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A Fair and Scalable Mechanism for Resource Allocation: The Multilevel QPQ Approach

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ABSTRACT In this paper the problem of distributing resources among a collection of users (or players) is explored. These players have independent preferences to get these resources and can be dishonest about their preferences in order to increase their utility (their preference for the resources they are allocated). The objective is design a mechanism to allocate resources to players so that all of them get the same amount of resources (fair), the total utility is maximized (optimal), and no player has incentive to be dishonest (strategy proof). Santos *et al.* proposed the Quid Pro Quo (QPQ) mechanism to solve this problem. In this paper a generalization of the QPQ mechanism is proposed that, in addition to the above properties, has a very high degree of scalability. The proposed *multilevel QPQ* mechanism divides the players into disjoint clusters and runs a mechanism similar to QPQ inside each cluster and across selected players in each cluster. As a consequence the amount of communication required is drastically reduced. Similarly, the storage used by the mechanism by each player is also significantly reduced, which in a practical setting can be used to improve the ability to detect dishonest players.

INDEX TERMS Resource allocation, mechanism design, fairness, scalability.

I. INTRODUCTION

A. MOTIVATION

Resources need to be assigned to users in many situations. A resource could be, for instance, the processing capacity of a computer system, the power of wireless transmitters or the bandwidth of communication paths. The way in which these resources are allocated to users determines the performance of the system. Therefore, a lot of research has been performed to propose mechanisms to achieve an efficient and fair resource allocation [1]–[5].

Resources could be assigned under the consideration of the existence of a central agent that establishes optimal allocation policies that the users follow (this can be also seen as a situation where users coordinate). However, current telecommunication networks are decentralized, and users that are present in the system very often take self-interested decisions [6]–[8]. Therefore, from the practical point of view, it is crucial to design mechanisms that assign resources to selfish users or players in an efficient manner.

The *Quid Pro Quo* (QPQ) mechanism [9] is a distributed resource allocation algorithm without payments with which a set of resources is allocated to a set of users. Players declare their preferences for the resources and each resource is assigned to the user with the largest value. The main particularity of that model is that each player can misreport its preferences, e.g., users can cheat to get more resources. To prevent that, the mechanism checks that the preferences declared by a player follow a uniform distribution. If this test fails, the declared preference is replaced with a random value. One of the main results of [9] states that this mechanism is fair in the sense that the expected utility of a player that

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declares true preferences is larger than the expected utility of a player that declares false preferences. This implies that, even though users can cheat on the declared values, they do not have benefit by doing so. The main disadvantage of the QPQ mechanism is that the exchange of information is very large when the number of players is large. This is because the implementation of this mechanism requires that each user sends its preference for each resource to the rest of the players, and these messages have to be processed. As a result, the QPQ mechanism [9] presents a clear scalability problem, i.e., its implementation is very expensive in communication and computation when the number of players is large.

To overcome the aforementioned problem, a multilevel approach of the QPQ mechanism is studied in this article. The idea behind this model is presented now. The players are divided into clusters and, in each cluster, the QPQ mechanism of [9] is implemented to determine the winner independently, i.e., the player with the largest declared value for each resource. Then, it is applied again the QPQ mechanism to the set of players formed by the winners of the clusters to determine who gets the resource. The mechanism used among winners has to be adapted, because all the players of the same cluster operate as a single player whose declared preferences follow a Beta distribution. Using this approach, the preferences need to be exchanged only among the players that belong to the same cluster and with the winners of the rest of the clusters. This is a clear advantage with respect to the mechanism of [9] since the amount of information to be shared by the users is much smaller.

B. CONTRIBUTIONS

In this work, the goal is to show that the nice properties of QPQ mechanism of [9] are complemented in the multilevel QPQ approach with some additional new features. Specifically, the main contributions of this article are the following:

- First, the features of the multilevel QPQ are studied when the cheating detection is perfect. It is shown that multilevel QPQ is optimal in the sense that, when players are honest (i.e., they declare their true preferences), no mechanism achieves larger total utility.
- Second, it is also shown that the multilevel QPQ mechanism is strategy proof, i.e., any cheating strategy leads to an expected utility that is not larger than being honest. Finally, it guarantees fairness in the assignment of resources and in their expected value.
- The benefits in the multilevel QPQ approach with respect to QPQ are also studied when all the players are honest. First, the communication cost is analyzed, and it is shown that the reduction on the volume of communication is largest when the number of clusters equals the square root of the number of players. It is also shown that, providing the same memory to multilevel QPQ and QPQ leads to a significant improvement in the number of values that are used to detect a dishonest behaviour and in the expected utility.
- In the numerical experiments, a system with honest and dishonest players is considered (i.e., players that do not declare their real preferences) in which the detection of cheating behavior is not perfect, and evaluate its performance. The goal is to compare the utilities achieved by the multilevel QPQ approach with the utilities of the QPQ mechanism of [9]. The main conclusion from this analysis is that, for the honest players, the expected utility of both mechanisms is very similar, whereas for the dishonest players the expected utility of the multilevel approach is smaller than that of the QPQ mechanism of [9]. This means that the multilevel QPQ approach penalizes more the dishonest players.

C. STRUCTURE

The rest of the article is organized as follows. First, the QPQ mechanism is described in Section II and in Section III the multilevel QPQ. Then, the main results of our work are presented: In Section IV, it is shown that multilevel QPQ is strategy proof and obtain analytically the utility of honest and dishonest players under different assumptions. In Section V, an analytical study of the benefits of multilevel QPQ with respect to QPQ [9] is presented. In Section VI, the simulations that have been carried out are presented. Finally, the related work is explained in Section VII and the main conclusions of this work in Section VIII.

II. THE QPQ MECHANISM

We define the following resource allocation problem:

Definition 1 [9]: The resource allocation problem is a tuple $\langle \mathcal{R}, \mathcal{N}, \Theta \rangle$ *where,*

1) $\mathcal{R} = \{r_1, r_2, \ldots\}$ *is a (potentially* ∞*) ordered list of resources.*
2) $\mathcal{N} = \{1, 2, \ldots, n\}$ *is a set nodes or players, where* $n > 1$ *is assumed to be finite. We assume that players are well identified.*
3) $\Theta = (\theta_j)_{j\in\mathcal{N}}$ *is a vector of continuous random variables where* θ_j *represents the preferences of player* j *for the different resources. This information is private, i.e., it is only known by player* j*. The preference of* j *for a resource* r *is denoted as* $\theta_j(r)$*.*[1]

A solution of a resource allocation problem $\langle \mathcal{R}, \mathcal{N}, \Theta \rangle$ assigns each resource to a single player. The utility of a player j is then the sum of the preferences $\theta_j(r)$ of the resources it gets, and the utility of the resource allocation system is the sum of the utility of all players. The objective is to find solutions that maximize the system utility.

We assume that the player's preferences are independent. This means that (1) for a resource r the preferences $\theta_1(r), \ldots, \theta_n(r)$ are mutually independent, and (2) the preferences $\theta_j(r_s)$ and $\theta_j(r_t)$ for different resources r_s and r_t by the same player j are also independent. Moreover, it may not be possible to compare these preferences among each other,

[1]Throughout the document, we write θ_j for the preference of player j when the resource r is clear from the context.

as each player could measure this parameter in her own metric and units. This may imply that the system utility could not be computed or its value be meaningless. (There could be many factors than can influence a player's preference, and they can affect in a different way depending on her personality.) To solve this problem, we apply the Probability Integral Transformation (PIT) function to the random variables $\theta_1, \ldots, \theta_n$ [9].

Definition 2 [10]: (Probability Integral Transformation) Let X be a continuous random variable with a Cumulative Distribution Function (CDF) F; that is, $X \sim F$. Then, the Probability Integral Transformation (PIT) defines a new random variable Y as $Y = F(X)$.

An interesting property of the PIT is that it follows a uniform distribution in [0, 1], independently of the distribution of X. Let us define the *normalized preference of player* $j \in \mathcal{N}$ as $\bar{\theta}_j = PIT_j(\theta_j)$. Then, by construction, $\bar{\theta}_j$ follows a uniform distribution in [0,1], and this fact makes possible the comparison among $\bar{\theta}_1, \ldots, \bar{\theta}_n$.

Moreover, the normalized preferences are independent, because the original players' preferences θ_j are independent and the normalized ones are obtained by applying to each θ_j its respective PIT, which is continuous (therefore, it is measurable). This is an immediate consequence of the following result, related to independent variables.

Proposition 1 [11]: Let

$$\begin{gathered}\mathcal{X}_{1,1}, \ldots, \mathcal{X}_{1,\eta_1},\\ \mathcal{X}_{2,1}, \ldots, \mathcal{X}_{2,\eta_2},\\ \vdots\\ \mathcal{X}_{\kappa,1} \ldots, \mathcal{X}_{\kappa,\eta_\kappa}\end{gathered}$$

be a set of $\sum_{i=1}^{\kappa} \eta_i$ independent random variables over a probability space $(\Omega, \mathcal{F}, P)$. Then, the random variables

$$\begin{gathered}f_1(\mathcal{X}_{1,1}, \ldots, \mathcal{X}_{1,\eta_1})\\ f_2(\mathcal{X}_{2,1}, \ldots, \mathcal{X}_{2,\eta_2})\\ \vdots\\ f_\kappa(\mathcal{X}_{\kappa,1} \ldots, \mathcal{X}_{\kappa,\eta_\kappa})\end{gathered}$$

where $f_i : \mathbb{R} \to \mathbb{R}$ are measurable functions, are independent.

From the properties of the PIT and Proposition 1, we have the following facts.

Proposition 2: The normalized preferences $\bar{\theta}_1, \ldots, \bar{\theta}_n$ of the players

- *follow a uniform distribution in* [0, 1]*, and*
- *are independent; i.e., (1) for a resource r the preferences $\bar{\theta}_1(r), \ldots, \bar{\theta}_n(r)$ are mutually independent, and (2) the preferences $\bar{\theta}_j(r_s)$ and $\bar{\theta}_j(r_t)$ for different resources r_s and r_t by the same player j are also independent.*

Given these properties, in the rest of the paper we will use only normalized preferences. Using the theory of Mechanism Design [12], note that the Resource Allocation Problem could be reformulated as a mechanism whose message space is the space of normalized preferences $\bar{\theta}_j(r)$ and whose decision function D is defined as $D : \bar{\Theta} \to \mathcal{N}$. That is, for each resource r the mechanism will ask to each player j her normalized preference, and assigns this resource to the player that the decision function returns from the preferences declared by the players.

Since the normalized preference is private information of a player j, she may choose (strategically) to declare a value different from $\bar{\theta}_j$. We denote the value declared by player j for resource r as $\dot{\theta}_j(r)$. For each resource r the values $\dot{\theta}_1(r), \ldots, \dot{\theta}_n(r)$ are common knowledge once they are declared. If player j is honest, $\dot{\theta}_j(r)$ will coincide with the real normalized preference $\bar{\theta}_j(r)$. Otherwise, if the player is not honest, the value $\dot{\theta}_j(r)$ declared by j may not represent the real normalized preference. Moreover, the set of declared values $\dot{\theta}_j$ may not even follow a uniform distribution in [0, 1].

We assume that the mechanism used to declare the $\dot{\theta}_j(r)$ values for a given resource r will guarantee that players do not have access to the declared values of other players before they declare their own value. Hence, for every resource r, the preferences $\dot{\theta}_1(r), \ldots, \dot{\theta}_n(r)$ are mutually independent. Moreover, we assume that the players' strategies and declared values cannot change the belief of other players. Intuitively, this means that the declared values of a player j are independent with respect to the previous preferences of j and previous declared values of the other players. Formally, a strategy for the player j is any map $\sigma_j : \Theta_j \to \Delta(\Theta_j)$, where $\sigma_j(\bar{\theta}_j, \dot{\theta}_j)$ is the conditional probability that the player reports $\dot{\theta}_j$ when her true type is $\bar{\theta}$. Observe that this formulation assumes that dishonest players do not collude.

Using the normalized preferences $\bar{\theta}(r)$, the declared preferences $\dot{\theta}(r)$, and the decision function $D(\cdot)$, we can define the normalized utility of a mechanism (for player j with respect to resource r).

$$\dot{u}_j(r) = \begin{cases} \bar{\theta}_j(r), & \text{if } j \text{ gets resource } r,\\ \\ 0, & \text{otherwise.}\end{cases} \tag{1}$$

Hence, we want to find mechanisms that maximize the utility of all players. When the player j is honest we denote this utility as $\bar{u}_j$.

The authors in [9] present the QPQ mechanism that solve this resource allocation problem. QPQ is based on the Linking Mechanism Design proposed by [13]. This solution has some nice properties like approximate truthfulness, expected utilities that converge to an efficient allocation, and no-payments. In Table 1, we present the QPQ mechanism applied to our family of resource allocation problems.

As can be seen in Line 6 of Table 1, the basic QPQ algorithm executed by player i applies a *Goodness of Fit* (GoF) test[2] to each of the declared value $\dot{\theta}_j(r)$ (with the aid of a repository $History_j$ in which the values previously used for player j are stored). This test evaluates whether the declared value $\dot{\theta}_j(r)$ matches the appropriate probability distribution

[2]We can use any Goodness-of-Fit test, as for example Kolmogorov-Smirnov or χ^2 test.

TABLE 1. The QPQ mechanism for distributions of resources.

Code for player i and a generic resource r.

01: Estimate the preference $\theta_i(r)$
02: Calculate the normalized preference $\bar{\theta}_i(r) = PIT_i(\theta_i(r))$
03: Declare a value $\dot{\theta}_i(r)$ that represents her normalized preference.
04: Wait to receive the published normalized preferences $\dot{\theta}_j(r)$ from all players
05: **For all** $j \in \mathcal{N}$ **do**
06: **if not** GoF_Test($\dot{\theta}_j(r)$, $History_j$) **then**
07: $\ddot{\theta}_j(r) \leftarrow \hat{\theta}_j(r)$, where $\hat{\theta}_j(r) :=$ Pseudorandom($\dot{\theta}_{-j}(r)$)
08: **else**
09: $\ddot{\theta}_j(r) \leftarrow \dot{\theta}_j(r)$
10: **end if**
11: $History_j \leftarrow History_j \cup \{\dot{\theta}_j(r)\}$
12: **end for**
13: Let $d = \arg\max_{j \in \mathcal{N}} \{\ddot{\theta}_j(r)\}$
14: **if** $d = i$ **then**
15: Player i gets resource r
16: **end if**

of values (a uniform distribution in [0, 1]). In the analytical sections of this paper we will assume that this GoF test is perfect, i.e., $\dot{\theta}_j(r)$ passes the test if and only if it has been extracted from the appropriate probability distribution.

The outcome of the GoF evaluation (Lines 6-10) is the value $\ddot{\theta}_j(r)$ for each player j. If a value $\dot{\theta}_j(r)$ passes the test, then simply $\ddot{\theta}_j(r) = \dot{\theta}_j(r)$. Otherwise, $\ddot{\theta}_j(r) = \hat{\theta}_j(r)$, where $\hat{\theta}_j(r)$ is a pseudo-random value generated from the rest of declared values [TODO: give a practical example of how this would be done, eg, using a cryptographically secure random generator], so that it follows the appropriate probability distribution. Since all players use the same GoF test and the same pseudo-random function, the values $\ddot{\theta}_j(r)$ are all the same in all players.

Finally, the values $\ddot{\theta}_j(r)$ are used to select the player that receives the resource r (Lines 13-16). The player d with the largest such value gets the resource. Since all players have the same set of values $\ddot{\theta}_j(r)$, they all agree on the decision. Hence, the decision function used is $D(\ddot{\theta}_j(r)) = \arg\max_{j \in \mathcal{N}} \{\ddot{\theta}_j(r)\}$.

III. THE MULTILEVEL QPQ APPROACH

We observe that the QPQ mechanism of Table 1 presents issues when the number of players is large because all the players must know the declared values of the rest. First, the number of communications that must be established for each resource is of the order of n^2. Secondly, each player must store in $History_j$ all the values declared by player j, for every $j \in \mathcal{N}$. This requires an overall amount of memory of n^2 times the number of resources. This clearly indicates that this mechanism does not scale properly and we deal with this issue by splitting the set of players into clusters organized hierarchically.

In the multilevel QPQ mechanism, we consider that the set of players $\mathcal{N}$ is partitioned into $k > 1$ subsets, or *clusters*, $\mathcal{N}_1, \ldots, \mathcal{N}_k$. The size of cluster c is $n_c = |\mathcal{N}_c|$, and the membership is $\mathcal{N}_c = \{(c, 1), \ldots, (c, n_c)\}$. Hence, $\mathcal{N} = \mathcal{N}_1 \cup \cdots \cup \mathcal{N}_k$, $\mathcal{N}_i \cap \mathcal{N}_j = \emptyset$ for $i \neq j$, and $n = \sum_{j=1}^{k} n_j$. Each of the clusters $\mathcal{N}_1, \ldots, \mathcal{N}_k$ behave as one player of a supercluster.

We consider that the clusters are fixed, that is, each player belongs to the same cluster in all the rounds. In addition to this, we assume that the size and membership of all the clusters are known by all the players.

We now explain how we adapt the notation of the previous section to the multilevel QPQ mechanism. We denote by (c, i) the player i of cluster c. The preference of player (c, i) for a given resource r is denoted by $\theta_{c,i}(r)$. Besides, (c, i) applies the PIT function to the preferences and, thus, its normalized preference for a resource r is denoted by $\bar{\theta}_{c,i}(r)$ and the declared preference by $\dot{\theta}_{c,i}(r)$.

The multilevel QPQ mechanism for player (c, i) and resource r is presented in Table 2 and we describe it briefly here. First, from Line 5 to Line 13, the QPQ algorithm of [9] is applied to the players of cluster c. Hence, for each player in cluster c, if the declared value passes the test, $\ddot{\theta}_{c,i}(r)$ equals the declared value, that is, $\ddot{\theta}_{c,i}(r) = \dot{\theta}_{c,i}(r)$, and $\ddot{\theta}_{c,i}(r)$ gets a uniform random value otherwise. We denote by $\mathcal{N}^{(1)}$ the set of winners of each cluster. Then, from Line 19 to Line 27, the QPQ algorithm of [9] is applied to the set of winners, that is, to $\mathcal{N}^{(1)}$. Hence, for each player (w, b_w) of $\mathcal{N}^{(1)}$, if the declared value passes the test, then $\dddot{\theta}_{w,b_w}(r) = \ddot{\theta}_{w,b_w}(r)$, and $\dddot{\theta}_{w,b_w}(r)$ gets a pseudorandom value that follows an appropriate distribution otherwise. Finally, the resource is allocated to player $(c, i) \in \mathcal{N}^{(1)}$ if $\dddot{\theta}_{c,i}(r)$ is the maximum value.

We define for each player (c, i) the following random variable to quantify her normalized utility associated to resource $r \in \mathcal{R}$.

$$\dot{u}_{c,i}(r) = \begin{cases} \bar{\theta}_{c,i}(r), & \text{if } D(\dddot{\theta}(r)) = (c, i), \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Note that the above value depends on the real normalized preference $\bar{\theta}_{c,i}(r)$ of the player.[3]

The authors in [9] showed that the QPQ mechanism is optimal in the sense that, if all the players are honest, the total utility generated is maximized. In the following result, we generalize this result to a system with clusters. That is, we show that, when the player are honest, the total utility of any mechanism that assigns a set of resources to players (divided in clusters or not) is smaller or equal than the total utility of the QPQ multilevel mechanism.

Proposition 3: Assume that all players are honest. For any resource $r \in \mathcal{R}$ and any normalized preferences $\bar{\theta}(r)$, every mechanism M (which may be probabilistic) verifies that

$$\sum_{j \in \mathcal{N}} \bar{u}_j^M(r) \leq \sum_{j \in \mathcal{N}} \bar{u}_j(r),$$

where $\bar{u}_j^M$ is any realization of the normalized utility of player j with respect to resource r and normalized preferences $\bar{\theta}(r)$ when mechanism M is applied.

Proof: Let us assume by way of contradiction that there exists a resource r and a mechanism M such that the above

[3] If it is clear from the context, we refer to the real normalized preferences as normalized preferences.

TABLE 2. **The multilevel QPQ mechanism for distributions of resources.**

Code for player i of cluster c and a generic resource r.

01: Estimate the preference $\theta_{c,i}(r)$
02: Calculate the normalized preference $\bar{\theta}_{c,i}(r) = PIT_{c,i}(\theta_{c,i}(r))$
03: Declare a value $\dot{\theta}_{c,i}(r)$.
04: Wait to receive the declared preferences from all players
05: **For all** $j \in \mathcal{N}_c$ **do**
06: **if not** GoF_Test($\dot{\theta}_{c,j}(r)$, $CHistory_{c,j}$) **then**
07: $\ddot{\theta}_{c,j}(r) \leftarrow \hat{\theta}_{c,j}(r)$, where $\hat{\theta}_{c,j}(r) :=$ Pseudorandom($\{\dot{\theta}_{c,m}(r)\}_{m \in \mathcal{N}_c \setminus \{j\}}$)
08: **else**
09: $\ddot{\theta}_{c,j}(r) \leftarrow \dot{\theta}_{c,j}(r)$
10: **end if**
11: $CHistory_{c,j} \leftarrow CHistory_{c,j} \cup \{\ddot{\theta}_{c,j}(r)\}$
12: **end for**
13: Let $d = \arg\max_{j \in \mathcal{N}_c}\{\ddot{\theta}_{c,j}(r)\}$
14: **if** $d = i$ **then** // Player (c, i) is the winner of cluster c;
15: Declare value $\ddot{\theta}_{c,i}(r)$ to all the players.
16: **end if**
17: Wait to receive the value $\ddot{\theta}_{w,b_w}(r)$ from the winners of the rest of the clusters.
18: Let $\mathcal{N}^{(1)} = \{(1, b_1), \ldots, (k, b_k)\}$ be the set of winners
19: **For all** $(w, b_w) \in \mathcal{N}^{(1)}$ **do**
20: **if not** GoF_Test($\ddot{\theta}_{w,b_w}(r)$, $SCHistory_w$) **then**
21: $\dddot{\theta}_{w,b_w}(r) \leftarrow \hat{\ddot{\theta}}_{w,b_w}(r)$, where $\hat{\ddot{\theta}}_{w,b_w}(r) :=$ Pseudorandom($\{\ddot{\theta}_{x,b_x}(r)\}_{(x,b_x) \in \mathcal{N}^{(1)} \setminus \{(w,b_w)\}}$)
22: **else**
23: $\dddot{\theta}_{w,b_w}(r) \leftarrow \ddot{\theta}_{w,b_w}(r)$
24: **end if**
25: $SCHistory_w \leftarrow SCHistory_w \cup \{\dddot{\theta}_{w,b_w}(r)\}$
26: **end for**
27: Let $d = \arg\max_{(w,b_w) \in \mathcal{N}^{(1)}}\{\dddot{\theta}_{w,b_w}(r)\}$
28: **if** $d = (c, i)$ **then**
29: Player i gets resource r
30: **end if**

inequality is not satisfied. Therefore, it holds that

$$\sum_{j \in \mathcal{N}} \bar{u}_j^M(r) > \sum_{j \in \mathcal{N}} \bar{u}_j(r).$$

When the multilevel QPQ mechanism of Table 2 is used, the following holds. Since all players (c, i) are honest, $\dot{\theta}_{c,i}(r) = \bar{\theta}_{c,i}(r)$. Since the GoF test is perfect, $\ddot{\theta}_{c,i}(r) = \bar{\theta}_{c,i}(r)$. Hence, the winner of each cluster is the player in the cluster with largest normalized utility. Similarly, for each winner it holds that $\dddot{\theta}_{w,b_w}(r) = \bar{\theta}_{w,b_w}(r)$. This implies that resource r is assigned in Lines 27-30 to the player $w^* = \arg\max_{j \in \mathcal{N}}\{\bar{\theta}_j(r)\}$. Let us assume that, when we apply mechanism M, the resource r is assigned to player w^M. With both mechanisms, resource r is assigned to a single player, and hence all players have a zero normalized utility except for that player (w^* and w^M, for Multilevel QPQ and M, respectively). Thus, it results that

$$\bar{u}_{w^M}^M(r) = \bar{\theta}_{w^M}(r) > \bar{u}_{w^*}(r) = \arg\max_{j \in \mathcal{N}}\{\bar{\theta}_j(r)\},$$

and we have found a contradiction. □

The following proposition follows from the definition of normalized utility and the fact that a resource r is assigned to only one player.

Proposition 4: For any resource $r \in \mathcal{R}$, any normalized preferences $\bar{\theta}(r)$, and any declared preferences $\dot{\theta}(r)$,

$$\sum_{j \in \mathcal{N}} \dot{u}_j(r) \leq \sum_{j \in \mathcal{N}} \bar{u}_j(r) \leq \max_{j \in \mathcal{N}}\{\bar{\theta}_j(r)\},$$

where $\dot{u}_j(r)$ is the real utility of player j when the declared preferences are $\dot{\theta}(r)$.

IV. THE BENEFITS OF BEING HONEST

In this section, we prove that, for any player, being honest is the strategy that maximizes its utility. Prior to presenting the analysis we have done to prove this result, we give the following result:

Proposition 5: The preferences of players (c, i) that the multilevel QPQ mechanism uses to assign the resource r, $\ddot{\theta}_{c,i}(r)$, are drawn from independent and uniform distributions in $[0, 1]$.

Proof: We distinguish the following three cases, depending on the behavior of the players:

- **The player** (c, i) **is honest.** In this case, we have that the preferences are $\ddot{\theta}_{c,i}(r) = \bar{\theta}_{c,i}(r)$ and, therefore, they follow a uniform normalized distribution, from Proposition 2. Besides, they are independent since players declare their preferences before receiving values of the others.

- **The player declares values that do not follow a uniform distribution.** In this case, the declared value $\dot{\theta}_{c,i}(r)$ does not pass the GoF test and the algorithm assigns a random value $\hat{\theta}_{c,i}(r)$ to the player (c, i), which is uniformly distributed in [0, 1] and independent from the preferences of the others.
- **A player is dishonest and she passes the *GoF* test.** This occurs when the declared values $\dot{\theta}_{c,i}(r)$ follow a uniform distribution in [0, 1], but they are different from $\bar{\theta}_{c,i}(r)$. Besides, the declared values are independent from other player's preferences, since the value is sent before the others are received.

Since, in all the cases, the values are independent and uniformly distributed in [0, 1], the desired result follows. □

The authors in [9] use the notion of aggregated player to compute the expected utility of a player. The idea is the following: they consider the rest of the players as a single fictitious player whose preference is the maximum of all of them. As a result, the computation of the expected utility of a player gets simplified since it is reduced to calculate the probability that its preference is larger than that of the aggregated player. In the following section, we show how we adapt the concept of aggregated player to the multilevel QPQ mechanism.

A. AGGREGATED PLAYER

In the multilevel QPQ mechanism, we consider three aggregated players. The first one is inside the clusters, i.e., we consider that each player (c, i) is competing against a fictitious aggregated player, denoted $(c, -i)$, whose preference is the maximum of the preferences of the rest of the players in cluster c. We denote by $\ddot{\theta}_{c,-i}$ the preference of this aggregated player in the cluster level, i.e.,

$$\ddot{\theta}_{c,-i} = \max\{\ddot{\theta}_{c,j} : j \in \mathcal{N}_c \backslash \{i\}\}. \tag{3}$$

Hence, player (c, i) is the winner of cluster c when $\ddot{\theta}_{c,i} > \ddot{\theta}_{c,-i}$.

When player (c, i) is the winner of cluster c (i.e., $b_c = i$) its preference $\dddot{\theta}_{c,i}$ is compared with that of the winners of the other clusters. Therefore, the second aggregated player we consider, denoted $(-c)$, is for the cluster winners, i.e., when the player (c, i) is the winner of the cluster c, it is in competition against the aggregated player that is formed by the rest of the winners, whose preference of the aggregated player of the winner is denoted by $\dddot{\theta}_{-c}$, and defined as

$$\dddot{\theta}_{-c} = \max\{\dddot{\theta}_{w,b_w} : (w, b_w) \in \mathcal{N}^{(1)} \backslash \{(c, i)\}\}. \tag{4}$$

We also define the aggregated player of (c, i) of the entire system, denoted by $-(c, i)$, as the player whose preference is the maximum between $\dddot{\theta}_{-c}$ and $\ddot{\theta}_{c,-i}$, i.e.,

$$\dddot{\theta}_{-(c,i)} = \max\{\dddot{\theta}_{-c}, \ddot{\theta}_{c,-i}\}. \tag{5}$$

The objective of this section is to calculate the distribution of preferences $\ddot{\theta}_{c,-i}$, $\dddot{\theta}_{-c}$ and $\dddot{\theta}_{-(c,i)}$. The next result is the key to quantify them.

Proposition 6: Let $\mathcal{X}_1, \ldots, \mathcal{X}_\kappa$, $\kappa > 1$, be independent continuous random variables such that they follow, respectively, a $Beta(p_j, 1)$ distribution, $p_j \in \mathbb{N}\ \forall j \in \{1, \ldots, \kappa\}$. Then, the random variable

$$\mathcal{X} = \max\{\mathcal{X}_j | 1 \leq j \leq \kappa\}$$

follows a $Beta(\sum_{j=1}^{\kappa} p_j, 1)$ distribution.

Proof: First, from Proposition 1, it follows that $\mathcal{X}$ is a continuous random variable, since the maximum is a measurable function. As a result, we consider $F_\mathcal{X}$ the CDF of $\mathcal{X}$ and we calculate its value for any $y \in \mathbb{R}$ as follows:

$$\begin{aligned} F_\mathcal{X}(y) &= P[\mathcal{X} \leq y] \\ &= P[\left(\max\{\mathcal{X}_j | 1 \leq j \leq \kappa\}\right) \leq y] \\ &= P[\mathcal{X}_1 \leq y, \ldots, \mathcal{X}_\kappa \leq y]. \end{aligned}$$

Since the random variables $\mathcal{X}_1, \ldots, \mathcal{X}_\kappa$ are independent and $\mathcal{X}_j$ follows a $Beta(p_j, 1)$ distribution, it results that

$$\begin{aligned} P[\mathcal{X}_1 \leq y, \ldots, \mathcal{X}_\kappa \leq y] &= \prod_{j=1}^{\kappa} P[\mathcal{X}_j \leq y] \\ &= \prod_{j=1}^{\kappa} \left(\int_{-\infty}^{y} f_j(x) dx\right), \end{aligned}$$

where $f_j(x)$ is the density function associated to a $Beta(p_j, 1)$ distribution. That is,

$$f_j(x) = \begin{cases} p_j \cdot x^{p_j - 1} & \text{if } x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Depending on the value of y, we differentiate three cases:

- If $y \leq 0$, then
$$F_\mathcal{X}(y) = \prod_{j=1}^{\kappa} \left(\int_{-\infty}^{y} 0\, dx\right) = 0.$$
- If $y \in [0, 1]$, then
$$\begin{aligned} F_\mathcal{X}(y) &= \prod_{j=1}^{\kappa} \left(\int_{-\infty}^{0} 0\, dx + \int_{0}^{y} p_j \cdot x^{p_j - 1} dx\right) \\ &= y^{\left(\sum_{j=1}^{\kappa} p_j\right)}. \end{aligned}$$
- If $y \geq 1$, then
$$\begin{aligned} F_\mathcal{X}(y) &= \prod_{j=1}^{\kappa} \left(\int_{-\infty}^{0} 0\, dx + \int_{0}^{1} p_j \cdot x^{p_j - 1} dx + \int_{1}^{y} 0\, dx\right) \\ &= 1. \end{aligned}$$

If we derive $F_\mathcal{X}$ with respect to y, we obtain the density function of $\mathcal{X}$:

$$f_\mathcal{X}(y) = \begin{cases} \left(\sum_{j=1}^{\kappa} p_j\right) \cdot y^{\left(\sum_{j=1}^{\kappa} p_j\right) - 1} & \text{if } y \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Let's recall that the density function associated to a *Beta*(p, q) distribution is:

$$f_\beta(y) = \begin{cases} \frac{1}{\beta(p,q)} \cdot y^{p-1} \cdot (1-y)^{q-1} & \text{if } y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

When $p = \sum_{j=1}^{\kappa} p_j$ and $q = 1$, it results that, if $y \in [0, 1]$,

$$\begin{aligned} f_\beta(y) &= \frac{1}{\beta\left(\sum_{j=1}^{\kappa} p_j, 1\right)} y^{\left(\sum_{j=1}^{\kappa} p_j\right)-1} \\ &= \frac{\Gamma(1+\sum_{j=1}^{\kappa} p_j)}{\Gamma(1)\Gamma(\sum_{j=1}^{\kappa} p_j)} y^{\left(\sum_{j=1}^{\kappa} p_j\right)-1} \\ &= \left(\sum_{j=1}^{\kappa} p_j\right) y^{\left(\sum_{j=1}^{\kappa} p_j\right)-1} \end{aligned}$$

and, otherwise, $f_\beta(y) = 0$. Therefore, the desired result follows since $f_{\mathcal{X}}$ is equal to f_β when $p = \sum_{j=1}^{\kappa} p_j$ and $q = 1$. □

From Proposition 5, we know that the preferences $\ddot{\theta}_{c,i}$ of the players follow independent uniform distributions in [0, 1]. A uniform distributions in [0, 1] coincides with the *Beta*(1, 1) distribution. Therefore, from Proposition 6, we derive the preferences' distribution of the first aggregated player in the following result.

Corollary 1: The preferences $\ddot{\theta}_{c,-i}$ of the aggregated player $(c, -i)$ follow a Beta$(n_c - 1, 1)$ distribution. Moreover, this distribution in independent from the distribution of $\ddot{\theta}_{c,i}$, which follows a uniform distribution in [0, 1].

Using the same arguments as in the above result, one can easily provide the distribution that the preferences of the winner of cluster c follow.

Corollary 2: The preferences $\ddot{\theta}_{c,b_c}$ of the winner b_c of cluster c follow a Beta$(n_c, 1)$ distribution. Moreover, this distribution is independent from the distribution of $\ddot{\theta}_{c',b_{c'}}$, for all cluster $c' \neq c$.

As a result of this corollary and Proposition 6, the next result follows.

Corollary 3: The preferences $\dddot{\theta}_{-c}$ of the aggregated player $(-c)$ follow a Beta$(n - n_c, 1)$ distribution. This distribution is independent of the distribution of preferences $\ddot{\theta}_{c,b_c}$ of the winner b_c of cluster c.

As a result of Corollary 3 and Corollary 1, we have the following result.

Corollary 4: The preferences $\dddot{\theta}_{-(c,i)}$ of the aggregated player $-(c, i)$ follow a Beta$(n - 1, 1)$ distribution. Moreover, this distribution is independent from the distribution of $\ddot{\theta}_{c,i}$, which follows a uniform distribution in [0, 1].

B. HONEST STRATEGY

When a player (c, i) declares its real preference for a resource r, since we assume that the GoF test is perfect, we have that $\ddot{\theta}_{c,i} = \bar{\theta}_{c,i}$ and, if it is the winner of cluster c, we also have that $\dddot{\theta}_{c,i} = \bar{\theta}_{c,i}$. Hence, the obtained utility of player (c, i) for resource $r \in \mathcal{R}$ (omitted from now on) can be written as follows (combining Eq. 2, 3, and 4, and the decision function $D(\cdot)$),

$$\bar{u}_{c,i} = \begin{cases} \bar{\theta}_{c,i}, & \text{if } (\ddot{\theta}_{c,-i} < \bar{\theta}_{c,i}) \wedge (\dddot{\theta}_{-c} < \bar{\theta}_{c,i}), \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Observe that, from the definition of $\dddot{\theta}_{-(c,i)}$ (Eq. 5), this is equivalent to

$$\bar{u}_{c,i} = \begin{cases} \bar{\theta}_{c,i}, & \text{if } \dddot{\theta}_{-(c,i)} < \bar{\theta}_{c,i}, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

We use the above expression to compute the expected profit of an honest player in the following result.

Proposition 7: The expected normalized utility of an honest player (c, i) is $E[\bar{u}_{c,i}] = \frac{1}{1+n}$.

Proof: As explained before, a player (c, i) gets resource $r \in \mathcal{R}$ if and only if $(\ddot{\theta}_{-(c,-i)} < \bar{\theta}_{c,i}) \wedge (\dddot{\theta}_{-c} < \bar{\theta}_{c,i})$ holds. This is equivalent to $\dddot{\theta}_{-(c,i)} < \bar{\theta}_{c,i}$. Therefore, from Corollary 4, the expected normalized utility of player (c, i) is

$$\begin{aligned} E[\bar{u}_{c,i}] &= \int_0^1 x \cdot P[(\ddot{\theta}_{c,-i} < x) \wedge (\dddot{\theta}_{-c} < x)]dx \\ &= \int_0^1 x \cdot P[\dddot{\theta}_{-(c,i)} < x]dx \\ &= \int_0^1 x \int_0^x (n-1)y^{n-2} dy dx \\ &= \int_0^1 x \left[x^{n-1}\right] dx \\ &= \int_0^1 x^n dx \\ &= \frac{1}{1+n}. \end{aligned}$$

□

C. ARBITRARY RATIONAL STRATEGIES

In this section we show that if *all players are rational,* no strategy will allow a player to have a expected real normalized utility higher than being honest. From Proposition 7 and 4 we have the following corollary, which gives an upper bound on the sum of the real normalized utilities of all players.

Corollary 5: For any set of strategies $\dot{\sigma}$ used by the players, where $\dot{\theta}$ are the declared preferences of the players with $\dot{\sigma}$,

$$E[\sum_j \dot{u}_j] \leq E[\sum_j \bar{u}_j] = \frac{n}{n+1}$$

The following result shows that no rational player will ever have less expected utility than the obtained being honest.

Corollary 6: Let $\dot{\sigma}_{c,i}$ be a strategy used by a rational player (c, i). Let $\dot{\theta}_{c,i}$ be the preferences declared by (c, i) with $\dot{\sigma}_{c,i}$, and $\dot{u}_{c,i}$ the corresponding utility. Then,

$$E[\dot{u}_{c,i}] \geq E[\bar{u}_{c,i}] = \frac{1}{n+1}$$

Proof: Since player (c, i) is rational, it is going to use the strategy that maximizes its expected utility. Since one possible such strategy is being honest, $E[\dot{u}_{c,i}]$ cannot be smaller than $E[\bar{u}_{c,i}]$. The equality follows from Proposition 7. □

With this, we can show the following theorem.

Theorem 1: Assume all players are rational. Let $\dot{\sigma}_{c,i}$ be a strategy used by a rational player (c, i). Player (c, i) never obtains more expected normalized utility with $\dot{\sigma}_{c,i}$ than by being honest,

$$E[\bar{u}_{c,i}] \geq E[\dot{u}_{c,i}].$$

Proof: Let us assume player (c, i) uses a strategy $\dot{\sigma}_{c,i}$ that allows it to have a expected normalized utility $E[\dot{u}_{c,i}]$ that is larger than the expected utility $E[\bar{u}_{c,i}]$ it gets being honest. I.e., $E[\dot{u}_{c,i}] > E[\bar{u}_{c,i}] = \frac{1}{n+1}$. From Corollary 6, the sum of the expected utilities for the other rational players satisfies,

$$E[\sum_{j\neq(c,i)} \dot{u}_j] = \sum_{j\neq(c,i)} E[\dot{u}_j] \geq \sum_{j\neq(c,i)} E[\bar{u}_j] = \frac{n-1}{n+1}.$$

Hence,

$$\begin{aligned} E[\sum_j \dot{u}_j] &= E[\dot{u}_{c,i}] + E[\sum_{j\neq(c,i)} \dot{u}_j] \\ &> \frac{1}{n+1} + \frac{n-1}{n+1} \\ &= \frac{n}{n+1}, \end{aligned}$$

which violates Corollary 5. □

D. DISHONEST STRATEGIES THAT DO NOT PASS THE GOF TEST

In this section, we prove that the utility of a player when it declares the real preferences is larger than when the declared preferences do not pass the GoF test.

We study the utility of a dishonest player whose the declared value does not pass the GoF test. This occurs when the player declares non uniform values. For this case, the player is assigned a new random preference, i.e., $\ddot{\theta}_{c,i} = \hat{\theta}_{c,i}$. We assume that the generator of the random values is perfect and, hence, as shown in Proposition 5, the preferences $\ddot{\theta}_{c,i}$ are independent and follow a uniform distribution in [0, 1]. As a result, the GoF test which is applied to the winners of the clusters (see Line 06 of Table 2) never fails. Hence, if (c, i) is the winner of cluster c, it holds that $\dddot{\theta}_{c,i} = \ddot{\theta}_{c,i}$ and $\dddot{\theta}_{-c} = \ddot{\theta}_{-c}$.

The normalized utility, $\hat{u}_{c,i}$, of player (c, i) when it declares values that do not pass the GoF test is given by

$$\hat{u}_{c,i} = \begin{cases} \bar{\theta}_{c,i}, & \text{if } (\ddot{\theta}_{c,-i} < \hat{\theta}_{c,i}) \wedge (\dddot{\theta}_{-c} < \hat{\theta}_{c,i}), \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Notice that, from the definition of $\dddot{\theta}_{-(c,i)}$ (Eq. 5), this is equivalent to

$$\hat{u}_{c,i} = \begin{cases} \bar{\theta}_{c,i}, & \text{if } \dddot{\theta}_{-(c,i)} < \hat{\theta}_{c,i}, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

In the next proposition, we compute the expected utility of a player when its declared values do not pass the GoF test.

Proposition 8: The expected normalized utility of a player (c, i) whose declared values do not pass the GoF test is

$$E[\hat{u}_{c,i}] = \frac{1}{2n}$$

Proof: When the value declared by the player (c, i) does not pass the *GoF*, from (9), it follows that the expected utility of a player (c, i) associated to the resource r is given by

$$E[\hat{u}_{c,i}] = \int_0^1 \int_0^1 x \cdot P[\dddot{\theta}_{-(c,i)} < z]dzdx,$$

where z represents the value $\hat{\theta}_{c,i}$ assigned to player (c, i). From Corollary 4, we know that the distribution of $\dddot{\theta}_{-(c,i)}$ is the $Beta(n-1, 1)$ and therefore

$$\begin{aligned} E[\hat{u}_{c,i}] &= \int_0^1 \int_0^1 x \cdot P[\dddot{\theta}_{-(c,i)} < z]dzdx \\ &= \int_0^1 x \int_0^1 \left(\int_0^z (n-1)y^{n-2}dy \right) dzdx \\ &= \int_0^1 x \int_0^1 z^{n-1}dzdx \\ &= \int_0^1 \frac{x}{n}dx \\ &= \frac{1}{2n}. \end{aligned}$$

And the desired result follows. □

From Proposition 7, the utility of a player when it declares real values is $\frac{1}{n+1}$ and from Proposition 8, the utility of a player when it declares values that do not pass the GoF test is $\frac{1}{2n}$. Hence, using that for all $n > 1$

$$\frac{1}{2n} < \frac{1}{n+1},$$

the next result follows.

Corollary 7: Let $(c, i) \in \mathcal{N}$ an arbitrary player. The expected utility of player (c, i) when it is honest is greater than when it declares values that do not pass the GoF test, i.e.,

$$E[\bar{u}_{c,i}] > E[\hat{u}_{c,i}].$$

E. DISHONEST STRATEGIES THAT PASS THE GOF TEST

In Section IV-D we have shown that it is preferable for a player to be honest than to declare values that do not pass the GoF test. However, players can declare values that pass the GoF test but are different from the real preferences. This occurs, for instance, if the declared values follow a uniform distribution different from the real uniform distribution of the player. Therefore, in this section, we generalize the result of Corollary 7 to any dishonest strategy.

In the remainder of this section we use the following concepts. The strategy of player (c, i) is realized by its declared values $\dot{\theta}_{c,i}$. Since these values pass the GoF test and therefore $\dot{\theta}_{c,i} = \ddot{\theta}_{c,i} = z$, they follow a uniform distribution in [0, 1]. We assume that they are defined by a bi-variate density function $\sigma_j(x, z) = \sigma_j(\bar{\theta}_j, \dot{\theta}_j)$ that relates the real preferences

x and the declared values z. I.e., $\dot{\theta}_{c,i}(r)$ is chosen randomly from a distribution with density function $\sigma_j(\bar{\theta}_{c,i}(r), z)$ on z.

Since these declared values $\dot{\theta}_{c,i}$ pass the GoF test, it holds that the marginal distribution $f_z(z)$ of $\sigma_j(x, z)$ on z satisfies

$$f_z(z) = \int_0^1 \sigma_j(x, z)dx = 1.$$

Lemma 1: For any player (c, i) and any $y \in [0, 1]$,

$$P[\bar{\theta}_{c,i} < y] = P[\dot{\theta}_{c,i} < y] = y.$$

Proof: Since the real preferences of player (c, i) follow a uniform distribution, it follows that

$$P[\bar{\theta}_{c,i} < y] = \int_0^y 1\, dx = y.$$

For the declared values we have that

$$\begin{aligned} P[\dot{\theta}_{c,i} < y] &= \int_0^1 \left(\int_0^y \sigma_j(x, z)\, dz \right) dx \\ &= \int_0^y \left(\int_0^1 \sigma_j(x, z)\, dx \right) dz \\ &= \int_0^y f_z(z)\, dz, \end{aligned}$$

where $f_z(z)$ is the marginal distribution of $\sigma_j(x, z)$ on z, as defined above. Since $f_z(z) = 1$, therefore,

$$P[\dot{\theta}_{c,i} < y] = \int_0^y 1\, dz = y.$$

□

A similar result could be obtained for the aggregated player $-(c, i)$. From Corollary 4, given that z is the maximum of $n - 1$ independent uniform random variables, we conclude that $z \sim Beta(n - 1, 1)$. Hence, we have the following result.

Lemma 2: For any strategy of the players in the set of the aggregated player $-(c, i)$ and for any $y \in [0, 1]$,

$$P[\ddot{\theta}_{-(c,i)} < y] = (n - 1)\, y^{n-1}.$$

The expected utility of the aggregated player $-(c, i)$ when all its players declare their real preferences is defined as follows:

$$\bar{u}_{-(c,i)} = \begin{cases} \bar{\theta}_{-(c,i)} & \text{if } \ddot{\theta}_{-(c,i)} > \ddot{\theta}_{c,i} \\ 0 & \text{otherwise} \end{cases}$$

where $\bar{\theta}_{-(c,i)} = \max\{\bar{\theta}_j : j \in \mathcal{N} \setminus \{(c, i)\}\}$.

We now provide an interesting result of the expected utility $\bar{u}_{-(c,i)}$ of the aggregated player $-(c, i)$ as defined.

Proposition 9: The expected utility $\bar{u}_{-(c,i)}$ of the aggregated player $-(c, i)$ when all its players declare their real preferences does not depend on the strategy of the player (c, i).

Proof: We know from Corollary 4 that $\ddot{\theta}_{-(c,i)}$ follows the $Beta(n - 1, 1)$ distribution. Let $f(y) = (n - 1)y^{n-2}$ be the density function of the $Beta(n - 1, 1)$ distribution. From Lemma 1 and the fact that $\ddot{\theta}_{(c,i)} = \dot{\theta}_{(c,i)}$ (since $\dot{\theta}_{(c,i)}$ passed the GoF test), we have that

$$\begin{aligned} E[\bar{u}_{-(c,i)}] &= \int_0^1 y f(y)\, P[\ddot{\theta}_{(c,i)} < y]\, dy \\ &= \int_0^1 y f(y)\, P[\dot{\theta}_{(c,i)} < y]\, dy \\ &= \int_0^1 y f(y)\, y\, dy \\ &= \int_0^1 y\, (n - 1)y^{n-2}\, y\, dy \\ &= \int_0^1 (n - 1)y^n\, dy \\ &= \frac{n - 1}{n + 1}. \end{aligned}$$

□

We also show an equivalent result for the player (c, i).

Proposition 10: The expected utility $\dot{u}_{(c,i)}$ of the player (c, i) does not depend on the strategy of the aggregated player $-(c, i)$.

Proof: The expected utility can be computed as

$$\begin{aligned} E[\dot{u}_{(c,i)}] &= \int_0^1 x \int_0^1 \sigma(x, z)P[\ddot{\theta}_{-(c,i)} < z]dz\, dx \\ &= \int_0^1 x \int_0^1 \sigma(x, z)(n - 1)z^{n-2}dz\, dx, \end{aligned}$$

where the equality follows from Lemma 2 and the independence between the values $\dot{\theta}_{(c,i)}$ (i.e., z) declared by (c, i) and $\ddot{\theta}_{-(c,i)}$. Since this expression does not depend on the strategies of the players in the aggregated player $-(c, i)$, the claim follows. □

Finally, using the above results, we prove the main result of this section now.

Theorem 2: A player (c, i) never obtains less normalized utility (in expectation) by being honest, i.e.,

$$E[\bar{u}_{c,i}] \geq E[\tilde{u}_{c,i}],$$

where $\tilde{u}_{c,i}$ is the utility obtained with the declared values $\tilde{\theta}_{c,i}$ (which can be different from $\bar{\theta}_{c,i}$). Moreover, this is true for any number of clusters and any number of players in each cluster.

Proof: Let us assume that the values $\tilde{\theta}_{c,i}$ declared by (c, i) do not pass the *GoF* test. Then, from Corollary 7, the desired result follows.

We now consider that the values $\tilde{\theta}_{c,i}$ declared by (c, i) pass the *GoF* test. Let us suppose that there exists a set of declared values $\tilde{\theta}_{c,i}$ (different from the true preferences $\bar{\theta}_{c,i}$) such that player (c, i) gets less utility by being honest than by declaring values $\tilde{\theta}_{c,i}$, i.e.

$$E[\bar{u}_{c,i}] < E[\tilde{u}_{c,i}].$$

The intuition of the rest of the proof is as follows. In this scenario, since $\tilde{\theta}_{c,i}$ is a uniform distribution independent from the other players, this strategy used by (c, i) does not affect the utility of the other players. Let us assume the other players are all honest. Then, the total utility is higher than when all players are honest. The next step is to create a mechanism M that reproduces the allocation of multilevel QPQ with (c, i) using $\tilde{\theta}_{c,i}$ and all other players honest when also (c, i)

is honest. The existence of this mechanism would lead to a contradiction with completes the proof that the strategy $\tilde{\theta}_{c,i}$ does not exist.

More formally, let us assume values $\tilde{\theta}_{c,i}$ follow strategy $\sigma(\bar{\theta}, \tilde{\theta})$, which is the density probability function of announcing $\tilde{\theta}$ when the real preference is $\bar{\theta}$. From Proposition 10, this inequality holds for any strategy of the aggregated player $-(c, i)$ (i.e., any strategy of the rest of players). For that reason, we consider in the rest of the proof that, except for player (c, i), all the players behave honestly. Moreover, by Proposition 9, we know that the aggregated player $-(c, i)$ (i.e., each of the other players) obtains the same expected utility independently of whether the player (c, i) is honest or dishonest. As a result, total utility in a system in which all players are honest is smaller than the total utility in a system in which player (c, i) declares preferences $\tilde{\theta}_{c,i}$ and the rest of players are honest, i.e.,

$$E[\bar{u}_{c,i}] + E[\bar{u}_{-(c,i)}] < E[\tilde{u}_{c,i}] + E[\bar{u}_{-(c,i)}]. \tag{10}$$

Now, we define a mechanism M that behaves exactly like Multilevel QPQ but assigning to player (c, i) a preference $\dot{\theta}'_{c,i}$ chosen with a density $\sigma(\dot{\theta}, \dot{\theta}')$ when it declares $\dot{\theta}_{c,i}$ in Line 3. With this transformation, the probability that the resource r is assigned to player (c, i) when it is honest coincides with the probability that our mechanism allocates the resource to the player when declares the values $\tilde{\theta}_{c,i}$. Observe that this new mechanism M does not alter the assignment to other players and the values $\dot{\theta}'_{c,i}$ are still independent from the preferences of the other players. Therefore,

- $E[\tilde{u}_{c,i}] = E[\bar{u}^M_{c,i}]$
- $E[\bar{u}_{-(c,i)}] = E[\bar{u}^M_{-(c,i)}]$.

Replacing these values in (10), we obtain that

$$E[\bar{u}_{c,i}] + E[\bar{u}_{-(c,i)}] < E[\bar{u}^M_{c,i}] + E[\bar{u}^M_{-(c,i)}].$$

However, the above expression contradicts Proposition 3, that shows that our algorithm is optimal for honest players. Therefore, the strategy that maximizes the expected profit of the player (c, i) player is to be honest. □

Remark 1: Observe that this theorem generalizes the result of Thm 10 of [9] since the model of QPQ is a particular case of the multilevel QPQ.

V. THE BENEFITS OF CLUSTERING IN PRACTICE

In this section we compare multilevel QPQ with QPQ in several dimensions: communication cost, memory used, and the expected utility of honest players. For simplicity, unless otherwise stated, we will assume in this section that all clusters have the same size n/k.

A. BENEFITS IN COMMUNICATION COST

The first dimension in which multilevel QPQ improves versus QPQ is in the total communication volume per resource assignment that are required. Let us assume, for instance, that the declared values are sent to a central relay R, which then sends them to the players. QPQ has the following sequence of actions:

- Each player i sends its declared value to R.
- Relay R sends the set of values declared by all players to every player.

The total volume of communication is $V_{QPQ} = n + n^2$ values in $2n$ messages.

In multilevel QPQ the sequence is as follows.

- Each player (c, i) sends its declared value to R.
- Relay R send the values declared by all the players in cluster c to all the players in cluster c, for each cluster.
- The winners (c, b_c) from all clusters send their declared values in Line 15 of Table 2 to R.
- Relay R sends the set of values declared by all winners to all the players.

The total volume is $V_{mQPQ} = n + \sum_{j=1}^{k} n_j^2 + k + nk$, in a total of $3n + k$ messages. Let us consider the case in which all cluster have the same size $n_j = n/k, \forall j$. Then,

$$V_{mQPQ} = n + k(n/k)^2 + k + nk = n + n^2/k + k + nk.$$

Let us obtain the value of k that minimizes V_{mQPQ}. The derivative of the above expression with respect to k is

$$\frac{\partial V_{mQPQ}}{\partial k} = -\frac{n^2}{k^2} + 1 + n$$

and this is zero when

$$-n^2 + k^2 + nk^2 = 0$$

which, when n is large enough, the optimal number of clusters k^* is approximately

$$k^* \approx \sqrt{n}.$$

This leads to an asymptotic improvement in the complexity of the volume of communication as follows.

Proposition 11: Multilevel QPQ with k clusters of the same size n/k reduces the volume of data communication with respect to QPQ by

$$\frac{V_{QPQ}}{V_{mQPQ}} = \Omega(\frac{n}{n/k + k}),$$

which becomes $\Omega(n^{1/2})$ for $k = n^{1/2}$.

B. BENEFITS DUE TO IMPROVED MEMORY USE

In the analysis of Section IV, we have assumed that the *GoF* test is perfect. However, in practice, the *GoF* test is not perfect. This means that it can accept values that do not follow an adequate distribution (i.e., there are false positives), and can reject values that follow the distribution (i.e., there are false negatives). We claim that the performance of the *GoF* test improves with the length of the available history. This means that the *GoF* test has fewer false positives and negatives. Ideally, all players would maintain the full history of values used in all prior resource assignment rounds. However, this may not be possible since the required memory would grow without bound.

1) INCREASE OF THE HISTORY LENGTH

Let us first assume that the available memory to store the history is fixed, and compare the length of the history used in a *GoF* test with multilevel QPQ versus the original single-level QPQ.

Let S_1 be the memory available to store the history values at each player in the single-level QPQ. Thus, if H_1 is the length of the history used for each *GoF* test, it follows that

$$S_1 = H_1 n,$$

since in the one level model, a player performs the *GoF* test to n players.

In the multilevel QPQ, let the memory available to store history values be S_2. Then, when the history length used per player at the cluster level is H_c and at the upper level is H_u, we have that

$$S_2 = \frac{n}{k} H_c + k H_u.$$

Let us assume that $H_u = \alpha H_c$, for a constant value $\alpha > 0$. Then, we have that

$$S_2 = \left(\frac{n}{k} + \alpha k\right) H_c.$$

We are interested in studying the relationship between H_1, H_u, and H_c when $S_1 = S_2$. If we equalize the memories, we have that

$$H_c = \frac{nH_1}{\alpha k + \frac{n}{k}} = \frac{nk}{\alpha k^2 + n} H_1. \tag{11}$$

$$H_u = \alpha H_c = \frac{nk}{k^2 + n/\alpha} H_1. \tag{12}$$

In the following result, we present the relation between H_1, H_u and H_c for $\alpha = 1$.

Proposition 12: Consider $\alpha = 1$. Then, H_c and H_u are, at most, k times larger than H_1. Moreover,

- *When $n/k = 1$, H_c and H_u are smaller than H_1.*
- *For all k such that $n/k \geq 2$, H_c and H_u are always larger than H_1.*

Proof: For a fixed value of k, we consider the function $g(n) = \frac{n}{\frac{n}{k}+k}$. Since $g(n)$ is increasing with n and $\lim_{n\to\infty} g(n) = k$, we have that H_c is, at most, k times larger than H_1. For the lower bound, we observe that $g(k) < 1$, whereas $g(mk) > 1$, for all $m = 2, 3, \ldots$. □

We now fix $k = \sqrt{n}$, which we proved in the previous section is the choice of k that minimizes the communication volume. For this case, (11) and (12) give respectively

$$H_c = \frac{n^{3/2}}{n(\alpha+1)} H_1 = \frac{n^{1/2}}{\alpha+1} H_1$$

and

$$H_u = \frac{n^{3/2}}{n(1+1/\alpha)} H_1 = \frac{n^{1/2}}{1+1/\alpha} H_1.$$

Hence, we have the following result.

Proposition 13: Consider $k = \sqrt{n}$. Then, H_u and H_c are larger than H_1 when

$$k > \min\{\alpha + 1, 1 + \frac{1}{\alpha}\}.$$

Moreover, since α is a constant, $H_u = \Omega(\sqrt{n} H_1)$ and $H_c = \Omega(\sqrt{n} H_1)$.

2) INCREASE OF EXPECTED UTILITY OF HONEST PLAYERS

We now focus on a system with only honest players. As we just showed, it is possible to have a longer history using multilevel QPQ than with QPQ [9]. This means that, in practice, honest players will suffer of less false negatives in the GoF test (by definition a honest player can never have a false positive) using multilevel QPQ. We show here that this leads to a higher practical expected utility of honest players with multilevel QPQ than with QPQ.

Let us first provide the expression of the expected utility of QPQ [9]. We denote by q that probability of false negative. Hence, the expect utility for a given honest player j is given by

$$E[\bar{u}_j^{QPQ}] = \frac{1-q}{n+1} + \frac{q}{2n}. \tag{13}$$

The computation of the above expression is available in the Appendix.

For the multilevel QPQ, the test of GoF is carried out in the cluster level and in the upper level. We denote by p_c and p_u, respectively, the probability of false negative in the cluster level and in the upper level.

We now introduce the following notation: U_H is the expected utility when a player is honest and both GoF test do not fail; U_u the expected utility when the declared value passes the test of the cluster level, but not in the upper level; U_c the expected utility when the declared value does not pass the test of the cluster level, but it does in the upper level test; $U_{u,c}$ the expected utility when the declared value does not pass the test in the cluster level and in the upper level. Therefore, for the multilevel QPQ, since the GoF is done in the cluster and in the upper level, the utility of a player is given by

$$E[\bar{u}_j^{mQPQ}] = (1-p_c)(1-p_u)U_H + (1-p_c)p_u U_u + p_c(1-p_u)U_c + p_c p_u U_{u,c}.$$

The values of U_H, U_u, U_c and $U_{u,c}$ are given in the Appendix.

When all the clusters are of the same size, it results

$$\begin{aligned} E[\bar{u}_j^{mQPQ}] &= \frac{(1-p_c)(1-p_u)}{n+1} + \frac{p_c(1-p_u)}{2n} + \\ &\quad + \frac{(1-p_c)p_u \frac{n}{k}}{n(\frac{n}{k}+1)} + \frac{p_c p_u}{2n} \\ &= \frac{(1-p_c)(1-p_u)}{n+1} + \frac{p_c}{2n} + \frac{(1-p_c)p_u}{n+k}. \end{aligned} \tag{14}$$

We now show that (14) is, at most, two times (13). Therefore, we study the ratio of the utility of the two levels model over the utility of the

$$\frac{E[\bar{u}_j^{mQPQ}]}{E[\bar{u}_j^{QPQ}]} = \frac{\frac{(1-p_c)(1-p_u)}{n+1} + \frac{p_c}{2n} + \frac{(1-p_c)p_u}{n+k}}{\frac{1-q}{n+1} + \frac{q}{2n}}. \quad (15)$$

We note that (14) decreases with p_u and p_c and (13) decreases with q. Therefore, the maximum over p_u, p_c and q of (15) is given when $p_u = p_c = 0$ and $q = 1$.

When $p_u = 0$, $p_c = 0$ and $q = 1$, (15) gives

$$\frac{\frac{1}{n+1}}{\frac{1}{2n}} = \frac{2n}{n+1}.$$

We observe that the above ratio increases with n and it is equal to one when $n = 1$ and equal to 2 when $n \to \infty$. As a consequence, we have the following result:

Proposition 14: The utility of the two level system is, at most, two times higher than the utility of the one level system.

Let us now show that in fact multilevel QPQ can achieve higher expected utility than QPQ. We show a relation between p_c, p_u, and q that guarantees this property

Proposition 15: If $n > k$ and $(1-p_c)(1-p_u) \geq 1-q$ then an honest player achieves higher expected utility with multilevel QPQ than with QPQ, i.e.,

$$\frac{E[\bar{u}_j^{mQPQ}]}{E[\bar{u}_j^{QPQ}]} > 1.$$

Proof: From the assumption that $(1-p_c)(1-p_u) \geq 1-q$ we have that

$$\frac{(1-p_c)(1-p_u)}{n+1} \geq \frac{1-q}{n+1}.$$

Then, to have $\frac{E[\bar{u}_j^{mQPQ}]}{E[\bar{u}_j^{QPQ}]} > 1$ it is enough to have (see Eq. (15))

$$\frac{p_c}{2n} + \frac{(1-p_c)p_u}{n+k} > \frac{q}{2n}.$$

Since $\frac{1}{n+k} > \frac{1}{2n}$ when $n > k$, then

$$\frac{p_c}{2n} + \frac{(1-p_c)p_u}{n+k} > \frac{p_c}{2n} + \frac{(1-p_c)p_u}{2n}.$$

Hence, $\frac{E[\bar{u}_j^{mQPQ}]}{E[\bar{u}_j^{QPQ}]} > 1$ holds if

$$\frac{p_c}{2n} + \frac{(1-p_c)p_u}{2n} \geq \frac{q}{2n},$$

i.e., $p_c + (1-p_c)p_u \geq q$. This is always true from $(1-p_c)(1-p_u) \geq 1-q$, which implies

$$\begin{aligned} q &\leq 1 - (1-p_c)(1-p_u) \\ &= 1 - 1 + p_c + (1-p_c)p_u \\ &= p_c + (1-p_c)p_u. \end{aligned}$$

□

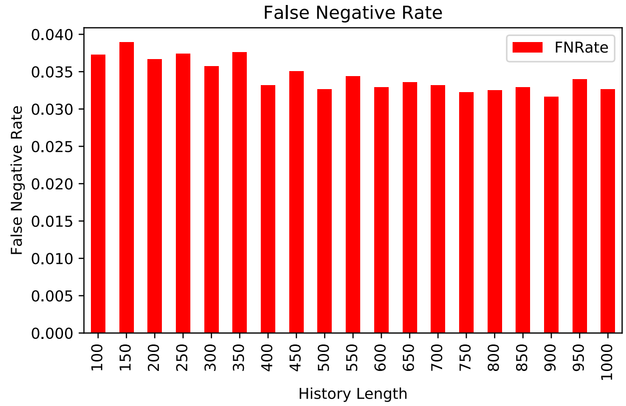

FIGURE 1. False negative probability when the history length changes from 100 to 1000.

VI. SIMULATION RESULTS

We now focus on the simulations we have carried out in this work.[4] The goal of this study is to compare the performance of multilevel QPQ with QPQ for several parameters and its impact on the utility of honest and dishonest players. While there are other mechanisms without payments that could be used to solve the problem [15], [16], they are designed for system models more general than the one considered here, and are equivalent to QPQ when deployed under the model and assumptions used in this paper.

In the experiments that we have conducted, we have used Kolmogorov-Smirnov as the *GoF* test. The Kolmogorov-Smirnov GoF test is given a history of values with a new value θ to test, and the reference distribution with which to compare. Then, it returns a p-value. This value p is compared with a threshold τ to determine whether θ passes the GoF test or not. If the p-value is smaller that the threshold τ, the value θ is considered to fail the GoF test.

When the history provided is in fact extracted from the reference distribution, the p-values returned are drawn from a [0, 1] uniform distribution. By construction, when the θ values provided are the preferences of an honest player, the probability of the GoF test failing (i.e., a false negative) is τ, independently of the history length (see Figure 1).

If τ_q is the threshold of QPQ, and τ_c and τ_u are the thresholds respectively in the cluster level and in the upper level of multilevel QPQ, by definition $\tau_q = q$, $\tau_c = p_c$, and $\tau_u = p_u$. In our experiments, we have set these thresholds of the *GoF* test of multilevel QPQ and QPQ such that the impact on the utility due to the false negatives in both systems is the same. Then, following Proposition 15, we consider that

$$\begin{aligned} (1-q) = (1-p_c)(1-p_u) \iff (1-\tau_q) \\ = (1-\tau_c)(1-\tau_u). \end{aligned}$$

Additionally, to simplify the analysis, we have considered in our simulations that $p_c = \tau_c = p_u = \tau_u$.

[4]Python libraries have been used for the simulations including numpy (version 1.18.1) and scipy (1.4.1). The results we present in this section can be reproduced using the code of [14].

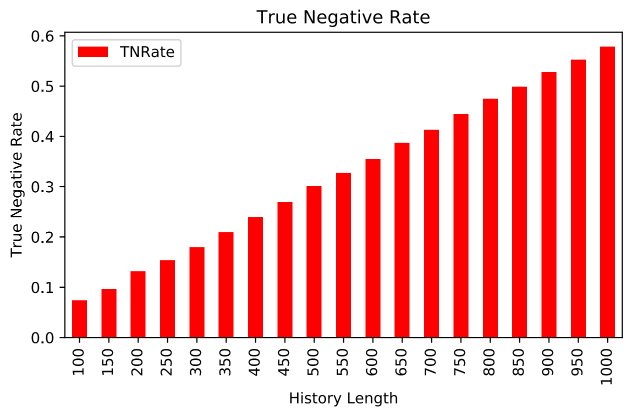

FIGURE 2. True negative probability when the history length changes from 100 to 1000.

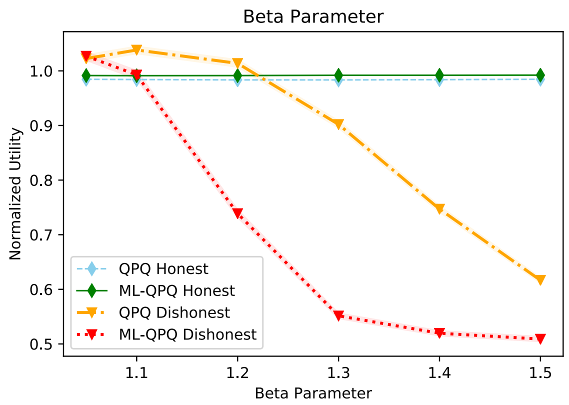

FIGURE 3. Normalized utility for different strategies of the dishonest player. $\tau_q = 0.03$.

In the Kolmogorov Smirnov GoF test, the probability of detecting that the declared preferences are different from the reference distribution (i.e., true negatives) increases with the length of the history (i.e., with the number of values used to perform GoF test; see Figure 2). These values require some memory space. (The false positive rate is not affected significantly by the history length; see Figure 1.) Therefore, the structure of the multilevel QPQ (i.e., the number of cluster and the number of players in each cluster) determines the memory available in the system. We fix the memory in both systems (QPQ and multilevel QPQ) to be the same.

Taking into account the previous considerations (thresholds and length of history), we define a baseline scenario with $n = 64$ players and, following Proposition 11, we consider that $k = \sqrt{n} = 8$ clusters. We assume a single dishonest player whose strategy consists of declaring preferences that follow a $Beta(1.2, 1)$ distribution. The threshold value is $\tau_q = 0.03$ and the length of the history of the GoF test in QPQ for each player is $H_1 = 100$ values (so the total memory is $S_1 = 100n$). In multilevel QPQ the total memory available $S_2 = S_1$ is distributed among the cluster level and at the upper level. We assume $\alpha = 1$ (see Section V-B1), so that the history length per player at the cluster level H_c is the same as the history length per cluster H_u at the upper level. The value of H_c and H_u is obtained with Eqs. 11 and 12, and from Proposition 12 they are larger than H_1 since in all the considered cases $n/k > 1$. Moreover, for each set of experiments, we vary one of the parameters while the rest of the parameters are fixed. In all the experiments, we consider the following utilities: (i) the utility under the QPQ approach of the honest players (labelled as QPQ honest) and of the dishonest players (labelled as QPQ dishonest), and (ii) the utility under the multilevel QPQ approach of the honest players (labelled as ML-QPQ honest) and of the dishonest players (labelled as ML-QPQ dishonest). In both cases, we represent in each plot the mean and the 95% confidence interval of, at least, 500 values we obtained for the normalized utility, which is:

$$\frac{u^*}{E[\bar{u}_{c,i}]},$$

where $E[\bar{u}_{c,i}] = \frac{1}{1+n}$ is the expected utility of an honest player (c, i) and u^* is each of the aforementioned utilities.

In Figure 3, we compare the performance of multilevel QPQ with QPQ for different strategies of the dishonest players. The strategies under consideration consist of declaring values that follow a $Beta(b, 1)$ distribution, where b changes from 1.05 to 1.5. We note that the $Beta(b, 1)$ distribution with $b = 1$ coincides with the uniform distribution in $[0, 1]$. We observe that the utility of the honest players does not change substantially under multilevel QPQ and QPQ. This is as expected, since we have fixed $\tau_q = 0.03$, and hence $(1 - \tau_q) = (1 - q) = (1 - p_c)(1 - p_u) = 0.97$. However, the normalized utility of the multilevel QPQ of the dishonest player is smaller than that of obtained with the basic QPQ. This means that the multilevel QPQ approach penalizes more the dishonest player than the QPQ.

In this figure, it can be also seen that for the multilevel QPQ the utility of a dishonest player is larger than that of an honest player when the parameter of the beta distribution is smaller or equal than 1.1, whereas for QPQ this occurs when this parameter is smaller than 1.2. The main reason for this is that the parameter of the beta distribution is very close to one, in which case the beta distribution and the uniform distribution are very similar. We also observe that when the strategy of the dishonest player is far from the uniform distribution (beta parameter equal to 1.5), the utility of the dishonest players in QPQ and multilevel QPQ are very close. Besides, the GoF test is not perfect. We have seen that this can be solved by changing the threshold τ_q. In fact, in Figure 4, we consider $\tau_q = 0.1$ and we note that for this case the utility of the dishonest players for multilevel QPQ and QPQ is almost always smaller than the utility of the honest players in the scenarios considered (except for the parameter 1.05). This comes at the cost of reducing the expected utility of the honest players, since they suffer more false negatives in the GoF test. Observe that honest players incur a smaller reduction with the increase of τ_q with multilevel QPQ than with QPQ. Another interesting property of considering a larger the value of τ_q the utility of dishonest player with QPQ and multilevel QPQ are closer, specially when the beta parameter is large.

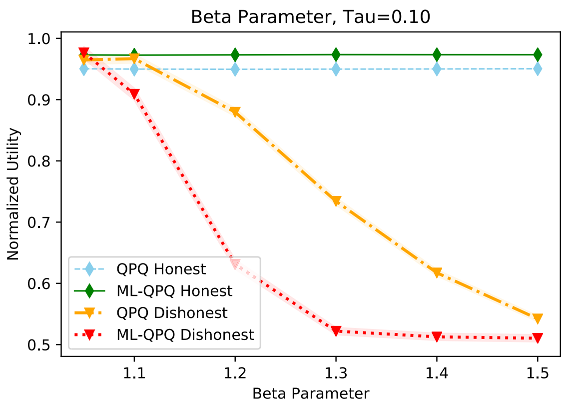

FIGURE 4. Normalized utility for different strategies of the dishonest player. $\tau_q = 0.1$.

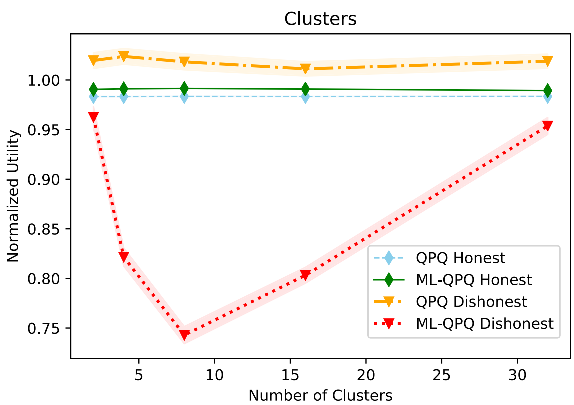

FIGURE 5. Normalized utility for different number of clusters and $n = 64$.

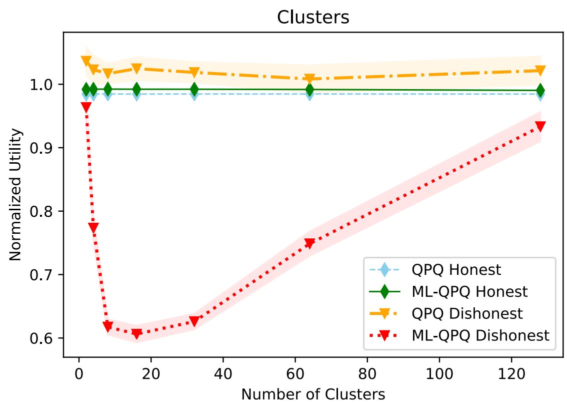

FIGURE 6. Normalized utility for different number of clusters and $n = 256$.

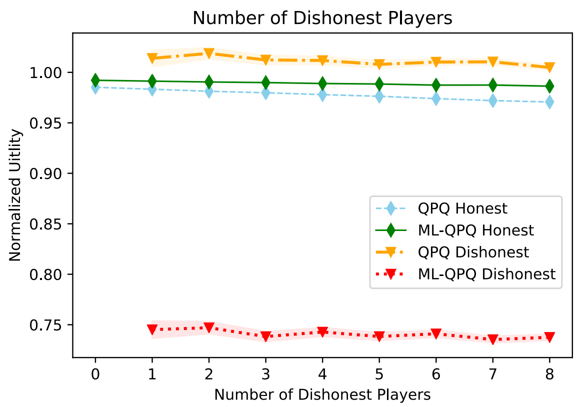

FIGURE 7. Normalized utility for different number of dishonest players.

We now study the utility when we vary the number of clusters k. In Figure 5, we observe that the utility of the honest players is again very similar for multilevel QPQ and QPQ and the multilevel QPQ approach penalizes more the dishonest player than the QPQ. Furthermore, we observe that the multilevel QPQ penalizes more the dishonest player than the basic QPQ and the penalty that the dishonest player suffers in multilevel QPQ is maximum when the number of clusters is $k = 8$, which is the square root of the number of players. In Figure 6, we consider a similar scenario with $n = 256$ players and we observe that the utility of the dishonest player follows the same pattern, that is, when $k = \sqrt{n} = 16$, the penalty suffered by the dishonest player in multilevel QPQ is the largest. Therefore, from these simulations, we conclude that, for $k = \sqrt{n}$, not only the volume of data communication is reduced as stated in Proposition 11, but also the penalty suffered by the dishonest player is maximized.

We now focus on the utilities when we vary the number of dishonest players from 0 to 8. We show in Figure 7 that the utilities of multilevel QPQ and QPQ are very similar for the honest players, whereas the dishonest players are more penalized for the multilevel QPQ. Moreover, we also see that the utility of the honest and dishonest players do not change substantially with the number of dishonest players, as expected. We also observe in this plot that the utility of honest player under multilevel QPQ is slightly larger than the utility of honest player under QPQ.

We also analyze the influence of the utility when we vary the number of players from 16 to 256 in Figure 8. We remark that in all the scenarios, the number of clusters is set to the square root of the number of players. As it can be seen in this illustration, for the honest players, the utility of the multilevel QPQ and the utility of QPQ are again very similar. However, for the dishonest player, the utility for multilevel QPQ is smaller than QPQ, i.e., the multilevel QPQ penalizes more the dishonest player than QPQ. We observe that the utility of the dishonest player for QPQ does not vary substantially with the number of players. Moreover, the utility of the dishonest player in multilevel QPQ decreases with the number of players because the total available memory increases and can be used to improve the GoF test.

We study the influence of the memory used to perform the *GoF* test (or history length) on the utility of honest and dishonest player for multilevel QPQ and QPQ in Figure 9. For this case, we consider three different values of H_1, which are 100, 300 and 1000. We observe that the utility of the honest players does not change with the history length H_1, whereas that of the dishonest player decreases with the history length for both QPQ and multilevel QPQ, as expected. The main reason for this is that, with a larger memory, the *GoF* test performs better the task of detecting a dishonest behaviour,

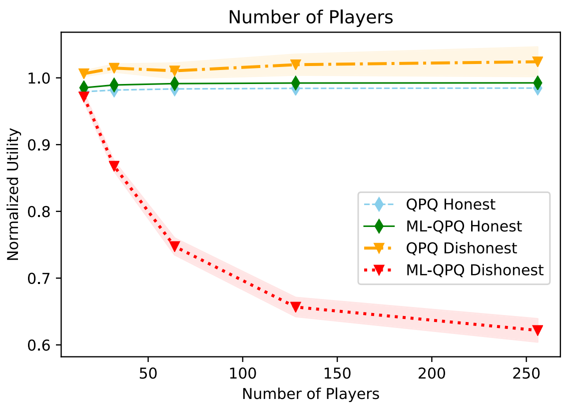

FIGURE 8. Normalized utility for different number of players.

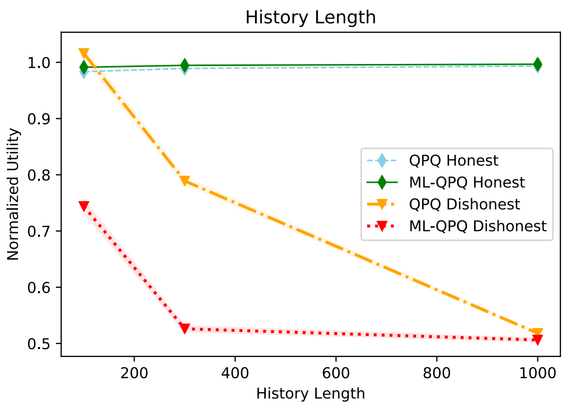

FIGURE 9. Normalized utility for different history length H_1 when the declared values of the dishonest players follow a *Beta*(1.2, 1) distribution.

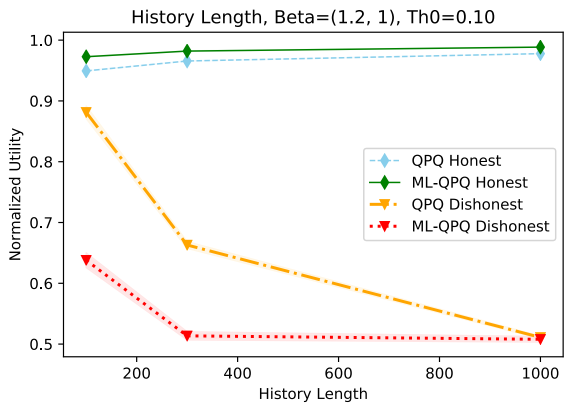

FIGURE 10. Normalized utility for different history length H_1 when the declared values of the dishonest players follow a *Beta*(1.2, 1) distribution. $\tau_q = 0.10$.

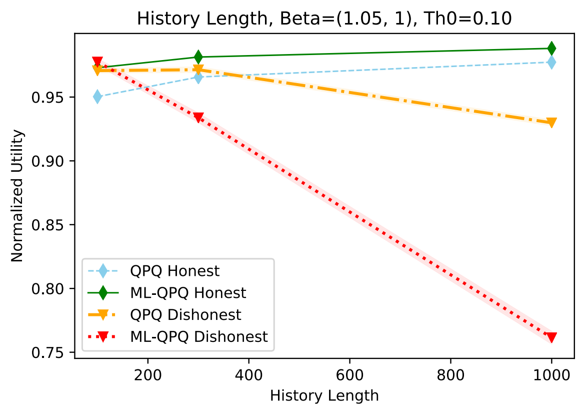

FIGURE 11. Normalized utility for different history length H_1 when the declared values of the dishonest players follow a *Beta*(1.05, 1) distribution. $\tau_q = 0.10$.

which leads to a smaller utility for the dishonest player. Observe that for history length 1000 the dishonest player has normalized utility roughly 0.5. This occurs because its utility is very close to $1/2n$, which is the expected utility of a dishonest when the GoF test is perfect, from Proposition 8. We also remark that, for QPQ, the utility of the dishonest player when H_1 is 1000 approximates $1/(2n)$ (i.e., the expected utility of a dishonest when the GoF test is perfect) and, therefore, we conclude that the utility of the dishonest player when the history length is large for QPQ and multilevel QPQ are very close.

In Figure 9 it can be observed that, with QPQ, for $H_1 = 100$ the dishonest player has higher utility than the honest players. As was shown in Figure 4, it is possible to deal with dishonest players that follow a strategy close to uniform by increasing the threshold τ_q, at the cost of reducing the utility of the honest players. We observe in Figure 10 that this reduction can be compensated by increasing the memory. The figure shows the impact of memory size when a threshold $\tau_q = 0.10$ is used. We observe that, when H_1 is small, the utility of the honest players is larger than that of the dishonest player for QPQ and multilevel QPQ. However, when the history length is large, for QPQ and multilevel QPQ, the utility of the dishonest player is close to $1/(2n)$.

We also consider the effect of the history length on the utilities for a threshold value $\tau_q = 0.1$ and a parameter of the beta distribution of the dishonest players equal to 1.05. As we saw in Figure 4, the utility of honest and dishonest players for this case is very similar. In Figure 11, we show that, for a larger value of the memory available, the utility of the dishonest player in multilevel QPQ is smaller than in QPQ and the utility of the honest players is still larger in multilevel QPQ than in QPQ.

VII. RELATED WORK

The problem of how to assign a set of resources to a fixed number of agents that act rationally has been studied in the context of Mechanism Design by different authors. In [12], [17], mechanisms based on payment systems are considered. We remark that the QPQ mechanism does not consider payments among users (as in [18], [19]). Other models in the literature such as [13], [20]–[22] assume the existence of a central agent that handles the probability distribution that characterizes the rational behaviour of the agents. This assumption has already been criticized by [23] arguing that it is not applicable in real environments.

In this work, we study an extension of the QPQ mechanism, which has been introduced in [9] and further analyzed in [15] and [16]. In [15], the authors relax the assumption of the preferences of the players to be i.i.d by considering that there is a correlation between the preferences of the players for the resources. On the other hand, in [16] they consider that in each round there are k resources to be assigned to the set of players (whereas in the work of [9] a single resource is considered in each round). The main feature of these models is that the properties of efficiency, fairness, etc. given in [9] are also achieved in the extensions under consideration.

VIII. CONCLUSIONS AND FUTURE WORK

In this article, we generalize the QPQ approach of [9] to a system with two levels. In the multilevel QPQ technique, players are divided in clusters and the QPQ approach is applied at the intra-cluster level and then at the inter-cluster level. More precisely, the multilevel QPQ firstly uses the QPQ approach to determine the winner of each cluster independently; then, secondly, the multilevel QPQ uses again the QPQ mechanism among the winners of all the clusters to determine who gets a resource.

We show that the positive properties of QPQ extend to the multilevel QPQ. First, we show that the utility of an honest player is always larger that the utility of a player that declares values different from its real distribution if a perfect detection mechanism is available. We also show that the multilevel QPQ has several advantages with respect to QPQ in terms of reduction of communication cost and use of memory, which means that multilevel QPQ is more scalable. We also study with simulations the performance of multilevel QPQ (and QPQ) when the detection of a dishonest player is not perfect. We show that in most cases dishonest players have lower utility then honest players, and that with similar amount of memory multilevel QPQ has this property in more cases.

For future work, we would like to generalize the results of this article to a multilevel QPQ mechanism with an arbitrary number of levels. We would also like to consider correlated players in multilevel QPQ. Another line worth exploring in practice is adapting the parameters of the GoF test (e.g., τ_q) to the evolution of the system (for instance the balance of resources assigned among players). Finally, we are extremely interested in studying QPQ and multilevel QPQ for unknown and variable player sets.

APPENDIX. AUXILIARY COMPUTATIONS

A. COMPUTATION OF UTILITIES OF QPQ

In the QPQ, a player j is in competition against an aggregated player formed by the rest of the players. In the remainder of this section, we denote by $\ddot{\theta}_{-j}$ the preference of the aggregated player of player j.

The expected utility of an honest player in the QPQ whose declared value passes the GoF test is denoted by $E[U_H^{QPQ}]$. We first show that $E[U_H^{QPQ}] = 1/(n+1)$.

Proposition 16: The utility of the expected utility of an honest player j whose declared value passes the GoF test is $\frac{1}{n+1}$.

Proof: We first note that the utility is given by

$$U_H^{QPQ} = \begin{cases} \bar{\theta}_j & \text{if } \ddot{\theta}_{-j} < \ddot{\theta}_j \\ 0 & \text{otherwise} \end{cases}$$

Let $f(x) = 1$ be the density function of the declared values by player j. Using Proposition 6, we conclude that the preferences $\ddot{\theta}_{-j}$ follow a $Beta(n-1, 1)$ distribution. Hence, the utility of player j is

$$\begin{aligned} E[U_H^{QPQ}] &= \int_0^1 xf(x)P[\ddot{\theta}_{-j} < x]dx \\ &= \int_0^1 x \int_0^x (n-1)y^{n-2}dydx \\ &= \int_0^1 xx^{n-1}dx \\ &\quad \frac{1}{n+1}. \end{aligned}$$

And the desired result follows. ☐

The expected utility of an honest player in the QPQ whose declared value does not pass the GoF test is denoted by $E[\bar{U}_H^{QPQ}]$. We now show that $E[\bar{U}_H^{QPQ}] = 1/(2n)$.

Proposition 17: The utility of the expected utility of an honest player j whose declared value does not pass the GoF test is $\frac{1}{2n}$.

Proof: First, we see that the utility is given by

$$\bar{U}_H^{QPQ} = \begin{cases} \bar{\theta}_j & \text{if } \ddot{\theta}_{-j} < \hat{\theta}_j \\ 0 & \text{otherwise} \end{cases}$$

where $\hat{\theta}_j$ is the regenerated value.

Let $f(x) = 1$ be the density function of the declared values by player j and $g(z) = 1$ the density function of the regenerated values assigned to player j. Using Proposition 6, we conclude that the preferences $\ddot{\theta}_{-j}$ follow a $Beta(n-1, 1)$ distribution. Hence, the utility of player j is

$$\begin{aligned} E[U_H^{QPQ}] &= \int_0^1 x \int_0^1 f(x)g(z)P[\ddot{\theta}_{-j} < z]dzdx \\ &= \int_0^1 x \int_0^1 \int_0^z (n-1)y^{n-2}dydzdx \\ &= \int_0^1 x \int_0^1 z^{n-1}dzdx \\ &= \int_0^1 x\frac{1}{n}dx \\ &= \frac{1}{2n}. \end{aligned}$$

And the desired result follows. ☐

B. COMPUTATION OF UTILITIES OF MULTILEVEL QPQ

We now compute the values of U_H, U_u, U_c and $U_{u,c}$ of player (c, i). We observe that, from Proposition 7, we have that $U_H = \frac{1}{n+1}$ and from Proposition 8, that $U_c = \frac{1}{2n}$. The rest of the expressions are given below.

Proposition 18: The utility of an honest player (c, i) that passes the cluster level GoF test but not the upper one is

$$U_u = \frac{n_c}{n\cdot(n_c+1)}.$$

Proof: When the preference declared by (c, i) passes the cluster level test and not the upper level test, its expected utility

$$\bar{u}_{c,i} = \begin{cases} \bar{\theta}_{c,i} & \text{if } \ddot{\theta}_{c,-i} < \ddot{\theta}_{c,i} \text{ and } \dddot{\theta}_{-c} < \hat{\hat{\theta}}_{c,i} \\ 0 & \text{otherwise} \end{cases}$$

where $\hat{\hat{\theta}}_{c,i}$ is the value regenerated in the upper level test. The regenerated value follows a $Beta(n_c, 1)$ distribution and the preference of the winner of cluster c follows a $Beta(n_c, 1)$.

We can now express the utility of the player (c, i) as follows

$$U_u = \int_0^1 x\cdot f(x) \int_0^1 g_u(z_u) \cdot P[(\ddot{\theta}_{c,-i} < x) \wedge (\dddot{\theta}_{-c} < z_u)] dz_u dx,$$

where z_u represents the regenerated value in the upper level test, $f(x) = 1$ is the density function associated to the random variable $\bar{\theta}_{c,i}$, and $g_u(z_u)$ is the density function associated to the random variable $\hat{\hat{\theta}}_{c,i}$.

The preferences $\ddot{\theta}_{c,-i}$ of the player $(c, -i)$ and the preferences $\dddot{\theta}_{-c}$ of the player $(-c)$ are independent and they follow, respectively, a $Beta(n_c - 1, 1)$ and a $Beta(n - n_c, 1)$ distribution. Additionally, they are independent of the values $\hat{\hat{\theta}}_{c,i}$, which follow a $Beta(n_c, 1)$ distribution. Therefore, the utility of player (c, i) when its passes the cluster level test but not the upper one is

$$\begin{aligned} U_u &= \int_0^1 x \int_0^1 g_u(z_u)\cdot P[\ddot{\theta}_{c,-i} < x]\cdot P[\dddot{\theta}_{-c} < z_u] dz_u dx \\ &= \int_0^1 x\cdot P[\ddot{\theta}_{c,-i} < x] \int_0^1 g_u(z_u)\cdot P[\dddot{\theta}_{-c} < z_u] dz_u dx \\ &= \int_0^1 x\cdot \left[\int_0^x (n_c-1)\cdot y^{n_c-2} dy\right] \\ &\quad \cdot \left[\int_0^1 n_c\cdot z_u^{n_c-1}\cdot \left(\int_0^{z_u} (n-n_c)\cdot y^{n-n_c-1} dy\right) dz_u\right] dx \\ &= \int_0^1 x\cdot \left[x^{n_c-1}\right]\cdot\left[\int_0^1 n_c\cdot z_u^{n_c-1}\cdot z_u^{n-n_c} dz_u\right] dx \\ &= \int_0^1 x^{n_c}\cdot\left[\frac{n_c}{n}\right] dx \\ &= \frac{n_c}{n\cdot(n_c+1)} \end{aligned}$$

□

Proposition 19: The utility of an honest player (c, i) that does not pass neither the cluster level GoF test nor the upper level one is

$$U_{u,c} = \frac{1}{2n}.$$

Proof: We consider that the preference of player (c, i) does not pass any test. In that case, its expected utility is

$$\bar{u}_{c,i} = \begin{cases} \bar{\theta}_{c,-i} & \text{if } \ddot{\theta}_{c,-i} \leqslant \hat{\theta}_{c,i} \text{ and } \dddot{\theta}_{-c} \leqslant \hat{\hat{\theta}}_{c,i} \\ 0 & \text{otherwise} \end{cases}$$

where $\hat{\theta}_{c,i}$ and $\hat{\hat{\theta}}_{c,i}$ represent the values regenerated in the cluster and in the upper level test, respectively. By Proposition 1 and Corollary 2, we know that these new values follow, respectively, a $U(0, 1)$ and a $Beta(n_c, 1)$ distribution.

We can now express the utility of the player as follows

$$\begin{aligned} U_{u,c} = &\int_0^1 x\cdot f(x) \int_0^1 g_c(z_c) \\ &\int_0^1 g_u(z_u)\cdot P[(\ddot{\theta}_{c,-i} < z_c) \wedge (\dddot{\theta}_{-c} < z_u)] dz_u dz_c dx \end{aligned}$$

where z_c and z_u represent the regenerated value in the cluster level and in the upper one, and $f(x) = 1$ is the density function associated to the normalized preferences $\bar{\theta}_{c,i}$, which follow a $U(0, 1)$ distribution, $g_c(z_c) = 1$ is the density function associated to the random variable $\hat{\theta}_{c,i}$, and $g_u(z_u) = n_c \cdot z_u^{n_c-1}$ is the density function associated to the random variable $\hat{\hat{\theta}}_{c,i}$.

The preferences $\ddot{\theta}_{c,-i}$ of the aggregated player inside cluster c of player (c, i) and the preferences $\dddot{\theta}_{-c}$ of the aggregated player of the winner of cluster c are independent and they follow, respectively, a $Beta(n_c - 1, 1)$ and a $Beta(n - n_c, 1)$ distribution (see Corollary 1 and Corollary 3). Then, the utility of player (c, i) when its does not pass any test is

$$\begin{aligned} U_{u,c} &= \int_0^1 x \int_0^1 \int_0^1 g_u(z_u)\cdot P[\ddot{\theta}_{c,-i} < z_c]\cdot P[\ddot{\theta}_{-c} < z_u] \\ &\quad dz_u dz_c dx \\ &= \int_0^1 x\, dx \int_0^1 P[\ddot{\theta}_{c,-i} < z_c] dz_c \\ &\quad \int_0^1 g_u(z_u)\cdot P[\ddot{\theta}_{-c} < z_u] dz_u \\ &= \frac{1}{2}\int_0^1 \left(\int_0^{z_c} (n_c-1)\cdot y^{n_c-2}\, dy\right) dz_c \\ &\quad \cdot \int_0^1 n_c\cdot z_u^{n_c-1}\cdot\left(\int_0^{z_u} (n-n_c)\cdot y^{n-n_c-1} dy\right) dz_u \\ &= \frac{1}{2}\int_0^1 z_c^{n_c-1} dz_c \int_0^1 n_c z_u^{n_c-1} z_u^{n-n_c} dz_u \\ &= \frac{1}{2}\int_0^1 z_c^{n_c-1} dz_c \int_0^1 n_c z_u^{n-1} dz_u \\ &= \frac{1}{2}\cdot\frac{1}{n_c}\cdot\frac{n_c}{n} \\ &= \frac{1}{2n} \end{aligned}$$

□

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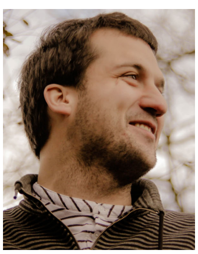

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