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Some Dynamical Behaviors of Fractional-Order Commutative Quaternion-Valued Neural Networks via Direct Method of Lyapunov

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ABSTRACT Some dynamical behaviors of fractional-order commutative quaternion-valued neural networks (FCQVNNs) are studied in this paper. First, because the commutative quaternion does not satisfy Schwartz triangle inequality, the FCQVNNs are divided into four real-valued neural networks (RVNNs) through quaternion commutative multiplication rules. Furthermore, several types of dynamical behaviors including global Mittag-Leffler stability, the boundedness with bounded disturbances, complete synchronization and quasi-synchronization of FCQVNNs are studied. Simultaneously, several conditions for these dynamical behaviors are driven by fractional-order Lyapunov direct method, some inequality techniques and fractional differential equation theory. At last, the effectiveness and feasibility of the obtained theoretical results are verified by several numerical simulation examples.

INDEX TERMS Global mittag-leffler stability, boundedness, synchronization, lyapunov direct method, fractional-order commutative quaternion-valued neural networks.

I. INTRODUCTION

Recently, fractional calculus and fractional-order neural networks (FNNs) have become hot research topics. Fractional calculus is an important mathematical concept, originated in the 17th century, to extend ordinary differential and integral to any non-integer order. As an important tool and valuable method, fractional differential equations are widely used in the fields of viscoelasticity, materials science, fluid mechanics, medicine, bioengineering, and biological models. And fractional-order models have many advantages that the traditional integer-order model does not have. For example, fractional-order equation is suitable for materials and processes with memory and genetic properties, as this approach yields more accurate description of neural networks behaviors. In 1994, Austrasio used fractional calculus $\frac{d^\alpha x}{dt^\alpha}(\alpha > 0)$ for the first time to round the biological vestibular neurons of the organism [1], and pointed out that the eye movement may be a fractional differential system which depends on the speed and position of the eyeball.

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In 1998, Austrasio gave a class of FNNs composed of 32 neurons [2]. Compared with the previous single-neuron or double-neuron models, the kinematic characteristics of FNNs are more consistent with the actual observed values. Then, based on the integer-order calculus, fractional calculus has achieved a breakthrough in theory. The ''infinite memory'' feature provides a more accurate tool for describing actual models of biological systems and viscoelastic systems. There is also evidence that the neural network system described by fractional calculus is more consistent with the actual biological experimental data. Introducing fractional calculus into artificial neural network can improve the accuracy and freedom of calculation, which ensures that FNNs have great application potential in calculation optimization and control. Subsequently, Arena, Fortuna and others took the lead in introducing fractional calculus into the cellular neural network system [3], [4], and analyzed its bifurcation, chaos and other dynamic properties. In 2008, Boroomand replaced the capacitor in the traditional integer-order Hopfield neural networks (IHNNs) for the first time with a division impedance (a fractional-order circuit element), and the fractional-order Hopfield neural networks (FHNNs) are proposed [5]. Compared with the traditional IHNNs, the FHNNs have the characteristics of unlimited memory, which has important practical significance and application value in powerful computing and signal processing capabilities.

In 2011, several dynamic properties of FHNNs are studied by Kaslik and Sivasundaram [6], [7], such as stability, bifurcation and chaos so on. Chaotic system is a class of the deterministic nonlinear system with pseudo-random characteristics which is widely used in secure communications and other fields. Since then, research on FNNs with chaotic system has emerged [8], and these studies have proposed numerical parameter combination condition of FNNs with chaotic system. Then the stability and application of FNNs including the stability analysis [9]–[16], [18]–[21] and synchronization methods [22]–[30] are studied. Among them, fractional-order real-valued neural networks (FRVNNs) and fractional-order complex-valued neural networks (FCVNNs) have been investigated in these years. These systems also succeed in delayed neural networks via pinning control, discrete-time neural networks, static digital images, signal processing, biological applications, associative memory and other fields.

However, the FRVNNs do not perform well when performing geometric transformations such as two-dimensional (2D) affine transformations, while the FCVNNs cannot process 3D data directly when applied to complex models, such as intelligent speech and special images. It is worth noting that, the quaternion-valued neural networks (QVNNs), a generalization of RVNNs and CVNNs, can be used to deal with multidimensional problems. The QVNNs are also used to restrict or optimize the structure of the network, and deeply mine certain signal features (such as speech and images) for specific data and applications to improve some performance. For speech recognition, picture recognition, color night vision [31]–[33], especially color picture processing involving multi-dimensional features, QVNNs have better performance and better processing capabilities. In addition, the QVNNs have advantages in handling optimization or estimation problems, and have a wide range of potential applications in engineering. At present, the research on fractional-order model and quaternion-valued neural networks has achieved some new results [36]–[57]. In [36], [37], the robust stability analysis for the QVNNs based on LMI conditions and uncertain parameters was researched. What's more, the global exponential stability of QVNNs under continuous and discrete conditions were studied in [38]. Yang and Li [39] studied global Mittag-Leffler stability and synchronization analysis of FQVNNs with linear threshold neurons on the basis of Chen. In [45], [46] the existence, uniqueness and exponential stability criteria of solutions for the quaternion-valued delayed Hopfield neural networks (QVDHNNs) are mainly investigated by means of the definitions of ξ-norms and a line impedance cooperative stability region identification method for grid-tied inverters under weak grids was proposed. And the stability and passivity problems for a class of memristor-based fractional-order competitive neural networks (MBFOCNNs) are studied by using Caputo's fractional derivation in [49].

In particular, compared with complex and real numbers, the QVNNs cannot handle complex images or models in practical applications and do not satisfy the commutative law. Therefore, one cannot directly analyze the QVNNs with the techniques and methods of studying CVNNs or RVNNs. Fortunately, in 1892, based on Hamilton's research, Segri proposed a modified quaternion, which satisfies the commutative law for multiplication [58]. The optimized quaternion is still a 4-dimensional number system, called commutative quaternions or reduced biquaternions, and can be divided into two complex number parts for analysis [60]–[62]. So far, commutative quaternions have attracted the attention of scholars [60]–[62], which have been extensively studied in practical application. In [61], the author analyzed the properties of CQVNNs and used them well to color images and complex signals processing. Currently, the authors have studied multi-state CQVNNs in [63] and [64].

In the related research of the existing fractional commutative quaternion-valued neural networks (FCQVNNs), the theoretical tools used in different studies in the literature are not the same, and there is no unified and effective analysis method. In fact, the fractional Lyapunov direct method [65] is a direct and effective research method. Therefore, inspired by the above research, this paper generalizes the fractional Lyapunov method, which can be used to study the global dynamic behavior of FCQVNNs. Since the commutative quaternion has three conjugates and does not satisfy the triangle inequality, there are some major differences between the quaternions and the commutative quaternions. In order to analyze these systems more conveniently, the FCQVNNs will be divided into real-valued systems, then based on fractional-order Lyapunov direct method, some conditions of stability and boundness are proposed. Using this approach, we prove the uniqueness and existence of the equilibrium point for the FCQVNNs after transformation, and discuss the dynamic conclusions of the global Mittag-Leffler stability and boundedness. Some different types of synchronization of FCQVNNs including complete synchronization and quasi-synchronization will be studied through the above conclusions. Finally, several numerical simulation experiments in MATLAB will be supplied to substantiate the feasibility and validity for the results. The main work in the rest of this paper is summarized as follows:

1. To overcome the non-commutativity of fractional-order quaternion-valued neural networks, a FCQVNN model is proposed.

2. Via the fractional-order Lyapunov direct method, the global Mittag-Leffler stability and boundedness criterion of the FCQVNNs are studied by dividing the FCQVNNs into four RVNNs models.

3. Based on the proposed dynamics conclusions, several different types of synchronization of FCQVNNs are achieved by designing different drive-response models.

Then, the effectiveness and feasibility of these conclusions will be demonstrated by MATLAB numerical simulation experiments.

II. PRELIMINARIES

In this paper, D^α denotes the α-order fractional derivative operator. $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}$ denote the real numbers, complex numbers, and the commutative quaternions, respectively. $\mathbb{R}^{n\times m}$, $\mathbb{C}^{n\times m}$ and $\mathbb{Q}^{n\times m}$, or simply, $\mathbb{R}^n$, $\mathbb{C}^n$, and $\mathbb{Q}^n$ when $m = 1$, denote the $n \times m$ matrices with entries from $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}$ respectively. For the commutative quaternion, the three conjugates for z can be described as $z \in \mathbb{Q}$, $z^{(1)}$, $z^{(2)}$, $z^{(3)}$. The norm for z can be described as $|z| = \sqrt[4]{qq^{(1)}q^{(2)}q^{(3)}}$.

Commutative quaternion will be defined as follows when it is used as a generalized complex number:

$$z = z^r + z^i \imath + z^j \jmath + z^k \kappa \in \mathbb{Q},$$

where $z^r, z^i, z^j, z^k \in \mathbb{R}$, $\imath, \jmath, \kappa$ represents three hypothetical units that meet the following rules:

$$\begin{cases} \imath\jmath = \jmath\imath = \kappa, \\ \jmath\kappa = \kappa\jmath = \imath, \\ \kappa\imath = \imath\kappa = -\jmath, \\ \imath^2 = \kappa^2 = -1, \\ \jmath^2 = 1, \imath\jmath\kappa = -1. \end{cases}$$

For the commutative quaternion $z = z^r + z^i\imath + z^j\jmath + z^k\kappa$, it has three conjugates, as shown below:

$$\begin{cases} z^{(1)} = z^r - z^i\imath + z^j\jmath - z^k\jmath, \\ z^{(2)} = z^r + z^i\imath - z^j\jmath + z^k\jmath, \\ z^{(3)} = z^r - z^i\imath - z^j\jmath + z^k\jmath. \end{cases}$$

The norm is given by

$$\begin{aligned} |z| &= \sqrt[4]{zz^{(1)}z^{(2)}z^{(3)}} \\ &= ([(z^r + z^j)^2 + (z^i + z^k)^2] \times [(z^r - z^j)^2] \\ &\quad + (z^i - z^k)^2)^{\frac{1}{4}} \\ &= ([(z^r)^2 + (z^i)^2 + (z^j)^2 + (z^k)^2]^2) - 4(z^r z^j \\ &\quad + z^i z^k)^{\frac{1}{4}} \geq 0. \end{aligned}$$

In addition, for any $z = (z_1, z_2, \ldots, z_n)^T \in \mathbb{Q}^n$, and the modulus of z can be denoted as $|z| = (|z_1|, |z_2|, \ldots, |z_n|)^T \in \mathbb{R}^n$. The norm of z is $\|z\| = \sqrt{\sum_{i=1}^n |z|^2}$. For another commutative quaternion $e = e^r + e^i\imath + e^j\jmath + e^k\kappa$, the product of z and e is defined as:

$$\begin{aligned} ze = ez &= (z^r e^r - z^i e^i + z^j e^j - z^k e^k), \\ &= (z^r e^i + z^i e^r + z^j e^k + z^k e^j)\imath, \\ &= (z^r e^j + z^j e^r - z^i e^k - z^k e^i)\jmath, \\ &= (z^r e^k + z^k e^r + z^i e^j + z^j e^i)\kappa. \end{aligned}$$

We will apply the following definitions and lemmas about commutative quaternion and FRVNNs in this paper.

Remark 1: In actual applications, the FRVNNs and the FCVNNs cannot process 3D data directly when it faced with complex models, such as intelligent speech and special images. In addition, compared with RVNNs and CVNNs, the QVNNs do not satisfy the commutative law. Therefore, one cannot directly analyze the QVNNs with the techniques and methods of studying CVNNs or RVNNs. Based on Hamilton's research, Segri proposed a modified quaternion, which satisfies the commutative law for multiplication. The FCQVNNs are still a 4-dimensional number system also called commutative quaternions or reduced biquaternions. In this paper, we introduce the commutative quaternions into the fractional neural networks, and divide the commutative quaternion into four real-valued parts, which avoids the above problems well. So far, the commutative quaternion has been widely developed both in theory and applications. Then, the properties of commutative quaternions are well summarized and used in signal and image processing.

Definition 1 [17]: The Caputo fractional-order derivative of order $\alpha > 0$ for a function $f(t) \in C^{n+1}([t_0, +\infty), \mathbb{H})$ can be described as

$${}_{t_0}^{C}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau,$$

where $\Gamma(\cdot)$ means the gamma function and n represents a positive integer such that $n-1 < \alpha < n$. Particularly, when $0 < \alpha < 1$,

$${}_{t_0}^{C}D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau.$$

Definition 2 [17]: With two parameters $\alpha > 0$ and $\beta > 0$, the Mittag-Leffler function can be rewritten as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

When $\beta = 1$, its one-parameter form will be defined as

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)},$$

where $z \in \mathbb{H}$. Particularly, $E_{1,1}(z) = e^z$.

Consider a class of FCQVNNs of the following form:

$$D^\alpha z(t) = -c_p z_p(t) + \sum_{q=1}^{n} a_{pq} f_q(z_q(t)) + l_p, \tag{1}$$

or the vector form

$$D^\alpha z(t) = -Cz(t) + Af(z(t)) + L, \tag{2}$$

where $z(t) = (z_1(t), z_2(t), \ldots, z_n(t)) \in \mathbb{Q}^n$, $\alpha \in (0,1)$, $z_p(t)$ means the state vector of the p th neuron, and C=diag$(c_1, c_2, \ldots, c_n) \in R^{n\times n}$ with $c_p > 0 (p = 1, 2, \ldots, n)$ represents the self-feedback connection matrix; $A = (a_{pq})_{n\times n} \in \mathbb{Q}^{n\times n} (p = 1, 2, \ldots, n, q = 1, 2, \ldots, n)$ represents the connection weight matrix, L=$(l_1, l_2, \ldots, l_n)^T \in \mathbb{Q}^n$ means the external input vector; $f(z(t))$=$(f_1(z_1(t)), f_2(z_2(t)), \ldots, f_n(z_n(t))) \in \mathbb{Q}^n$ represents the neuron activation, where $f_q(z_q(t)) (q = 1, 2, \ldots, n)$ means the commutative quaternion-valued linear activation function that can be written in the following form:

$$f_q(z_q) = \max\{0, z_q^R\} + \max\{0, z_q^I\}\imath + \max\{0, z_q^J\}\jmath + \max\{0, z_q^K\}\kappa, \tag{3}$$

where $z_q = z_q^R + z_q^I\imath + z_q^J\jmath + z_q^K\kappa$, $z_q^R, z_q^I, z_q^J, z_q^K \in \mathbb{R}$, $q = 1, 2, \ldots, n$.

In addition, the following equations are the initial conditions of FCQVNNs system (2)

$$z(0) = z_0 \in \mathbb{Q}^n, \tag{4}$$

where $z_0 = z(0)^R + z(0)^I\imath + z(0)^J\jmath + z(0)^K\kappa$.

Remark 2: The activation function is one of the important factors affecting the artificial neural networks. The activation function usually satisfies the Lipschitz condition. Hence, choosing a suitable quaternion-valued activation functions for FCQVNNs will be a difficult task. Recently, in [38] and [35], the neural networks were analyzed and designed through quaternion and complex-valued linear threshold activation functions. It should be noted that one advantage of the linear threshold neurons is that the analyzed network can be seen as a linear system if each neurons output is always greater than or less than zero, and linear threshold neurons will change one linear system to another similar linear system only when some neurons outputs switch on the zero boundary. Therefore, the stability, boundedness and synchronization of the models are studied by the quaternion linear threshold neurons in this paper.

Lemma 1 [39]: Let $f(z) = f^R(z^R) + f^I(z^I)\imath + f^J(z^J)\jmath + f^K(z^K)\kappa \in \mathbb{Q}^n$, *where* $f_q^R(z_q^R = \max\{0, z_q^R\}, f_q^I(z_q^I = \max\{0, z_q^I\}, f_q^J(z_q^J = \max\{0, z_q^J\}, f_q^K(z_q^R = \max\{0, z_q^K\}$, $q = 1, 2, \ldots, n$. *Then the following inequalities hold:*

$$\begin{aligned}
|f^R(z^R) - f^R(\tilde{z}^R)| &\le |z^R - \tilde{z}^R|,\\
|f^I(z^I) - f^I(\tilde{z}^I)| &\le |z^I - \tilde{z}^I|,\\
|f^J(z^J) - f^J(\tilde{z}^J)| &\le |z^J - \tilde{z}^J|,\\
|f^K(z^K) - f^K(\tilde{z}^K)| &\le |z^K - \tilde{z}^K|,\\
|f(z) - f(\tilde{z})| &\le |z - \tilde{z}|,\\
(f(z) - f(\tilde{z}))^*\Lambda(f(z) - f(\tilde{z})) &\le (z - \tilde{z})^*\Lambda(z - \tilde{z}),
\end{aligned}$$

where Λ *means a real positive diagonal matrix, for any* $z^R, \tilde{z}^R, z^I, \tilde{z}^I, z^J, \tilde{z}^J, z^K, \tilde{z}^K \in \mathbb{R}^n$.

Under Lemma 1 and FCQVNNs (2), we have

$$\begin{aligned}
D^\alpha z(t) &= D^\alpha z^R + D^\alpha z^I\imath + D^\alpha z^J\jmath + D^\alpha z^K\kappa\\
&= -Cz(t) + Af(z(t)) + L\\
&= -C(z^R + z^I\imath + z^J\jmath + z^K\kappa)\\
&\quad + (A^R + A^I\imath + A^J\jmath + A^K\kappa)\\
&\quad \times(f^R(z^R) + f^I(z(I))\imath + f^J(z(J))\jmath\\
&\quad + f^K(z(K))\kappa) + (L^R + L^I\imath + L^J\jmath + L^K\kappa),
\end{aligned} \tag{5}$$

by the commutative theory of quaternion multiplication resulting: $\imath^2 = \kappa^2 = \imath\jmath\kappa = -1$, $\jmath^2 = 1$, $\imath\jmath = \jmath\imath = \kappa$, $\jmath\kappa = \kappa\jmath = \imath$, and $\kappa\imath = \imath\kappa = \jmath$, the equation (5) will be rewritten as the following four equations:

$$\begin{cases}
D^\alpha z^R(t) = -Cz^R(t) + A^R f^R(z^R(t)) - A^I f^I(z^I(t))\\
\qquad + A^J f^J(z^J(t)) - A^K f^K(z^K(t)) + L^R,\\
D^\alpha z^I(t) = -Cz^I(t) + A^R f^I(z^I(t)) + A^I f^R(z^R(t))\\
\qquad + A^J f^K(z^K(t)) + A^K f^J(z^J(t)) + L^I,\\
D^\alpha z^J(t) = -Cz^J(t) + A^R f^J(z^J(t)) - A^I f^K(z^K(t))\\
\qquad + A^J f^R(z^R(t)) - A^K f^I(z^I(t)) + L^J,\\
D^\alpha z^K(t) = -Cz^K(t) + A^R f^K(z^K(t)) + A^I f^J(z^J(t))\\
\qquad + A^J f^I(z^I(t)) + A^K f^R(z^R(t)) + L^K,
\end{cases} \tag{6}$$

where $A = A^R + A^I\imath + A^J\jmath + A^K\kappa$, $L = L^R + L^I\imath + L^J\jmath + L^K\kappa$. Let $\hat{C} = \mathrm{diag}(C, C, C, C)$, $Z(t) = (z^R(t)^T, z^I(t)^T, z^J(t)^T, z^K(t)^T)^T$, $\hat{f}(Z(t)) = (f^R(z^R(t))^T, f^I(z^I(t))^T, f^J(z^J(t))^T, f^K(z^K(t)^T))^T$ $\hat{L} = ((L^R)^T, (L^I)^T, (L^J)^T, (L^K)^T)$ and

$$\hat{A} = \begin{pmatrix} A^R & -A^I & A^J & -A^K\\ A^I & A^R & A^K & A^J\\ A^J & -A^K & A^R & -A^I\\ A^K & A^J & A^I & A^R \end{pmatrix}. \tag{7}$$

Then Eq. (6) can be rewritten as

$$D^\alpha Z(t) = -\hat{C}Z(t) + \hat{A}\hat{f}(Z(t)) + \hat{L}. \tag{8}$$

Definition 3: The constant vector $\bar{z} = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)^T$ *is an equilibrium point of FCQVNNs (2), if*

$$D^\alpha \bar{z} = -C\bar{z} + Af(\bar{z}) + L = 0. \tag{9}$$

Definition 4: An equilibrium point $\tilde{z}$ *of system (2) is said to be Mittag-Leffler stable if there exist positive constants* β, λ, *and* $h(z) \ge 0$ $(h(0) = 0)$ *satisfies locally Lipschitz condition for any* $z \in \mathbb{Q}^n$ *with Lipschitz constant* h_0, *such that for any solution* $z(t)$ *of the FCQVNNs (2) with initial value* z_0, *one has*

$$\|z(t)\| \le \left[h(z_0)E_\alpha(-\lambda t^\alpha)\right]^\beta, \; t \ge 0.$$

Remark 3: In general, all definitions and theorems are given for situations where the equilibrium point is the origin, that is, $\bar{z} = 0$, *because any equilibrium point can be converted to the origin through variable changes based on the following Property 1 and Property 2,*

Property 1: $D^\alpha c = 0$,
Property 2: $D^\alpha[af(t) + bh(t))] = aD^\alpha f(t) + bD^\alpha h(t)$,
where $a, b, c \in \mathbb{R}$.

If the equilibrium point in (9) is $\bar{z} \ne 0$, then as the variable $h(t) = z(t) - \bar{z}$ changes, the system (9) can be rewritten as

$$\begin{aligned}
D^\alpha(h(t)) &= D^\alpha(z(t) - \bar{z}) = f(t, z(t)) = f(t, h(t) + \bar{z})\\
&= u(t, h),
\end{aligned}$$

where $u(t, 0) = 0$, then, the new system has equilibrium point at the origin of the new variable h.

Remark 4: Mittag-Leffler stability ensures asymptotic stability, and

$$\lim_{t\to+\infty}\|z(t)\| = 0.$$

Li, Chen, and Podlubny [65] gave a sufficient condition for the Mittag-Leffler stability of the non-linear fractional system, using fractional Lyapunov direct method (or Lyapunov second method). Similar to the mathematical form of the integer-order Lyapunov direct method, this approach is simple and intuitive, and provides a good idea and tool for analyzing the global stability with nonlinear fractional systems. According to the above methods, this paper will be extended to a certain extent, so that it can be used to a greater extent to analyze and solve the dynamics problems of FCQVNNs.

Lemma 2: If $\widetilde{z} = 0$ defines the equilibrium point of the FCQVNNs (2) and $\mathbb{D} \subset \mathbb{Q}^n$ stands for the domain containing the origin, $V(t, z(t)) : [t_0, +\infty) \times \mathbb{D} \to \mathbb{Q}$ is a continuously differentiable function that is locally Lipschitz with respect to z such that

$$a_1\|z(t)\|^a \le V(t, z(t)) \le a_2\|z(t)\|^{ab}, \tag{10}$$
$$D^\alpha V(t^+, z(t^+)) \le -a_3\|z(t)\|^{ab}, \tag{11}$$

where $t \ge 0$, $\alpha \in (0, 1)$, a_1, a_2, a_3, a, and b are arbitrary positive constants. $\dot{V}(z, z(t))$ is piecewise continuous, $\lim_{\iota\to t^+} \dot{V}(\iota, z(\iota))$ exists for any $t \subset [0,\infty)$, and $V(t^+, z(t^+)) = \lim_{\iota\to t^+} \dot{V}(\iota, z(\iota))$. Then $\widetilde{z}$ is Mittag-Leffler stable. If the assumptions hold globally on $\mathbb{Q}^n$, then $\widetilde{x}$ is globally Mittag-Leffler stable.

Proof: By (10) and (11), we obtain the following inequality

$$D^\alpha V(t^+, z(t^+)) \le -\omega V(t, z(t)),$$

where $\omega = a_2^{-1}a_3$ and there exists $k(t) \ge 0$ such that

$$D^\alpha V(t^+, z(t^+)) + k(t) = -\omega V(t, z(t)). \tag{12}$$

Through the Laplace transform to (12), the following equation can be achieved,

$$s^\alpha \mathcal{V}^+(s) - V_0^+ s^{\alpha-1} + K(s) = -\omega\mathcal{V}(s). \tag{13}$$

where $V_0^+ = V^+(z_0)$, $\mathcal{V}^+(s) = \$\{V(t^+, z(t^+))\}$, $K(s) = \$\{k(t)\}$, $\$\{\cdot\}$ means the Laplace transform. For the continuity of $V(t, z(t))$, $V(t^+, z(t)^+) = V(t, z(t))$, and $\mathcal{V}^+(s) = \mathcal{V}(s)$. According to (13), we have

$$\mathcal{V}(s) = \frac{V_0 s^{\alpha-1} - K(s)}{s^\alpha + \omega}. \tag{14}$$

On the one hand, when $z_0 = \widetilde{z}$, then, $z(t) = \widetilde{z}$ is the solution of (2) and $V_0 = 0$. On the other hand, if $z_0 \neq \widetilde{z}$, then $V_0 > 0$. Since $V(z)$ is locally Lipschitz, we obtain

$$V(z(t)) = V_0 E_\alpha(-\omega t^\alpha) - k(t) * [t^{\alpha-1} E_{\alpha,\alpha}(-\omega t^\alpha)],$$

after the inverse Laplace transform of (14). Then, the following equation will be obtained

$$V(z(t)) \le V_0 E_\alpha(-\omega t^\alpha)$$

where $t^{\alpha-1} \ge 0$ and $E_{\alpha,\alpha}(-\omega t^\alpha) \ge 0$ for all $t \in [0, +\infty]$. According to (10), we have

$$\|(z(t))\| \le [a_1^{-1} V_0 E_\alpha(-\omega t^\alpha)]^{\frac{1}{a}},$$

where $a_1^{-1}V_0 > 0(x(0) = 0)$. Let $h^* = a_1^{-1}V_0 \ge 0$, we have

$$\|(z(t))\| \le [h^* E_\alpha(-\omega t^\alpha)]^{\frac{1}{a}},$$

where $h^* = 0$ holds if and only if $z(0) = 0$. Then $h^* = a_1^{-1}V_0$ still satisfies local Lipschitz condition on $z(0) = 0$. Therefom, $\widetilde{z} = 0$ of FCQVNNs (2) is Mittag-Leffler stable by Definition 4. □

Remark 5: According to Lemma 2, the extended Lyapunov direct method reduces the difficulty of constructing Lyapunov functions. There is no need to satisfy the condition of continuous differentiability when constructing Lyapunov function, which is more suitable for studying FCQVNNs than the traditional Lyapunov method. For example, 1-norm of z which fractional differential dissatisfaction exists everywhere, it can be used as Lyapunov function under Lemma 2.

In practical applications, the stability analysis of some noisy systems is hard to study. Therefore, other methods are needed to analyze the dynamic behavior of such systems. Consider the following method of bounded analysis.

Lemma 3: Let $\widetilde{z} = 0$ be an equilibrium point of the system (2) and $\mathbb{D} \subset \mathbb{Q}^n$ be a domain containing the origin. Let $V(t, z(t)) : [t_0, +\infty) \times \mathbb{D} \to \mathbb{Q}$ be a continuously differentiable function that is locally Lipschitz with respect to z such that

$$a_1\|z(t)\|^a \le V(t, z(t)) \le a_2\|z(t)\|^{ab}, \tag{15}$$
$$D^\alpha V(t^+, z(t^+)) \le -a_3\|z(t)\|^{ab} + a_4, \tag{16}$$

where $t \ge 0$, $\alpha \in (0, 1)$, a_1, a_2, a_3, a_4, a, and b are arbitrary positive constants if there exists a constant $T \ge 0$, when $t \ge T$ such that for all solutions $z(t)$ of FCQVNNs (2) satisfies

$$\|(z(t))\| \le (\frac{a_2 a_4}{a_1 a_3} + \frac{\varepsilon}{a_1}).$$

Then, for any $0 < \ell \ll 1$, the solution of FCQVNNs (2) is uniformly bounded on $\mathbb{D}$.

Proof: By (15) and (16), we obtain,

$$D^\alpha V(t^+, z(t^+)) \le -\delta V(t, z(t)) + a_4,$$

where $\delta = a_2^{-1}a_3$ and let $\varphi(t) = V(t^+, z(t^+)) - a_4^{-1}\delta$, clearly,

$$D^\alpha \varphi(t) \le -\delta\varphi(t).$$

Hence,

$$D^\alpha \varphi(t) + k(t) = -\delta\varphi(t), \tag{17}$$

where $k(t) \ge 0$. Through the Laplace transform to (17), we get

$$s^\alpha \bar{\varphi}(s) - \varphi(0)s^{\alpha-1} + K(s) = -\delta\bar{\varphi}(s),$$

where $\bar{\varphi}(s) = \$\{\varphi(t)\}$, $K(s) = \$\{k(t)\}$, $\$\{\cdot\}$ means the Laplace transform. That is,

$$\bar{\varphi}(s) = \frac{\varphi(0)s^{\alpha-1} - K(s)}{s^\alpha + \delta}, \tag{18}$$

then, $\varphi(t)$ is locally Lipschitz. Thus, after the inverse Laplace transform of (18), we obtain

$$\varphi(t) = \varphi(0)E_{\alpha}(-\delta t^{\alpha}) - m(t) * [t^{\alpha-1}E_{\alpha,\alpha}(-\delta t^{\alpha})].$$

Since $t^{\alpha-1} \geq 0$ and $E_{\alpha,\alpha}(-\delta t^{\alpha}) \geq 0$, when $t \to +\infty$, we have

$$\varphi(t) \leq \varphi(0)E_{\alpha}(-\delta t^{\alpha}) \to 0.$$

Substituting any $\ell > 0$, there exists a constant $T \geq 0$ such that for all $t \geq T$ and any solutions $V(t, z(t))$,

$$V(t, z(t)) - a_4^{-1}\delta = V(t^+, z(t^+)) - a_4^{-1}\delta \leq \ell.$$

Through the above equation, we get

$$a_1\|z(t)\|^a = V(z(t)) \leq a_4^{-1}\delta + \ell.$$

Thus

$$\|z(t)\| \leq (\frac{a_2a_4}{a_1a_3} + \frac{\ell}{a_1})^{\frac{1}{a}},$$

and the solutions of FCQVNNs (2) are uniformly bounded. □

Remark 6: From the arbitrariness of ℓ, if $\ell \to 0$ and there exists $T \to +\infty$, when $t > T$ such that for all solutions $z(t)$, we have

$$\|z(t)\| \leq (\frac{a_2a_4}{a_1a_3} + \frac{\ell}{a_1})^{\frac{1}{a}} \to (\frac{a_2a_4}{a_1a_3})^{\frac{1}{a}}, t \to +\infty,$$

the unanimously bounded conclusion is helpful to study the dynamic behavior analysis and control of the system with bounded noise. In the following, the FCQVNNs will be used as an example to introduce its application method.

III. MAIN RESULTS

In this part, several dynamic behaviors including the global Mittag-Leffler stability and boundedness with the disturbances of the FCQVNNs are discussed, and the completely synchronization and quasi-synchronization will also be studied.

A. GLOBAL MITTAG-LEFFLER STABILITY OF FCQVNNs

We need to make the following assumption for getting the stability conditions of the fractional-order commutative quaternion-valued neural networks:

Define $\tilde{Z} = ((\tilde{Z})^R, (\tilde{Z})^I, (\tilde{Z})^J, (\tilde{Z})^K)$ is an equilibrium point of FCQVNNs (8). By varying $M(t) = Z(t) - \tilde{Z}$, the FCQVNNs (8) can be transformed to

$$D^{\alpha}M(t) = -\hat{C}M(t) + \hat{A}\hat{F}(M(t)), \tag{19}$$

where $\hat{F}(M(t)) = \hat{f}(M(t) + \tilde{Z}) - \hat{f}(\tilde{Z}), M(t) = ((m^R(t))^T, (m^I(t))^T, (m^J(t))^T, (m^K(t))^T)^T$, $\hat{F}(M(t)) = ((\hat{F}^R(m^R(t))^T), (\hat{F}^I(m^I(t))^T), (\hat{F}^J(m^J(t))^T), (\hat{F}^K(m^K(t))^T))^T$, $\hat{C}$ and $\hat{A}$ are the same as that in (8).

Assumption 1: For $p = 1, 2, \ldots, n$, there exists a positive constants k_p such that

$$k_p = c_p - \sum_{q=1}^{n}(|a_{qp}^R| + |a_{qp}^I| + |a_{qp}^J| + |a_{qp}^K|).$$

Theorem 1: Under Assumption 1, there exists a unique equilibrium point for FCQVNNs (2).

Proof: Denote $\|z\| = \|z\|_1$ in the rest of this paper, i.e.,

$$\|z\| = \sum_{p=1}^{n}|z_p|,$$

for any $z = (z_1, z_2, \ldots, z_n)^T \in \mathbb{Q}^n$. Consider a mapping $H(m) = (H_1(m), H_2(m), \ldots, H_n(m))^T$, where $m = (m_1, m_2, \ldots, m_n)^T \in \mathbb{Q}^n$. Then

$$\begin{aligned} H_p(m) = &\sum_{q=1}^{n}(a_{pq}^R + a_{pq}^I + a_{pq}^J + a_{pq}^K) \\ &\times(f_q^R(\frac{m_q^R}{c_q}) + f_q^I(\frac{m_q^I}{c_q}) + f_q^J(\frac{m_q^J}{c_q}) + f_q^K(\frac{m_q^K}{c_q})) \\ &+(l_p^R + l_p^I + l_p^J + l_p^K), (p = 1, 2, .., n). \end{aligned}$$

Due to Lemma 1, for any $m, n \in \mathbb{Q}^n$, we have

$$\begin{aligned} &|H_p(m) - H_p(n)| \\ &= |\sum_{q=1}^{n}(a_{pq}^R + a_{pq}^I + a_{pq}^J + a_{pq}^K) \\ &\quad\times\{f_q^R(\frac{m_q^R}{c_q}) - f_q^R(\frac{n_q^R}{c_q}) + f_q^I(\frac{m_q^I}{c_q}) - f_q^I(\frac{n_q^I}{c_q}) \\ &\quad+(f_q^J(\frac{m_q^J}{c_q}) - f_q^J(\frac{n_J^R}{c_q}) + f_q^K(\frac{m_q^K}{c_q}) - f_q^K(\frac{n_q^K}{c_q})\}| \\ &\leq \sum_{q=1}^{n}\frac{|(a_{pq}^R + a_{pq}^I + a_{pq}^J + a_{pq}^K)|}{c_p}(|(m_q^R - n_q^R)| \\ &\quad+|(m_q^I - n_q^I)| + |(m_q^J - n_q^J)| + |(m_q^K - n_q^K)|). \end{aligned}$$

Then, from Assumption 1, we obtain

$$\begin{aligned} &\|H_p(m) - H_p(n)\| \\ &= \sum_{p=1}^{n}|H_p(m) - H_p(n)| \\ &\leq \sum_{p=1}^{n}\sum_{q=1}^{n}\frac{|(a_{pq}^R + a_{pq}^I + a_{pq}^J + a_{pq}^K)|}{c_p}(|(m_q^R - n_q^R)| \\ &\quad+|(m_q^I - n_q^I)| + |(m_q^J - n_q^J)| + |(m_q^K - n_q^K)|) \\ &= \sum_{p=1}^{n}(\sum_{q=1}^{n}\frac{|(a_{pq}^R + a_{pq}^I + a_{pq}^J + a_{pq}^K)|}{c_p})(|(m_q^R - n_q^R)| \\ &\quad+|(m_q^I - n_q^I)| + |(m_q^J - n_q^J)| + |(m_q^K - n_q^K)|) \\ &< \sum_{p=1}^{n}\rho(|(m_q^R - n_q^R)| + |(m_q^I - n_q^I)| + |(m_q^J - n_q^J)| \\ &\quad+|(m_q^K - n_q^K)|) \\ &= \rho\|m - n\|_1. \end{aligned}$$

where $\rho = \max\{1 - \frac{k_p}{c_p}|p = 1, 2, \ldots, n\} < 1$. So the mapping H is a contraction mapping. Then there exists a unique fixed point $\bar{m} \in Q^n$, such that $H(\bar{m}) = \bar{m}$, i.e.,

$$\bar{m}_p = \sum_{q=1}^{n}(a_{pq}^R + a_{pq}^I + a_{pq}^J + a_{pq}^K)(f_q^R(\frac{\overline{m}_q^R}{c_q}) + f_q^I(\frac{\overline{m}_q^I}{c_q}) + f_q^J(\frac{\overline{m}_q^J}{c_q}) + f_q^K(\frac{\overline{m}_q^K}{c_q})) + (l_p^R + l_p^I + l_p^J + l_p^K), (p = 1, 2, .., n).$$

Denote $\bar{z}_p = (\frac{\overline{m}_q}{c_q})$, then

$$-c_p(\bar{z}_p^R + \bar{z}_p^I + \bar{z}_p^J + \bar{z}_p^K) + \sum_{q=1}^{n}(a_{pq}^R + a_{pq}^I + a_{pq}^J + a_{pq}^K) \times(f_q^R(\bar{z}_q^R) + f_q^I(\bar{z}_q^I) + f_q^J(\bar{z}_q^J) + f_q^K(\bar{z}_q^K)) + (l_p^R + l_p^I + l_p^J + l_p^K) = 0, (p = 1, 2, .., n).$$

Thus, $\bar{z} = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)^T$ is the unique equilibrium of the FCQVNNs (2). □

Remark 7: In this paper, the existence of a unique equilibrium point for FCQVNNs is proved in Theorem 1. It is very important for the follow-up study of the Mittag-Leffler stability and boundedness of the FCQVNNs.

Theorem 2: If Assumption 1 holds, the unique equilibrium point of FCQVNNs (2) *is Mittag-Leffler stable.*

Proof: Consider a class of Lyapunov function which will be defined as the following:

$$V(t) = \|e(t)\|_1 = \sum_{p=1}^{n}(|e_p^R(t)| + |e_p^I(t)| + |e_p^J(t)| + |e_p^K(t)|), \quad (20)$$

we can obtain from Assumption 1 and Lemma 1 that

$$\begin{aligned}
D^\alpha V(t) &= \sum_{p=1}^{n} D^\alpha |e_p^R(t)| + \sum_{p=1}^{n} D^\alpha |e_p^I(t)| + \sum_{p=1}^{n} D^\alpha |e_p^J(t)| + \sum_{p=1}^{n} D^\alpha |e_p^K(t)| \\
&\leq \sum_{p=1}^{n} \text{sgn}(e_p^R(t)) D^\alpha e_p^R(t) + \sum_{p=1}^{n} \text{sgn}(e_p^I(t)) D^\alpha e_p^I(t) + \sum_{p=1}^{n} \text{sgn}(e_p^J(t)) D^\alpha e_p^J(t) + \sum_{p=1}^{n} \text{sgn}(e_p^K(t)) D^\alpha e_p^K(t) \\
&= \sum_{p=1}^{n} \text{sgn}(e_p^R(t))[-c_p e_p^R(t) + \sum_{q=1}^{n} a_{pq}^R \hat{F}_q^R(e_q^R(t)) - \sum_{q=1}^{n} a_{pq}^I \hat{F}_q^I(e_q^I(t)) + \sum_{q=1}^{n} a_{pq}^J \hat{F}_q^J(e_q^J(t)) \\
&\quad - \sum_{q=1}^{n} a_{pq}^K \hat{F}_q^K(e_q^K(t))] + \sum_{p=1}^{n} \text{sgn}(e_p^I(t))[-c_p e_p^I(t) + \sum_{q=1}^{n} a_{pq}^R \hat{F}_q^I(e_q^I(t)) + \sum_{q=1}^{n} a_{pq}^I \hat{F}_q^R(e_q^R(t)) \\
&\quad + \sum_{q=1}^{n} a_{pq}^J \hat{F}_q^K(e_q^K(t)) + \sum_{q=1}^{n} a_{pq}^K \hat{F}_q^J(e_q^J(t))] \\
&\quad + \sum_{p=1}^{n} \text{sgn}(e_p^J(t))[-c_p e_p^J(t) + \sum_{q=1}^{n} a_{pq}^R \hat{F}_q^J(e_q^J(t)) - \sum_{q=1}^{n} a_{pq}^I \hat{F}_q^K(e_q^K(t)) + \sum_{q=1}^{n} a_{pq}^J \hat{F}_q^R(e_q^R(t)) \\
&\quad - \sum_{q=1}^{n} a_{pq}^K \hat{F}_q^I(e_q^I(t))] + \sum_{p=1}^{n} \text{sgn}(e_p^K(t))[-c_p e_p^K(t) + \sum_{q=1}^{n} a_{pq}^R \hat{F}_q^K(e_q^K(t)) + \sum_{q=1}^{n} a_{pq}^I \hat{F}_q^J(e_q^J(t)) \\
&\quad + \sum_{q=1}^{n} a_{pq}^J \hat{F}_q^I(e_q^I(t)) + \sum_{q=1}^{n} a_{pq}^K \hat{F}_q^R(e_q^R(t))] \\
&\leq \sum_{p=1}^{n}[-c_p|e_p^R(t)| + \sum_{q=1}^{n} |a_{pq}^R||e_q^R(t)| + \sum_{q=1}^{n} |a_{pq}^I||e_q^I(t)| + \sum_{q=1}^{n} |a_{pq}^J||e_q^J(t)| + \sum_{q=1}^{n} |a_{pq}^K||e_q^K(t)|] \\
&\quad + \sum_{p=1}^{n}[-c_p|e_p^I(t)| + \sum_{q=1}^{n} |a_{pq}^R||(e_q^I(t))| + \sum_{q=1}^{n} |a_{pq}^I||e_q^R(t)| + \sum_{q=1}^{n} |a_{pq}^J||e_q^K(t)| + \sum_{q=1}^{n} |a_{pq}^K||e_q^J(t)|] \\
&\quad + \sum_{p=1}^{n}[-c_p|e_p^J(t)| + \sum_{q=1}^{n} |a_{pq}^R||(e_q^J(t))| + \sum_{q=1}^{n} |a_{pq}^I||e_q^K(t)| + \sum_{q=1}^{n} |a_{pq}^J||e_q^R(t)| + \sum_{q=1}^{n} |a_{pq}^K||e_q^I(t)|] \\
&\quad + \sum_{p=1}^{n}[-c_p|e_p^K(t)| + \sum_{q=1}^{n} |a_{pq}^R||(e_q^K(t))| + \sum_{q=1}^{n} |a_{pq}^I||e_q^J(t)| + \sum_{q=1}^{n} |a_{pq}^J||e_q^I(t)| + \sum_{q=1}^{n} |a_{pq}^K||e_q^R(t)|] \\
&= \sum_{p=1}^{n}[-c_p(|e_p^R(t)| + |e_p^I(t)| + |e_p^J(t)| + |e_p^K(t)|) + \sum_{q=1}^{n} |a_{pq}^R|(|e_p^R(t)| + |e_p^I(t)| + |e_p^J(t)| + |e_p^K(t)|)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{q=1}^{n}|a_{pq}^{I}|(|e_p^R(t)|+|e_p^I(t)|+|e_p^J(t)|+|e_p^K(t)|)\\
&+\sum_{q=1}^{n}|a_{pq}^{J}|(|e_p^R(t)|+|e_p^I(t)|+|e_p^J(t)|+|e_p^K(t)|)\\
&+\sum_{q=1}^{n}|a_{pq}^{K}|(|e_p^R(t)|+|e_p^I(t)|+|e_p^J(t)|+|e_p^K(t)|)\\
=&\sum_{p=1}^{n}[-c_p(|e_p^R(t)|+|e_p^I(t)|+|e_p^J(t)|+|e_p^K(t)|)\\
&+\sum_{q=1}^{n}(|a_{pq}^R|+|a_{pq}^I|+|a_{pq}^J|+|a_{pq}^K|)(|e_p^R(t)|\\
&+|e_p^I(t)|+|e_p^J(t)|+|e_p^K(t)|)]\\
=&-\sum_{p=1}^{n}[c_p-\sum_{q=1}^{n}(|a_{pq}^R|+|a_{pq}^I|+|a_{pq}^J|+|a_{pq}^K|)]\\
&\times(|e_p^R(t)|+|e_p^I(t)|+|e_p^J(t)|+|e_p^K(t)|)\\
=&-k(|e_p^R(t)|+|e_p^I(t)|+|e_p^J(t)|+|e_p^K(t)|)\\
\le&-k\|e(t)\|_1,
\end{aligned}
$$

where $k=\min\{k_1,k_2,\ldots,k_n\}$. Then, Lyapunov function $V(t)$ satisfies inequality the inequation (11). According to Lemma 2, we have

$$\|e(t)\|_1\le V(0)E_\alpha(-\lambda t^\alpha).$$

It can be shown that:

$$\|z(t)-\bar{z}\|\le V(z(0)-\bar{z})E_\alpha(-\lambda t^\alpha).$$

Hence, the equilibrium point of (19) is global Mittag-Leffler stable by definition 4, which means the unique equilibrium point of FCQVNNs (2) is global Mittag-Leffler stable. □

B. BOUNDEDNESS ANALYSIS OF FCQVNNs

In fact, due to noise disturbance, the external input l_p may be time-varying in practice. We now analyze the dynamic behaviors of the FCQVNNs with external input disturbance.

Consider the following FCQVNNs with the time-varying external input vector:

$$D^\alpha z(t)=-c_pz_p(t)+\sum_{q=1}^{n}a_{pq}f_q(z_q(t))+l_p(t),\tag{21}$$

where $l_p(t)$ is bounded time-varying and $|l_p(t)|\le M_p$. It is hard to find an equilibrium point of FCQVNNs (2) because $l_p(t)$ is time-varying. Therefore, we can discuss bounded of the solutions.

Definition 5: The system (2) is said to be uniformly bounded, if for each ϵ there exists $T\ge0$ such that

$$\|z(t)\|\le\frac{H}{k}+\epsilon,\tag{22}$$

for all $t\ge T$ and any solution $z(t)$ of FCQVNNs (21), where $k=\min\{k_1,k_2,\ldots,k_n\}$, $H=\sum_{p=1}^{n}[\sum_{q=1}^{n}(|a_{pq}^R|+|a_{pq}^I|+|a_{pq}^J|+|a_{pq}^K|)(|f_q^R(0)|+|f_q^I(0)|+|f_q^J(0)|+|f_q^K(0)|)+M_p]$, *and $1\gg\epsilon>0$ is any small constant.*

Theorem 3: If Assumption 1 holds, FCQVNNs (21) is uniformly bounded on $\mathbb{Q}^n$.

Proof: Based on the solutions of system (21), consider a class of Lyapunov is any:

$$
\begin{aligned}
V(t,z(t))&=\|z(t)\|_1\\
&=\sum_{p=1}^{n}(|z_p^R(t)|+|z_p^I(t)|+|z_p^J(t)|+|z_p^K(t)|).
\end{aligned}\tag{23}
$$

We can obtain from Assumption 1 and system (6) that

$$
\begin{aligned}
&D^\alpha V(t)\\
&=\sum_{p=1}^{n}D^\alpha|z_p^R(t)|+\sum_{p=1}^{n}D^\alpha|z_p^I(t)|\\
&\quad+\sum_{p=1}^{n}D^\alpha|z_p^J(t)|+\sum_{p=1}^{n}D^\alpha|z_p^K(t)|\\
&\le\sum_{p=1}^{n}\mathrm{sgn}(z_p^R(t))D^\alpha z_p^R(t)+\sum_{p=1}^{n}\mathrm{sgn}(z_p^I(t))D^\alpha z_p^I(t)\\
&\quad+\sum_{p=1}^{n}\mathrm{sgn}(z_p^J(t))D^\alpha z_p^J(t)+\sum_{p=1}^{n}\mathrm{sgn}(z_p^K(t))D^\alpha z_p^K(t)\\
&=\sum_{p=1}^{n}\mathrm{sgn}(z_p^R(t))[-c_pz_p^R(t)+\sum_{q=1}^{n}a_{pq}^R(f_q^R(z_q^R(t))-f_q^R(0)\\
&\quad+f_q^R(0))-\sum_{q=1}^{n}a_{pq}^I(f_q^I(z_q^I(t))-f_q^I(0)+f_q^I(0))\\
&\quad+\sum_{q=1}^{n}a_{pq}^J(f_q^Jz_q^J(t)-f_q^J(0)+f_q^J(0))\\
&\quad-\sum_{q=1}^{n}a_{pq}^K(f_q^K(z_q^K(t))-f_q^K(0)+f_q^K(0))]+l_p^R(t)\\
&\quad+\sum_{p=1}^{n}\mathrm{sgn}(z_p^I(t))[-c_pz_p^I(t)+\sum_{q=1}^{n}a_{pq}^R(f_q^I(z_q^I(t))\\
&\quad-f_q^I(0)+f_q^I(0))+\sum_{q=1}^{n}a_{pq}^I(f_q^R(z_q^R(t))-f_q^R(0)\\
&\quad+f_q^R(0))+\sum_{q=1}^{n}a_{pq}^J(f_q^Kz_q^K(t)-f_q^K(0)+f_q^K(0))\\
&\quad+\sum_{q=1}^{n}a_{pq}^K(f_q^J(z_q^J(t))-f_q^J(0)+f_q^J(0))]+l_p^I(t)\\
&\quad+\sum_{p=1}^{n}\mathrm{sgn}(z_p^J(t))[-c_pz_p^J(t)+\sum_{q=1}^{n}a_{pq}^R(f_q^J(z_q^J(t))\\
&\quad-f_q^I(0)+f_q^I(0))-\sum_{q=1}^{n}a_{pq}^I(f_q^K(z_q^K(t))-f_q^K(0)
\end{aligned}
$$

$$
\begin{aligned}
&+f_q^K(0))+\sum_{q=1}^{n}a_{pq}^J(f_q^R z_q^R(t)-f_q^R(0)+f_q^R(0))\\
&-\sum_{q=1}^{n}a_{pq}^K(f_q^I(z_q^I(t))-f_q^I(0)+f_q^I(0))]+l_p^J(t)\\
&+\sum_{p=1}^{n}\mathrm{sgn}(z_p^K(t))[-c_p z_p^K(t)+\sum_{q=1}^{n}a_{pq}^R(f_q^K(z_q^K(t))\\
&-f_q^K(0)+f_q^K(0))+\sum_{q=1}^{n}a_{pq}^I(f_q^{KJ}(z_q^J(t))-f_q^J(0)\\
&+f_q^J(0))+\sum_{q=1}^{n}a_{pq}^J(f_q^I z_q^I(t)-f_q^I(0)+f_q^I(0))\\
&+\sum_{q=1}^{n}a_{pq}^K(f_q^R(z_q^R(t))-f_q^R(0)+f_q^R(0))]+l_p^K(t)\\
\le&\sum_{p=1}^{n}[-c_p|z_p^R(t)|+(\sum_{q=1}^{n}|a_{pq}^R||z_q^R(t)|+\sum_{q=1}^{n}|a_{pq}^R||f_q^R(0)|)\\
&+(\sum_{q=1}^{n}|a_{pq}^I||z_q^I(t)|+\sum_{q=1}^{n}|a_{pq}^I||f_q^0(t)|)\\
&+(\sum_{q=1}^{n}|a_{pq}^J||z_q^J(t)|+\sum_{q=1}^{n}|a_{pq}^J||f_q^J(0)|)\\
&+(\sum_{q=1}^{n}|a_{pq}^K||z_q^K(t)|+\sum_{q=1}^{n}|a_{pq}^K||f_q^K(0)|)]+l_p^R(t)\\
&+\sum_{p=1}^{n}[-c_p|z_p^I(t)|+(\sum_{q=1}^{n}|a_{pq}^R||(z_q^I(t)|+\sum_{q=1}^{n}|a_{pq}^R||f_q^I(0)|)\\
&+(\sum_{q=1}^{n}|a_{pq}^I||z_q^R(t)|+\sum_{q=1}^{n}|a_{pq}^I||f_q^R(0)|)\\
&+(\sum_{q=1}^{n}|a_{pq}^J||z_q^K(t)|+\sum_{q=1}^{n}|a_{pq}^J||f_q^K(0)|)\\
&+(\sum_{q=1}^{n}|a_{pq}^K||z_q^J(t)|+\sum_{q=1}^{n}|a_{pq}^K||f_q^J(0)|)]+l_p^I(t)\\
&+\sum_{p=1}^{n}[-c_p|e_p^J(t)|+(\sum_{q=1}^{n}|a_{pq}^R||(z_q^J(t))|\\
&+\sum_{q=1}^{n}|a_{pq}^R|(|f_q^J(0))|)+(\sum_{q=1}^{n}|a_{pq}^I||z_q^K(t)|\\
&+\sum_{q=1}^{n}|a_{pq}^I|(|f_q^K(0)|)+(\sum_{q=1}^{n}|a_{pq}^J||z_q^R(t)|\\
&+\sum_{q=1}^{n}|a_{pq}^J|(|f_q^R(0)|)+(\sum_{q=1}^{n}|a_{pq}^K||z_q^I(t)|)\\
&+\sum_{q=1}^{n}|a_{pq}^K|(|f_q^I(0)|)]+l_p^J(t)\\
&+\sum_{p=1}^{n}[-c_p|z_p^K(t)|+(\sum_{q=1}^{n}|a_{pq}^R||z_q^K(t)|\\
&+\sum_{q=1}^{n}|a_{pq}^R||f_q^K(0)|)+(\sum_{q=1}^{n}|a_{pq}^I||z_q^J(t)|\\
&+\sum_{q=1}^{n}|a_{pq}^I||f_q^J(0)|)+(\sum_{q=1}^{n}|a_{pq}^J||z_q^I(t)|\\
&+\sum_{q=1}^{n}|a_{pq}^J||f_q^I(0)|)+(\sum_{q=1}^{n}|a_{pq}^K||z_q^R(t)|\\
&+\sum_{q=1}^{n}|a_{pq}^K||f_q^R(0)|)]+l_p^K(t)\\
=&\sum_{p=1}^{n}[-c_p(|z_p^R(t)|+|z_p^I(t)|+|z_p^J(t)|+|z_p^K(t)|)\\
&+\sum_{q=1}^{n}|a_{pq}^R|((|z_p^R(t)|+|z_p^I(t)|+|z_p^J(t)|+|z_p^K(t)|)\\
&+(|f_p^R(0)|+|f_p^I(0)|+|f_p^J(0)|+|f_p^K(0)|))\\
&+\sum_{q=1}^{n}|a_{pq}^I|((|z_p^R(t)|+|z_p^I(t)|+|z_p^J(t)|+|z_p^K(t)|)\\
&+(|f_p^R(0)|+|f_p^I(0)|+|f_p^J(0)|+|f_p^K(0)|))\\
&+\sum_{q=1}^{n}|a_{pq}^J|((|z_p^R(t)|+|z_p^I(t)|+|z_p^J(t)|+|z_p^K(t)|)\\
&+(|f_p^R(0)|+|f_p^I(0)|+|f_p^J(0)|+|f_p^K(0)|))\\
&+\sum_{q=1}^{n}|a_{pq}^K|((|z_p^R(t)|+|z_p^I(t)|+|z_p^J(t)|+|z_p^K(t)|)\\
&+(|f_p^R(0)|+|f_p^I(0)|+|f_p^J(0)|+|f_p^K(0)|))\\
&+(l_p^R(t)+l_p^I(t)+l_p^J(t)+l_p^K(t))]\\
\le&-\sum_{p=1}^{n}[c_p-\sum_{q=1}^{n}(|a_{pq}^R|+|a_{pq}^I|+|a_{pq}^J|+|a_{pq}^K|)]\\
&\cdot(|z_p^R(t)|+|z_p^I(t)|+|z_p^J(t)|+|z_p^K(t)|)\\
&+\sum_{p=1}^{n}[\sum_{q=1}^{n}(|a_{pq}^R|+|a_{pq}^I|+|a_{pq}^J|+|a_{pq}^K|)(|f_p^R(0)|\\
&+|f_p^I(0)|+|f_p^J(0)|+|f_p^K(0)|)]+M_p\\
\le&\, k\|z(t)\|_1+H.
\end{aligned}
$$

According to Definition 4 and Lemma 3, the system (21) satisfies the inequation (23) which is uniformly bounded. □

C. SYNCHRONIZATION ANALYSIS OF FCQVNNs

We now discuss the synchronization problem between neural networks. Then the response system for the drive system (2) is depicted as:

$$D^{\alpha}h(t)=-Ch(t)+Af(h(t))+L+\mu(t), \tag{24}$$

where $h_q = h_q^R + h_q^I\imath + h_q^J\jmath + h_q^K\kappa$, $f(h(t)) = (f_1(h_1(t)), f_2(h_2(t)), \ldots, f_n(h_n(t)))^T, f_q(h_q(t))(q=1,2,\ldots,n)$ stands for a commutative quaternion-valued linear activation function defined as:

$$f_q(h_q) = \max\{0, h_q^R\} + \max\{0, h_q^I\}\imath + \max\{0, h_q^J\}\jmath + \max\{0, h_q^K\}\kappa, \quad (25)$$

and $\mu_q = \mu_q^R + \mu_q^I\imath + \mu_q^J\jmath + \mu_q^K\kappa$ represents the appropriate controller to be designed.

Based on the rules witch is about the commutativity of quaternion multiplication, the FCQVNNs (24) can be rewritten as:

$$\begin{cases} D^\alpha h^R(t) = -Ch^R(t) + A^R f^R(h^R(t)) - A^I f^I(h^I(t)) \\ \quad + A^J f^J(h^J(t)) - A^K f^K(h^K(t)) + L^R + \mu_q^R(t), \\ D^\alpha h^I(t) = -Ch^I(t) + A^R f^I(h^I(t)) + A^I f^R(h^R(t)) \\ \quad + A^J f^K(h^K(t)) + A^K f^J(h^J(t)) + L^I + \mu_q^I(t), \\ D^\alpha h^J(t) = -Ch^J(t) + A^R f^J(h^J(t)) - A^I f^K(h^K(t)) \\ \quad + A^J f^R(h^R(t)) - A^K f^I(h^I(t)) + L^J + \mu_q^J(t), \\ D^\alpha h^K(t) = -Ch^K(t) + A^R f^K(h^K(t)) + A^I f^J(h^J(t)) \\ \quad + A^J f^I(h^I(t)) + A^K f^R(h^R(t)) + L^K + \mu_q^K(t), \end{cases} \quad (26)$$

which is equivalent to

$$D^\alpha H(t) = -\hat{C}H(t) + \hat{A}\hat{f}(H(t)) + \hat{L} + U(t),$$

where $H(t) = ((h^R(t))^T, (h^I(t))^T, (h^J(t))^T, (h^K(t))^T)^T$, $\hat{f}(H(t)) = ((\hat{f}^R(h^R(t))^T), (\hat{f}^I(h^I(t))^T), (\hat{f}^J(h^J(t))^T), (\hat{f}^K(h^K(t))^T))^T$, $U(t) = ((\mu^R(t))^T, (\mu^I(t))^T, (\mu^J(t))^T, (\mu^K(t))^T)^T$. Then the controller $\mu(t)$ will be defined as

$$\mu(t) = -d(h(t) - z(t)), \quad (27)$$

where $d = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$, $\sigma_p > 0$, $p = (1, 2, \ldots, n)$ is a constant representing the control gain.

Let $e(t) = h(t) - z(t)$. Then, the error system between the FCQVNNs (2) and (24) will be denoted as

$$D^\alpha e(t) = -(C + d)e(t) + A\delta(e(t)), \quad (28)$$

where $\delta(e(t)) = f((e(t) + z(t)) - f(z(t)))$.

Similarly, according to the commutativity of quaternion multiplication theory, the FCQVNNs (28) is given by:

$$D^\alpha E(t) = -\hat{W}E(t) + \hat{A}\hat{\delta}(E(t)),$$

where $\hat{W} = \mathrm{diag}(C + d, C + d, C + d, C + d)$, $E(t) = ((e^R(t))^T, (e^I(t))^T, (e^J(t))^T, (e^K(t))^T)^T$, $\delta(E(t)) = (\delta^R(h^R(t))^T, \delta^I(h^I(t))^T, \delta^J(h^J(t))^T, \delta^K(h^K(t))^T)^T$.

Definition 6: If the error system $e(t) = h(t) - z(t)$ converges to 0, that is

$$\lim_{t\to+\infty} |e(t)| = \lim_{t\to+\infty} |h(t) - z(t)| = 0,$$

then, FCQVNN (2) and FCQVNN (24) are completely synchronized.

Theorem 4: The FCQVNNs (2) and (24) are completely synchronized, if there exists a constant k_p that satisfies

$$k_p = d_p + c_p - \sum_{q=1}^{n}(|a_{qp}^R| + |a_{qp}^I| + |a_{qp}^J| + |a_{qp}^K|) > 0. \quad (29)$$

Proof: According to the error system (28), the proof of Theorems 4 is similar to that of Theorems 2. Hence, the proof is omitted. □

Now, we consider a special kind of synchronization. Quasi-synchronization is a special control method, which only requires driving the error between the response systems to gradually approach a bounded region, and does not require its convergence to 0. Hence, the drive-response system with different external inputs can be depicted as:

$$D^\alpha z_p(t) = -c_p z_p(t) + \sum_{q=1}^{n} a_{pq} f_q(z_q(t)) + l_p(t), \quad (30)$$

$$D^\alpha h_p(t) = -c_p h_p(t) + \sum_{q=1}^{n} a_{pq} f_q(h_q(t)) + i_p(t) + \mu_p(t), \quad (31)$$

where $l_p(t)$, $i_p(t)$ are different external inputs, which satisfy $|l_p(t)| \leq M_p$, $|i_p(t)| \leq N_p, p = 1, 2, \ldots, n$, $\mu_p(t) = -d_p(h_p(t) - z_p(t))$, d_p is a constant.

Let $e(t) = h(t) - z(t)$. Then, the error system between the FCQVNNs (30) and (31) is

$$D^\alpha e_p(t) = -(c_p + d_p)e_p(t) + \sum_{q=1}^{n} a_{pq}(f_q(h_q(t)) - f_q(z_q(t)) + (i_p(t) - l_p(t)). \quad (32)$$

Since the external inputs of the two systems are different, so the complete synchronization of the FCQVNNs (30) and the FCQVNNs (31) cannot be realized under the current linear feedback controller. The following defines and discusses the implementation of its quasi-synchronization.

Definition 7: For a positive constant $\psi \geq 0$, if there exists a $T \geq 0$ such that for all $t \geq T$ we have

$$\|e(t)\| = \|h(t) - z(t)\| \leq \psi,$$

then, the drive-response systems (30) and (31) are quasi-synchronous with ψ bound.

Theorem 5: The drive-response systems (30) and (31) can be defined as quasi-synchronous with $\psi = \frac{G}{k} + \epsilon$ bound, if there exists the constant k_p that satisfies

$$k_p = d_p + c_p - \sum_{q=1}^{n}(|a_{qp}^R| + |a_{qp}^I| + |a_{qp}^J| + |a_{qp}^K|) > 0,$$

where $k = \min\{k_1, k_2, \ldots, k_n\}$, $G = \sum_{p=1}^{n}(M_p + N_p)$, and $0 < \epsilon \ll 1$ is any small constant.

Proof: According to the error system (32), the proof of Theorems 5 is similar to that of Theorem 3. Hence, the proof is omitted. □

Remark 8: *If larger control parameters d_p are selected, the bound*

$$\psi = \frac{G}{k} + \epsilon$$

of the error system will be smaller. Therefore, the bounds of quasi-synchronous errors can be reduced to the actual requirements by selecting appropriate control parameters.

Remark 9: *Complete synchronization is one of the simplest types of synchronization, and the error state tends to 0, which indicates complete synchronization between drive system and response system. Compared with complete synchronization, only the external input is different between quasi-synchronization and complete synchronization, quasi-synchronization is a special control method which is only required that the error between the drive-response system gradually tends to a bounded area, and it is not required to converge to 0. Quasi-synchronization is often used to control some systems with unknown disturbances, thereby reducing the difficulty of control and enhancing practicability. In order to distinguish between completely synchronized system and quasi-synchronous system, (1) and (21) are used to replace the completely synchronized and quasi-synchronized systems, which is convenient for readers to study and understand.*

IV. NUMERICAL EXAMPLES

In this part, several simulation numerical examples will be used to prove the feasibility and effectiveness of the obtained results.

Example 1: *Consider the following 2-neuron FCQVNNs*:

$$D^{\alpha} z(t) = -Cz(t) + Bf(z(t)) + L \tag{33}$$

where $\alpha = 0.95$, $z(t) = z^R + z^i \imath + z^j \jmath + z^k \kappa = (z_1(t), z_2(t))^T$, $C = \mathrm{diag}(4, 4)$, $f(z) = \max\{0, z^R\} + \max\{0, z^i\}\imath + \max\{0, z^j\}\jmath + \max\{0, z^k\}\kappa$,

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

$$b_{11} = -0.26 - 0.32\imath - 0.27\jmath - 0.41\kappa,$$
$$b_{12} = -0.59 + 0.11\imath - 0.30\jmath + 0.37\kappa,$$
$$b_{21} = -0.39 + 0.15\imath - 0.22\jmath - 0.13\kappa,$$
$$b_{22} = -0.51 - 0.25\imath - 0.28\jmath + 0.14\kappa.$$

Case 1: For $L = (0.4 - 0.3\imath - 0.6\jmath + 0.7\kappa, -0.5 + 0.4\imath + 0.5\jmath - 0.4\kappa)^T$ *be the constant external inputs of system (33). Note that* $\rho = \max\{1 - \frac{k_p}{c_p} | p = 1, 2\} = \max\{\frac{2.15}{4}, \frac{2.55}{4}\} = \frac{2.55}{4} < 1$. *Hence, on the basis of Theorem 1, the FCQVNNs (33) has a unique equilibrium point. Then, the Assumption 1 will be used to prove the global Mittag-Leffler stability for the unique equilibrium point, where the constant* $\{k_p | p = 1, 2\} = \min\{k_1, k_2\} = \min\{1.85, 1.45\} = 1.45 > 0$. *Hence, Assumption 1 holds. Therefore, from Theorem 2, the unique equilibrium point of the FCQVNNs (33) is globally Mittag-Leffler stable. In the MATLAB numerical simulation experiment, the initial conditions are* $Z_{10} = 1.5 + 0.5\imath + 2.5\jmath + 3.5\kappa$ *and* $Z_{20} = -1.5 - 0.5\imath - 2.5\jmath - 3.5\kappa$. *Fig.1 shows the trajectories of the state variables of the four parts for the FCQVNNs (33) with the constant external inputs.*

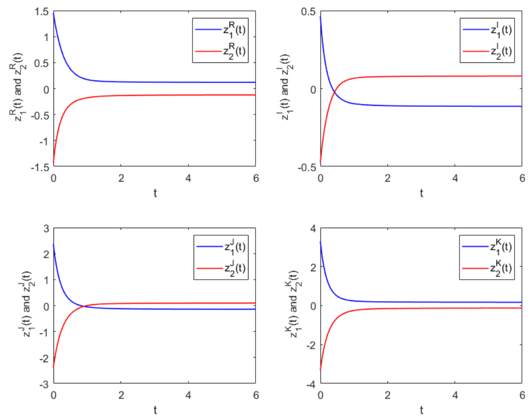

FIGURE 1. **Transient states for case 1 of the four parts of FCQVNNs (33) with the constant external inputs.**

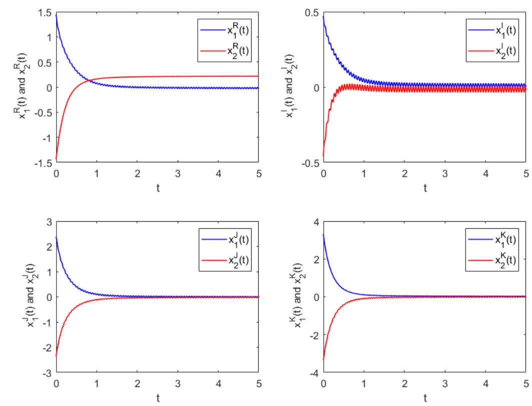

FIGURE 2. **Transient states for case 2 of the four parts of FCQVNNs (33) with the time-dependent external inputs.**

Case 2: Let $L = (1.5cos(t) - 1.2sin(t)\imath + +2.7sin(t)\jmath - 0.6sin(t)\kappa, \frac{t-1.4}{t+1} + 1.8sin(t)\imath + 1.6sin(t)\jmath - 2.8sin(t)\kappa)$ *be the time-dependent external inputs of system (33). After calculating, Theorem 1 applies. Hence, by Theorem 3, the FCQVNNs (33) is uniformly bounded,and there exists a* $T \geq t_0$ *such that* $|z_p(t)| \leq \|z_p(t)\| \leq 2.01$ *for all* $t \geq T$ *and any solution* $z(t) = (z_1(t), z_2(t))$, *where* $p = 1, 2$, $\frac{H}{k} = 2$, $\epsilon = 0.01$. *In the MATLAB numerical simulation experiment, the initial conditions are* $Z_{10} = 1.5 + 0.5\imath + 2.5\jmath + 3.5\kappa$ *and* $Z_{20} = -1.5 - 0.5\imath - 2.5\jmath - 3.5\kappa$. *Fig.2 shows the trajectories of the state variables of the four parts for the FCQVNNs (33) with the time-dependent external inputs.*

Remark 10: *In fact, for all fractional coefficients* $0 < \alpha < 1$, *the convergence speeds for different* α *are not equal. In the reference [13], the author proposes that the larger* α *is, the faster the converges.*

Example 2: *Consider the following 2-neuron FCQVNNs as the drive system*:

$$D^{\alpha} z(t) = -Cz(t) + Bf(z(t)) + L \tag{34}$$

where $\alpha = 0.95$, $z(t) = z^R + z^i\imath + z^j\jmath + z^k\kappa = (z_1(t), z_2(t))^T$, $C = \text{diag}(0.05, 0.05)$, $f(z) = \max\{0, z^R\} + \max\{0, z^i\}\imath + \max\{0, z^j\}\jmath + \max\{0, z^k\}\kappa$,

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

$$b_{11} = -0.35 + 0.55\imath - 0.42\jmath + 0.11\kappa,$$
$$b_{12} = 0.55 + 0.25\imath - 0.56\jmath + 0.25\kappa,$$
$$b_{21} = -0.42 + 0.14\imath - 0.26\jmath + 0.55\kappa,$$
$$b_{22} = 0.25 + 0.24\imath - 0.29\jmath + 0.66\kappa.$$

And the corresponding response system will be defined as:

$$D^{\alpha}h(t) = -Ch(t) + Bf(h(t)) + J + \mu(t), \qquad (35)$$

where $\alpha = 0.95$, $h(t) = h^R + h^i\imath + h^J\jmath + h^K\kappa = (h_1(t), h_2(t))^T$, $f(h) = \max\{0, h^R\} + \max\{0, h^I\}\imath + \max\{0, h^J\}\jmath + \max\{0, h^K\}\kappa$, *C and A are the same as that in system (35). Then, the controller is* $\mu(t) = -d(h(t) - z(t))$ *with gain matrix* $d = diag(4, 4)$.

Case 3: Let $L = J = (0.8 - 0.5\imath + 0.3\jmath + 0.2\kappa, 0.6 + 0.4\imath - 0.3\jmath - 0.3\kappa)^T$ *be the constant external inputs of system (34) and (35), whereupon, the error system between the FCQVNNs (34) and (35)will be defined as*:

$$D^{\alpha}e(t) = -(C + d)e(t) + B\delta(e(t)), \qquad (36)$$

where $e(t) = h(t) - z(t)$, $\delta(e(t)) = f(e(t) + z(t)) - f(z(t))$. *After calculation,* $\rho = \max\{1 - \frac{k_p}{c_p}|p = 1, 2\} = \max\{-24, -19\} = -19 < 1$. *Hence, on the basis of Theorem 1, system (36) has a unique equilibrium point. After that, the above completely synchronization for the drive-response systems (34) and (35) can be verified by proving the global Mittag-Leffler stability of the unique equilibrium point of the error system (36), at the same time, the conditions (29) will be verified. After calculation, the constant* $k_p = \min\{k_1, k_2\} = \min\{1.25, 1\} = 1 > 0$. *Therefore, though Theorem 4, the equilibrium point of the FCQVNNs (36) is the globally Mittag-Leffler stable which imply that the complete synchronization between the drive-response system (34) and (35) will be proved. For the numerical simulation experiment, the initial conditions of the FCQVNNs (34) and FCQVNNs (35) are* $Z_{10} = 1.5 + 0.5\imath + 2.5\jmath + 3.5\kappa$, $Z_{20} = -1.5 - 0.5\imath - 2.5\jmath - 3.5\kappa$ *and* $h_{10} = -3.0 - 2.5\imath - 2.5\jmath - 1.0\kappa$, $h_{20} = 5.0 + 2.0\imath + 2.0\jmath + 1.5\kappa$ *respectively. Figs.3-6 depict the state trajectories graph of the four part for Case 3 of drive-response systems (34) and (35) with and without control. The phase plot of the FCQVNNs (34) and (35) with and without control are depicted in Fig.7 and Fig.8. Meanwhile, the transient states of four parts of the synchronization error system is shown in Fig.9.*

Case 4: Let $L = (0.4\cos(t) - 0.9\sin(t)\imath + 0.7\sin(t)\jmath + 0.6\sin(t)\kappa, 1.6\cos(t) + 0.8\sin(t)\imath - 0.7\sin(t)\jmath - 0.7\sin(t)\kappa)^T$ *and* $J = (0.9\cos(t) - 0.8\sin(t)\imath + 0.8\sin(t)\jmath + 0.7\sin(t)\kappa, 1.1\cos(t) + 0.6\sin(t)\imath - 0.5\sin(t)\jmath - 0.2\sin(t)\kappa)^T$ *be the time-varying external inputs of the FCQVNNs (34) and (35), Then, the error system between drive-response*

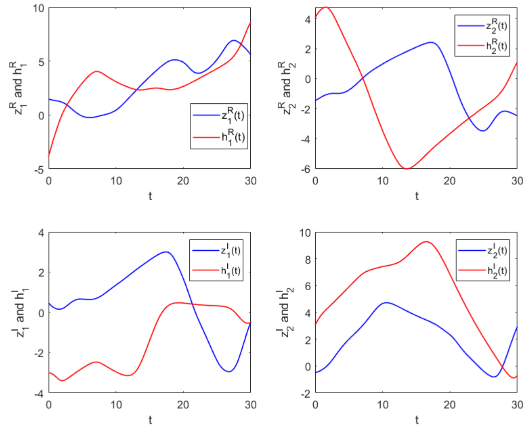

FIGURE 3. Transient states for case 3 of the first and second part of drive-response systems (34) and (35) without control.

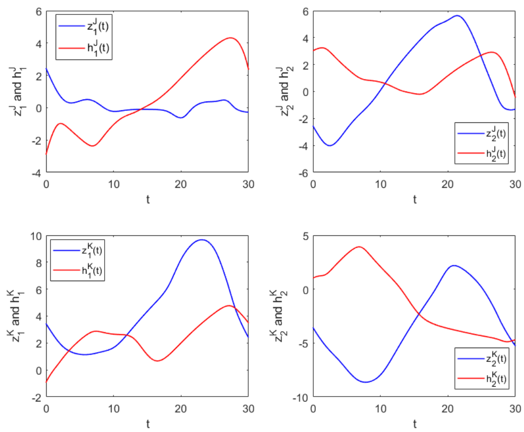

FIGURE 4. Transient states for case 3 of the third and fourth part of drive-response systems (34) and (35) without control.

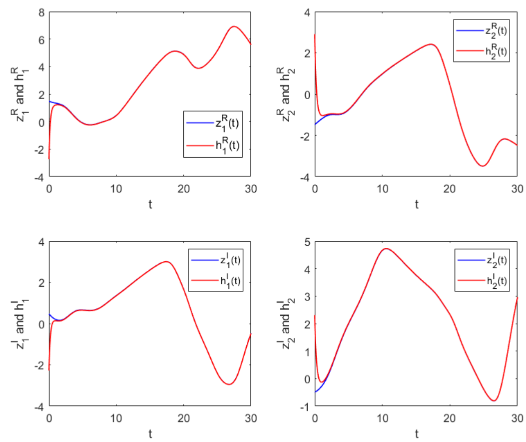

FIGURE 5. Transient states for case 3 of the first and second part of drive-response systems (34) and (35) with control.

systems (34) and (35) are shown as:

$$D^{\alpha}e(t) = -(C + d)e(t) + B\delta(e(t)) + (J - L), \qquad (37)$$

where $e(t) = h(t) - z(t)$, $\delta(e(t)) = f(e(t) + z(t)) - f(z(t))$. *By calculation, Theorem 1 applies. Under Theorem (5),*

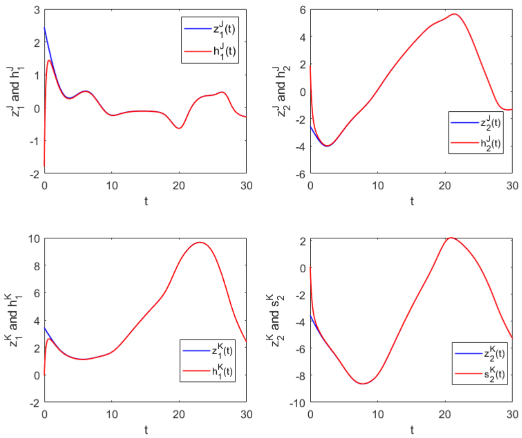

FIGURE 6. Transient states for case 3 of the third and forth part of drive-response systems (34) and (35) with control.

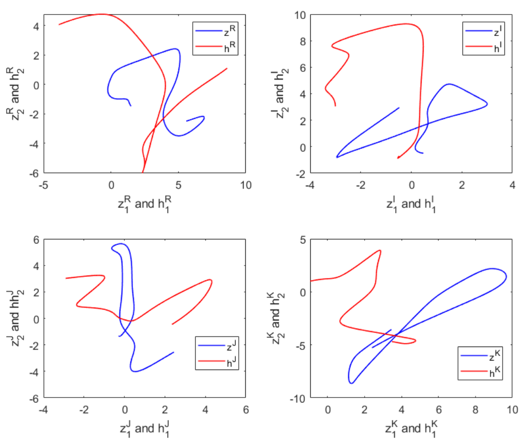

FIGURE 7. Phase plot for case 3 of drive-response systems (34) and (35) without control.

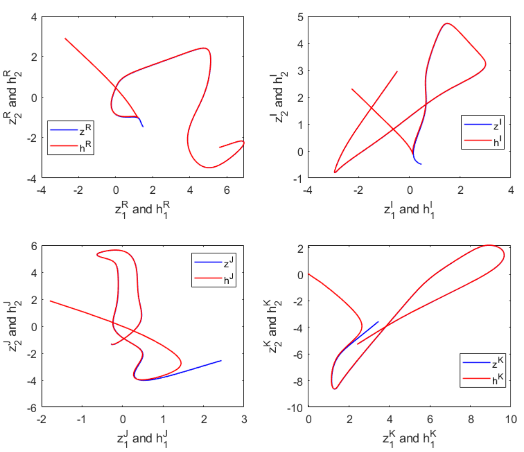

FIGURE 8. Phase plot for case 3 of drive-response systems (34) and (35) with control.

the quasi-synchronization between the drive-response systems (34) and (35) will be achieved with error bound $\psi = 4.01$, *where* $\frac{G}{k} = 4$, $\epsilon = 0.01$. *Then, there exists a* $T \geq t_0$ *such*

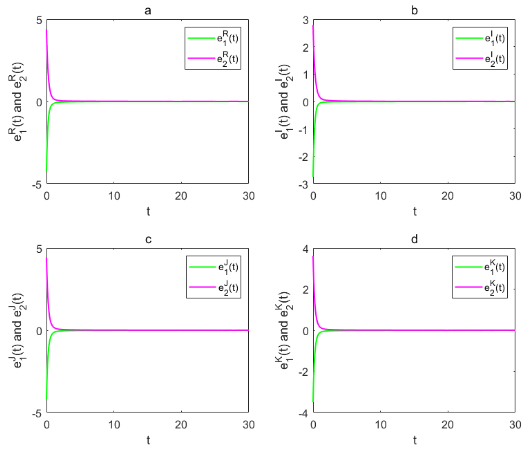

FIGURE 9. Transient states for case 3 of the four parts of the synchronization error system between (34) and (35).

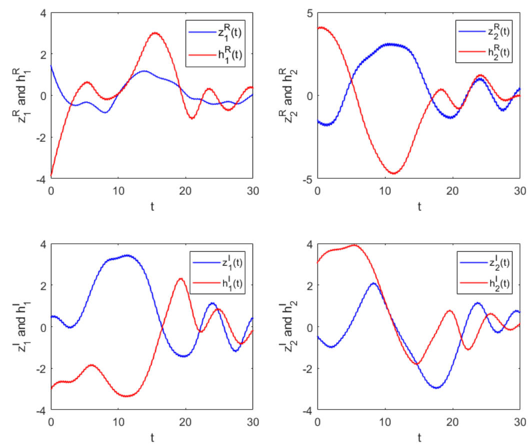

FIGURE 10. Transient states for case 4 of the first and second part of drive-response systems (34) and (35) without control.

that $|e_i(t)| \leq \|e_i(t)\| \leq 4.01$ *for all* $t \geq T$, *the initial values for the drive system (34) and response system (35) are* $Z_{10} = 1.5 + 0.5\imath + 2.5\jmath + 3.5\kappa$, $Z_{20} = -1.5 - 0.5\imath - 2.5\jmath - 3.5\kappa$ *and* $h_{10} = -3.0 - 2.5\imath - 2.5\jmath - 1.0\kappa$, $h_{20} = 5.0 + 2.0\imath + 2.0\jmath + 1.5\kappa$. *Then, the trajectory diagrams of the four state variable parts of the time responses for systems (34) and (35) with and without control are described in Figs.10-13. Fig.14 and Fig.15 describe the phase plot for the FCQVNNs (34) and (35) without and with control. Moreover, the states trajectory of four parts for the synchronization error system between the drive-response systems (34) and (35) will be shown in Fig.16.*

Remark 11: In fact, due to noise disturbance, the external input of fractional-order commutative quaternion-valued neural networks may be time-varying in practice, then the dynamic behaviors of the fractional-order commutative quaternion-valued neural networks have time-varying external input disturbance will be analyzed. Therefore, in the numerical examples, several cases including the time-varying external inputs and the constant external inputs are considered in this paper.

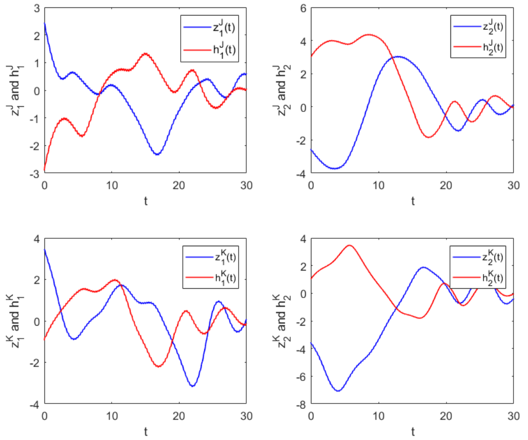

FIGURE 11. Transient states for case 3 of the third and fourth part of the synchronization error system between (34) and (35).

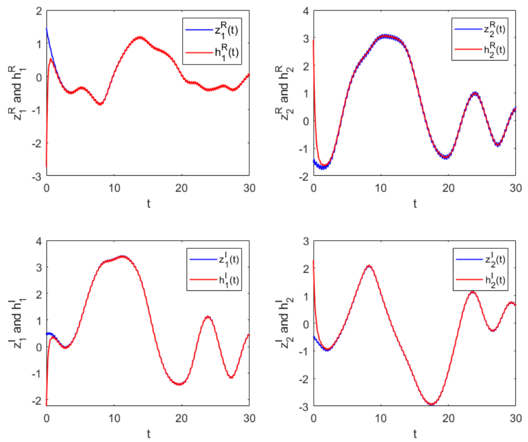

FIGURE 12. Transient states for case 4 of the first and second part of drive-response systems (34) and (35) with control.

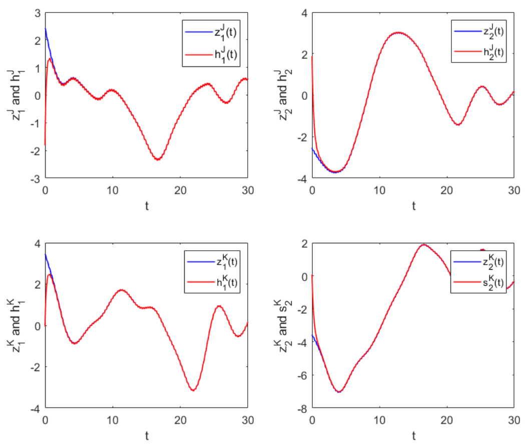

FIGURE 13. Transient states for case 4 of the third and fourth part of drive-response systems (34) and (35) with control.

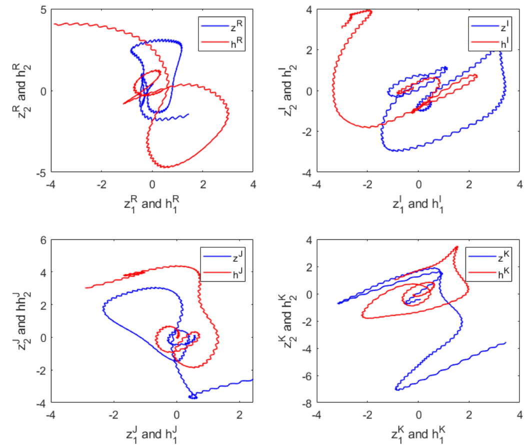

FIGURE 14. Phase plot for case 4 of drive-response systems (34) and (35) without control.

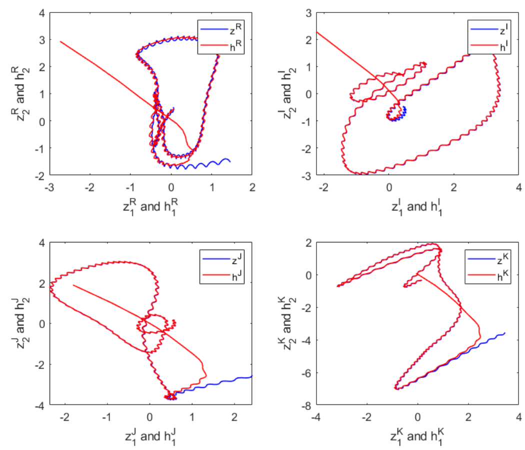

FIGURE 15. Phase plot for case 4 of drive-response systems (34) and (35) with control.

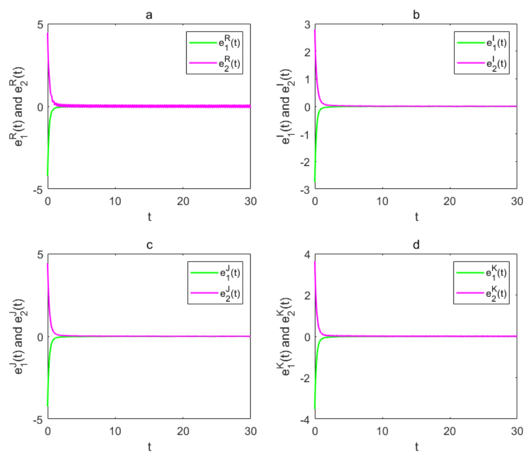

FIGURE 16. Transient states for case 4 of the four parts of the synchronization error system between (34) and (35).

Remark 12: In this paper, the calculation results are mainly based on Theorems 1-5 and the calculation process is simple. Although the fractional-order commutative quaternion-valued neural networks model uses commutative quaternion values, the stability and boundedness criteria expressed in the forms (20), (23), and (29) in the calculations *are real values. For the FCQVNNs with different external inputs, we constructed a Lyapunov function (1 norm of z) that does not satisfy the traditional fractional-order Lyapunov direct method, and designed different numerical experiments to prove the validity and feasibility of the results. Therefore, the results calculated in the article are easy to implement and calculate.*

V. CONCLUSION

In this paper, several dynamic behaviors are studied, including the global Mittag-Leffler stability, the bounded stability and some types of synchronizations, for FCQVNNs based on direct method of Lyapunov. Then, we achieved the condition of FCQVNNs for Mittag-Leffler and the bounded stability by Lyapunov direct method and commutative quaternion theory. On this basis, the synchronization problems of FCQVNNs are studied, including complete synchronization and quasi-synchronization, and good results are obtained. Finally, a few examples illustrate the validity of the theory. Moreover, based on the advantages of Lyapunov direct method, the stability and synchronization of FCQVNNs can be studied more in-depth, and our further works will research the application of FCQVNNs in complex signal processing and image processing.

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