.3 & -0.5 & 0 & 0.2 & 0 & 0 & 0 \\ 0.1 & -0.2 & 0.3 & 0.3 & 0.1 & 0 & 0 \\ 0.2 & -0.1 & 0.2 & 0.1 & 0.4 & 0 & 0 \\ 0.3 & -0.1 & 0.1 & 0.1 & 0.4 & 0 & 0 \\ 0 & -0.6 & 0 & 0 & 0.3 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.7 & 0.3 \end{bmatrix}.$$

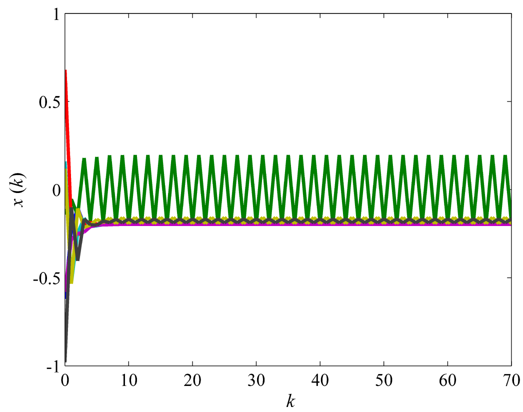

FIGURE 8. **The system (1) does not converge in Example 2.**

Obviously, it can be verified that $A(4)$ and $A(5)$ belong to Ω. But because the condition of symmetry is not satisfied, we find that even if $A(4)$ and $A(5)$ appear frequently, the convergence of the system can not be guaranteed. For example, for switching signal $\sigma(2k) = 4, \sigma(2k+1) = 5, k = 0, 1, 2, \ldots$, the evolution of opinion is shown in FIGURE 8. By FIGURE 8, it can be easily found that the system does not converge. It implies that the condition $A^T(p) = A(p)$ for all $p \in \mathfrak{M}$ is essential in Theorem 4.

V. CONCLUSION

In this paper, the convergence of modified DeGroot model, in which "competition", "switching" and "confidence" are considered simultaneously, has been investigated. If all network topologies containing a spanning tree with the positive diagonal entries are structurally balanced, opinions may present polarization, neutralization or fluctuation due to the switching and competition. Meanwhile, for the situation where the structurally balanced and unbalanced networks simultaneously exist, a consensus condition is obtained where all agents ultimately remain neutral to topics. Finally, two numerical examples have been given to illustrate the effectiveness of our results.

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