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Single Server Repairable Queueing System With Variable Service Rate and Failure Rate

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ABSTRACT This paper studies the single server repairable queueing system with variable service rate and failure rate. The rule of variation of the service rate is that the service rate changes when the number of customers in the system reaches a certain value. The server may fail at any time, and the failure rates of idle periods and busy periods are different. The system has one reliable repairman to repair the failure server. The steady-state joint distribution of the customers number and server states is obtained by the matrix geometric solution method, the steady-state availability of the server and significant queue performances measures are evaluated. Two special cases are analyzed, and some numerical experiments are given for the illustrations of the parameters effect.

INDEX TERMS Repairable queueing system, service rate, failure rate, availability, matrix geometric solution.

I. INTRODUCTION

Repairable queuing model is an important model of queuing theory research, it can be applied to a variety of real situations, such as computer networks, telecommunications, aircraft maintenance, high-speed railway maintenance, and many others. The latest studies usually assume that the system parameters such as the service rate, repair rate and failure rate are invariable, but the fact of the system parameters are variable or adjustable is an actual universal phenomenon in many service systems, even multiple parameters are variable or adjustable simultaneously in one system. For repairable queuing system, the models with fixed parameters and other different characteristics of conditions were studied in [1]–[3], and rich research achievements obtained. In addition, many more flexible service and maintenance strategies reflect the characteristics of the actual systems that were applied to the models of repairable queueing systems [4]–[8]. Recently, Gao *et al.* [9] analyzed a retrial queue with two-type breakdowns and delayed repairs, one type of the breakdowns cannot be repaired immediately. Sikha and Manivasakan [10] deal with a two-type vacation queue system with queue-length dependent service, the two types of vacation distributions are deterministic and exponential. Lyu *et al.* [11] studied the M/M/2 queue system with flexible service policy, they assumed that the two servers can service one customer collectively at the same time. Lv [12] considered a machine repairable model with flexible repair policy that the two repairmen can repair one failure machine collectively at the same time.

In many real service systems, the service rate may be affected by the number of customers in the system. On the one hand, the managers try to improve the efficiency when large amounts of customers are waiting in the system, so they can take the initiative to improve the service rate if the working conditions permit. On the other hand, passive fall of service rate may happen as a result of the change of working environment due to a large number of customers in the backlog.

For the failure rate, modern control or service systems have been becoming extensively integrated and complex, along with growth of the running time, components in the service systems are degrading with age and the failure rate will increase. On the other hand, the development of fault detection and diagnosis techniques will enhance the incipient fault diagnosis level and reduce the risk of serious accidents in the entire systems [13], [14], so the the rate of minor failures will increase and the rate of major failures will decrease. In short, the failure rate is variable because of objective reasons or subjective reasons is a general phenomenon in every kind of service or control system. Further, an obvious

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fact is that the service systems are more prone to failure due to various external factors like overloading, unexpected disturbances and environmental changes when they are in working condition, while the system is relatively stable during the idle period. Therefore, the equipment has different failure rates at working time and idle time is quite a common phenomenon.

From the above discussion, we know that system parameters are variable is a general fact [15], but few researchers considered this characteristic in the past. Although some researchers have considered this feature, they assume that only one parameter is changeable in the service or control systems, and the research about systems with two variable parameters is very rare.

In this paper, we study a single server repairable system with variable service rate and failure rate. The server may fail at any time, but the failure rates in idle periods and busy periods are different. This kind of change of the failure rate is passive generally. Further, the service rate is variable depending on the customers' number in the system. This kind of change of service rate can be active or passive. A reliable repairman is responsible for the repair of the server failure, and the repairman begins to repair immediately after the server failed. Such a repairable queue system arises in various practical fields such as communication networks and manufacturing systems. We obtain analytical solution in term of closed form expression, and evaluate the reliability and the queue performance measures of the considered system which may be suited to many practical service systems. The basic findings of the paper and their significance are outlined as follows:

- We introduce a new repairable queue model with variable service rate and failure rate, in which the service rate and failure rate depend on the working condition. Such models reflect the features of many actual systems.
- We give the stable condition of the system, the steady-state joint distribution of the number of customers and the server's states.
- We give the expressions of important steady-state performances of the system such as steady-state availability of the server and the steady-state queueing length.
- We analysed two special cases and gave the explicit results of the two special cases, and verified that the model of this paper is a more general model which can reflect the actual conditions of the service system more factually.

The steady-state joint distribution of the customers' number and the server's states is obtained by the matrix geometric solution method of quasi-birth-and-death(QBD) processes, which are a class of two-dimensional Markov processes arising in many fields of science, engineering and business. The QBD processes are of particular theoretical and practical importance, often occurring directly or through decomposition of higher-dimensional processes. It is well known that the invariant distributions of QBD processes to have a matrix-geometric form under appropriate conditions. Based on the steady-state joint distribution of the system, the other steady-state performances are obtained. The influences of the system parameters on the system steady-state indexes are illustrated by some numerical experiments.

The rest of this paper is organized as follows. Section 2 gives the system description and the transfer rate matrix of the QBD process. Section 3 analyses the stable condition of the system. Section 4 focuses on the steady-state probability of the system. Section 5 presents the steady-state performance indexes of the system. Section 6 gives the explicit results of two special cases and carries out the general analysis of our model. Section 7 gives some numerical examples to illustrate the features of our model.

II. MODEL DESCRIPTION

The repairable system has a repairable server and a reliable repairman. The arrival of the customers is a Poisson process with the parameter of $\lambda(\lambda > 0)$. The server service time is exponential distribution, the service rate changes depend on the number of customers in the system. The service rate is $\mu_1(\mu_1 > 0)$ when the number of customers in the system is less than $m(m > 0)$. The service rate is $\mu_2(\mu_2 > 0)$ when the number of customers is greater than or equal to m. The server will be idle when the server is not at failure state and there is no customer in the system. The server may fail at any time, the occurrence of the failure is a Poisson process with variable failure rate. The failure rate is $\xi_1(\xi_1 > 0)$ when the server is idle, and the failure rate is $\xi_2(\xi_2 > 0)$ when the server is busy. The customer whose service is interrupted by the server failure becomes the head of the queue, and will be serviced again immediately after the server is repaired.

Let $N(t)$ denote the number of customers at the time t in the system, and $J(t)$ denote the number of available server at the time t, then $\{N(t), J(t)\}$ is a QBD process rely to the arrival interval, the service time, the failure time and the maintenance time are exponentially distributed. Thus the space of the system states is

$$\Omega = \{(i, j), i = 0, 1, \cdots ; j = 0, 1\}.$$

Arranging the states in lexicographic order [17], the state transfer rate matrix of block tridiagonal is obtained as follows:

$$Q = \begin{matrix} 1 \\ \\ \vdots \\ \\ m \\ \\ \vdots \end{matrix} \begin{bmatrix} A_0 & C & & & & & \\ B_1 & A_1 & C & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & B_1 & A_1 & C & & \\ & & & B & A & C & \\ & & & & B & A & C \\ & & & & & \ddots & \ddots & \ddots \end{bmatrix},$$

where

$$A_0 = \begin{bmatrix} -(\lambda + \eta) & \eta \\ \xi_1 & -(\lambda + \xi_1) \end{bmatrix}, \quad C = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & \mu_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & \mu_2 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -(\lambda+\eta) & \eta \\ \xi_2 & -(\lambda+\mu_1+\xi_2) \end{bmatrix},$$
$$A = \begin{bmatrix} -(\lambda+\eta) & \eta \\ \xi_2 & -(\lambda+\mu_2+\xi_2) \end{bmatrix}.$$

III. THE STEADY-STATE CONDITION

Theorem 1: The matrix equation $R^2B + RA + C = 0$ has the least nonnegative solution

$$R = \begin{bmatrix} \frac{\lambda\xi_2+\lambda\mu_2}{(\lambda+\eta)\mu_2} & \frac{\lambda}{\mu_2} \\ \frac{\lambda\xi_2}{(\lambda+\eta)\mu_2} & \frac{\lambda}{\mu_2} \end{bmatrix}.$$

Proof: We set

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix},$$

substitute it into the matrix equation $R^2B + RA + C = 0$, we obtain

$$\begin{cases} -(\lambda+\eta)r_{11} + \xi_2 r_{12} + \lambda = 0, \\ \mu_2(r_{11}r_{12} + r_{12}r_{22}) + \eta r_{11} - (\lambda+\mu_2+\xi_2)r_{12} = 0, \\ -(\lambda+\eta)r_{21} + \xi_2 r_{22} = 0, \\ \mu_2(r_{12}r_{21} + r_{22}^2) + \eta r_{21} - (\lambda+\mu_2+\xi_2)r_{22} + \lambda = 0. \end{cases}$$

Mathematica software is used to solve the above equations, and the minimum non-negative solution is R.

Further, the QBD process $\{N(t), J(t)\}$ is positive recurrent, if and only if the spectral radius of R denoted by $sp(R)$ meets the condition of $sp(R) < 1$ [17].

Theorem 2: Let R is the minimum nonnegative solution of matrix equation $R^2\,B + RA + C = 0$, $sp(R)$ is the spectral radius of R, $sp(R) < 1$ is equivalent to

$$\frac{\lambda}{\mu_2} < \frac{\eta}{\eta+\xi_2}. \tag{1}$$

Proof: We solve the following equations

$$|\phi E - R| = 0,$$

then

$$\begin{vmatrix} \phi - \frac{\lambda\xi_2+\lambda\mu_2}{(\lambda+\eta)\mu_2} & -\frac{\lambda}{\mu_2} \\ -\frac{\lambda\xi_2}{(\lambda+\eta)\mu_2} & \phi - \frac{\lambda}{\mu_2} \end{vmatrix}$$
$$= \phi^2 - [\frac{\lambda\xi_2+\lambda\mu_2}{(\lambda+\eta)\mu_2} + \frac{\lambda}{\mu_2}]\phi + \frac{\lambda^2}{(\lambda+\eta)\mu_2} = 0. \tag{2}$$

while

$$\Delta = [\frac{\lambda\xi_2+\lambda\mu_2}{(\lambda+\eta)\mu_2} + \frac{\lambda}{\mu_2}]^2 - \frac{4\lambda^2}{(\lambda+\eta)\mu_2}$$
$$= \frac{\lambda^2[(\lambda+\eta+\mu_2+\xi_2)^2 - 4(\lambda+\eta)\mu_2]}{(\lambda+\eta)^2{\mu_2}^2}.$$

Since

$$(\lambda+\eta)^2 + {\mu_2}^2 \geq 2(\lambda+\eta)\mu_2,$$

so we have

$$(\lambda+\eta)^2 + {\mu_2}^2 + 2(\lambda+\eta)\mu_2 + {\xi_2}^2 + 2(\lambda+\eta)\xi_2 + 2\xi_2\mu_2 \geq 4(\lambda+\eta)\mu_2,$$

it is

$$(\lambda+\eta+\mu_2+\xi_2)^2 \geq 4(\lambda+\eta)\mu_2.$$

Then the equation Eq.(2) has two different solutions as follows:

$$\phi_1 = \frac{[\frac{\lambda\xi_2+\lambda\mu_2}{(\lambda+\eta)\mu_2} + \frac{\lambda}{\mu_2}] + \Phi}{2},$$
$$\phi_2 = \frac{[\frac{\lambda\xi_2+\lambda\mu_2}{(\lambda+\eta)\mu_2} + \frac{\lambda}{\mu_2}] - \Phi}{2},$$

where

$$\Phi = \sqrt{[\frac{\lambda\xi_2+\lambda\mu_2}{(\lambda+\eta)\mu_2} + \frac{\lambda}{\mu_2}]^2 - \frac{4\lambda^2}{(\lambda+\eta)\mu_2}}.$$

So $sp(R) = \max\{|\phi_i|, i = 1, 2\} < 1$, is

$$\phi_1 < 1,$$
$$[\frac{\lambda\xi_2+\lambda\mu_2}{(\lambda+\eta)\mu_2} + \frac{\lambda}{\mu_2}]^2 - \frac{4\lambda^2}{(\lambda+\eta)\mu_2} < \frac{(\lambda\mu_2 + 2\eta\mu_2 - \lambda^2 - \lambda\eta - \lambda\xi_2)^2}{[(\lambda+\eta)\mu_2]^2},$$
$$\lambda^2(\lambda+\eta+\mu_2+\xi_2)^2 - 4\lambda^2(\lambda+\eta)\mu_2 < (\lambda\mu_2 + 2\eta\mu_2 - \lambda^2 - \lambda\eta - \lambda\xi_2)^2,$$
$$\lambda(\eta+\xi_2) < \mu_2\eta,$$
$$\frac{\lambda}{\mu_2} < \frac{\eta}{\eta+\xi_2}.$$

Since the QBD process $\{N(t), J(t)\}$ process is invertible, we got the theorem and know that the steady-state condition is Eq.(1).

IV. THE STEADY-STATE PROBABILITY

We define the steady-state probability as follows:

$$\pi_{ij} = \lim_{t\to\infty} P\{N(t) = i, J(t) = j\}, (i, j) \in \Omega.$$

Corresponding to Q, the steady-state probability vector π is

$$\pi = (\pi_0, \quad \pi_1, \quad \pi_2, \quad \cdots),$$

where

$$\pi_i = (\pi_{i0}, \quad \pi_{i1})\ (i \geq 0).$$

If $sp(R) < 1$, we can calculate the steady-state probability vector π as follows [17]:

$$\begin{cases} (\pi_0, \pi_1, \pi_2, \cdots, \pi_{m-1}, \pi_m)B[R] = 0, \\ \sum_{k=0}^{m-1} \pi_k e + \pi_m (I-R)^{-1} e = 1, \\ \pi_k = \pi_m R^{k-m}, k \geq m, \end{cases} \tag{3}$$

where

$$B[R]=\begin{bmatrix} A_0 & C & & & & \\ B_1 & A_1 & C & & & \\ & B_1 & A_1 & C & & \\ & & \ddots & \ddots & \ddots & \\ & & & B_1 & A_1 & C \\ & & & & B & RB+A \end{bmatrix},$$

$(\pi_0, \pi_1, \pi_2, \cdots, \pi_{m-1}, \pi_m)$ is a vector of $2(m+1)$-dimension, $B[R]$ is irreducible and aperiodic.

Since

$$D_0 = A_0 = \begin{bmatrix} -(\lambda+\eta) & \eta \\ \xi_1 & -(\lambda+\xi_1) \end{bmatrix},$$

then

$$\begin{aligned} |D_0| = |A_0| &= \begin{vmatrix} -(\lambda+\eta) & \eta \\ \xi_1 & -(\lambda+\xi_1) \end{vmatrix} \\ &= (\lambda+\eta)(\lambda+\xi_1) - \eta\xi_1 \\ &= \lambda(\lambda+\xi_1+\eta) \neq 0, \end{aligned}$$

so D_0 is invertible. More over

$$\begin{aligned} D_1 &= A_1 - B_1 D_0^{-1} C \\ &= \begin{bmatrix} -(\lambda+\eta) & \eta \\ \xi_2 + \dfrac{\xi_1\mu_1}{\eta+\lambda+\xi_1} & \dfrac{\mu_1(\lambda+\eta)}{\eta+\lambda+\xi_1} - (\lambda+\mu_1+\xi_2) \end{bmatrix}, \end{aligned}$$

then

$$|D_1| = \lambda(\lambda+\eta+\xi_2) + \frac{\lambda\xi_1\mu_1}{\eta+\lambda+\xi_1} \neq 0.$$

So D_1 is invertible. Further, all $D_i = A_1 - B_1 D_{i-1}^{-1} C (1 \le i \le m-1)$ are invertible by recursion. From Eq. (3) we obtain:

$$\begin{aligned} \pi_i &= -\pi_{i+1} B_1 D_i^{-1}, (0 \le i < m-1), \\ \pi_{m-1} &= \pi_m B D_{m-1}^{-1}, \\ \pi_m &= -\pi_{m-1} C (RB+A)^{-1}, \end{aligned}$$

then

$$\begin{aligned} \pi_0 &= -\pi_1 B_1 D_0^{-1} = (-1)^2 \pi_2 B_1 D_1^{-1} B_1 D_0^{-1} \\ &= \cdots = (-1)^m \pi_m B D_{m-1}^{-1} \prod_{j=0}^{m-2} B_1 D_j^{-1}, \\ \pi_1 &= -\pi_2 B_1 D_1^{-1} = (-1)^2 \pi_3 B_1 D_2^{-1} B_1 D_1^{-1} \\ &= \cdots = (-1)^{m-1} \pi_m B D_{m-1}^{-1} \prod_{j=1}^{m-2} B_1 D_j^{-1}, \\ &\cdots \\ \pi_i &= (-1)^{m-i} \pi_m B D_{m-1}^{-1} \prod_{j=i}^{m-2} B_1 D_j^{-1}, (2 \le i \le m-2), \\ &\cdots \\ \pi_{m-1} &= -\pi_m B D_{m-1}^{-1}. \end{aligned}$$

Let

$$M_i = (-1)^{m-i} B D_{m-1}^{-1} \prod_{j=i}^{m-2} B_1 D_j^{-1}, (0 \le i \le m-2),$$

then

$$\pi_i = \pi_m M_i, \ (0 \le i \le m-2).$$

If $\dfrac{\lambda}{\mu_2} < \dfrac{\eta}{\eta+\xi_2}$, the boundary probability vector is

$$\begin{cases} \pi_i = \pi_m M_i (0 \le i \le m-2), \\ \pi_{m-1} = -\pi_m B D_{m-1}^{-1}, \end{cases}$$

where π_m is obtained from the following equations:

$$\begin{cases} \pi_m (\sum_{i=0}^{m-2} M_i - B D_{m-1}^{-1} + (I-R)^{-1} e) = 1, \\ \pi_{m-1} C + \pi_m (RB+A) = 0. \end{cases}$$

V. STEADY-STATE PERFORMANCE INDEXES OF THE SYSTEM

A. STEADY-STATE DISTRIBUTION OF QUEUEING LENGTH

$$P(L=k) = \begin{cases} \pi_m M_k e, & 0 \le k \le m-2, \\ -\pi_m B D_{m-1}^{-1} e, & k = m-1, \\ \pi_m R^{k-m} e, & k \ge m, \end{cases}$$

where $e = (1, 1)^T$.

B. STEADY-STATE AVAILABILITY OF THE SERVER

$$\begin{aligned} A &= P(J=1) \\ &= \sum_{i=0}^{m-2} \pi_m M_i e_2 + \pi_{m-1} e_2 + \pi_m (I-R)^{-1} e_2, \end{aligned} \quad (4)$$

where $e_2 = (0, 1)^T$.

C. MEAN QUEUEING LENGTH

$$\begin{aligned} E(Q) &= \sum_{i=0}^{m-2} i\pi_m M_i e + (m-1)\pi_{m-1} e \\ &\quad + \sum_{i=0}^{\infty} [(m+i)\pi_m R^i] e \\ &= \sum_{i=0}^{m-2} i\pi_m M_i e + (m-1)\pi_{m-1} e \\ &\quad + \pi_m [m \sum_{i=0}^{\infty} R^i + R \sum_{i=1}^{\infty} i R^{i-1}] e \\ &= \sum_{i=0}^{m-2} i\pi_m M_i e + (m-1)\pi_{m-1} e \\ &\quad + \pi_m [m(I-R)^{-1} + R(\sum_{i=1}^{\infty} R^i)'] e \\ &= \sum_{i=0}^{m-2} i\pi_m M_i e + (m-1)\pi_{m-1} e \\ &\quad + \pi_m [m(I-R)^{-1} + R(I-R)^{-2}] e. \end{aligned} \quad (5)$$

D. MEAN WAITING QUEUEING LENGTH

$$E(Q_W) = E(Q) - 1 + \pi_0 e.$$

VI. SPECIAL CASES

A. THE CASE OF $m = 1$

Letting $m = 1$, the service rate of the server is fixed as μ_2, and the model of this paper turns into the model of M/M/1 repairable queue system with variable failure rate.

1) THE AVAILABILITY OF THE SERVER

From Eq. (4) we obtain the availability of the server as follows:

$$A = \frac{\eta\mu_2 + \lambda(\xi_1 - \xi_2)}{\mu_2(\eta + \xi_1)},$$

this result is consistent with the conclusion obtained in literature [7].

If $\xi_1 = \xi_2$, then we have

$$A = \frac{\eta}{\eta + \xi_1}, \tag{6}$$

this is the availability of the general model of the M/M/1 repairable queue system [18].

2) THE MEAN QUEUE LENGTH

For the mean queue length, from Eq. (5) we obtain:

$$\begin{aligned} E(Q) &= \pi_1[(I-R)^{-1} + R(I-R)^{-2}]e \\ &= \sum_{i=1}^{\infty} \pi_i e + \pi_1 R(I-R)^{-2}e = 1 - \pi_0 e + \pi_1 R(I-R)^{-2}e \\ &= \frac{\lambda(\lambda(\xi_2 - \xi_1) + \mu_2\xi_1 + (\xi_1 + \eta)(\xi_2 + \eta))}{(\eta + \xi_1)(\mu_2\eta - \lambda\eta - \lambda\xi_2)}. \end{aligned}$$

This result is consistent with the conclusion obtained in literature [7].

B. THE CASE OF $m = 2$

Letting $m = 2$, the service rate will change when the number of customers in the system reaches 2. The model of this case is the simplest model of M/M/1 repairable queue system with variable failure rate and service rate.

1) THE AVAILABILITY OF THE SERVER

From Eq. (4), we obtain the availability of the server as follows:

$$\begin{aligned} A = &\{\eta\lambda\mu_2(\eta + \lambda + \xi_1) - \mu_1[-\eta(\eta + \lambda)\mu_2 \\ &+ \lambda\left(-\lambda\xi_1 + \eta^2 + \xi_2\eta + \lambda\eta + \lambda\xi_2\right)]\} \\ &\times \{\lambda\mu_2(\eta + \xi_2)(\eta + \lambda + \xi_1) \\ &+ \mu_1[\mu_2(\eta + \lambda)(\eta + \xi_1) \\ &- \lambda(\eta + \xi_2)(\eta + \lambda + \xi_1)]\}^{-1} \end{aligned} \tag{7}$$

If $\mu_1 = \mu_2$ and $\xi_1 = \xi_2$, Eq.(7) becomes as follows:

$$A = \frac{\eta}{\eta + \xi_1},$$

it is the same as Eq.(6).

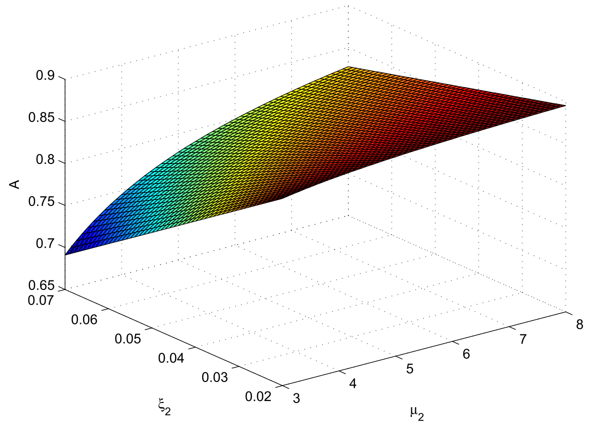

FIGURE 1. The availability of the server for $m = 1(\mu_1 = 0.5, \xi_1 = 0.01, \eta = 0.1, \lambda = 1.2)$.

2) THE MEAN QUEUE LENGTH

For the mean queue length, from Eq. (5) we obtain:

$$\begin{aligned} E(Q) &= \pi_1 e + \pi_2[2(I-R)^{-1} + R(I-R)^{-2}]e \\ &= \pi_1 e + 2\pi_2(I-R)^{-1}e + \pi_2 R(I-R)^{-2}e \\ &= \pi_1 e + 2\sum_{i=2}^{\infty}\pi_i e + \pi_2 R(I-R)^{-2}e \\ &= \pi_1 e + 2(1 - \pi_0 - \pi_1)e + \pi_2 R(I-R)^{-2}e \\ &= 2 - 2\pi_0 e - \pi_1 e + \pi_2 R(I-R)^{-2}e \\ &= 1 + \frac{(\mu_2\eta - \lambda\eta - \lambda\xi_2)\Psi_1 + \lambda^2\Psi_2}{(\mu_2\eta - \lambda\eta - \lambda\xi_2)\Psi}, \end{aligned}$$

where

$$\begin{aligned} \Psi_1 &= \lambda(\eta + \mu_2 + \xi_2)[\lambda(\lambda + \eta + \mu_1 + \xi_2) - \frac{\lambda\mu_1(\lambda + \eta)}{\lambda + \eta + \xi_1}] \\ &\quad - \lambda^2\mu_2(\lambda + \eta) - \mu_1(\lambda + \eta)(\mu_2\eta - \lambda\eta - \lambda\xi_2), \\ \Psi_2 &= [\mu_2(\eta + \mu_2 + \xi_2 - \lambda) + (\eta + \xi_2)^2 + \mu_2\xi_2] \\ &\quad \cdot[\lambda(\lambda + \eta + \mu_1 + \xi_2) - \frac{\lambda\mu_1(\lambda + \eta)}{\lambda + \eta + \xi_1}] \\ &\quad - \lambda\mu_2(\lambda + \eta)(\eta + \mu_2 + \xi_2 - \lambda), \\ \Psi &= \mu_2(\lambda + \eta)[\lambda(\eta + \mu_1 + \xi_2) - \frac{\lambda\mu_1(\lambda + \eta)}{\lambda + \eta + \xi_1}] \\ &\quad + \mu_1(\lambda + \eta)(\mu_2\eta - \lambda\eta - \lambda\xi_2). \end{aligned}$$

VII. NUMERICAL EXPERIMENTS

Letting $\mu_1 = 0.5, \xi_1 = 0.01, \eta = 0.1$, the numerical experiments were carried out in different groups for other parameters' values of $m = 1, 2, \lambda = 1.2, 1.4$ and $\xi_2 = 0.02, 0.03$. The rang of the service rate μ_2 is $3 \leq \mu_2 \leq 8$. All parameters values in the set ranges satisfy the steady-state condition.

First, we give the numerical results of the steady-state availability of the server. In Fig.1 and Fig.2, the availability of the server decreases with the increase of ξ_2, it is in line with our intuition. On the other hand, the availability of the server increases with the increase of μ_2, that is because of the idle period failure rate $\xi_1(= 0.01)$ is less than the busy period failure rate $\xi_2(> 3)$, so the more idle time the higher

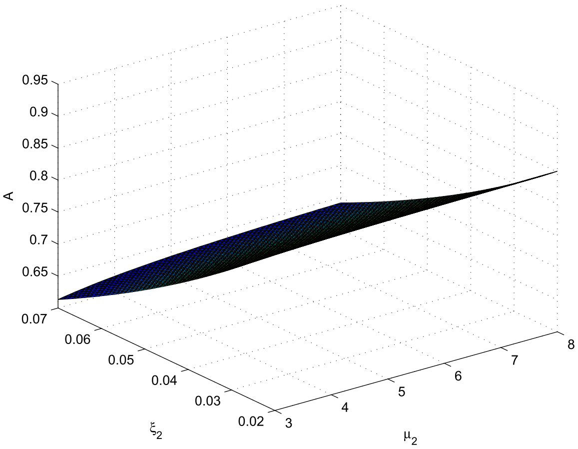

FIGURE 2. The availability of the server for $m = 2(\mu_1 = 0.5, \xi_1 = 0.01, \eta = 0.1, \lambda = 1.2)$.

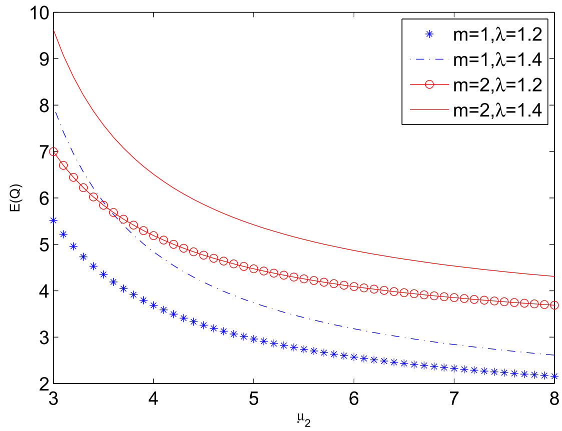

FIGURE 3. The mean queue length for $\xi_2 = 0.02(\xi_1 = 0.01, \eta = 0.1)$.

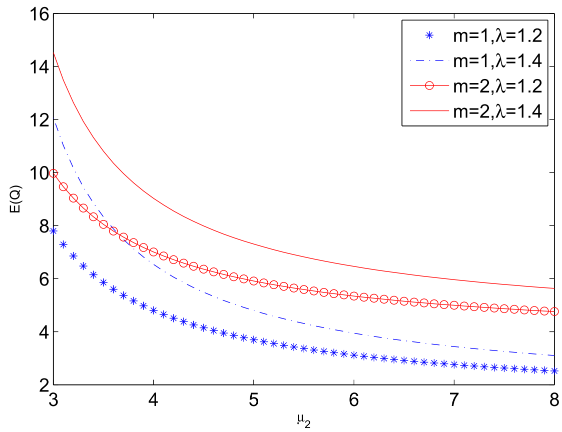

FIGURE 4. The mean queue length for $\xi_2 = 0.03(\xi_1 = 0.01, \eta = 0.1)$.

availability due to the idle period failure rate is low, and the higher the service rate the longer the idle period.

Secondly, we give the numerical results of the steady-state performance measures of the queueing length. In Fig.3 and Fig.4, the mean queueing length decreases and tends to be stable with the increase of μ_2. In the same figure, for the

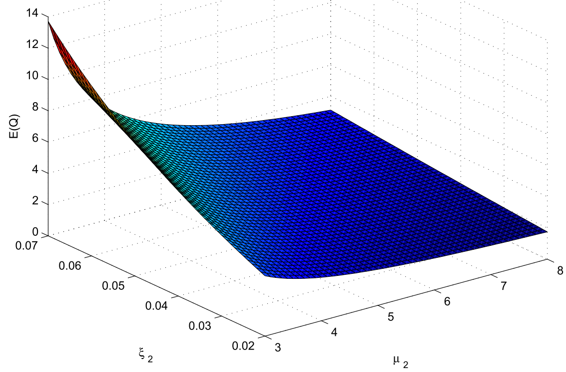

FIGURE 5. The mean queue length for $m = 1(\mu_1 = 0.5, \xi_1 = 0.01, \eta = 0.1, \lambda = 1.2)$.

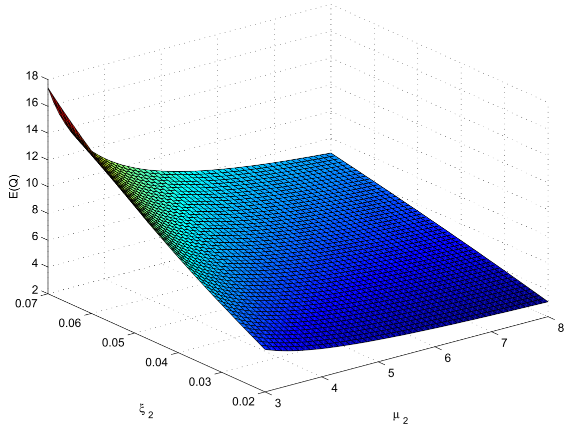

FIGURE 6. The mean queue length for $m = 2(\mu_1 = 0.5, \xi_1 = 0.01, \eta = 0.1, \lambda = 1.2)$.

same value of busy period failure rate ξ_2 and the same value of m, the mean queueing length is positively correlated with the value of input rate λ, that is consistent with our intuitive. As well, for the same value of input rate λ, the mean queueing length is positively correlated with the threshold value of m, that is because of the value of $\mu_1 = 0.5$, it is less than the least value of $\mu_2(3 \leq \mu_2 \leq 8)$, and the larger value of m means the less working time of the service rate μ_2. On the contrary, we should note that if the value of μ_1 is greater than the maximum value of μ_2, the mean queueing length will be negatively correlated with the value of m. We know that μ_1 is greater than μ_2 is possible in actual systems for objective reasons.

Comparing Fig.3 and Fig.4, it can be seen that the mean queueing length is positively correlated with the busy period failure rate ξ_2, that is consistent with our intuitive. In addition, Fig.5 shows the joint effect of μ_2 and ξ_2 on the mean queue length for the other parameters' values of $m = 1(\mu_1 = 0.5, \xi_1 = 0.01, \eta = 0.1, \lambda = 1.2)$. Fig.6 shows the joint effect of μ_2 and ξ_2 on the mean queue length for the other parameters' values of $m = 2(\mu_1 = 0.5, \xi_1 = 0.01, \eta = 0.1, \lambda = 1.2)$. The curved surface of Fig.5 and Fig.6 coincide with the curves of Fig.3 and Fig.4 respectively.

VIII. CONCLUSION

In this paper, the $M/M/1$ repairable queueing model with variable service rate and failure rate was studied. For this model, we analyzed the stable condition for the system, and obtained the steady-state probability distribution of the system states by the matrix geometric solution method. Based on the steady-state probability distribution of the system states, some significant performance indexes such as the steady-state availability and steady-state queue length indexes were obtained. For the special cases of the service rate thresholds $m = 1$ and $m = 2$, the explicit results of the availability of the server and the mean queueing length of the system were given. Some numerical examples were given to study the effect of the parameters on the reliability index and the queue length of the model. The experimental results showed that the variety of the service rate and failure rate had a significant effect on the system indexes. Therefore, the research in this paper can provide theoretical basis and data analysis reference for the design and optimization of some relevant service systems in practice. As one direction of further future research, it is very interesting to develop the model which the repairable rate is variable simultaneously, the reason is that three parameters are variable is more feasible to some actually service systems. Another direction of future research, one can consider the optimization design of the model. Since the variety of the service rate is controllable in many practical systems, and different rates of service incur different costs, to find the balance of the cost of service and the loss of customers' delay is an important issue.

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