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Exponential Synchronization of Partially Coupled Heterogeneous Networks With Time-Delays and Heterogeneous Impulses

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ABSTRACT This work focuses on exponential synchronization for a class of partially coupled heterogeneous networks with time-delays and heterogeneous impulses. The synchronization targets are selected as the common equilibrium solution and the average trajectory, respectively. Some synchronization criteria are deduced by using Lyapunov function and comparison principle.

INDEX TERMS Exponential synchronization, heterogeneous networks, partially coupled, comparison principle, impulse.

I. INTRODUCTION

As an important mathematical model to describe the real world, complex dynamic networks have aroused strong research interests from scholars at home and abroad in recent years. Research on the cooperative collective behavior of complex networks is a mainstream direction of current complex network research. Cooperative collective behavior of networks mainly includes consistency [1], [2], stability [3]–[5] and synchronization [6], [7].

For a long time, research of complex networks has mainly focused on homogeneous networks that consist of a single type of objects and links, and the state of each network node follows the same evolutionary laws. Homogeneous networks cannot completely reveal the differences in individual state evolution and the corresponding changes in network couplings. Conversely, heterogeneous network models have advantages in revealing these differences. Therefore, the study of heterogeneous complex networks has great practical importance and application values.

When many individuals with different state evolution laws are coupled together to form a self-organizing and collaborative heterogeneous network, their cooperative collective behavior has great uncertainty. In the process of information transmission, a time-delay phenomenon inevitably occurs due to the influence of factors such as spatial distance, limited speed of information transmission, node competition, channel congestion and memory effect. The existence of time delay causes instability and oscillation of dynamic systems. Network nodes exchange information through multiple channels, which may be different for different nodes [8]. Therefore, studying the cooperative collective behavior of partially coupled heterogeneous networks with time-delay effects is important. Research on synchronization is widely investigated.

Synchronization refers to the process that all network nodes starting from a certain initial state reach the same state through the dynamic evolution of nodes own state, the coupling between the nodes, and the external control operations on the network. Only a few networks can achieve synchronization by adjusting system parameters, whereas most of the networks need to use control strategies to realize synchronization in accordence with the specific characteristics of the network [9]–[11]. For heterogeneous networks, the difference in node evolution makes it difficult to achieve full synchronization by virtue of static linear controllers. Some new control strategies have been proposed to realize synchronization. Reference [12] used a state-feedback control strategy to achieve synchronization by adding constraints or controllers to each node for compensating the differences between nodes. Some new synchronization concepts were introduced to replace the full synchronization.

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References [13]–[15] proposed the concept of output synchronization. Cooperative control was added to achieve synchronization by using the output information of neighboring nodes. References [16] and [17] introduced the concept of quasi-synchronization that enables the network to achieve synchronization within a certain error bound.

Compared with continuous control strategies, impulsive control has obvious advantages because it only needs to design the controller in sparse time sequence, and has been successfully applied to many synchronization problems [12], [18]–[22]. In recent years, research on the synchronization of heterogeneous networks based on impulsive strategy has attracted considerable interests. He *et al.* analyzed the synchronization problem of master-slave heterogeneous network on the basis of distributed impulsive control [23], and presented a method to optimize the synchronization error bound and controller design. Reference [24] discussed the synchronization of heterogeneous networks with impulsive effects and coupling delays. This process compensates for the differences between nodes by adding additional constraints and control to each node.

This paper studies the synchronization of a class of heterogeneous networks. Different from the existing results that only consider time delays [12] or only consider partial coupling [8], [25], the network model in this paper simultaneously considers the effects of time delays and partial coupling on the cooperative collective behavior of heterogeneous networks. Under the action of a class of heterogeneous impulses that depend on time and node states, the heterogeneous network achieves exponential synchronization. The main results of this study are twofold. First, the exponential synchronization of the partially coupled time-delay network with heterogeneous impulses is proven by means of the comparison theorem and Lyapunov function if the nodes of the heterogeneous network have a common equilibrium state; Second, the partially coupled time-delay network with heterogeneous impulses exponentially synchronizes to the average state of the nodes if they do not have a common equilibrium state.

Notations: Throughout this paper, $\mathcal{N}$ is the set of positive integers, I_n is the $n \times n$ identity matrix, $\mathcal{R}^n$ is the $n-$dimensional column vector space, $\mathcal{R}^{n\times n}$ is the set of all $n \times n$ matrices. For $A \in \mathcal{R}^{n\times n}$, let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ be the minimum eigenvalue and maximum eigenvalue of A, respectively. The norm of A is denoted by $\|A\|$, $\|A\| = \sqrt{\lambda_{\max}(A^TA)}$, where A^T is the transpose of A. For $A \in \mathcal{R}^{n\times n}$, $A < 0$ implies $x^TAx < 0$ for any $x \in \mathcal{R}^n, x \neq 0$. For two matrices $A = (a_{ij})_{m\times n}$ and B, their Kronecker product $A\otimes B = (a_{ij}B)_{m\times n}$. For a continuously differentiable function $f : \mathcal{R}^n \to \mathcal{R}^n$, Df denotes the Jacobian.

II. MODEL DESCRIPTIONS AND PRELIMINARIES

We consider the following dynamical networks

$$\begin{aligned}\dot{x}_i(t) &= A_ix_i(t) + B_if_1(x_i(t)) + C_if_2(x_i(t-\tau(t))) \\ &\quad + c\sum_{j=1,j\neq i}^{N} d_{ij}RH_{ij}(x_j(t)-x_i(t)), \quad i = 1,\cdots,N, \qquad (1)\end{aligned}$$

where $x_i(t) = [x_{i1}(t),\cdots,x_{in}(t)]^T \in \mathcal{R}^n$ is the state vector of the ith node at time t; $A_i, B_i, C_i \in \mathcal{R}^{n\times n}$ are constant matrices. $\tau(t)$ is the time-varying delay satisfying $0 \le \tau(t) \le \tau$; $f_i(x) = [f_{i1}(x_1),\cdots,f_{in}(x_n)]^T$ for $x = [x_1,\cdots,x_n]^T \in \mathcal{R}^n$, $i = 1, 2$; $c > 0$ is the coupling strength; $R = diag\{r_1,\cdots,r_n\}(r_i > 0)$ is the inner coupling matrix; $D = (d_{ij})_{1\le i,j\le N} \in \mathcal{R}^{N\times N}$ is a symmetric matrix which denotes connection weight of the dynamical networks: $d_{ij} = d_{ji} > 0$ if there is a connection from node j to node i $(i \neq j)$, otherwise $d_{ij} = 0$; $H_{ij} = diag\{h^1_{ij},\cdots,h^n_{ij}\}$ is the channel matrix defined as follows: if there is an information transmission in the kth $(1 \le k \le n)$ channel from node j to node i, then $h^k_{ij} = 1$, otherwise, $h^k_{ij} = 0$.

Let $\Gamma_{ij} = diag\{\gamma^1_{ij},\cdots,\gamma^n_{ij}\} = d_{ij}H_{ij}$ for $1 \le i, j \le N, i \neq j$, i.e., $\gamma^k_{ij} = d_{ij}h^k_{ij}, k = 1,\cdots,n$. Suppose $\Gamma_{ii} = -\sum_{j=1,j\neq i}^{N}\Gamma_{ij}$, $1 \le i \le N$. Then the network (1) can be rewritten as

$$\begin{aligned}\dot{x}_i(t) &= A_ix_i(t) + B_if_1(x_i(t)) + C_if_2(x_i(t-\tau(t))) \\ &\quad + c\sum_{j=1}^{N} R\Gamma_{ij}x_j(t), \quad i = 1,\cdots,N. \qquad (2)\end{aligned}$$

We consider (1) or (2) with the following impulsive effects

$$x_i(t_k^+) = x_i(t_k^-) + \mu_{ik}x_i(t_k^-), \qquad (3)$$

where μ_{i_k} denotes the impulsive strength; $\{t_k\}$ is an impulsive sequence satisfying $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots, \lim t_k = \infty (k \to \infty)$; $x_i(t_k^+)$ and $x_i(t_k^-)$ denote the limit from the left and the right at time t_k, respectively. In this paper, we assume that the solution to (2) is right-hand continuous, then $x_i(t_k^+) = x_i(t_k)$, $i = 1, 2,\cdots,N$ and $k \in \mathcal{N}$.

Remark 1: (2) is heterogeneous because A_i, B_i, C_i vary with node state x_i. The impulse in (3) is also heterogeneous because μ_{i_k} depends on impulsive time t_k and node state x_i.

We make the following assumptions:

(A1) f_1 and f_2 satisfy the Lipschitz conditions: $\|f_i(x) - f_i(y)\| \le l_i\|x - y\|, \quad i = 1, 2.$

(A2) The impulsive sequence $\{t_k\}$ satisfies that there exists $N_0 \in \mathcal{Z}^+$ and $T_a > 0$ such that

$$\frac{T-t}{T_a} - N_0 \le N_\zeta(T,t) \le \frac{T-t}{T_a} + N_0$$

for any $t_0 \le t \le T$, where $N_\zeta(T,t)$ is the times of impulses in the interval $[t, T]$.

Definition 1: The upper-right Dini derivative $D^+u(t)$ is defined as

$$D^+u(t) = \overline{\lim}_{h\to 0^+}(u(t+h) - u(t))/h.$$

Lemma 1 [9]: Suppose that $0 \le \tau_i(t) \le \tau$, $i = 1, 2,\cdots,m$, $F(t, u, \bar{u}, \bar{u}_1, \bar{u}_2,\cdots,\bar{u}_m) : \mathcal{R}^+ \times \mathcal{R}^{m+1} \to \mathcal{R}$ is nondecreasing in $\bar{u}_i$ for each fixed $(t, u, \bar{u}_1,\cdots,\bar{u}_{i-1}, \bar{u}_{i+1},\cdots,\bar{u}_m)$, $i = 1, 2,\cdots,m$, and $I_k(u) : \mathcal{R} \to \mathcal{R}$ is nondecreasing in u, $k \in \mathcal{N}^+$. If

$$\begin{cases} D^+u(t) \le F(t, u(t), u(t-\tau_1(t)),\cdots,u(t-\tau_m(t))), t \neq t_k, \\ u(t_k^+) \le I_k(u(t_k^-)), k \in \mathcal{N}^+, \end{cases}$$

and

$$\begin{cases} D^+v(t) \geq F(t, v(t), v(t-\tau_1(t)), \cdots, v(t-\tau_m(t))), & t \neq t_k, \\ v(t_k^+) \geq I_k(v(t_k^-)), & k \in \mathcal{N}^+, \end{cases}$$

then $u(t) \leq v(t)$ for $-\tau \leq t \leq 0$ implies that $u(t) \leq v(t)$ for $t \geq 0$.

Lemma 2: Let $Q \in \mathcal{R}^{n\times n}$ be a positive semi-definite matrix, then the following inequality holds

$$2x^T Qy \leq x^T Qx + y^T Qy$$

for $x, y \in \mathcal{R}^n$.

III. SYNCHRONIZATION CRITERIA

This section aims to supply some synchronization criteria for the network (1). Difference is found in achieving synchronization when different synchronization targets are considered, and the network synchronization to the anticipated states often fails. Therefore, the choice of target states is extremely crucial to achieve synchronization, especially for heterogeneous networks. In this paper, we focus on two types of target synchronization states of (1). The first type is the common equilibrium solution, and the second type is the average trajectory.

A. SYNCHRONIZATION TO THE COMMON EQUILIBRIUM SOLUTION

We assume that all the isolated nodes in the network (1) (or (2)) have a common equilibrium solution $s(t)$, that is $s(t)$ satisfies

$$\dot{s}(t) = A_i s(t) + B_i f_1(s(t)) + C_i f_2(s(t-\tau(t))), \quad i = 1, \cdots, N, \tag{4}$$

and the network (1) (or (2)) is anticipated to synchronize to $s(t)$.

Let $e_i(t) = x_i(t) - s(t)$, then the error system is

$$\dot{e}_i(t) = A_i e_i(t) + B_i g_1(e_i(t)) + C_i g_2(e_i(t-\tau(t))) + c\sum_{j=1}^{N} R\Gamma_{ij} e_j(t), \quad i = 1, \cdots, N, \tag{5}$$

where $g_1(e_i(t)) = f_1(x_i(t)) - f_1(s(t)), g_2(e_i(t-\tau(t))) = f_2(x_i(t-\tau(t))) - f_2(s(t-\tau(t)))$.

Considering the effect of heterogeneous impulses (3), we obtain the following network:

$$\begin{cases} \dot{e}_i(t) = A_i e_i(t) + B_i g_1(e_i(t)) + C_i g_2(e_i(t-\tau(t))) \\ \qquad + c\sum_{j=1}^{N} R\Gamma_{ij} e_j(t), \quad t \neq t_k, k \in \mathcal{N}^+ \\ e_i(t_k^+) = e_i(t_k^-) + \mu_{i_k} e_i(t_k^-). \end{cases} \tag{6}$$

The network (6) subject to the initial condition can be rewritten as

$$\begin{cases} \dot{e}(t) = Ae(t) + BG_1(e(t)) + CG_2(e(t-\tau(t))) \\ \qquad +He(t), \quad t \neq t_k, k \in \mathcal{N}^+, \\ e(t_k^+) = U_k e(t_k^-), \\ e(t) = \Phi(t), \qquad t \in [-\tau, 0], \end{cases} \tag{7}$$

where $e(t) = (e_1^T(t), e_2^T(t), \cdots, e_N^T(t))^T$, $A = diag\{A_1, A_2, \cdots, A_N\}, B = diag\{B_1, B_2, \cdots, B_N\}, C = diag\{C_1, C_2, \cdots, C_N\}$, $G_1(e(t)) = (g_1^T(e_1(t)), g_1^T(e_2(t)), \cdots, g_1^T(e_N(t)))^T$, $G_2(e(t-\tau(t))) = (g_2^T(e_1(t-\tau(t))), g_2^T(e_2(t-\tau(t))), \cdots, g_2^T(e_N(t-\tau(t))))^T$, $\Gamma = (\Gamma_{ij})_{N\times N}$, $H = c(I_N \otimes R)\Gamma$, $U_k = diag\{1+\mu_{1_k}, 1+\mu_{2_k}, \cdots, 1+\mu_{N_k}\}\otimes I_n$, $\Phi(t) \in PRC([-\tau, 0], \mathcal{R}^{nN}) = \{\varphi : [-\tau, 0] \to \mathcal{R}^{nN} | \varphi$ is piecewise right continuous$\}$, endowed with norm $\|\cdot\|_\tau : \|\varphi\|_\tau = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$.

Theorem 1: Suppose that (A_1) and (A_2) hold, there exist a positive definite matrix-valued function $P(t)$ and positive numbers $\bar{\lambda}$, α, β and $\mu \in (0, 1)$ such that

$$\beta + \mu^{N_0+1}(\alpha + \frac{\ln \mu}{T_a}) < 0. \tag{8}$$

For $t \in [t_k, t_{k+1}), k \in \mathcal{N}$,

$$P(t) \leq \bar{\lambda} I_{nN}, \tag{9}$$

$$P(t)A + A^T P(t) + P(t)H + H^T P(t) + (2-\alpha)P(t) + \bar{\lambda} l_1^2 \|B\|^2 I_{nN} + \dot{P}(t) \leq 0, \tag{10}$$

and

$$\bar{\lambda} l_2^2 \|C\|^2 I_{nN} - \beta P(t-\tau(t)) \leq 0. \tag{11}$$

For $t = t_k, k \in \mathcal{N}$,

$$U_k^T P(t_k) U_k < \mu P(t_k^-), \tag{12}$$

then (1) exponentially synchronizes to $s(t)$.

Proof: Consider the following Lyapunov function

$$V(t) = e^T(t)P(t)e(t).$$

Taking the derivative of $V(t)$ along the trajectories of (7), we obtain

$$\begin{aligned} D^+V(t) = {} & e^T[A^T P(t) + P(t)A + H^T P(t) + P(t)H]e(t) \\ & + G_1^T(e(t))B^T P(t)e(t) + e^T(t)P(t)BG_1(e(t)) \\ & + G_2^T(e(t-\tau(t)))C^T P(t)e(t) + e^T(t)P(t)CG_2 \\ & *(e(t-\tau(t))) + e^T \dot{P}(t)e(t), \ t \in [t_k, t_{k+1}). \end{aligned}$$

By Lemma 2 and (A_1),

$$\begin{aligned} & G_1^T(e(t))B^T P(t)e(t) + e^T(t)P(t)BG_1(e(t)) \\ & \quad \leq e^T(t)P(t)e(t) + G_1^T(e(t))B^T P(t)BG_1(e(t)) \\ & \quad \leq e^T(t)P(t)e(t) + \bar{\lambda} l_1^2 \|B\|^2 e^T(t)e(t); \\ & G_2^T(e(t-\tau(t)))C^T P(t)e(t) \\ & \qquad + e^T(t)P(t)CG_2(e(t-\tau(t))) \\ & \quad \leq e^T(t)P(t)e(t) \\ & \qquad + G_2^T(e(t-\tau(t)))C^T P(t)CG_2(e(t-\tau(t))) \\ & \quad \leq e^T(t)P(t)e(t) \\ & \qquad + \bar{\lambda} l_2^2 \|C\|^2 e^T(t-\tau(t))e(t-\tau(t)). \end{aligned}$$

Then, we obtain

$$\begin{aligned} D^+V(t) = {} & e^T[P(t)A + A^T P(t) + P(t)H + H^T P(t) \\ & + (2-\alpha)P(t) + \bar{\lambda} l_1^2 \|B\|^2 I_{nN} + \dot{P}(t)]e(t) \end{aligned}$$

$$\begin{aligned}&+e^T(t-\tau(t))[\bar{\lambda}l_2^2\|C\|^2I_{nN}-\beta P(t-\tau(t))]\\&*e(t-\tau(t))+\alpha V(t)+\beta V(t-\tau(t))\\&\leq \alpha V(t)+\beta V(t-\tau(t)),\quad t\in[t_k,t_{k+1}).\end{aligned}$$

By (30), we have

$$\begin{aligned}V(t_k)&=e^T(t_k)P(t_k)e(t_k)=e^T(t_k^-)U_k^TP(t_k)U_ke(t_k^-)\\&<\mu e^T(t_k^-)P(t_k^-)e(t_k^-)=\mu V(t_k^-).\end{aligned}\tag{13}$$

For any $\epsilon>0$, let $v(t)$ be the unique solution to the following comparing system

$$\begin{cases}\dot{v}(t)=\alpha v(t)+\beta v(t-\tau(t))+\epsilon, & t\neq t_k,\\ v(t_k^+)=\mu v(t_k^-), & k\in\mathcal{N}^+\\ v(t)=V(t), & t\in[-\tau,0],\end{cases}\tag{14}$$

By (13),(13) and (14), using Lemma1, we obtain

$$V(t)\leq v(t)\quad for\ t\geq 0.\tag{15}$$

In the following, we estimate the solution of (14). We start from solving the linear system

$$\begin{cases}\dot{v}(t)=\alpha v(t), & t\neq t_k,\\ v(t_k^+)=\mu v(t_k^-), & k\in\mathcal{N}^+\\ v(t)=V(t), & t\in[-\tau,0].\end{cases}\tag{16}$$

The solution to (16) is

$$v(t)=V(0)\mu^m e^{\alpha t},\quad t\in[t_m,t_{m+1}),m\in\mathcal{N}.$$

Using the idea of variation of parameters, we suppose that (14) has a solution of the form

$$v(t)=C(t)\mu^m e^{\alpha t},\quad t\in[t_m,t_{m+1}),m\in\mathcal{N},$$

where $C(t)$ is undetermined.
Substituting $v(t)$ into (14), we obtain

$$C'(t)=[\beta v(t-\tau(t))+\epsilon]e^{-\alpha t}\mu^{-m},\ t\in[t_m,t_{m+1}),m\in\mathcal{N}.$$

Therefore, for any $m\in\mathcal{N}$, and $t\in[t_m,t_{m+1})$, we have

$$\begin{aligned}C(t)-C(0)&=C(t)-C(t_m)+\sum_{k=1}^{m}[C(t_k)-C(t_{k-1})]\\&=\int_{t_m}^{t}[\beta v(s-\tau(s))+\epsilon]e^{-\alpha s}\mu^{-m}ds\\&\quad+\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}[\beta v(s-\tau(s))+\epsilon]e^{-\alpha s}\mu^{-k+1}ds\\&\doteq\int_0^t[\beta v(s-\tau(s))+\epsilon]e^{-\alpha s}\mu_s^{-k}ds,\end{aligned}$$

where $\mu_s^{-k}=\mu^{-k}$ when $s\in[t_k,t_{k+1}),0\leq k\leq m$.

Hence, for any $m\in\mathcal{N}$, and $t\in[t_m,t_{m+1})$,

$$v(t)=C(0)\mu^m e^{\alpha t}+\int_0^t[\beta v(s-\tau(s))+\epsilon]*e^{\alpha(t-s)}\mu_s^{m-k}ds,$$

where $\mu_s^{m-k}=\mu^{m-k}$ when $s\in[t_k,t_{k+1}),0\leq k\leq m$. $C(0)=V(0)=\Phi(0)^TP(0)\Phi(0)$.

By (A_2), for $t\in[t_m,t_{m+1})$ and $s\in[t_k,t_{k+1}),0\leq k\leq m$,

$$\frac{t-s}{T_a}-N_0\leq m-k+1,$$

and then

$$\mu_s^{m-k}\leq\mu^{\frac{t-s}{T_a}-N_0-1}=\frac{1}{\mu^{N_0+1}}e^{\frac{\ln\mu}{T_a}(t-s)},\tag{17}$$

$$\mu^m\leq\frac{1}{\mu^{N_0+1}}e^{\frac{\ln\mu}{T_a}t}.\tag{18}$$

Let $\eta=-(\alpha+\frac{\ln\mu}{T_a})$, $\xi=\frac{\bar{\lambda}}{\mu^{N_0+1}}\sup_{-\tau\leq t\leq 0}\Phi(t)^T\Phi(t)$, $\mu_0=\mu^{N_0+1}$. Substituting (17) and (18) into (17), we obtain

$$v(t)<\xi e^{-\eta t}+\int_0^t\frac{e^{-\eta(t-s)}}{\mu_0}[\beta v(s-\tau(s))+\epsilon]ds.\tag{19}$$

Let $\phi(x)=\beta e^{\tau x}-\mu_0(\eta-x)$, then $\phi'(x)>0$. By (8), $\phi(0)=\beta-\mu_0\eta=\beta+\mu^{N_0+1}(\alpha+\frac{\ln\mu}{T_a})<0$. Obviously, $\phi(+\infty)=+\infty$.

Therefore, there exists a unique λ such that $\phi(\lambda)=0$, which derives

$$\beta e^{\lambda\tau}=\mu_0(\eta-\lambda).\tag{20}$$

Since $\mu_0\eta-\beta>0$, we have

$$v(t)\leq\xi<\xi e^{-\lambda t}+\frac{\epsilon}{\mu_0\eta-\beta},\quad t\in[-\tau,0].\tag{21}$$

We claim that (21) holds for all $t\geq 0$.

If not, there exists a $t^*>0$ such that

$$v(t^*)\geq\xi e^{-\lambda t^*}+\frac{\epsilon}{\mu_0\eta-\beta},\tag{22}$$

and

$$v(t)<\xi e^{-\lambda t}+\frac{\epsilon}{\mu_0\eta-\beta},\quad t<t^*.\tag{23}$$

By(19) and (23), we obtain

$$\begin{aligned}v(t^*)&<\xi e^{-\eta t^*}+\int_0^{t^*}\frac{e^{-\eta(t^*-s)}}{\mu_0}[\beta v(s-\tau(s))+\epsilon]ds\\&<e^{-\eta t^*}\{\xi+\frac{\epsilon}{\mu_0\eta-\beta}+\frac{\beta\xi e^{\lambda\tau}}{\mu_0}\int_0^{t^*}e^{(\eta-\lambda)s}ds\\&\quad+\frac{\eta\epsilon}{\mu_0\eta-\beta}\int_0^{t^*}e^{\eta s}ds\}\\&=e^{-\eta t^*}\{\xi+\frac{\epsilon}{\mu_0\eta-\beta}+\frac{\beta\xi e^{\lambda\tau}}{\mu_0(\eta-\lambda)}(e^{(\eta-\lambda)t^*}-1)\\&\quad+\frac{\epsilon}{\mu_0\eta-\beta}(e^{\eta t^*}-1)\}\\&=e^{-\eta t^*}\{\xi e^{\eta t^*}e^{-\lambda t^*}+\frac{\epsilon}{\mu_0\eta-\beta}e^{\eta t^*}\}\\&=\xi e^{-\lambda t^*}+\frac{\epsilon}{\mu_0\eta-\beta}.\end{aligned}$$

This contracicts (22), and then (21) holds for all $t\geq 0$.

Letting $\epsilon\to 0$, we obtain from (15) and (21) that

$$V(t)\leq v(t)\leq\xi e^{-\lambda t}.$$

Therefore, (1) exponentially synchronizes to the common equilibrium solution.

Remark 2: The positive definite matrix $P(t)$ is used to design the Lyapunov function $V(t)$. In [24], a linear matrix-valued function $P(t)$ connecting two fixed matrices in each impulsive interval is designed to construct the Lyapunov function. Compared with the synchronization criteria in [24], Theorem 1 in this work has a wider application.

In Theorem 1, if $P(t)$ is constant P, then we can easily obtain the following result.

Corollary 1: Suppose that (A_1) and (A_2) hold, there exist a positive definite matrix P and positive numbers $\bar{\lambda}$, α, β and $\mu \in (0, 1)$ such that (9) holds and

$$\frac{\bar{\lambda}}{\beta} l_2^2 \|C\|^2 \leq \lambda_{\min}(P) \leq \lambda_{\max}(P) \leq \bar{\lambda}, \tag{24}$$

$$P(A+H) + [A^T + H^T + (2-\alpha) I_{nN}]P + \bar{\lambda} l_1^2 \|B\|^2 I_{nN} \leq 0, \tag{25}$$

and

$$U_k^T P U_k < \mu P, \tag{26}$$

then (1) exponentially synchronizes to $s(t)$.

Remark 3: In Corollary 1, inequality $U_k^T P U_k < \mu P$ implies that the impulses in (1) are synchronizing up to $|1 + \mu_k| < 1$. Theorem 1 can be applied to investigate the synchronization problem of dynamical networks when the impulses are synchronizing, desynchronizing ($|1 + \mu_k| > 1$) or inactive ($|1 + \mu_k| = 1$). If the impulses are inactive or desynchronizing, then $P(t)$ is used to offset the negative impulsive effects, as shown in (12).

Remark 4: The impulsive intervals are usually assumed to be bounded when the synchronization problem with impulsive effects is considered, especially with inactive or desynchronizing impulsive effects. This assumption limits the frequency of these negative impulses in a fixed time period. This work uses the average impulsive interval to limit the impulsive interval bounds. Thus, the results in this work are less conservative.

B. SYNCHRONIZATION TO THE AVERAGE TRAJECTORY

In this subsection, we consider the following average state

$$s(t) = \frac{1}{N} \sum_{k=1}^{N} x_k(t).$$

Let $e_i(t) = x_i(t) - s(t)$. Obviously, $\sum_{i=1}^{N} e_i(t) = 0$. Since

$$\begin{aligned} \dot{s}(t) &= \frac{1}{N} \sum_{k=1}^{N} [A_k x_k(t) + B_k f_1(x_k(t)) \\ &\quad + C_k f_2(x_k(t-\tau(t))) + c \sum_{j=1}^{N} R\Gamma_{kj} x_j(t)] \\ &= \frac{1}{N} \sum_{k=1}^{N} [A_k(s(t)+e_k(t)) + B_k f_1(s(t)+e_k(t)) \\ &\quad + C_k f_2(s(t-\tau(t)) + e_k(t-\tau(t)))], \end{aligned}$$

we obtain

$$\begin{aligned} \dot{e}_i(t) &= A_i(s(t)+e_i(t)) + B_i f_1(s(t)+e_i(t)) \\ &\quad + C_i f_2(s(t-\tau(t)) + e_i(t-\tau(t))) + c \sum_{j=1}^{N} R\Gamma_{ij} e_j(t) \\ &\quad - \frac{1}{N} \sum_{k=1}^{N} [A_k(s(t)+e_k(t)) + B_k f_1(s(t)+e_k(t)) \\ &\quad + C_k f_2(s(t-\tau(t)) + e_k(t-\tau(t)))] \\ &= A_i(s(t)+e_i(t)) \\ &\quad + B_i(\int_0^1 Df_1(s(t)+\omega e_i(t)) d\omega) e_i(t) \\ &\quad + B_i f_1(s(t)) + C_i(\int_0^1 Df_2(s(t-\tau(t))) \\ &\quad + \omega e_i(t-\tau(t))) d\omega) e_i(t-\tau(t)) + C_i f_2(s(t-\tau(t))) \\ &\quad + c \sum_{j=1}^{N} R\Gamma_{ij} e_j(t) - \frac{1}{N} \sum_{k=1}^{N} A_k(s(t)+e_k(t)) \\ &\quad - \frac{1}{N} \sum_{k=1}^{N} B_k(\int_0^1 Df_1(s(t)+\omega e_k(t)) d\omega) e_k(t) \\ &\quad - \frac{1}{N} \sum_{k=1}^{N} B_k f_1(s(t)) - \frac{1}{N} \sum_{k=1}^{N} C_k(\int_0^1 Df_2(s(t-\tau(t)) \\ &\quad + \omega e_k(t-\tau(t))) d\omega) e_k(t-\tau(t)) \\ &\quad - \frac{1}{N} \sum_{k=1}^{N} C_k f_2(s(t-\tau(t))). \end{aligned}$$

Then

$$\begin{aligned} \dot{e}(t) &= He(t) + diag\{A_1 + B_1 \int_0^1 Df_1(s(t)+\omega e_1(t) d\omega, \\ &\quad \cdots, A_N + B_N \int_0^1 Df_1(s(t)+\omega e_N(t)) d\omega\} e(t) \\ &\quad + diag\{C_1 \int_0^1 Df_2(s(t-\tau(t)) + \omega e_1(t-\tau(t))) d\omega, \\ &\quad \cdots, C_N \int_0^1 Df_2(s(t-\tau(t)) + \omega e_N(t-\tau(t)) d\omega)\} \\ &\quad * e(t-\tau(t)) \\ &\quad - \frac{1}{N} \begin{bmatrix} \wedge_1, & \cdots & , \wedge_N \\ & \cdots\cdots & \\ \wedge_1, & \cdots & , \wedge_N \end{bmatrix} e(t) \\ &\quad - \frac{1}{N} \begin{bmatrix} \vee_1, & \cdots & , \vee_N \\ & \cdots\cdots & \\ \vee_1, & \cdots & , \vee_N \end{bmatrix} e(t-\tau(t)) \\ &\quad + \begin{bmatrix} \Delta_1 \\ \vdots \\ \Delta_N \end{bmatrix} \\ &\quad + \begin{bmatrix} \nabla_1 \\ \vdots \\ \nabla_N, \end{bmatrix} \end{aligned}$$

where

$$\bigwedge_i = A_i + B_i \int_0^1 Df_1(s(t) + \omega e_i(t)d\omega,$$

$$\bigvee_i = C_i \int_0^1 Df_2(s(t-\tau(t)) + \omega e_i(t-\tau(t)))d\omega,$$

$$\Delta_i = (A_i - \frac{1}{N}\sum_{k=1}^{N} A_k)s(t) + (B_i - \frac{1}{N}\sum_{k=1}^{N} B_k)f_1(s(t)),$$

$$\nabla_i = = (C_i - \frac{1}{N}\sum_{k=1}^{N} C_k)f_2(s(t-\tau(t))),$$

$$i = 1, 2, \cdots, N.$$

Let

$$\Theta(t) = diag\{A_1 + B_1 \int_0^1 Df_1(s(t) + \omega e_1(t)d\omega, \cdots, A_N + B_N \int_0^1 Df_1(s(t) + \omega e_N(t))d\omega\},$$

$$\Xi(t-\tau(t)) = diag\{C_1 \int_0^1 Df_2(s(t-\tau(t)) + \omega e_1(t-\tau(t)))d\omega, \cdots, C_N \int_0^1 Df_2(s(t-\tau(t)) + \omega e_N(t-\tau(t))d\omega)\},$$

$$\Pi(t) = \frac{1}{N}\begin{bmatrix} \bigwedge_1, & \cdots & , \bigwedge_N \\ & \cdots\cdots & \\ \bigwedge_1, & \cdots & , \bigwedge_N \end{bmatrix},$$

$$\Upsilon(t-\tau(t)) = \frac{1}{N}\begin{bmatrix} \bigvee_1, & \cdots & , \bigvee_N \\ & \cdots\cdots & \\ \bigvee_1, & \cdots & , \bigvee_N \end{bmatrix},$$

$$\Psi(t) = \begin{bmatrix} \Delta_1 \\ \vdots \\ \Delta_N \end{bmatrix}$$

and

$$\Omega(t-\tau(t)) = \begin{bmatrix} \nabla_1 \\ \vdots \\ \nabla_N, \end{bmatrix},$$

then the error system with initial condition reads as

$$\begin{cases} \dot{e}(t) = He(t) + \Theta(t)e(t) + \Xi(t-\tau(t))e(t-\tau(t)) - \Pi(t)e(t) \\ \quad -\Upsilon(t-\tau(t))e(t-\tau(t)) + \Psi(t) + \Omega(t-\tau(t)), \\ e(t_k^+) = U_k e(t_k^-). \end{cases}$$

Theorem 2: Suppose that (A_1) and (A_2) hold, there exists a positive definite matrix $P(t)$, positive numbers $\bar{\lambda}$, α, β and $\mu \in (0, 1)$ such that

$$2\beta + \mu^{N_0+1}(\alpha + \frac{\ln \mu}{T_a}) < 0, \tag{27}$$

and for $t \in [t_k, t_{k+1})$, $k \in \mathcal{N}$,

$$P(t) \le \bar{\lambda} I_{nN}, \tag{28}$$

$$\begin{aligned} &H^T P(t) + P(t)H + \Theta^T(t)P(t) + P(t)\Theta(t) \\ &-\Pi^T(t)P(t) - P(t)\Pi(t) + (4-\alpha)P(t) + \dot{P}(t) \le 0, \\ &\Xi^T(t-\tau(t))P(t)\Xi(t-\tau(t)) \\ &+\Upsilon^T(t-\tau(t))P(t)\Upsilon(t-\tau(t)) \\ &\le \beta P(t-\tau(t)). \end{aligned} \tag{29}$$

Let $\eta = -(\alpha + \frac{\ln \mu}{T_a})$, $\xi = \frac{\bar{\lambda}}{\mu^{N_0+1}} \sup_{-\tau \le t \le 0} \Phi^T(t)\Phi(t)$ and λ be the unique solution to $2\beta e^{\tau x} - \mu^{N_0+1}(\eta - x) = 0$, satisfy

$$\Psi^T(t)P(t)\Psi(t) + \Omega^T(t-\tau(t))P(t)\Omega(t-\tau(t)) \le \beta \xi e^{-\lambda(t-\tau(t))}.$$

For $t = t_k$, $k \in \mathcal{N}$,

$$U_k^T P(t_k) U_k < \mu P(t_k^-), \tag{30}$$

then (1) synchronizes to the average state $s(t) = \frac{1}{N}\sum_{k=1}^{N} x_k(t)$.

Proof: Let $V(t) = e^T(t)P(t)e(t)$. Then

$$\begin{aligned} D^+V(t) = {} & e^T(t)[H^T P(t) + P(t)H + \Theta^T(t)P(t) \\ &+ P(t)\Theta(t) - \Pi^T(t)P(t) - P(t)\Pi(t) + \dot{P}(t)]e(t) \\ &+ e^T(t-\tau(t))\Xi^T(t-\tau(t))P(t)e(t) \\ &+ e^T(t)P(t)\Xi(t-\tau(t))e(t-\tau(t)) \\ &- e^T(t-\tau(t))\Upsilon^T(t-\tau(t))P(t)e(t) \\ &- e^T(t)P(t)\Upsilon(t-\tau(t))e(t-\tau(t)) \\ &+ \Psi^T(t)P(t)e(t) + e^T(t)P(t)\Psi(t) \\ &+ \Omega^T(t-\tau(t))P(t)e(t) + e^T(t)P(t)\Omega(t-\tau(t)). \end{aligned}$$

By Lemma 2,

$$\begin{aligned} & e^T(t-\tau(t))\Xi^T(t-\tau(t))P(t)e(t) \\ &\quad + e^T(t)P(t)\Xi(-\tau(t)t)e(t-\tau(t) \\ &\le e^T(t)P(t)e(t) \\ &\quad + e^T(t-\tau(t))\Xi^T(t-\tau(t))P(t)\Xi(t-\tau(t))e(t-\tau(t)), \\ & e^T(t-\tau(t))\Upsilon^T(t-\tau(t))P(t)e(t) \\ &\quad - e^T(t)P(t)\Upsilon(t-\tau(t))e(t-\tau(t)) \\ &\le e^T(t)P(t)e(t) \\ &\quad + e^T(t-\tau(t))\Upsilon^T(t-\tau(t))P(t)\Upsilon(t-\tau(t))e(t-\tau(t)), \\ & \Psi^T(t)P(t)e(t) + e^T(t)P(t)\Psi(t) \\ &\le e^T(t)P(t)e(t) + \Psi^T(t)P(t)\Psi(t), \\ & \Omega^T(t-\tau(t))P(t)e(t) + e^T(t)P(t)\Omega(t-\tau(t)) \\ &\le e^T(t)P(t)e(t) + \Omega^T(t-\tau(t))P(t)\Omega(t-\tau(t)). \end{aligned}$$

Therefore,

$$\begin{aligned} D^+V(t) \le {} & e^T(t)[H^T P(t) + P(t)H + \Theta^T(t)P(t) \\ &+ P(t)\Theta(t) - \Pi^T(t)P(t) - P(t)\Pi(t) \\ &+ \dot{P}(t) + 4P(t)]e(t) \\ &+ e^T(t-\tau(t))[\Xi^T(t-\tau(t))P(t)\Xi(t-\tau(t)) \\ &+ \Upsilon^T(t-\tau(t))P(t)\Upsilon(t-\tau(t))]e(t-\tau(t)) \\ &+ \Psi^T(t)P(t)\Psi(t) \\ &+ \Omega^T(t-\tau(t))P(t)\Omega(t-\tau(t)). \end{aligned}$$

Then, we obtain

$$D^+V(t) \leq \alpha V(t) + \beta V(t-\tau(t)) + \beta\xi e^{\lambda(\tau - t)}. \quad (31)$$

Obviously,

$$V(t_k) < \mu V(t_k^-). \quad (32)$$

Similar to the proof of Theorem 1, we consider the following modified comparing system

$$\begin{cases} \dot{v}(t) = \alpha v(t) + \beta v(t-\tau(t)) \\ \qquad + \beta\xi e^{\lambda(\tau-t)} + \epsilon, & t \neq t_k, \\ v(t_k^+) = \mu v(t_k^-), & k \in \mathcal{N}^+ \\ v(t) = V(t), & t \in [-\tau, 0], \end{cases} \quad (33)$$

By Lemma1,

$$V(t) \leq v(t) \quad \textit{for } t \geq 0. \quad (34)$$

We use the method of variation of parameters to estimate the solution of (33), and suppose that the solution of (33) has the form of

$$v(t) = C(t)\mu^m e^{\alpha t}, \quad t \in [t_m, t_{m+1}), m \in \mathcal{N}.$$

Substituting $v(t)$ into (33), we obtain, for any $m \in \mathcal{N}$, $t \in [t_m, t_{m+1})]$

$$C(t) = C(0) + \int_0^t [\beta v(s-\tau(s)) + \beta\xi e^{\lambda(\tau-t)} + \epsilon] e^{-\alpha s}\mu_s^{-k} ds,$$

where $\mu_s^{-k} = \mu^{-k}$ when $s \in [t_k, t_{k+1})$, $0 \leq k \leq m$.

Then, using the same estimation in the proof of Theorem 1, we have, for any $m \in \mathcal{N}$ and $t \in [t_m, t_{m+1})$,

$$\begin{aligned} v(t) &= C(0)\mu^m e^{\alpha t} \\ &\quad + \int_0^t [\beta v(s-\tau(s)) + \beta\xi e^{\lambda(\tau-s)} + \epsilon] e^{\alpha(t-s)}\mu_s^{m-k} ds, \\ &< \xi e^{-\eta t} \\ &\quad + \int_0^t \frac{e^{-\eta(t-s)}}{\mu_0}[\beta v(s-\tau(s)) + \beta\xi e^{\lambda(\tau-s)} + \epsilon] ds, \end{aligned}$$

where $\mu_s^{m-k} = \mu^{m-k}$ when $s \in [t_k, t_{k+1})$, $0 \leq k \leq m$, $\mu_0 = \mu^{N_0+1}$ and ξ, η and λ are defied as in Theorem 2.

We show that

$$v(t) < \xi e^{-\lambda t} + \frac{\epsilon}{\mu_0\eta - 2\beta} \quad (35)$$

holds for all $t \geq 0$.

If not, there exists a $t^* > 0$ such that

$$v(t^*) \geq \xi e^{-\lambda t^*} + \frac{\epsilon}{\mu_0\eta - 2\beta}, \quad (36)$$

and

$$v(t) < \xi e^{-\lambda t} + \frac{\epsilon}{\mu_0\eta - 2\beta}, \quad t < t^*. \quad (37)$$

By(35) and (37), we obtain

$$\begin{aligned} v(t^*) &< \xi e^{-\eta t^*} \\ &\quad + \int_0^{t^*} \frac{e^{-\eta(t^*-s)}}{\mu_0}[\beta v(s-\tau(s)) + \beta\xi e^{\lambda(\tau-s)} + \epsilon] ds \\ &< e^{-\eta t^*}\{\xi + \frac{\epsilon}{\mu_0\eta - 2\beta} \\ &\quad + \int_0^{t^*} \frac{e^{\eta s}}{\mu_0}[\beta\xi e^{-\lambda(s-\tau(s))} + \beta\xi e^{\lambda(\tau-s)}] ds \\ &\quad + \frac{\eta\epsilon}{\mu_0\eta - 2\beta}\int_0^{t^*} e^{\eta s} ds \\ &\leq e^{-\eta t^*}\{\xi + \frac{\epsilon}{\mu_0\eta - 2\beta} + \frac{2\beta\xi e^{\lambda\tau}}{\mu_0(\eta-\lambda)}(e^{(\eta-\lambda)t^*} - 1) \\ &\quad + \frac{\epsilon}{\mu_0\eta - 2\beta}(e^{\eta t^*} - 1)\} \\ &= e^{-\eta t^*}\{\xi e^{\eta t^*} e^{-\lambda t^*} + \frac{\epsilon}{\mu_0\eta - 2\beta} e^{\eta t^*}\} \\ &= \xi e^{-\lambda t^*} + \frac{\epsilon}{\mu_0\eta - 2\beta}, \end{aligned}$$

which is contradictory to (1), and the proof is completed.

Remark 5: An error system is usually nonstandard when a heterogeneous network is considered to synchronize a non-common equilibrium point. Thus, investigating the full synchronization for heterogeneous network is difficult. The quasi-synchronization is raised, thereby allowing all node states tend to a manifold rather than a fixed point [23]. This work transforms the error system into a standard form when the average trajectory is selected as the target state and gives a full synchronization criterion.

Remark 6: Linear matrix inequality (LMI) is unsuitable to solve (32) and (33) due to the presence of $\int_0^1 Df_i(s + \omega e_j)d\omega$ and $\int_0^1 Df_i(s(t-\tau(t)) + \omega e_j(t-\tau(t)))d\omega$. However, finding solutions to (32) and (33) is possible in some special cases. The details can be found in [26].

Remark 7: The time-dependent Lyapunov function plays an important role when heterogeneous impulses are considered. Reference [24] investigated the exponential synchronization of time-delay homogeneous networks with heterogeneous impulses by virtue of Lyapunov function and the comparison principle. Different from [24], here we consider time-delay heterogeneous networks with heterogeneous impulses and deduce a synchronization criterion such that all nodes synchronize to the average trajectory. Reference [26] used matrix decomposition techniques to synchronize a heterogeneous network to the average trajectory. However, this method is unsuitable for time-delay heterogeneous networks. In this work, we use the Lyapunov function combined with the comparison principle to investigate the exponential synchronization for a class of time-delay heterogeneous networks with heterogeneous impulses.

Remark 8: we need carefully analyze the decay rate of Lyapunov function over time when we consider exponential synchronization of networks. Estimating the Lyapunov function is difficult when time-delays and impulses are simultaneously considered. The reason lies in that time-delays and impulsive intervals are usually mixed up. The comparison theorem supplies us a method of applying ODE theory to investigate synchronization of networks. After obtaining some preliminary estimation to the Lyapunov function, comparison theorem allows us to consider an ODE. Then we can

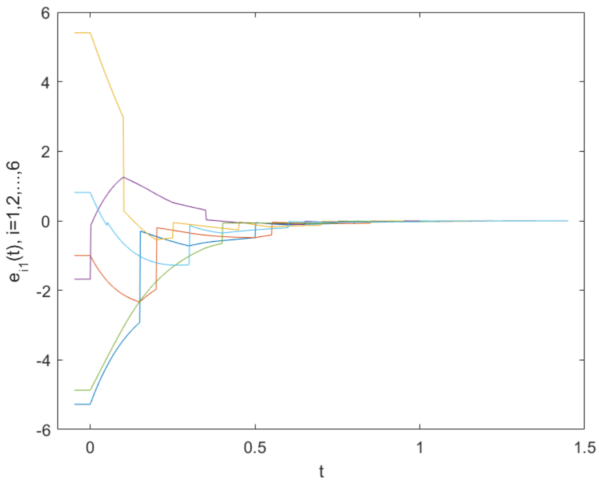

FIGURE 1. Synchronization error $e_{i1}(t), i = 1, 2, \ldots, 6$.

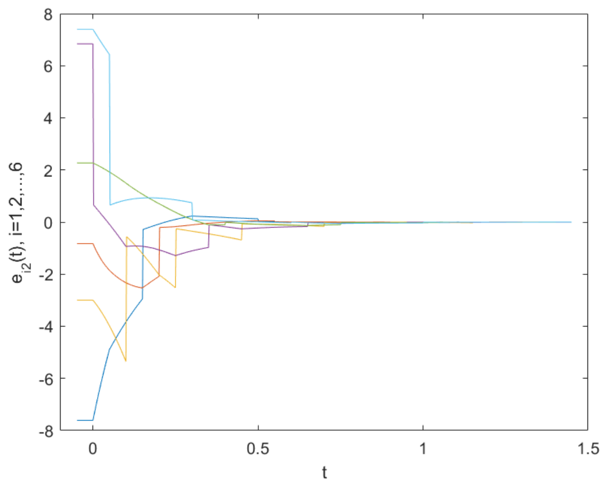

FIGURE 2. Synchronization error $e_{i2}(t), i = 1, 2, \ldots, 6$.

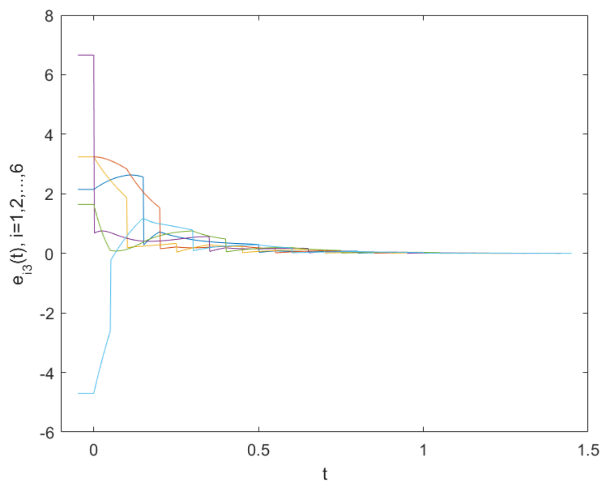

FIGURE 3. Synchronization error $e_{i3}(t), i = 1, 2, \ldots, 6$.

use the method of variation of parameters to investigate the effects of time-delays and impulses, and obtain an accurate estimation of the Lyapunov function. In recent years, Lyapunov function combined with the comparison principle has been widely applied in impulsive systems [9]–[11], [21], [27].

IV. NUMERICAL EXAMPLES

In this section, a numerical example is presented to illustrate the effectiveness of the theoretical results.

The common equilibrium solution $s(t)$ satisfies the following system described by

$$\dot{s}(t) = A_i s(t) + B_i f_1(s(t)) + C_i f_2(s(t - \tau(t)))$$

where $s(t) = (s_1(t), s_2(t), s_3(t))^T$, $A_1 = -0.1I_3$, $B_1 = \begin{bmatrix} 0.09 & 0 & -0.1 \\ 0 & 0.02 & 0 \\ -0.06 & 0 & -0.1 \end{bmatrix}$, $C_1 = \begin{bmatrix} 0.1 & 0 & -0.2 \\ 0 & 0.2 & 0 \\ -0.5 & 0 & -0.1 \end{bmatrix}$, $A_2 = 0.6A1$, $B_2 = 0.8B1$, $C_2 = 0.8C1$, $A_3 = 0.7A1$, $B_3 = 0.9B1$, $C_3 = 0.8C1$, $A_4 = 0.5A1$, $B_4 = 0.7B1$, $C_4 = 0.7C1$, $A_5 = 0.4A1$, $B_5 = 0.4B1$, $C_5 = 0.6C1$, $A_6 = 0.3A1$, $B_6 = 0.5B1$, $C_6 = 0.5C1$, $f_1(s(t)) = (0.5(|s_1 + 1| - |s_1 - 1|), 0, 0)^T$, and $f_2(s(t - \tau(t))) = (0.5(|s_1(t - \tau(t)) + 1| - |s_1(t - \tau(t)) - 1|), 0, 0)^T$. Then, Lipschitz constants can be got $l_1 = l_2 = 1$.

Referring to the topology of the heterogeneous partial couple network in the work of Lu *et al.* [28], the network is described by

$$\dot{x}_i(t) = A_i x_i(t) + B_i f_1(x_i(t)) + C_i f_2(x_i(t - \tau(t)) + c \sum_{j=1, j\neq i}^{N} d_{ij} R H_{ij}(x_j(t) - x_i(t)), i = 1, \cdots, 6,$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))^T$, $c = 1$, $R = 0.02I_3$,

$$D = \begin{bmatrix} -11.3 & 7 & 0 & 0.2 & 0 & 4.1 \\ 4.2 & -11.2 & 7 & 0 & 0 & 0 \\ 0 & 4.1 & -11.1 & 7 & 0 & 0 \\ 0 & 0.1 & 4.1 & -11.2 & 7 & 0 \\ 0 & 0 & 0 & 3.9 & -10.9 & 7 \\ 7 & 0 & 0 & 0 & 4 & -11 \end{bmatrix},$$

The channel matrix of H_{ij} can be seen in the work [28].

Let $N_0 = 2$, $0 < \tau(t) < \tau$, $\tau = 0.25$, according to Corollary 1, by solving the linear matrix inequalities (9), (27), and (28), we can find a feasible solution with $T_a = 0.01$, $\alpha = 11.2$, $\beta = 1.55$, $\mu = 0.9$, $P = 0.1I_{18}$.

Consequently, by Corollary 1, we can obtain exponential synchronization of the heterogeneous systems. Fig.1, Fig.2 and Fig.3 depict the trajectory of the error state $e_i(t) = x_i(t) - s(t)$, $i = 1, 2, 3, 4, 5, 6$.

V. CONCLUSION

In this paper, we investigate the synchronization for a class of partially coupled heterogeneous impulsive networks with time-delays, and obtain some sufficient conditions to realize exponential synchronization. The main results are divided into two parts, where the networks are anticipated to synchronize to the common equilibrium solution and the average trajectory. The method is based on Lyapunov function and the comparison principle. The results serve as a useful supplement to the full synchronization of heterogeneous impulsive networks with time-delays. The heterogeneity of networks makes the form of the error system unstandard if the nodes have no common equilibrium point. Fortunately, we obtain some standard error system by linearization when we select the average trajectory as the synchronization target. The conditions in Theorem 2 are difficult to solve by using LMI. Thus, we will modify or simplify these conditions in future works.

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